MacLane Homology and Topological Hochschild Homology

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1 Introduction

In 1985 Bökstedt [2] introduced the topological Hochschild homology functor THH which is related to Waldhausen's algebraic K-theory of certain A_∞ rings [25] in a way similar to the relationship of Hochschild homology of rings to Quillen K-theory: there is a generalization of the trace map. If R is a genuine ring, the trace map factors

$$\begin{tikzcd}
K_*(R) \arrow[r] & K^s_*(R) \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & HH_*(R) \\
& & THH_*(R) \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

through Bökstedt's map β. $K^s_*(R)$ is Waldhausen's stable K-theory of R [26].

It was a surprise when Pirashivili brought MacLane homology H^{ML}_* into the picture, a homology theory for rings developed 30 years earlier by Mac Lane [15]. He observed [19] that α factors as

$$\begin{tikzcd}
& H^{ML}_*(R) \arrow[dr] & \\
K^s_*(R) \arrow[ur, "\delta"] \arrow[rr, "\alpha"] & & HH_*(R)
\end{tikzcd}$$

and in all calculated examples δ turned out to be an isomorphism. This motivated him to conjecture that MacLane homology is stable K-theory. Waldhausen changed this conjecture to "MacLane homology is topological Hochschild homology" based on the "brave new algebra" argument we will explain in Section 3 below. The two conjectures were related at the time by the long standing conjecture that β is an isomorphism. (This last conjecture has recently been proved for simplicial rings [3], see also [22].) Subsequently Pirashvili and Waldhausen could show [20]

(1.1) Theorem: For rings R there is a natural isomorphism $THH_*(R) \cong H^{ML}_*(R)$

In fact, they proved a more general result allowing coefficients in a functor from the category $\mathcal{P}(R)$ of finitely generated projective R-modules to the category of all R-modules.

The result is established indirectly by showing that $THH_*(R)$ satisfies a set of axioms characterizing the homology of $\mathcal{P}(R)$ with coefficients in the Hom-functor and the fact, due to

Jibladze and Pirashvili [14], that the same is true for $H^{ML}(R)$. This situation is somewhat unsatifactory because it does not really explain the relationship of the two functors.

The purpose of this paper is threefold. Firstly we want to construct an explicit map between MacLane homology and topological Hochschild homology. Secondly we want to analyze the intriguing Q-construction of Eilenberg and Mac Lane used in the definition of MacLane homology, an analysis necessary for the construction of the comparison map but also of separate interest. Thirdly we want to introduce the reader to arguments in "brave new algebra" which in our case really illuminate why MacLane homology and topological Hochschild homology ought to be the same.

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2 Definitions and the Main Result

(2.1) Hochschild homology: For a ring R and an R-bimodule M let $B_*(R,M)$ denote the simplicial abelian group $B_*(R,M) = M \otimes R^{\otimes n}$ with boundary maps

$$\begin{aligned} d^i(m \otimes r_1 \otimes \ldots \otimes r_n) &= mr_1 \otimes r_2 \otimes \ldots \otimes r_n & i=0 \\ &= m \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_n & 0<i<n \\ &= r_n m \otimes r_1 \otimes \ldots \otimes r_{n-1} & i=n \end{aligned}$$

and degeneracies

$$s_i(m \otimes r_1 \otimes \ldots \otimes r_n) = (m \otimes \ldots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \ldots \otimes r_n).$$

Hochschild homology $HH_*(R,M)$ is defined to be the homology of the normalized chain complex $N(B_*(R,M))$ of the simplicial group $B_*(R,M)$ (e.g. see [17, Cor. 22.3]).

To extend this definition to differential graded algebras A we replace R by A, consider M as a trivial complex concentrated in 0, replace $\otimes$ by the tensor product of complexes and arrive at a simplicial chain complex. Regard this as a chain complex of complexes with differential $\partial = \sum_{k=0}^{n_i} (-1)^k \delta^k$.

Hochschild homology of a chain algebra A with coefficients in an A-module M, $HH_*(A,M)$ is the homology of the total complex of this bicomplex (the homology of the condensation in the sense of [16]).

(2.2) MacLane homology: Let C_n denote the set of vertices of the n-dimensional unit cube $I^n = [0,1]^n$. For an abelian group G let $G[C_n]$ denote the set of all maps $g : C_n \to G$. Think of g as the n cube with an element of G at each vertex. Define maps

$$U_i, L_i, S_i : G[C_n] \to G[C_{n-1}]$$

$i = 1, \ldots, n$ as follows: U_i is the restriction to the ith upper face, L_i the restriction to the ith lower face, while S_i adds the ith upper face vertexwise to the ith lower face, i.e.

$$\begin{aligned} U_i(g)(\varepsilon_1, \ldots, \varepsilon_{n-1}) &= g(\varepsilon_1, \ldots, \varepsilon_{i-1}, 1, \varepsilon_i, \ldots, \varepsilon_{n-1}) \\ L_i(g)(\varepsilon_1, \ldots, \varepsilon_{n-1}) &= g(\varepsilon_1, \ldots, \varepsilon_{i-1}, 0, \varepsilon_i, \ldots, \varepsilon_{n-1}) \\ S_i(g)(\varepsilon_1, \ldots, \varepsilon_{n-1}) &= g(\varepsilon_1, \ldots, \varepsilon_{i-1}, 1, \varepsilon_i, \ldots, \varepsilon_{n-1}) + g(\varepsilon_1, \ldots, \varepsilon_{i-1}, 0, \varepsilon_i, \ldots, \varepsilon_{n-1}) \end{aligned}$$

The *cubical construction* of Eilenberg and Mac Lane [7] is the chain complex $(Q'_*(G), \partial')$ defined by

$$Q'_n(G) = \mathbb{Z}[G[C_n]]$$

the free abelian group generated by $G[C_n]$, and

$$\partial' = \sum_{i=1}^{n}(-1)^i(S_i - U_i - L_i) : Q'_n(G) \to Q'_{n-1}(G).$$

An element $g \in G[C_n]$ is called a *slab* if $g = 0 \in G[C_0] = G$ or an $(n-1)$-dimensional face has all vertices $0 \in G$, and an *i-diagonal* if

$$g(\varepsilon_1, \dots, \varepsilon_n) = 0 \qquad \text{for all } (\varepsilon_1, \dots, \varepsilon_n) \text{ with } \varepsilon_i \neq \varepsilon_{i+1}$$

(here $n \geq 2$, $1 \leq i < n$). Let $DQ'_n(G) \subset Q'_n(G)$ denote the subgroup generated by all diagonals and all slabs. Define

$$Q_*(G) = (Q_n(G), \partial) = (Q'_n(G)/DQ'_n(G), \partial)$$

with the induced boundary ∂. Eilenberg and Mac Lane proved that $H_*(Q_*(G))$ is the stable homology of the Eilenberg-MacLane spaces $K(G,n)$, i. e.

(2.3) $$H_k(Q_*(G)) = colim_n H_{k+n}(K(G,n)) \cong H_{k+q}(K(G,q)) \quad q > k.$$

(2.4) There is an augmentation $Q_*(G) \to G$ sending generators of degree > 0 to 0 and mapping generators in dimension 0 identically.

(2.5) Dixmier defined a pairing $Q_*(G) \otimes Q_*(H) \to Q_*(G \otimes H)$ sending $g \in G[C_m]$ and $h \in H[C_n]$ to

$$g \otimes h : C_{m+n} \to G \otimes H, \qquad (\varepsilon_1, \dots, \varepsilon_{m+n}) \mapsto g(\varepsilon_1, \dots, \varepsilon_m) \otimes h(\varepsilon_{m+1}, \dots, \varepsilon_{m+n})$$

If R is a ring the Dixmier pairing turns $Q_*(R)$ into a chain algebra and the augmentation into a map of chain algebras.

(2.6) Definition: If R is a ring and M an R-bimodule, *MacLane homology of R with coefficients in M* is defined to be Hochschild homology of $Q_*(R)$ with coefficients in M:

$$H_*^{ML}(R, M) = H_*(Q_*(R), M).$$

(2.7) Topological Hochschild homology: Let $\mathcal{SS}ets_*$ denote the category of pointed simplicial sets. If R is a ring, the functor

$$\tilde{R} : \mathcal{SS}ets_* \longrightarrow \mathcal{SS}ets_*, \quad X \mapsto R[X]/R[*]$$

is a functor with smash product in the sense of [2]. Let

$$\begin{array}{rcl} \mu_{X,Y} : \tilde{R}(X) \wedge \tilde{R}(Y) & \longrightarrow & \tilde{R}(X \wedge Y) \\ 1_X : X & \longrightarrow & \tilde{R}(X) \end{array}$$

denote its multiplication and unit.

If M is a R-bimodule, the functor

$$M : \mathcal{SS}ets_* \longrightarrow \mathcal{SS}ets_*, \quad X \mapsto M \Box \tilde{\mathbb{Z}}(X)$$

is a bimodule over $\tilde{R}$ in the sense of [20] with structure maps

$$\begin{array}{rrcl} l_{X,Y}: & \tilde{R}(X) \wedge M(Y) & \longrightarrow & M(X \wedge Y) \\ r_{X,Y}: & M(X) \wedge \tilde{R}(Y) & \longrightarrow & M(X \wedge Y) \end{array}$$

defined by

$$\tilde{R}(X) \wedge M(Y) \to \tilde{R}(X) \Box M(Y) \cong R \Box M \Box \tilde{\mathbb{Z}}(X) \Box \tilde{\mathbb{Z}}(Y) \to M \Box \tilde{\mathbb{Z}}(X) \Box \tilde{\mathbb{Z}}(Y) = M(X \wedge Y)$$

for $l_{X,Y}$ and similarly for $r_{X,Y}$, where $\Box$ is the dimensionwise tensor product.

Let $\mathcal{I}$ be the category of natural numbers considered as ordered sets $n = \{1, 2, \ldots, n\}$, $n \geq 0$, and order preserving injections. $\mathcal{I}$ has a monoidal structure $\sqcup$ induced by *concatenation.*

Topological Hochschild homology $THH(R, M)$ of the ring R with coefficients in the R-bimodule M is the simplicial space defined by

$$[k] \mapsto hocolim_{\mathcal{I}^{n+1}}\ F_k(\underline{n}),$$

where $F_k : \mathcal{I}^{k+1} \longrightarrow \mathcal{T}op_*$ sends $\underline{n} = (n_0, \ldots, n_k) \in \mathcal{I}^{k+1}$ to

$$\Omega^{n_0+\ldots+n_k}\ |M(S^{n_0}) \wedge \tilde{R}(S^{n_1}) \wedge \ldots \wedge \tilde{R}(S^{n_k})\ |$$

and each injection to the corresponding suspension. Here S^k is the k-fold smash product of $S^1 = \Delta(1)/\partial\Delta(1)$ where $\Delta(n)$ is the standard n-simplex with boundary $\partial\Delta(n)$. The boundary maps are induced by the natural transformations

$$d^i : F_k \longrightarrow F_{k-1} \circ \partial^i \qquad 0 \leq i \leq k,$$

where

$$\begin{array}{lll} \partial^i(n_0, \ldots, n_k) & = (n_0, \ldots, n_i \sqcup n_{i+1}, \ldots, n_k) & 0 \leq i < k \\ & = (n_k \sqcup n_0, n_1, \ldots, n_{k-1}) & i = k \end{array}$$

and for $f : S^{n_0+\ldots+n_k} \longrightarrow |\ M(S^{n_0}) \wedge \tilde{R}(S^{n_1}) \wedge \ldots \wedge \tilde{R}(S^{n_k})\ |$ in $F_k(\underline{n})$

$$\begin{array}{lll} d^0 f : S^{n_0+\ldots+n_k} \xrightarrow{f} & |\ M(S^{n_0}) \wedge \ldots \wedge \tilde{R}(S^{n_k})\ | \xrightarrow{r \wedge id} |\ M(S^{n_0+n_1}) \wedge \tilde{R}(S^{n_2}) \wedge \ldots \wedge \tilde{R}(S^{n_k})\ | & \\ d^i f : S^{n_0+\ldots+n_k} \xrightarrow{f} & |\ M(S^{n_0}) \wedge \ldots \wedge \tilde{R}(S^{n_k})\ | \xrightarrow{id \wedge \mu \wedge id} & \\ & |\ M(S^{n_0}) \wedge \ldots \wedge \tilde{R}(S^{n_i+n_{i+1}}) \wedge \ldots \wedge \tilde{R}(S^{n_k})\ | & \text{for } 0 < i < n \\ d^n f : S^{n_k+n_0+\ldots+n_{k-1}} & \xrightarrow{\rho} S^{n_0+\ldots+n_k} \xrightarrow{f}\ |\ M(S^{n_0}) \wedge \ldots \wedge \tilde{R}(S^{n_k})\ | \xrightarrow{(l \wedge id)\circ\tau} & \\ & |\ M(S^{n_k+n_0}) \wedge \ldots \wedge \tilde{R}(S^{n_{k-1}})\ |, & \end{array}$$

where ρ interchanges the first n_k copies of S^1 in order with the $n_0 + \ldots + n_{k-1}$ copies, and τ interchanges $\tilde{R}(S^{n_k})$ with $M(S^{n_0}) \wedge \tilde{R}(S^{n_1}) \wedge \ldots \wedge \tilde{R}(S^{n_{k-1}})$. The degeneracies are defined similarly.

In order to be able to compare MacLane homology and topological Hochschild homology we give a simplicial description of MacLane homology. Let $D : \mathcal{C}plx \to \mathcal{SA}b$ be the Dold-Kan functor from the category of chain complexes over $\mathbb{Z}$ which are 0 in negative dimensions to the category of simplicial abelian groups [5].

(2.8) Properties:(1) D is inverse (up to natural isomorphism) to the normalized chain complex functor $N : \mathcal{SAb} \to \mathcal{C}plx$.
(2) The Eilenberg-Zilber map in the generalized form of [6, (2.15), (2.16)] used in Section 5 below defines a natural transformation

$$EZ : N(-) \otimes N(-) \to N(-\Box-).$$

(3) EZ is an associative, commutative and unital (with respect to $\otimes$ and $\Box$) chain equivalence with the Alexander-Whitney map as homotopy inverse.

(2.9) Lemma: The simplicial space

$$H^{ML}(R, M) : [k] \mapsto |D(M \otimes Q_*(R)^{\otimes k})|$$

with the Hochschild structure maps is MacLane homology. More precisely,

$$\pi_* H^{ML}(R, M) \cong H_*^{ML}(R, M).$$

Proof: The realization of $([k] \mapsto |D(M \otimes Q_*(R)^{\otimes k})|)$ is the realization of the bisimplicial abelian group

$$([k], [q]) \mapsto D(M \otimes Q_*(R)^{\otimes k})_q$$

and the homotopy of the realization of a bisimplicial abelian group is the homology of the total normalized complex associated with it [17, Thm. 22.1]. Since D is inverse to the normalized complex functor up to isomorphism, the total normalized complex is isomorphic to the normal complex of $B(Q_*(R), M)$. $\Box$

(2.10) Theorem: There is a sequence of natural equivalences between the simplicial spaces $THH(R, M)$ and $H^{ML}(R, M)$.

(2.11) Remark: A priori, this result is stronger than Theorem 1.1 which only claims the existence of an abstract natural isomorphism of the homotopy groups of the simplicial spaces $THH(R, M)$ and $H^{ML}(R, M)$. By definition $|H^{ML}(R, M)|$ is homotopy equivalent to a product of Eilenberg-MacLane spaces, and so is $|THH(R, M)|$ according to Bökstedt [2]. Hence, a posteriori, (2.10) is equivalent to Theorem 1.1.

3 The "brave new algebra" argument

The purpose of this chapter is to give a heuristic argument in the terminology of brave new algebra why Theorem 2.10 should be true.

The idea of topological Hochschild homology is to take classical Hochschild homology and to substitute the classical notions by the corresponding "brave new" notions. Here is a short dictionary:

(3.1)

classical algebra	brave new algebra
abelian group	spectrum
non-commutative ring	A_∞ ring spectrum
commutative ring	E_∞ ring spectrum
module	module spectrum
ground ring $\mathbb{Z}$	sphere spectrum $\underline{S}$
$\otimes_{\mathbb{Z}}$	smash product $\wedge$

To define $THH_*(R,M)$ we consequently have to substitute the ring R by its Eilenberg-MacLane ring spectrum $\underline{\underline{R}}$ and M by the corresponding module spectrum $\underline{\underline{M}}$.

(3.2) Brave new algebra definition of THH(R,M): THH(R,M) is the realization in the category $\mathcal{S}p$ of spectra of the simplicial spectrum $[n] \to M \wedge R^{\wedge n}$ with the Hochschild structure maps.

(3.3) Remark: When Bökstedt defined THH there was no known category of spectra with a strictly associative smash product admitting strictly associative ring spectra. Bökstedt's construction is a smash product construction for spectra along with a minimal book keeping device for the coherence questions. Recently Elmendorf, Kriz and May discovered an associative, commutative and unital pairing which for cofibrant spectra is equivalent to the smash product [12]. So Definition (3.2) makes sense with this pairing as smash product. The equivalence of (3.2) with Bökstedt's construction is subject of [23].

MacLane homology as simplicial spectrum: There is an Eilenberg-MacLane spectrum functor

$$\mathcal{C}plx \longrightarrow \mathcal{S}p, \quad C \mapsto \underline{\underline{C}}$$

(3.4) Properties:

- (i) if R is a ring considered as a complex concentrated in dimension 0, then $\underline{\underline{R}}$ is a ring spectrum [18]
- (ii) if M is an R-module, $\underline{\underline{M}}$ is an $\underline{\underline{R}}$-module spectrum [21], [11]
- (iii) $C \mapsto \underline{\underline{C}}$ preserves homotopy colimits
- (iv) $\pi_*(\underline{\underline{C}}) \cong \tilde{H}_*(C)$
- (v) there is an equivalence $\underline{\underline{C \overset{L}{\otimes} C'}} \simeq \underline{\underline{C}} \wedge_{\underline{\underline{\mathbb{Z}}}} \underline{\underline{C'}}$ due to Robinson [21], natural up to homotopy where $\overset{L}{\otimes}$ is the derived tensor product and $\wedge_{\underline{\underline{\mathbb{Z}}}}$ is defined as a bar construction.

(Robinson works in Boardman's category of spectra [1]; his A_∞ ring spectra do not have coherent units. In view of Remark 3.3 we work in Elmendorf's category [10]. In this category Robinson's $\wedge_{\underline{\underline{\mathbb{Z}}}}$ corresponds to the pairing in [11] in the cofibrant case.)

The normalized chain complex of $B(Q_*(R), M)$ is the realization in $\mathcal{C}plx$ of the simplicial object

$$[n] \mapsto M \otimes Q_*(R)^{\otimes n}$$

with the Hochschild structure maps. Hence (3.4) implies

(3.5) Propositon: $H_*^{ML}(R,M)$ is the homotopy of the simplicial spectrum $H^{ML}(R,M)$

$$[n] \mapsto \underline{\underline{M \otimes Q_*(R)^{\otimes n}}}$$

with the Hochschild structure maps. □

(3.6) Brave new algebra version of Theorem 2.10: The simplicial spectra $THH(R,M)$ and $H^{ML}(R,M)$ are equivalent via a sequence of natural maps.

Warning: This is not a theorem. We only give a heuristic argument why this should be true based on a stronger version of (3.4(v)) and the Conjecture 3.7 below.

By (3.4(iv)), the definition of spectrum homology and the smash product of spectra

$$\pi_k(\underline{\underline{Q_*(R)}}) \cong H_k(Q_*(R)) = colim_n H_{k+n}(K(R,n)) = \pi_k(\underline{\underline{\mathbb{Z}}} \wedge \underline{\underline{R}}).$$

Since $\underline{\underline{Q_*(R)}}$ and $\underline{\underline{\mathbb{Z}}} \wedge \underline{\underline{R}}$ are products of Eilenberg-MacLane spectra, we have an equivalence

$$\underline{\underline{Q_*(R)}} \simeq \underline{\underline{\mathbb{Z}}} \wedge \underline{\underline{R}}$$

Both are A_∞ ring spectra whose multiplications are induced by the ring structures of R and $\mathbb{Z}$. Hence it would be surprising if

(3.7) Conjecture: $\underline{\underline{Q_*(R)}} \simeq \underline{\underline{\mathbb{Z}}} \wedge \underline{\underline{R}}$ is an equivalence of A_∞ ring spectra

were false. Since $Q_*(R)$ is a complex of free abelian groups the equivalence (3.4(v)) defines a map from the simplicial spectrum (3.5) to the simplicial spectrum

$$[n] \mapsto \underline{\underline{M}} \wedge_{\underline{\underline{\mathbb{Z}}}} \underline{\underline{Q_*(R)}} \wedge_{\underline{\underline{\mathbb{Z}}}} \underline{\underline{Q_*(R)}} \wedge_{\underline{\underline{\mathbb{Z}}}} \ldots \wedge_{\underline{\underline{\mathbb{Z}}}} \underline{\underline{Q_*(R)}}.$$

This is not a genuine simplicial spectrum unless we use the pairing of [12] mentioned above. Since $\underline{\underline{Q_*(R)}}$ and $\underline{\underline{\mathbb{Z}}}$ are A_∞ ring spectra coherence homotopies enter which are already incorporated in the pairing of [12].

(3.7) defines an equivalence to the simplicial spectrum (the remark just made applies again)

(3.8)

$$[n] \mapsto \underline{\underline{M}} \wedge_{\underline{\underline{\mathbb{Z}}}} (\underline{\underline{\mathbb{Z}}} \wedge \underline{\underline{R}}) \wedge_{\underline{\underline{\mathbb{Z}}}} \ldots \wedge_{\underline{\underline{\mathbb{Z}}}} (\underline{\underline{\mathbb{Z}}} \wedge \underline{\underline{R}}).$$

By the standard change of rings isomorphism, but for ring spectra as opposed to rings (details in [21] or [11]) (3.7) is equivalent to the simplicial spectrum

$$[n] \mapsto \underline{\underline{M}} \wedge \underline{\underline{R}}^{\wedge n},$$

i.e. to the brave new version of $THH(R,M)$. □

(3.9) Remark: (3.4(v)) is a statement about modules and does not say anything about the multiplicative structure if the inputs are chain algebras. This is the crucial question in Conjecture 3.7. Note also that the right side of (3.7) is symmetric in R and $\mathbb{Z}$ while the left side not. Recent calculations of the second author show that the A_∞ ring structures on $\underline{\underline{Q_*(R)}}$ and $\underline{\underline{R \overset{L}{\otimes} Q_*(\mathbb{Z})}}$ differ while Conjecture 3.7 supports their equivalence.

4 The Q-construction of Eilenberg and Mac Lane

(4.1)Segal's r-fold delooping (cf. [24]) of an abelian group G is the realization of the r-fold multisimplicial abelian group

$$[n_1, \dots, n_r] \longmapsto G_{n_1,\dots,n_r} = Map_*((\underline{n}_1)_+ \wedge \dots \wedge (\underline{n}_r)_+, G),$$

where Map_* is the set of based maps, G has base point 0, $\underline{n} = \{1, 2, ..., n\}$, and $\underline{0} = \phi$. The face maps of the i-th simplicial direction are the maps

$$\delta_i^k : G_{n_1,\dots,n_r} \longrightarrow G_{n_1,\dots,n_i-1,\dots,n_r} \qquad 0 \le k \le n_i, \quad n_i > 0$$

defined by

$$(\delta_i^k g)(j_1, \dots, j_r) = \begin{cases} g(j_1, \dots, j_r) & j_i < k \\ g(j_1, \dots, j_r) + g(j_1, \dots, j_i + 1, \dots, j_r) & j_i = k \\ g(j_1, \dots, j_i + 1, \dots, j_r) & j_i > k \end{cases}$$

The k-th degeneracies are obtained by composition with the maps

$$s^k : (\underline{n+1})_+ \to (\underline{n})_+,$$

sending $k+1 \in (\underline{n+1})_+$ to $+$ and the rest order preserving bijectively onto $(\underline{n})_+$.

Let $T_*^r(G)$ be the normalized total chain complex of the multisimplicial group $\tilde{\mathbb{Z}}[G_{*,\dots,*}]$, i.e.

$$\begin{aligned} T_p^r(G) &= \tilde{\mathbb{Z}}[\bigoplus_{n_1+\dots+n_r=p} G_{n_1,\dots,n_r}] \;/\; \text{degeneracies} \quad p > 0 \\ T_0^r(G) &= 0 \end{aligned}$$

with boundary operator $\partial[g] = \sum_{i=1}^{r} (-1)^{n_1+\dots+n_{i-1}} \partial^i[g]$ where $\partial^i[g] = \sum_{k=0}^{n_i} (-1)^k [\delta_i^k(g)]$ for generators $[g], g \in G_{n_1,\dots,n_r}$.

(4.2) Observe that $\tilde{R}[S^r]$ of Section 2 is the diagonal of the r-fold simplicial ring $R_{*,\dots,*}$.

(4.3) Let $\sigma Q_*(G)$ denote the *shift suspension*, i.e. $(\sigma Q_*(G))_n = Q_{n-1}(G)$ and $\partial_n^{\sigma Q} = \partial_{n-1}^{Q}$. It corresponds to the suspension from the right in the category of chain complexes. Here we use the standard sign convention for the differential in the tensor product $X \otimes Y$ of two chain complexes X and Y, namely $\partial = \partial_X \otimes id + (-1)^p id \otimes \partial_Y : X_p \otimes Y_q \to (X \otimes Y)_{p+q-1}$.

Eilenberg and Mac Lane constructed a chain map

$$f_* : T_*^1(G) \to \sigma Q_*(G),$$

which can be described as follows: define a map $\iota_n : (C_{n-1})_+ \to \underline{n}_+$ by sending the vertices $(0, \dots, 0, \overset{k}{1}, \dots, 1)$ to $k \in \underline{n}$ for $k = 1, \dots, n$ and all other vertices to the base point $+$. Then f_n is defined on generators by

$$G_n \to G[C_{n-1}], \qquad g \mapsto g \circ \iota_n.$$

(4.4) One verifies:

(1) if g is degenerate, $g = \hat{g} \circ s^k$, $0 \leq k \leq n-1$, then
$f_n(g)$ is a slab for $k = 0,\ n-1$
$f_n(g)$ is a k-diagonal for $0 < k < n-1$.

(2) $U_i \circ f_{n+1}$ is degenerate for $i < n$
$L_i \circ f_{n+1}$ is degenerate for $1 < i$

(3) $S_i \circ f_{n+1} = f_n \circ \delta_1^i \qquad 0 < i < n+1$
$L_1 \circ f_{n+1} = f_n \circ \delta_1^0$
$U_n \circ f_{n+1} = f_n \circ \delta_1^{n+1}$

(4.4(i)) explains the introduction of diagonals as degenerate elements in the Q-construction.

¿From (4.4) one deduces

(4.5) $\partial_n^Q \circ f_{n+1} = f_n \circ \partial_{n+1}^{T^1}$

Following Eilenberg and Mac Lane we call an index i *critical* for $g \in G[C_n]$ if

$$g(\varepsilon_1, \ldots, \varepsilon_{i-1}, 1, 0, \varepsilon_{i+2}, \ldots, \varepsilon_n) \neq 0$$

for some choice of ε's.

Filter $Q_*(G)$ by subcomplexes $Q_*^r(G)$ generated by all g having at most r critical indices. Since $Q_p^r(G) = Q_p(G)$ for $p \leq r+1$ the inclusion $Q_*^r(G) \subset Q_*(G)$ is $(r+1)$-connected [7, (10)]. f_* can easily be shown to factor through a chain isomorphism

$$f_* : T_*^1(G) \longrightarrow \sigma Q_*^0(G).$$

so that $f_* : T_*^1(G) \longrightarrow \sigma Q_*(G)$ is 2-connected.

We extend f_* to an $2r$-connected chain map

$$f_*^r : T_*^r(G) \longrightarrow \sigma^r Q_*(G).$$

as follows: let

$$f_p^r = \sum_{n_1 + \ldots + n_r = p} (-1)^\rho \cdot f_{n_1, \ldots, n_r},$$

where $f_{n_1}, \ldots, {}_{n_r}$ is induced by the map $G_{n_1, \ldots, n_r} \rightarrow G[C_{p-r}]$ sending the generator $g : (\underline{n}_1)_+ \wedge \ldots \wedge (\underline{n}_r)_+ \rightarrow G$ to

$$g \circ (\iota_{n_1} \wedge \ldots \wedge \iota_{n_r}) : (C_{p-r})_+ = (C_{n_1-1})_+ \wedge \ldots \wedge (C_{n_r-1})_+ \rightarrow (\underline{n}_1)_+ \wedge \ldots \wedge (\underline{n}_r)_+ \rightarrow G.$$

and $\rho = \rho(n_1, \ldots, n_r) = \sum\limits_{i=1}^{r} (i-1)(n-1)$. Using (4.4) one verifies

(4.6) $\partial_{n-r}^Q \circ f_n^r = f_{n-1}^r \circ \partial_n^{T^r}$

The image of f_*^r is $Q_*^{r-1}(G)$: by definition of the ι_n the image is contained in $Q_*^{r-1}(G)$. Conversely, if $g \in G[C_{p-r}]$ has exactly $r-1$ critical indices $n_1 < \ldots < n_r$ it factors through $\iota_{n_1+1} \wedge \ldots \wedge \iota_{n_r+1}$. If it has less than $r-1$ critical indices it has more than one counterimage in $T_*^r(G)$.

(4.7) Proposition: $f_*^r : T_*^r(G) \rightarrow \sigma^r Q_*(G)$ is $(2r)$-connected.

Proof: $H_k(T^r_*(G)) = H_k(K(G,r)) \cong H_k(\sigma^r Q^{r-1}_*(G))$ by [7, Thm. 4]. Hence there exists a chain equivalence $g_* : T^r_*(G) \simeq \sigma^r Q^{r-1}_*(G)$ (e.g. see [4, II, 4.8]). Let $i_* : \sigma^r Q^{r-1}_*(G) \to \sigma^r Q_*(G)$ be the inclusion. Since $T^r_*(G)$ is generically augmented in the sense of [8, Sect. 5] and $Q_*(G)$ is generically acyclic [8, Thm. 12.1] f^r_* is homotopic to $i_* \circ g_*$ [8, Thm. 6.1]. Since i_* is $(2r)$-connected the result follows. $\square$

For $i = 1, \ldots, r+1$ we have inclusions

$$\sigma_i : \sigma T^r_*(G) \longrightarrow T^{r+1}_*(G)$$

sending the generator $g \in G_{n_1,\ldots,n_r}$ (hence $[g] \in (\sigma T^r(G))_{n_1+\ldots+n_r+1}$) to $(-1)^\rho[g]$ where g now is considered as element of $G_{n_1,\ldots,1,\ldots,n_r}$ with 1 in the i-th position and $\rho = n_i + \ldots + n_r$. We observe for later use

(4.8)

$$\begin{array}{ccc} \sigma T^r_*(G) & \xrightarrow{\quad \sigma_i \quad} & T^{r+1}_*(G) \\ \Big\downarrow{\scriptstyle \sigma f^r_*} & & \Big\downarrow{\scriptstyle f^{r+1}_*} \\ \sigma^{r+1}Q_*(G) & \xrightarrow{(-1)^{r+1-i}} & \sigma^{r+1}Q_*(G) \end{array}$$

commutes (the sign results from a permutation of suspensions).

(4.9) There is a pairing

$$p_* : T^r_*(G) \otimes T^s_*(H) \longrightarrow T^{r+s}_*(G \otimes H)$$

defined on generators by

$$p_* : G_{n_1,\ldots,n_r} \times H_{m_1,\ldots,m_s} \longrightarrow (G \otimes H)_{n_1,\ldots,n_r,m_1,\ldots,m_s}$$

$$p_*(g,h)(j_1,\ldots,j_{r+s}) = g(j_1,\ldots,j_r) \otimes h(j_{r+1},\ldots,j_{r+s}).$$

The Dixmier pairing $\overline{q} : Q_*(G) \otimes Q_*(H) \to Q_*(G \otimes H)$ induces a pairing

$$q_* : \sigma^r Q_*(G) \otimes \sigma^s Q_*(H) \longrightarrow \sigma^{r+s} Q_*(G \otimes H),$$

where

$$q_{k,l} : (\sigma^r Q_*(G))_k \otimes (\sigma^s Q_*(H))_l \longrightarrow (\sigma^{r+s} Q_*(G \otimes H))_{k+l}$$

is defined on generators by $q_{k,l}(g \otimes h) = (-1)^{r\cdot(l-s)}\overline{q}_*(g \otimes h)$, $g \in Q_{k-r}(G)$, $h \in Q_{l-s}(H)$. The diagram

$$\begin{array}{ccc} T^r_*(G) \otimes T^s_*(H) & \longrightarrow & T^{r+s}_*(G \otimes H) \\ \Big\downarrow{\scriptstyle f^r_* \otimes f^s_*} & & \Big\downarrow{\scriptstyle f^{r+s}_*} \\ \sigma^r Q_*(G) \otimes \sigma^s Q_*(H) & \longrightarrow & \sigma^{r+s} Q_*(G \otimes H) \end{array}$$

commutes.

(4.10) Consider $\tilde{\mathbb{Z}}(G)$ as a complex concentrated in dimension 0. We have an augmentation

$$T^r_*(G) \to \sigma^r \tilde{\mathbb{Z}}(G)$$

defined by the identity in dimension r. By definition f^r_* respects the augmentations.

5 Proof of Theorem 2.10

Let $\tilde{\mathbb{Z}}F_k : \mathcal{I}^{k+1} \longrightarrow \mathcal{T}op_*$ be the functor sending $\underline{n} = (n_0, \ldots, n_k)$ to

$$\Omega^{n_0+\ldots+n_k}|M(S^{n_0})\square\tilde{\mathbb{Z}}(\tilde{R}(S^{n_1}))\square\ldots\square\tilde{Z}(\tilde{R}(S^{n_k})))|$$

Since $\tilde{\mathbb{Z}}(\tilde{R}(S^p))\square\tilde{\mathbb{Z}}(\tilde{R}(S^q)) \cong \tilde{\mathbb{Z}}(\tilde{R}(S^p) \wedge \tilde{R}(S^q))$ in a natural way, these functors give rise to a simplicial space

$$\tilde{Z}F : [k] \mapsto hocolim\tilde{\mathbb{Z}}F_k$$

with the structure maps induced by the ones of the definition of THH in (2.7).

The Hurewicz map (inclusion of generators) defines maps of $\mathcal{I}^{k+1}$-diagrams

$$\alpha_k : F_k \longrightarrow \tilde{\mathbb{Z}}F_k.$$

(5.1) Change of rings lemma: α_k induces an equivalence $hocolimF_k \to hocolim(\tilde{\mathbb{Z}}F)_k$.

Proof: It suffices to show that the connectivity of

$$\Omega^{n_0+\ldots+n_k}|M(S^{n_0})\wedge\tilde{R}(S^{n_1})\wedge\ldots\wedge\tilde{R}(S^{n_k})| \to \Omega^{n_0+\ldots+n_k}|M(S^{n_0})\square\tilde{\mathbb{Z}}(\tilde{R}(S^{n_1}))\square\ldots\square\tilde{\mathbb{Z}}(\tilde{R}(S^{n_k}))|$$

tends to ∞ with $\min\{n_0, n_1, \ldots, n_k\}$. Now

$$\begin{aligned}
&\pi_i|M(S^{n_0})\square\tilde{\mathbb{Z}}(\tilde{R}(S^{n_1}))\square\ldots\square\tilde{\mathbb{Z}}(\tilde{R}(S^{n_k}))| \\
&\quad = \pi_i|M\square\tilde{\mathbb{Z}}(S^{n_0} \wedge \tilde{R}(S^{n_1}) \wedge \ldots \wedge \tilde{R}(S^{n_k}))| \\
&\quad = \tilde{H}_i(S^{n_0} \wedge K(R, n_1) \wedge \ldots \wedge K(R, n_k), M).
\end{aligned}$$

Following the argument of [20, Lemma 3.3] we have isomorphisms

$$\begin{array}{cl}
\pi_i|M(S^{n_0}) \wedge \tilde{R}(S^{n_1}) \wedge \ldots \wedge \tilde{R}(S^{n_k})| & = \pi_i(K(M, n_0) \wedge K(R, n_1) \wedge \ldots \wedge K(R, n_k)) \\
(1) & \cong \pi_{i+N}(S^N \wedge K(M, n_0) \wedge K(R, n_1) \wedge \ldots \wedge K(R, n_k)) \\
(2) & \cong \pi_{i+N}(K(M, n_0 + N) \wedge K(R, n_1) \wedge \ldots \wedge K(R, n_k)) \\
& \cong \lim\limits_N \pi_{i+N}(K(M, n_0 + N) \wedge K(R, n_1) \wedge \ldots \wedge K(R, n_k)) \\
& \cong \lim\limits_N \pi_{i-n_0+N}(K(M, N) \wedge K(R, n_1) \wedge \ldots \wedge K(R, n_k)) \\
& \cong \tilde{H}_{i-n_0}(K(R, n_1) \wedge \ldots \wedge K(R, n_k), M) \\
& \cong \tilde{H}_i(S^{n_0} \wedge K(R, n_1) \wedge \ldots \wedge K(R, n_k), M).
\end{array}$$

(1) is Freudenthal's theorem with isomorphisms for $i \leq 2(n_0 + \ldots + n_k) - 2$, while (2) uses the isomorphism $\pi_{j+N}(S^N \wedge K(M, n_0)) \cong \pi_{j+N}(K(M, n_0 + N))$ for $j \leq 2_{n_0} - 1$; hence (2) is an isomorphism for $i \leq 2n_0 + n_1 + \ldots + n_k - 2$. $\square$

For a pointed simplicial set X_* the loop space of its realization $\Omega|X_*|$ has the homotopy type of a CW-complex. Hence $\Omega|X_*|$ is equivalent to its CW-approximation which in turn is homeomorphic to the realization of

$$[q] \mapsto \mathcal{SS}ets_*(S^1 \wedge \Delta(q)_+, Sing|X_*|)$$

where $Sing$ is the singular functor. Finally, if X_* is a Kan complex this space is equivalent to the realization of

$$[q] \mapsto \mathcal{SS}ets_*(S^1 \wedge \Delta(q)_+, X_*)$$

.

If we replace the functor $\tilde{\mathbb{Z}}F_k : \mathcal{I}^{n+1} \longrightarrow \mathcal{T}op_* $ by $S\tilde{\mathbb{Z}}F_k : \mathcal{I}^{n+1} \longrightarrow \mathcal{T}op_*$ sending $\underline{n} = (n_0, \ldots, n_k) \in \mathcal{I}^{k+1}$ to the realization of

$$[q] \mapsto \mathcal{SS}ets_*(S^{n_0+\ldots+n_k} \wedge \Delta(q)_+ \, , M(S^{n_0})\Box\tilde{\mathbb{Z}}(\tilde{R}(S^{n_1}))\Box \ldots \Box\tilde{\mathbb{Z}}(\tilde{R}(S^{n_k}))) \tag{5.2}$$

we obtain an $\mathcal{I}^{n+1}$-diagram equivalent to $\tilde{\mathbb{Z}}F_k$ because simplicial groups are Kan complexes.

Let $\triangle$ denote the category of ordered sets $[n] = \{0, 1, \ldots, n\}$ and order preserving maps. The correspondence $[n] \mapsto \mathcal{I}^{n+1}$ with $\mathcal{I}^{n+1}$ of (2.7) defines a functor $\triangle^{op} \to \mathcal{C}at$ into the category of small categories. Let $\triangle^{op} \int \mathcal{I}$ denote the resulting Grothendieck construction. It is the category with objects $([k], (n_0, \ldots, n_k))$ and morphisms $(\alpha, f) : ([k], (n_0, \ldots, n_k)) \to ([l], (m_0, \ldots, m_l))$ with $\alpha : [k] \to [l]$ in $\triangle^{op}$ and $f : \alpha(n_0, \ldots, n_k) \to (m_0, \ldots, m_l)$ in $\mathcal{I}^{l+1}$. The $\tilde{\mathbb{Z}}F_k$ and $S\tilde{\mathbb{Z}}F_k$ together with the maps arising from the boundaries and degeneracies of the definition of THH in (2.7) combine to diagrams

$$\tilde{\mathbb{Z}}F, S\tilde{\mathbb{Z}}F : \triangle^{op}\textstyle\int\mathcal{I} \longrightarrow \mathcal{T}op_*.$$

Since $[k] \mapsto hocolim\tilde{\mathbb{Z}}F_k$ and $[k] \mapsto hocolimS\tilde{\mathbb{Z}}F_k$ are proper simplicial spaces their topological realizations are equivalent to their homotopy colimits as $\triangle^{op}$-diagram. From [13, 6.2] and our deduction so far we obtain $|THH(R, M)| \simeq hocolimS\tilde{\mathbb{Z}}F$, the homotopy colimit being taken over all of $\triangle^{op}\int\mathcal{I}$.

The following sequence of maps defines maps of $\triangle^{op}\int\mathcal{I}$-diagrams which connect $S\tilde{\mathbb{Z}}F$ with the simplicial space of Lemma 2.9 describing MacLane homology.

$$\begin{tikzcd}
\mathcal{SS}ets_* \left(S^{n_0+\ldots+n_k} \wedge \Delta(q)_+ \, , M(S^{n_0})\Box\tilde{\mathbb{Z}}(\tilde{R}(S^{n_1}))\Box \ldots \Box\tilde{\mathbb{Z}}(\tilde{R}(S^{n_k}))\right) \arrow[d, "A"', "\cong"] \\
\mathcal{SA}b \left(\tilde{\mathbb{Z}}(S^{n_0+\ldots+n_k})\Box\tilde{\mathbb{Z}}(\Delta(q)_+), M(S^{n_0})\Box\tilde{\mathbb{Z}}(\tilde{R}(S^{n_1}))\Box \ldots \Box\tilde{\mathbb{Z}}(\tilde{R}(S^{n_k}))\right) \arrow[d, "N"', "\cong"] \\
\mathcal{C}plx \left(N\left(\tilde{\mathbb{Z}}(S^{n_0+\ldots+n_k})\Box\tilde{\mathbb{Z}}(\Delta(q)_+)\right), N\left(M(S^{n_0})\Box\tilde{\mathbb{Z}}(\tilde{R}(S^{n_1}))\Box \ldots \Box\tilde{\mathbb{Z}}(\tilde{R}(S^{n_k}))\right)\right) \arrow[d, "(EZ)^*"'] \\
\mathcal{C}plx \left((N\tilde{\mathbb{Z}}(S^1))^{\otimes(n_0+\ldots+n_k)} \otimes N\tilde{\mathbb{Z}}(\Delta(q)_+), N\left(M(S^{n_0})\Box\tilde{\mathbb{Z}}(\tilde{R}(S^{n_1}))\Box \ldots \Box\tilde{\mathbb{Z}}(\tilde{R}(S^{n_k}))\right)\right) \\
\mathcal{C}plx \left((N\tilde{\mathbb{Z}}(S^1))^{\otimes(n_0+\ldots+n_k)} \otimes N\tilde{\mathbb{Z}}(\Delta(q)_+), M \otimes (N\tilde{\mathbb{Z}}(S^1))^{\otimes n_0} \otimes N\tilde{\mathbb{Z}}(\tilde{R}(S^{n_1})) \otimes N\tilde{\mathbb{Z}}(\tilde{R}(S^{n_k}))\right) \arrow[u, "(EZ)_*"] \\
\mathcal{C}plx \left((N\tilde{\mathbb{Z}}(S^1))^{\otimes(n_0+\ldots+n_k)} \otimes N\tilde{\mathbb{Z}}(\Delta(q)_+), M \otimes (N\tilde{\mathbb{Z}}(S^1))^{\otimes n_0} \otimes T_*^{n_1}(R) \otimes \ldots \otimes T_*^{n_k}(R)\right) \arrow[u, "\hat{EZ}_*"] \arrow[d, "f"'] \\
\mathcal{C}plx \left((N\tilde{\mathbb{Z}}(S^1))^{\otimes(n_0+\ldots+n_k)} \otimes N\tilde{\mathbb{Z}}(\Delta(q)_+), M \otimes (N\tilde{\mathbb{Z}}(S^1))^{\otimes n_0} \otimes \sigma^{n_1}Q_*(R) \otimes \ldots \otimes \sigma^{n_k}Q_*(R)\right) \\
\mathcal{C}plx \left(N\tilde{\mathbb{Z}}(\Delta(q)_+), M \otimes Q_*(R)^{\otimes k}\right) = D(M \otimes Q_*(R)^{\otimes k})_q \arrow[u, "\Sigma", "\cong"']
\end{tikzcd}$$

The map A is defined by the universal property of free abelian groups, N is application of the normalized chain complex functor, the structure maps of the second and third diagram are induced from the first.

The stabilization maps in the fourth diagram are given by sending $\phi : (N\tilde{\mathbb{Z}}(S^1))^{\otimes k} \to N(X_n)$ to

$$(N\tilde{\mathbb{Z}}(S^1))^{\otimes k} \otimes N\tilde{\mathbb{Z}}(S^1) \xrightarrow{\phi\otimes id} N(X_n) \otimes N\tilde{\mathbb{Z}}(S^1) \xrightarrow{EZ} N(X_n \Box \tilde{\mathbb{Z}}(S^1)) \xrightarrow{N(\sigma)} N(X_{n+1})$$

where σ is the appropriate stabilization on the right side (of course, $N\tilde{\mathbb{Z}}(S^1)$ has to be at the correct spot of the tensor product $N\tilde{\mathbb{Z}}(S^1)^{k+1}$ according to the particular injection $n_i \to n_i+1$ considered, and σ has to be the stabilization corresponding to this injection . Recall here that we use the n-fold smash product of S^1 as model for S^n in $\mathcal{SS}ets_*$.) The Hochschild boundaries and degeneracies are defined accordingly. Since EZ is associative, commutative and unital, the fourth row defines a $\triangle^{op}\int\mathcal{I}$-diagram. The map $(EZ)^*$ is precomposition with the Eilenberg-Zilber map. Since each morphism of the Hom set of the fourth row is of the form $N(\phi)$ the naturality of EZ and its property to be associative, commutative and unital imply that $(EZ)^*$ defines a map of $\triangle^{op}\int\mathcal{I}$- diagrams.

The structure maps of the fifth diagram are defined in the same way as in the fourth, e.g. suspension is defined using

$$N(X_n) \otimes N\tilde{\mathbb{Z}}(S^1)) \xrightarrow{EZ} N(X_n \Box \tilde{\mathbb{Z}}(S^1)) \xrightarrow{N(\sigma)} N(X_{n+1}).$$

$(EZ)_*$ is composition with the Eilenberg-Zilber map, and for the same reasons as before it is a map of $\triangle^{op}\int\mathcal{I}$-diagrams.

The structure maps of the sixth and seventh diagram are those of Section 4 with the action of $T^n_*(R)$ respectively $\sigma^n Q_*(R))$ on M given via the augmentations. $\hat{EZ}_*$ is induced by the Eilenberg-Zilber maps $EZ : T^n_*(R) \to N\tilde{\mathbb{Z}}(\tilde{R}(S^n))$ and f by the chain maps f^n_* of Section 4. The isomorphism Σ is shift suspension.

Since the Eilenberg-Zilber map is a homotopy equivalence of chain complexes and f^n_* is highly connected, all maps induce equivalences after passage to the homotopy colimits.

The eighth $\triangle^{op}\int\mathcal{I}$-diagram, more explicitly given by

$$(n_0, \ldots, n_k) \mapsto |[q] \mapsto D(M \otimes Q_*(R)^{\otimes k})_q|$$

is constant in the $\mathcal{I}^{k+1}$-directions. Hence the colimits along the $\mathcal{I}^{k+1}$ are $|D(M \otimes Q_*(R)^{\otimes k})| \times B(\mathcal{I}^{k+1})$ where B is the classifying space functor. Since $\mathcal{I}$ is contractible [2] the projection onto $|D(M \otimes Q_*(R)^{\otimes k})|$ is an equivalence. Since

$$[k] \mapsto |D(M \otimes Q_*(R)^{\otimes k})|$$

is the simplicial space describing MacLane homology (Lemma 2.9) this completes the proof of the theorem.

References

[1] J.M. Boardman, Stable homotopy theory; mimeographed notes; University of Warwick (1965).

[2] M. Bökstedt, Topological Hochschild homology, Preprint Universität Bielefeld

[3] B. I. Dundas and R. McCarthy, Stable K-theory and topological Hochschild homology (Preprint 1992).

[4] A. Dold, Lectures on Algebraic Topology; Springer-Verlag (1972).

[5] A. Dold, Homology of symmetric products and other functors of complexes; *Ann. of Math.* **68** (1958); 54 - 80.

[6] A. Dold and D. Puppe, Homologie nicht-additiver Funktoren. Anwendungen, *Ann. Inst. Fourier* **11** (1961), 201 - 312.

[7] S. Eilenberg and S. Mac Lane, Cohomology theory of abelian groups and homotopy theory I; *Proc. Nat. Acad. Sci.* USA **36** (1950), 443 - 447.

[8] S. Eilenberg and S. Mac Lane, Homology theories for multiplicative systems; *Trans. Amer. Math. Soc.* **71** (1951), 294 - 330.

[9] S. Eilenberg and S. Mac Lane, On the groups $H(\Pi, n)$ I, *Ann. of Math.* **68** (1953); 55 - 106.

[10] A. D. Elmendorf, The Grassmanian geometry of spectra; *J. Pure Appl. Algebra* **54** (1988), 37 - 94.

[11] A. D. Elmendorf, I. Kriz and J. P. May, E_∞ module spectra over E_∞ ring spectra, to appear.

[12] A. D. Elmendorf, I. Kriz and J. P. May, Associative smash products for spectra, Preprint August 1993.

[13] J. Hollender and R. M. Vogt, Modules of topological spaces, applications to homotopy limits and E_∞ structures; *Arch. Math.* **59** (1992), 115 - 129.

[14] M. Jibladze and T. Pirashvili, Cohomology of algebraic theories; *J. of Algebra* **137** (1991), 253 - 296.

[15] S. Mac Lane, Homologie des anneaux et des modules; *Coll. topologie algébrique*, Louvain (1956), 55 - 80.

[16] S. Mac Lane, Homology, Springer Verlag 1963.

[17] J.P. May, Simplicial objects in algebraic topology, Van Nostrand Math. Studies 11 (1967).

[18] J.P. May, Multiplicative infinite loop space theory, *J. Pure Appl. Algebra* **26** (1982), 1 - 69.

[19] T. Pirashvili, New homology and cohomology of rings; *Bull. Acad. Sci.* Georgia **133** (1989), 477 - 480.

[20] T. Pirashvili and F. Waldhausen, MacLane homology and topological Hochschild homology; *J. Pure Appl. Algebra* **82** (1992), 81 - 98.

[21] A. Robinson, Derived tensor products in stable homotopy theory, *Topology* **22** (1983), 1 - 18.

[22] R. Schwänzl, R. Staffeldt and F. Waldhausen, The equivalence of stable K-theory and topological Hochschild homology; Preprint Univ. Bielefeld (1994).

[23] R. Schwänzl and R. Vogt, On the equivalence of two definitions of topological Hochschild homology, in preparation.

[24] G. Segal, Categories and cohomology theories; *Topology* **13** (1974), 293-312.

[25] F. Waldhausen, Algebraic K-theory of topological spaces I; *Proc. Symp. Pure Math.* **32** (1978), 35-60.

[26] F. Waldhausen, Algebraic K-theory of topological spaces II; *Springer Lecture Notes in Mathematics* **763** (1979), 356-394.