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0. Introduction

Let $M_n = M_n(K)$ be the space of n by n matrices over an algebraically closed field K of characteristic 0. An important class of linear preserver problems deals with the investigation of linear transformations of the space M_n which leave invariant certain functions, subsets and/or relations on M_n. To be precise, let $\mathrm{End}(n^2)$ be the algebra of linear transformations of the vector space M_n, and $\mathrm{GL}(n^2)$ its group of invertible elements. If $X \subset M_n$, we define the *stabilizer monoids*

$$E_X = \{\sigma \in \mathrm{End}(n^2) : \sigma(X) \subset X\}$$

and

$$G_X = \mathrm{GL}(n^2) \cap E_X.$$

If θ is a binary relation on M_n, i.e., $\theta \subset M_n \times M_n$, one defines the monoids E_θ and G_θ similarly. For a function $f : M_n \to K$ the corresponding monoids are defined by :

$$E_f = \{\sigma \in E(n^2) : f \circ \sigma = f\}$$

and

$$G_f = \mathrm{GL}(n^2) \cap E_f.$$

One usually refers to the elements of E_X (E_θ, E_f) and G_X (G_θ, G_f) as the *linear preservers* of and the *nonsingular linear preservers*, respectively, of X (θ, f). It is often true that G_X, G_θ, or G_f is a subgroup of $\mathrm{GL}(n^2)$, and in that case we shall refer to it as the *stabilizer subgroup* of X, θ, or f. This is the case if X, θ are closed or open in the Zariski topology, or if f is a polynomial function.

We mention three typical examples. The first has been considered in [**Di**].

Example 1. If X is the set of singular matrices in M_n then

$$G_X = \{\sigma \in \mathrm{GL}(n^2) : \ \det(x) = 0 \ \Rightarrow \ \det(\sigma(x)) = 0\}.$$

Example 2. If X is the zero set in M_n of a polynomial $f \in K[t]$ then

$$G_X = \{\sigma \in \mathrm{GL}(n^2) : \ f(x) = 0 \ \Rightarrow \ f(\sigma(x)) = 0\}.$$

Example 3. If $\theta \subset M_n \times M_n$ consists of all commuting pairs of matrices then

$$G_\theta = \{\sigma \in \mathrm{GL}(n^2) : \ xy = yx \ \Rightarrow \ \sigma(x)\sigma(y) = \sigma(y)\sigma(x)\}.$$

During the last 25 years a large number of papers was published in this direction (see the survey paper [**LPP**] and its bibliography). In those papers, as a rule, the authors use a variety of *ad hoc* matrix methods even in the case of linear preserver problems of similar nature. Most of the papers consider only the nonsingular linear preservers. In this paper we shall present a universal method for solving linear preserver problems whose stabilizer group contains the group of similarity transformations and we demonstrate its effectiveness in a number of important cases.

Our method is based on the solution of the following well known problem about the natural n^2-dimensional (adjoint) representation G of the general linear group $\mathrm{GL}(n)$; namely, to describe all algebraic overgroups of G in the ambient group $\mathrm{GL}(n^2)$. The solution of this problem is given in Theorems A and B of Section 1 and is the main result of the paper. We remark that the proofs of these theorems are independent of the classification results for semisimple subgroups of simple groups.

In Section 2 we apply these results to obtain solutions to several new linear preserver problems. We also show that solutions to several classical linear preserver problems follow immediately from our Theorems A and B. Of course, the applicability of these theorems is not restricted only to the problems treated in this paper. As the proofs of Theorems 2, 4, and 8 indicate, the scope of applicability of our method can be considerably widened by studying the properties of algebraic varieties that arise.

A short exposition of the results of this paper was published in our note [**ĐP**]. In that note, in the case $n = 4$, an exceptional overgroup $\Lambda \cong \mathrm{SL}(6)/\langle -1 \rangle$ of G was overlooked. We thank C.-K. Li for discussing this problem with us, for supplying many references, and pointing out the above exceptional case (an observation due to R. Guralnick).

1. Overgroups of $\mathrm{PSL}(n)$ in $\mathrm{GL}(n^2)$

We have a direct decomposition

$$M_n = M_n^0 \oplus K \cdot 1, \tag{1.1}$$

where 1 denotes the identity matrix, and M_n^0 the subspace defined by $\mathrm{tr}\,(x) = 0$.

We denote by $\mathrm{GL}(n^2-1)$ the subgroup of $\mathrm{GL}(n^2)$ which fixes 1 and preserves M_n^0, and by G its subgroup consisting of all similarity transformations $x \mapsto axa^{-1}$ where $a \in \mathrm{GL}(n)$. Note that $G \cong \mathrm{PSL}(n)$.

We define $\tau \in \mathrm{GL}(n^2-1)$ to be the transposition map and we write often x' for $\tau(x)$.

The algebra $\mathrm{End}(n^2)$ will be identified with the tensor product algebra $M_n \otimes M_n^{\mathrm{opp}}$. If $a, b \in M_n$ then $a \otimes b$ acts as follows :

$$a \otimes b(x) = axb.$$

Note that $(a \otimes b) \cdot (c \otimes d) = ac \otimes db$ for $a, b, c, d \in M_n$. The group

$$\mathrm{SL}(n) \otimes \mathrm{SL}(n)^{\mathrm{opp}} = \{a \otimes b \ : \ a, b \in \mathrm{SL}(n)\}$$

is a maximal connected algebraic subgroup of $\mathrm{SL}(n^2)$, see [**Dy**]. Clearly it contains G.

Our main objective in this section is to determine all algebraic subgroups Γ of $\mathrm{GL}(n^2)$ containing G.

Let $\mathrm{O}(n^2)$ be the subgroup of $\mathrm{GL}(n^2)$ which preserves the non-degenerate symmetric bilinear form $\mathrm{tr}\,(xy)$, $x, y \in M_n$. We denote by $\mathrm{O}(n^2-1)$ the subgroup of $\mathrm{O}(n^2)$ consisting of operators which fix the identity matrix 1. Since the decomposition (1.1) is orthogonal with respect to this form, $\mathrm{O}(n^2-1)$ preserves

M_n^0. The identity components of $\mathrm{O}(n^2)$ and $\mathrm{O}(n^2-1)$ will be denoted by $\mathrm{SO}(n^2)$ and $\mathrm{SO}(n^2-1)$, respectively. As G also preserves the form $\mathrm{tr}\,(xy)$ and fixes 1, we have $G \subset \mathrm{SO}(n^2-1)$. We note that $\tau \in \mathrm{O}(n^2-1)$ and $\det(\tau) = (-1)^{n(n-1)/2}$. If $n = 2$, G and $\mathrm{SO}(3)$ have the same dimension, and so $G \subset \mathrm{SO}(3)$ implies that $G = \mathrm{SO}(3)$, and $G\langle\tau\rangle = \mathrm{O}(3)$.

Let P (resp. Q) be the subgroup of $\mathrm{GL}(n^2)$ which acts trivially on M_n^0 and M_n/M_n^0 (resp. $K \cdot 1$ and $M_n/K \cdot 1$). By T we denote the 2-dimensional subtorus of $\mathrm{SL}(n^2)$ which acts by scalar transformations on M_n^0 and $K \cdot 1$. This torus centralizes $\mathrm{SL}(n^2-1)$ and normalizes P and Q. We introduce three 1-dimensional subtori of T : T_0 which fixes 1, $T_1 = T \cap \mathrm{SL}(n^2)$, and T_2 which acts trivially on M_n^0. We denote by Z the center of $\mathrm{GL}(n^2)$, which is also a 1-dimensional subtorus of T.

If $n = 4$, there is an overgroup $\Lambda \cong \mathrm{SL}(6)/\langle -1 \rangle$ of G in $\mathrm{SL}(15)$. We shall now construct this overgroup. For that purpose, let $\mathfrak{so}(4)$ denote the space of 4 by 4 skew-symmetric matrices over K. We shall view $\mathfrak{so}(4)$ as an $\mathrm{SL}(4)$-module via the action $x \to axa'$, $x \in \mathfrak{so}(4)$, $a \in \mathrm{SL}(4)$. This module is irreducible and self-dual, and so there exists a vector space isomorphism $f : \mathfrak{so}(4) \to \mathfrak{so}(4)$, unique up to a scalar, such that

$$f(axa') = a'^{-1} f(x) a^{-1},$$

for all $x \in \mathfrak{so}(4)$ and all $a \in \mathrm{SL}(4)$. Such an f is given by

$$f(e_{12} - e_{21}) = e_{34} - e_{43}, \quad f(e_{34} - e_{43}) = e_{12} - e_{21},$$

$$f(e_{13} - e_{31}) = e_{42} - e_{24}, \quad f(e_{24} - e_{42}) = e_{31} - e_{13},$$

$$f(e_{14} - e_{41}) = e_{23} - e_{32}, \quad f(e_{23} - e_{32}) = e_{14} - e_{41}.$$

Note that $f^2 = 1$. Define a skew-symmetric bilinear map $\lambda : \mathfrak{so}(4) \times \mathfrak{so}(4) \to M_4$ by $\lambda(x, y) = xf(y) - yf(x)$. The corresponding linear map $\phi : \mathfrak{so}(4) \wedge \mathfrak{so}(4) \to M_4$ is a homomorphism of SL(4)-modules because

$$\begin{aligned}
\phi(a \cdot (x \wedge y)) &= \phi(axa' \wedge aya') \\
&= axa' \cdot f(aya') - aya' \cdot f(axa') \\
&= axa' \cdot a'^{-1} f(y) a^{-1} - aya' \cdot a'^{-1} f(x) a^{-1} \\
&= a(xf(y) - yf(x))a^{-1} \\
&= a\phi(x \wedge y)a^{-1},
\end{aligned}$$

holds for all $x, y \in \mathfrak{so}(4)$ and $a \in \mathrm{SL}(4)$. It follows that ϕ induces an isomorphism $\mathfrak{so}(4) \wedge \mathfrak{so}(4) \to M_4^0$.

Let SL(6) be the group of linear operators on $\mathfrak{so}(4)$ of determinant 1. By using ϕ we can transfer the natural action of SL(6) on $\mathfrak{so}(4) \wedge \mathfrak{so}(4)$ to M_4^0. Hence we obtain a representation $\rho : \mathrm{SL}(6) \to \mathrm{GL}(15)$ such that

$$\rho(a)(\lambda(x, y)) = \lambda(a(x), a(y))$$

for all $x, y \in \mathfrak{so}(4)$ and $a \in \mathrm{SL}(6)$. We shall write Λ for the image $\rho(\mathrm{SL}(6)) \cong \mathrm{SL}(6)/\langle -1 \rangle$. Since ϕ provides an isomorphism $\mathfrak{so}(4) \wedge \mathfrak{so}(4) \cong M_4^0$, of SL(4)-modules, we have

$$G \subset \Lambda \subset \mathrm{SL}(15).$$

Table 1

Some simple $\mathfrak{g}$-submodules of $\mathfrak{sl}(n^2)$

Module	ω	Dim	v
$\mathfrak{g}$	$\epsilon_1 - \epsilon_n$	$n^2 - 1$	$e_{1n} \otimes 1 - 1 \otimes e_{1n}$
V_1	$\epsilon_1 + \epsilon_2 - 2\epsilon_n$	$(n^2-1)(n^2-4)/4$	$e_{1n} \otimes e_{2n} - e_{2n} \otimes e_{1n}$
V_2	$2\epsilon_1 - \epsilon_{n-1} - \epsilon_n$	$(n^2-1)(n^2-4)/4$	$e_{1,n-1} \otimes e_{1n} - e_{1n} \otimes e_{1,n-1}$
V_3	$\epsilon_1 - \epsilon_n$	$n^2 - 1$	$\sum_i (e_{1i} \otimes e_{in} - e_{in} \otimes e_{1i})$
V_4	$\epsilon_1 - \epsilon_n$	$n^2 - 1$	$e_{1n} \otimes 1 + 1 \otimes e_{1n}$
V_5	$2\epsilon_1 - 2\epsilon_n$	$n^2(n-1)(n+3)/4$	$e_{1n} \otimes e_{1n}$
V_6	$\epsilon_1 + \epsilon_2 - \epsilon_{n-1} - \epsilon_n$	$n^2(n+1)(n-3)/4$	$e_{2,n-1} \otimes e_{1n} + e_{1n} \otimes e_{2,n-1} - e_{1,n-1} \otimes e_{2n} - e_{2n} \otimes e_{1,n-1}$
V_7	$\epsilon_1 - \epsilon_n$	$n^2 - 1$	$\sum_i (e_{1i} \otimes e_{in} + e_{in} \otimes e_{1i})$
V_8	0	1	$1 \otimes 1 - n \sum_{i,j} e_{ij} \otimes e_{ji}$
$\mathfrak{p}$	$\epsilon_1 - \epsilon_n$	$n^2 - 1$	$\sum_i e_{1i} \otimes e_{in}$
$\mathfrak{q}$	$\epsilon_1 - \epsilon_n$	$n^2 - 1$	$\sum_i e_{in} \otimes e_{1i}$
V_4'	$\epsilon_1 - \epsilon_n$	$n^2 - 1$	$n(e_{1n} \otimes 1 + 1 \otimes e_{1n}) - 2\sum_i (e_{1i} \otimes e_{in} + e_{in} \otimes e_{1i})$

We shall also need the fact that τ normalizes Λ. Indeed, for $x, y \in \mathfrak{so}(4)$ and $a \in \mathrm{SL}(6)$, we have

$$\begin{aligned}
\tau\rho(a)\tau(\lambda(x,y)) &= \tau\rho(a)\tau(xf(y) - yf(x)) \\
&= \tau\rho(a)(f(y)'x' - f(x)'y') \\
&= \tau\rho(a)(f(y)x - f(x)y) \\
&= \tau\rho(a)(f(y)f(f(x)) - f(x)f(f(y))) \\
&= \tau(af(y)a' \cdot f(af(x)a') - af(x)a' \cdot f(af(y)a')) \\
&= f(af(x)a') \cdot af(y)a' - f(af(y)a') \cdot af(x)a' \\
&= \lambda(f(af(x)a'), f(af(y)a')) \\
&= \rho(faf)(\lambda(x,y)).
\end{aligned}$$

If we fix a basis of M_n which is adapted to the direct sum in (1.1), then P, Q, T can be identified with the group of all matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

respectively, where x is arbitrary column vector, y arbitrary row vector, and α and β arbitrary nonzero scalars.

Theorem A. *Let Γ be a proper connected algebraic subgroup of $\mathrm{SL}(n^2)$ containing $G = \mathrm{PSL}(n)$. Then*

(i) *if reducible, Γ is one of the groups*

$$H,\ HP,\ HQ,\ HT_1,\ HPT_1,\ HQT_1 \tag{1.2}$$

where $H = G,\ \mathrm{SO}(n^2-1),\ \mathrm{SL}(n^2-1)$, or Λ if $n = 4$;

(ii) *if irreducible, Γ is one of the groups*

$$t\mathrm{SO}(n^2)t^{-1}, \quad t \cdot (\mathrm{SL}(n) \otimes \mathrm{SL}(n)^{\mathrm{opp}}) \cdot t^{-1}, \quad t \in T_1. \tag{1.3}$$

Proof. The Lie algebra $\mathfrak{gl}(n^2)$ of $\mathrm{GL}(n^2)$ can be identified with $M_n \otimes M_n^{\mathrm{opp}}$ with Lie bracket defined as

$$\begin{aligned}[a \otimes b, c \otimes d] &= (a \otimes b) \cdot (c \otimes d) - (c \otimes d) \cdot (a \otimes b) \\ &= ac \otimes db - ca \otimes bd.\end{aligned}$$

Then the Lie algebra $\mathfrak{g}$ of G is given by

$$\mathfrak{g} = \{x \otimes 1 - 1 \otimes x \ : \ x \in \mathfrak{sl}(n)\}$$

where $\mathfrak{sl}(n) = M_n^0$. The projector $\pi : M_n \to K \cdot 1$, having M_n^0 as kernel, is given by

$$\pi = \frac{1}{n} \sum_{i,j} e_{ij} \otimes e_{ji}, \tag{1.4}$$

where $e_{ij} \in M_n$ are the standard matrix units. Indeed it sends e_{rs} to 0 if $r \neq s$ and to $(1/n) \cdot 1$ if $r = s$.

By restricting the adjoint representation of $\mathfrak{gl}(n^2)$ to $\mathfrak{g}$, we may consider $\mathfrak{gl}(n^2)$ as a $\mathfrak{g}$-module. The decomposition of $M_n^0 \otimes M_n^0$ into simple $\mathfrak{g}$-modules is known in the sense that the highest weights of the simple summands and their multiplicities are known, see for example [**OV**, Table 5, p. 300]. With some additional computations we obtain the direct decomposition :

$$\mathfrak{sl}(n^2) = \mathfrak{g} + \sum_{i=1}^{8} V_i \tag{1.5}$$

where the highest weight ω, the dimension, and the highest weight vector v for each of the V_i's and some other simple modules is given in Table 1.

If $n = 2$ then $V_4 = V_7$ and the modules V_1, V_2, V_4, and V_6 should be omitted from (1.5). If $n = 3$ only V_6 should be omitted. The Lie algebras of P, Q, and T_1 are $\mathfrak{p}$, $\mathfrak{q}$, and V_8, respectively. It is easy to check that :

$$\mathfrak{so}(n^2-1) = \mathfrak{g} + V_1 + V_2, \quad \mathfrak{so}(n^2) = \mathfrak{so}(n^2-1) + V_3,$$

$$\mathfrak{sl}(n^2-1) = \mathfrak{so}(n^2-1) + V_4' + V_5 + V_6,$$
$$\mathfrak{sl}(n^2) = \mathfrak{sl}(n^2-1) + \mathfrak{p} + \mathfrak{q} + V_8.$$

We shall verify the above formula for $\mathfrak{so}(n^2)$. If $z = a \otimes b - b \otimes a \in \mathfrak{gl}(n^2)$, then z annihilates the symmetric bilinear form $\operatorname{tr}(xy)$ on $M_n \times M_n$. Indeed,

$$\operatorname{tr}(z(x)\cdot y) + \operatorname{tr}(x \cdot z(y)) = \operatorname{tr}(axby - bxy) + \operatorname{tr}(xayb - xbya) = 0,$$

for all $x, y \in M_n$. Consequently, the Lie algebra $\mathfrak{so}(n^2)$ of $\mathrm{SO}(n^2)$ is the Lie subalgebra of $\mathfrak{gl}(n^2)$ consisting of all skew-symmetric tensors. Since the highest weight vectors of V_1, V_2, and V_3 are skew-symmetric tensors, this justifies the above formula for $\mathfrak{so}(n^2)$.

Other formulas are proved similarly. For instance, the inclusion $V_4' \subset \mathfrak{sl}(n^2-1)$ follows from the fact that the highest weight vector of V_4' kills the identity matrix and commutes with the projector (1.4).

We now begin the proof of assertion (i). We claim that if L is a proper connected algebraic subgroup of $\mathrm{SL}(n^2-1)$ containing G properly then $L = \mathrm{SO}(n^2-1)$, $n > 2$, or $L = \Lambda$ and $n = 4$. Let $\mathfrak{l}$ be the Lie algebra of such a group.

Assume that $\mathfrak{l} \supset V_5$. The lowest weight vector of V_5 is $e_{n1} \otimes e_{n1}$. We shall prove this claim in detail, but we shall omit these arguments for similar assertions that occur in this proof. For each i in the range $1 \leq i < n$ we have

$$\begin{aligned}&[e_{i+1,i} \otimes 1 - 1 \otimes e_{i+1,i}, e_{n1} \otimes e_{n1}] \\ &\quad = e_{i+1,i}e_{n1} \otimes e_{n1} - e_{n1}e_{i+1,i} \otimes e_{n1} - e_{n1} \otimes e_{n1}e_{i+1,i} + e_{n1} \otimes e_{i+1,i}e_{n1} = 0.\end{aligned}$$

Therefore $e_{n1} \otimes e_{n1}$ is a lowest weight vector of the $\mathfrak{g}$-module $\mathfrak{gl}(n^2)$. If

$$z = \sum_{i=1}^{n} \alpha_i e_{ii}, \quad \alpha_1 + \cdots + \alpha_n = 0,$$

then we find that

$$\begin{aligned}&[z \otimes 1 - 1 \otimes z, e_{n1} \otimes e_{n1}] \\ &\quad = ze_{n1} \otimes e_{n1} - e_{n1}z \otimes e_{n1} - e_{n1} \otimes e_{n1}z + e_{n1} \otimes ze_{n1} \\ &\quad = 2(\alpha_n - \alpha_1)e_{n1} \otimes e_{n1}.\end{aligned}$$

Thus the weight of this vector is $2(\epsilon_n - \epsilon_1)$. Since $\mathfrak{gl}(n^2)$ contains only one simple $\mathfrak{g}$-submodule of highest weight $2(\epsilon_1 - \epsilon_n)$, namely V_5, it follows that $e_{n1} \otimes e_{n1}$ must be the lowest weight vector of V_5.

Hence $\mathfrak{l}$ contains the bracket of the highest and the lowest weight vectors of V_5 :

$$[e_{1n} \otimes e_{1n}, e_{n1} \otimes e_{n1}] = e_{11} \otimes e_{nn} - e_{nn} \otimes e_{11}.$$

Since $V_5 \subset \mathfrak{sl}(n^2 - 1)$, this element belongs to $\mathfrak{sl}(n^2 - 1)$. Since it is a skew-symmetric tensor, it also belongs to $\mathfrak{so}(n^2)$. Hence it belongs to $\mathfrak{sl}(n^2 - 1) \cap \mathfrak{so}(n^2) = \mathfrak{so}(n^2 - 1)$. On the other hand this element does not lie in $\mathfrak{g}$. As $\mathfrak{g}$ is a maximal subalgebra of $\mathfrak{so}(n^2 - 1)$ when $n > 2$, we infer that $\mathfrak{l} \supset \mathfrak{so}(n^2 - 1)$. Now, $\mathfrak{so}(n^2 - 1)$ is a maximal subalgebra of $\mathfrak{sl}(n^2 - 1)$ and $\mathfrak{l} \supset V_5$ by our assumption, and so $\mathfrak{l} \supset \mathfrak{sl}(n^2 - 1)$. This is a contradiction. Hence we conclude that $\mathfrak{l} \not\supset V_5$.

Next assume that $\mathfrak{l} \supset V_6$. The lowest weight vector of V_6 is :

$$e_{n1} \otimes e_{n-1,2} + e_{n-1,2} \otimes e_{n1} - e_{n-1,1} \otimes e_{n2} - e_{n2} \otimes e_{n-1,1}.$$

We compute the bracket w of the highest and the lowest weight vectors of V_6 :

$$\begin{aligned} w &= e_{11} \otimes e_{n-1,n-1} - e_{n-1,n-1} \otimes e_{11} + e_{22} \otimes e_{nn} - e_{nn} \otimes e_{22} \\ &+ e_{11} \otimes e_{nn} - e_{nn} \otimes e_{11} + e_{22} \otimes e_{n-1,n-1} - e_{n-1,n-1} \otimes e_{22} \\ &= (e_{11} + e_{22}) \otimes (e_{n-1,n-1} + e_{nn}) - (e_{n-1,n-1} + e_{nn}) \otimes (e_{11} + e_{22}) \\ &= (e_{11} + e_{22} - e_{n-1,n-1} - e_{nn}) \otimes (e_{11} + e_{22}) \\ &- (e_{11} + e_{22}) \otimes (e_{11} + e_{22} - e_{n-1,n-1} - e_{nn}). \end{aligned}$$

Note that

$$\begin{aligned} w \equiv (e_{11} + e_{22} - e_{n-1,n-1} - e_{nn}) \otimes (e_{11} + e_{22} - \tfrac{2}{n} \cdot 1) \\ -(e_{11} + e_{22} - \tfrac{2}{n} \cdot 1) \otimes (e_{11} + e_{22} - e_{n-1,n-1} - e_{nn}) \end{aligned}$$

modulo $\mathfrak{g}$. If $n > 4$, then $w \notin \mathfrak{g}$. Since $w \in \mathfrak{l}$, we obtain a contradiction by the same argument as in the case of V_5. In the case $n = 4$, $\mathfrak{g} + V_6$ is the Lie algebra of Λ. This follows from the decomposition (1.5) and a dimension argument using Table 1.

¿From now on we may assume that $\mathfrak{l} \not\supset V_6$.

If $\mathfrak{l} \supset V_1$ or $\mathfrak{l} \supset V_2$ then $\mathfrak{l} \supset \mathfrak{so}(n^2-1)$ because $\mathfrak{g}$ is maximal in $\mathfrak{so}(n^2-1)$. It follows that $\mathfrak{l} = \mathfrak{so}(n^2-1)$ since $\mathfrak{so}(n^2-1)$ is maximal in $\mathfrak{sl}(n^2-1)$.

It remains to consider the case when $\mathfrak{l}$ does not contain any of the modules V_1, V_2, V_5, V_6. This forces $\mathfrak{l} = \mathfrak{g} + V_4'$. The lowest weight vector of V_4' is

$$n(e_{n1} \otimes 1 + 1 \otimes e_{n1}) - 2\sum_i (e_{i1} \otimes e_{ni} + e_{ni} \otimes e_{i1}).$$

By computing the bracket of the highest and the lowest weight vectors of V_4', we conclude that $\mathfrak{l}$ contains the element :

$$n^2(e_{11} \otimes 1 - 1 \otimes e_{11} + 1 \otimes e_{nn} - e_{nn} \otimes 1) - 4n(e_{11} \otimes e_{nn} - e_{nn} \otimes e_{11}).$$

As $e_{11} \otimes e_{nn} - e_{nn} \otimes e_{11} \notin \mathfrak{g} + V_4'$, we infer that $\mathfrak{g} + V_4'$ is not a subalgebra. So we have a contradiction. The assertion (i) now follows easily.

In order to prove the assertion (ii), assume now that Γ is irreducible and let $\mathfrak{l}$ be its Lie algebra.

It is easy to verify that $[\mathfrak{p}, \mathfrak{q}] = \mathfrak{sl}(n^2-1) + V_8$. Indeed, $[\mathfrak{p}, \mathfrak{q}]$ is the linear span of all matrices

$$[\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}] = \begin{pmatrix} xy & 0 \\ 0 & -yx \end{pmatrix},$$

where we use the matrix representation mentioned before the statement of the theorem, and x and y are arbitrary column and row vectors, respectively. It follows that $\mathfrak{l} \not\supset \mathfrak{p} + \mathfrak{q}$.

We claim that $\mathfrak{l} \not\supset V_8$. The module V_4 in the decomposition (1.5) can be replaced by V_4'. The simple $\mathfrak{g}$-modules V_4', $\mathfrak{p}$, and $\mathfrak{q}$ are non-isomorphic as $(\mathfrak{g} + V_8)$-modules. Hence, if $\mathfrak{l} \supset V_8$, by using $\mathfrak{l} \not\supset \mathfrak{p} + \mathfrak{q}$, we conclude that $\mathfrak{l} \subset \mathfrak{sl}(n^2-1) + \mathfrak{p} + V_8$ or $\mathfrak{l} \subset \mathfrak{sl}(n^2-1) + \mathfrak{q} + V_8$. This is impossible, since Γ is irreducible. Thus our claim is proved.

We now claim that $\mathfrak{l}$ contains neither $\mathfrak{p}$ nor $\mathfrak{q}$. Assume, say, that $\mathfrak{l}$ contains $\mathfrak{p}$. Let $x \in \mathfrak{p}$ and $y \in \mathfrak{q}$ be arbitrary. Since Γ is irreducible, we have $\mathfrak{l} \not\subset \mathfrak{sl}(n^2-1) + \mathfrak{p} + V_8$. Therefore there exists $z \in \mathfrak{sl}(n^2-1) + \mathfrak{p} + V_8$ such that $y + z \in \mathfrak{l}$. As $[x, y+z] \in \mathfrak{l}$ and $[x, z] \in \mathfrak{p}$, we have $[x, y] \in \mathfrak{l}$. Hence $\mathfrak{l} \supset \mathfrak{p} + \mathfrak{q}$, a contradiction. This proves our claim. This claim can also be proved by using Proposition 1 from [**Pl**].

Assume that $\mathfrak{l}$ contains V_5. As in the proof of (i), we conclude that $\mathfrak{l} \supset \mathfrak{so}(n^2-1)$. Since V_5 is contained in $\mathfrak{sl}(n^2-1)$ and $\mathfrak{so}(n^2-1)$ is maximal in $\mathfrak{sl}(n^2-1)$, it follows that $\mathfrak{l} \supset \mathfrak{sl}(n^2-1)$. As Γ is irreducible, $\mathfrak{l}$ must contain $\mathfrak{p}$ and $\mathfrak{q}$, and so we have a contradiction. Hence we have shown that $\mathfrak{l}$ does not contain V_5.

Now assume that $\mathfrak{l}$ contains V_1 or V_2. As $\mathfrak{g}$ is a maximal subalgebra of $\mathfrak{so}(n^2-1)$, it follows that $\mathfrak{l} \supset \mathfrak{so}(n^2-1)$. As $\mathfrak{so}(n^2-1)$-module, $\mathfrak{sl}(n^2)$ is a direct sum of five irreducible modules, namely $\mathfrak{so}(n^2-1)$, $V_4' + V_5 + V_6$, $\mathfrak{p}$, $\mathfrak{q}$, and V_8. Among them only $\mathfrak{p}$ and $\mathfrak{q}$ are isomorphic. As $\mathfrak{l}$ contains neither V_5 nor V_8, it follows that $\mathfrak{l} \subset \mathfrak{so}(n^2-1) + \mathfrak{p} + \mathfrak{q}$. Hence we must have $\mathfrak{l} = \mathfrak{so}(n^2-1) \oplus W$ where $W \subset \mathfrak{p} + \mathfrak{q}$ is a simple $\mathfrak{g}$-module and $W \neq \mathfrak{p}, \mathfrak{q}$.

The map $\mathfrak{p} \longrightarrow \mathfrak{q}$ defined by

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ x' & 0 \end{pmatrix}$$

is an isomorphism of $\mathfrak{so}(n^2-1)$-modules, and in particular of $\mathfrak{g}$-modules. There is a constant $\lambda \in K^*$ such that W consists of all matrices

$$\begin{pmatrix} 0 & x \\ \lambda x' & 0 \end{pmatrix},$$

where x is arbitrary column vector. The torus T_1 consists of all matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

where $\alpha \in K^*$ and $\beta = \alpha^{1-n^2}$. By conjugating W by this matrix, we obtain matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 0 & x \\ \lambda x' & 0 \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \alpha\beta^{-1}x \\ \lambda\alpha^{-1}\beta x' & 0 \end{pmatrix}.$$

We can choose α such that $\lambda\alpha^{-1}\beta = -\alpha\beta^{-1}$, i.e., $\alpha^{2n^2} = -\lambda$. Hence W is T_1-conjugate to V_3, which consists of all matrices of the form

$$\begin{pmatrix} 0 & x \\ x' & 0 \end{pmatrix}.$$

Consequently, Γ is a T_1-conjugate of $\mathrm{SO}(n^2)$.

We may now assume that $\mathfrak{l}$ contains neither V_1 nor V_2. If $n > 4$, then we can show that $\mathfrak{l}$ does not contain V_6 by the same method as the one used above for V_5. Assume now that $n = 4$ and $\mathfrak{l}$ contains V_6. Then $\mathfrak{g} + V_6 \cong \mathfrak{sl}(6)$ is the Lie algebra of Λ. We have $\mathfrak{l} \subset \mathfrak{g} + V_6 + V_4 + \mathfrak{p} + \mathfrak{q}$ and the dimensions of V_4, $\mathfrak{p}$, and $\mathfrak{q}$ are all 15. The dimension of $\mathfrak{l}$ must be 35, 50, or 65. Since Γ is irreducible, $\mathfrak{l}$ is reductive. Since the centralizer of $\mathfrak{g}$ in $\mathfrak{l}$ is trivial, $\mathfrak{l}$ is in fact semisimple, and the subalgebra $\mathfrak{g} + V_6 \cong \mathfrak{sl}(6)$ has a nonzero projection on each of the simple ideals of $\mathfrak{l}$. It follows that each of these ideals has dimension at least 35. Hence we must have $\mathfrak{l} = \mathfrak{g} + V_6$, and so $\Gamma = \Lambda$.

It remains to consider the case $\mathfrak{l} \subset \mathfrak{g} + V_4 + \mathfrak{p} + \mathfrak{q}$. Assume that as a $\mathfrak{g}$-module $\mathfrak{l}$ has length 3, i.e., $\mathfrak{l} = \mathfrak{g} \oplus W_1 \oplus W_2$ where $W_1 \cong W_2 \cong \mathfrak{g}$ as $\mathfrak{g}$-modules. We may assume that $W_2 \subset \mathfrak{p} + \mathfrak{q}$. As $W_2 \neq \mathfrak{p}, \mathfrak{q}$, by replacing $\mathfrak{l}$ with a suitable T_1-conjugate, we may assume that $W_2 = V_3$. Since $[V_3, V_3] = \mathfrak{so}(n^2 - 1)$, we have a contradiction. It follows that $\mathfrak{l} = \mathfrak{g} \oplus W$ where $W \cong \mathfrak{g}$. By Lemma 1, $\mathfrak{l} \cong \mathfrak{g} \times \mathfrak{g}$ as Lie algebras. Since Γ is irreducible, an old result of Dynkin [**Dy**, Theorem 1.3] implies that $\Gamma \cong \mathrm{SL}(n) \otimes \mathrm{SL}(n)^{\mathrm{opp}}$ and in fact $u\Gamma u^{-1} = \mathrm{SL}(n) \otimes \mathrm{SL}(n)^{\mathrm{opp}}$ for some $u \in \mathrm{SL}(n^2)$. Now $uGu^{-1} \subset \mathrm{SL}(n) \otimes \mathrm{SL}(n)^{\mathrm{opp}}$ and $uGu^{-1} \cong G$ imply, see Lemma 2, that there exists $v \in \mathrm{SL}(n) \otimes \mathrm{SL}(n)^{\mathrm{opp}}$ such that $vuGu^{-1}v^{-1} = G$. Thus vu belongs to the normalizer N of G in $\mathrm{SL}(n^2)$. We have $\det(\tau) = (-1)^{n(n-1)/2}$ where τ is the transposition map $x \mapsto x'$. If $\det(\tau) = -1$ then $N = GT_1$ while in the case $\det(\tau) = 1$ we have $N = GT_1\langle\tau\rangle$. If $\det(\tau) = 1$ and $vu \in GT_1\tau$ we replace v with τv. Thus we may assume that $vu \in GT_1$. Then $gvu = t \in T_1$ for some $g \in G$ and $t\Gamma t^{-1} = \mathrm{SL}(n) \otimes \mathrm{SL}(n)^{\mathrm{opp}}$.

This completes the proof. ∎

We shall now prove the two lemmas which were used in the above proof.

Lemma 1. *Let* $\mathfrak{g}$ *be a simple subalgebra of a semisimple Lie algebra* $\mathfrak{l}$ *such that* $\mathfrak{l} = \mathfrak{g} \oplus W$ *where* W *is a simple* $\mathfrak{g}$*-submodule of* $\mathfrak{l}$ *isomorphic to* $\mathfrak{g}$. *Then* $\mathfrak{l} \cong \mathfrak{g} \times \mathfrak{g}$ *as Lie algebras.*

Proof. It suffices to show that $\mathfrak{l}$ is not simple. Let us fix an isomorphism $\mathfrak{g} \to W$, $x \mapsto x'$, of $\mathfrak{g}$-modules. Clearly such isomorphism is unique up to a multiplicative scalar. Since $\mathfrak{g}$ is simple, the exterior square $\wedge^2\mathfrak{g}$ contains only one copy of $\mathfrak{g}$, see for instance [**Dj**, Proposition 7]. This implies that there exist constants λ and μ such that

$$[x', y'] = \lambda[x, y] + \mu[x, y]'$$

for all $x, y \in \mathfrak{g}$. We may assume that $\lambda \neq 0$ since otherwise W would be an ideal of $\mathfrak{l}$. Fix constants α and β and let $\mathfrak{h}$ be the $\mathfrak{g}$-submodule of $\mathfrak{l}$ consisting of all elements $\alpha x + \beta x'$ with $x \in \mathfrak{g}$. For $x, y \in \mathfrak{g}$ we have

$$[x, \alpha y + \beta y'] = \alpha[x, y] + \beta[x, y]',$$

$$\begin{aligned} [x', \alpha y + \beta y'] &= \alpha[x, y]' + \beta(\lambda[x, y] + \mu[x, y]') \\ &= \lambda\beta[x, y] + (\alpha + \mu\beta)[x, y]'. \end{aligned}$$

Hence $\mathfrak{h}$ will be an ideal of $\mathfrak{l}$ if we can choose α and β, not both 0, such that $\lambda\beta^2 - \mu\alpha\beta - \alpha^2 = 0$. This we can do since K is algebraically closed. ∎

Lemma 2. *If H is an algebraic subgroup of* $\mathrm{SL}(n) \otimes \mathrm{SL}(n)^{\mathrm{opp}}$ *and* $H \cong G = \mathrm{PSL}(n)$ *then H is conjugate to G.*

Proof. Let

$$\phi : \mathrm{SL}(n) \times \mathrm{SL}(n)^{\mathrm{opp}} \to \mathrm{SL}(n) \otimes \mathrm{SL}(n)^{\mathrm{opp}}$$

be the surjective homomorphism defined by $\phi(a, b) = a \otimes b$. Let F be the identity component of $\phi^{-1}(H)$. Since $H \cong \mathrm{PSL}(n)$, F cannot be contained in just one of the direct factors $\mathrm{SL}(n)$. It follows that $F = \{(x, \theta(x)) : x \in \mathrm{SL}(n)\}$ where θ is an anti-automorphism of $\mathrm{SL}(n)$. By replacing F with a suitable conjugate, we may assume that θ is either the inversion map or the transposition map τ. In the former case we have $H = \phi(F) = G$. The latter case is impossible since the group $\{x \otimes \tau(x) : x \in \mathrm{SL}(n)\}$ is not isomorphic to $\mathrm{PSL}(n)$. ∎

Remark. Statement (i) of Theorem A can be proved by using the classification results from [**Dy**]. It can be extended to algebraically closed fields K of positive characteristic p by using the results of Seitz [**Se**]. If p does not divide n then it is only necessary to add to the list (1.2) the case $H = \mathrm{Sp}(n^2 - 1)$ when $p = 2$.

Corollary 1. *If Γ is a proper connected algebraic subgroup of* $\mathrm{SL}(n^2)$ *containing GP or GQ, then Γ is reducible.*

Proof. By Theorem A, it suffices to show that Γ does not contain any of the groups (1.3). As T_1 normalizes GP and GQ, we may assume that $t = 1$ in (1.3). Now it suffices to observe that the Lie algebras $\mathfrak{p}$ and $\mathfrak{q}$ are not contained in those of $\mathrm{SO}(n^2)$ and $\mathrm{SL}(n) \otimes \mathrm{SL}(n)^{\mathrm{opp}}$. ∎

For applications to linear preserver problems we need the description of all, not only connected, algebraic subgroups $\Gamma \subset \mathrm{GL}(n^2)$ containing G. This problem reduces now essentially to the computation of normalizers. We recall that τ is the transposition map $x \mapsto x'$ and Z the center of $\mathrm{GL}(n^2)$. In the proof of the next proposition we shall make use of the standard group theoretical notations for normalizers and centralizers of subgroups.

Theorem B. *Let Γ be one of the groups listed in Theorem A and N its normalizer in $\mathrm{GL}(n^2)$. Then*

(i) *if Γ is given by (1.2), then N is $\Gamma T\langle\tau\rangle$ if $\Gamma = G$, $\Gamma \cdot \mathrm{O}(n^2-1) \cdot T$ if $\Gamma = \mathrm{SO}(n^2-1)$, and $\Gamma \cdot T$ if $\Gamma = \mathrm{SL}(n^2-1)$;*

(ii) *if $\Gamma = \mathrm{SO}(n^2)$, $N = \mathrm{O}(n^2) \cdot Z$;*

(iii) *if $\Gamma = \mathrm{SL}(n) \otimes \mathrm{SL}(n)^{\mathrm{opp}}$, $N = \Gamma Z\langle\tau\rangle$.*

Proof. (i) Since H is a maximal semisimple subgroup of Γ, we have

$$N \subset \Gamma \cdot N_{\mathrm{GL}(n^2)}(H).$$

As $T = Z_{\mathrm{GL}(n^2)}(H)$, we have

$$N_{\mathrm{GL}(n^2)}(H) \subset N_{\mathrm{GL}(n^2)}(T) = \mathrm{GL}(n^2-1) \times T_2.$$

It follows that

$$N_{\mathrm{GL}(n^2)}(H) = N_{\mathrm{GL}(n^2-1)}(H) \times T_2.$$

It is easy to check that

$$N_{\mathrm{GL}(n^2-1)}(H) = \begin{cases} HT_0\langle\tau\rangle & \text{if } H = G, \text{ or } \Lambda \text{ if } n = 4 \ , \\ \mathrm{O}(n^2-1)T_0 & \text{if } H = \mathrm{SO}(n^2-1) \ , \end{cases}$$

and (i) follows. If $n = 4$ and $H = \Lambda$, one has to use the fact that M_4^0 is not self-dual as a Λ-module.

The assertion (ii) is well known, and for (iii) see [**ĐL**, Theorem 2.3]. ∎

Corollary 2. *Let $L \supset G = \mathrm{PSL}(n)$ be a connected algebraic subgroup of* $\mathrm{GL}(n^2)$.

(i) *If L is reducible, then $L = \Gamma S$ where Γ is as in* (1.2) *and S is a subtorus of T (including the trivial subtorus).*

(ii) *If L is irreducible, then $L = \Gamma$ or $L = \Gamma Z$ where Γ is* $\mathrm{SL}(n^2)$ *or one of the groups* (1.3).

Proof. Let Γ be the identity component of $L \cap \mathrm{SL}(n^2)$. If L is reducible, then Γ is reducible and so it is one of the groups (1.2). As L normalizes Γ, we have $\Gamma \subset L \subset \Gamma T$ and (i) follows. If L is irreducible, then Γ is irreducible and so it is one of the groups (1.3). Then $\Gamma \subset L \subset \Gamma Z$ and (ii) follows. ∎

2. Linear preserver problems

We shall apply the results of Section 1 to several natural linear preserver problems. *We denote by F an arbitrary field of characteristic* 0. By $\mathbf{Q}$ we denote the field of rational numbers, considered as a subfield of F. By $M_n(F)$ we denote the algebra of n by n matrices over F, and by $M_n^0(F)$ its subspace of matrices of trace 0. We shall need the following simple algebraic-geometrical fact.

Lemma 3. *Let $X_F \subset M_n(F)$ be Zariski closed and $Y_F =$*

$M_n(F) \setminus X_F$. *If* σ *is a nonsingular linear transformation of* $M_n(F)$, *then*

(i) $\sigma(X_F) \subset X_F \;\Rightarrow\; \sigma(X_F) = X_F$;

(ii) $\sigma(Y_F) \subset Y_F \;\Rightarrow\; \sigma(Y_F) = Y_F$.

Proof. For (i) see e.g. [**Dix**, Lemma 1]. If $\sigma(Y_F) \subset Y_F$, then $\sigma^{-1}(X_F) \subset X_F$, and (ii) follows from (i). ∎

For any algebraic group L defined over F, we denote by L_F the group of F-rational points of L. We write $\mathrm{GL}(n,F)$ for $GL(n)_F$, $\mathrm{SL}(n,F)$ for $\mathrm{SL}(n)_F$, etc. The following well known result will be used often. If $L = RS$ is a semidirect product of algebraic groups defined over F, then $L_F = R_F S_F$. For instance the groups G, Q, T from the previous section are all defined over F. As GQ and $(GQ)T$ are semidirect products, it follows that $(GQT)_F = G_F Q_F T_F$. The group G_F consists of all transformations $x \mapsto axa^{-1}$ with $a \in \mathrm{GL}(n,F)$, the group Q_F of all transformations $x \mapsto x + f(x) \cdot 1$, where f is a linear function on $M_n(F)$ such that $f(1) \neq -1$, and T_F consists of all transformations which act as nonzero scalars on $M_n^0(F)$ and $F \cdot 1$. It follows that the group $(GQT\langle\tau\rangle)_F$ consists of all nonsingular linear transformations σ of $M_n(F)$ of the form

$$\sigma(x) = \alpha a x a^{-1} + f(x) \cdot 1 \tag{2.1}$$

or

$$\sigma(x) = \alpha a x' a^{-1} + f(x) \cdot 1 \tag{2.2}$$

where $\alpha \in F^*$, $a \in \mathrm{GL}(n,F)$, and f is a linear function on $M_n(F)$ such that $f(1) \neq -\alpha$.

First in [**W1**] for $n \geq 4$ and then in [**Be**],[**PW**] for $n = 3$ the solution of the commutativity preserver problem (see Example 3) was obtained. We shall derive that result as a simple consequence of our Theorems A and B, and moreover we shall describe the stabilizer in the case $n = 2$.

Theorem 1. *Let θ_F be the set of all ordered pairs $(x, y) \in M_n(F) \times M_n(F)$ such that $xy = yx$ and σ a nonsingular linear transformation of $M_n(F)$. If $n > 2$, σ preserves θ_F iff it has the form* (2.1) *or* (2.2). *If $n = 2$, then σ preserves θ_F iff it leaves the subspace $F \cdot 1$ invariant.*

Proof. Let K be an algebraic closure of F and $\theta \subset M_n \times M_n$ the Zariski closure of θ_F. Thus θ is the set of ordered pairs of commuting matrices in M_n. It is easy to check that $G_\theta \supset GQT\langle\tau\rangle$. If $n = 2$ then every commuting pair of matrices in M_n^0 is linearly dependent and so $G_\theta \supset \mathrm{GL}(3)$. As $G_\theta \not\supset \mathrm{SL}(n^2)$, Corollary 1 implies that the identity component L of $\mathrm{SL}(n^2) \cap G_\theta$ is reducible. If $n > 2$, then $\mathrm{SO}(n^2 - 1) \not\subset G_\theta$ because, by Witt's theorem on quadratic forms, $\mathrm{SO}(n^2-1)$ is transitive on linearly independent pairs $(x, y) \in M_n^0 \times M_n^0$ with $\mathrm{tr}\,(x^2) = \mathrm{tr}\,(y^2) = \mathrm{tr}\,(xy) = 0$.

If $n = 4$, then $G_\theta \not\supset \Lambda$. In order to prove this claim, it suffices to show that there exist two linearly independent quadruples (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) in $\mathfrak{so}(4)$ such that the matrices $\lambda(x_1, x_2)$ and $\lambda(x_3, x_4)$ commute, while the matrices $\lambda(y_1, y_2)$ and $\lambda(y_3, y_4)$ do not. (See Section 1 for the definition of the map λ.) For instance, we can take $x_1 = e_{13} - e_{31}$, $x_2 = e_{14} - e_{41}$, $x_3 = e_{12} - e_{21}$, $x_4 = e_{34} - e_{43}$, $y_1 = x_1$, $y_2 = x_2$, $y_3 = e_{23} - e_{32}$, and $y_4 = x_4$.

It follows that $L = GQT_1$ for $n > 2$ and $L = \mathrm{SL}(3)QT_1$ for $n = 2$. By Theorem B, $G_\theta = GQT\langle\tau\rangle$ if $n > 2$, and $G_\theta =$

$\mathrm{GL}(3)QT_2$ if $n = 2$.

As σ preserves θ_F, it also preserves θ. Hence, if $n > 2$, $\sigma \in G_\theta = GQT\langle\tau\rangle$. Since σ is defined over F, we have $\sigma \in G_F Q_F T_F\langle\tau\rangle$, i.e., σ has the required form. If $n = 2$, it suffices to observe that G_θ is the subgroup of $\mathrm{GL}(4)$ which leaves $K \cdot 1$ invariant. ∎

Theorem 2. *Let $X_F \subset M_n(F)$, $X_F \not\subset F \cdot 1$, be a finite union of similarity classes. If σ is a nonsingular linear transformation of $M_n(F)$ and $\sigma(X_F) \subset X_F$, then*

(i) *if $X_F \subset M_n^0(F)$,*

$$\sigma(x) = \alpha(axa^{-1} - \frac{\mathrm{tr}\,(x)}{n} \cdot 1) + \mathrm{tr}\,(x)b, \tag{2.3}$$

or

$$\sigma(x) = \alpha(ax'a^{-1} - \frac{\mathrm{tr}\,(x)}{n} \cdot 1) + \mathrm{tr}\,(x)b, \tag{2.4}$$

where $\alpha \in F^*$, $a \in \mathrm{GL}_n(F)$, and $b \in M_n(F)$ with $\mathrm{tr}\,(b) \neq 0$;

(ii) *if $X_F \not\subset M_n^0(F)$,*

$$\sigma(x) = \alpha(axa^{-1} - \frac{\mathrm{tr}\,(x)}{n} \cdot 1) + \frac{\beta}{n}\mathrm{tr}\,(x) \cdot 1, \tag{2.5}$$

or

$$\sigma(x) = \alpha(ax'a^{-1} - \frac{\mathrm{tr}\,(x)}{n} \cdot 1) + \frac{\beta}{n}\mathrm{tr}\,(x) \cdot 1, \tag{2.6}$$

where $\alpha, \beta \in F^*$ are roots of unity, except in the case where every $x \in X_F$ is a sum of a nilpotent and a scalar matrix, in which case $\alpha \in F^*$ is arbitrary.

Proof. Let K be an algebraic closure of F. If X is the Zariski closure of X_F in M_n, then G_X is an algebraic group. We denote by L the identity component of $\mathrm{SL}(n^2) \cap G_X$.

If $n > 2$, we claim that $G_X \not\supset \mathrm{SO}(n^2-1)$. Choose $x = x_0 + \lambda \cdot 1 \in X_F$ with nonzero $x_0 \in M_n^0(F)$ and $\lambda \in F$. Since X is also a finite union of similarity classes and $n > 2$, we can choose $y_0 \in M_n^0$ such that $\mathrm{tr}\,(y_0^2) = \mathrm{tr}\,(x_0^2)$ and $y := y_0 + \lambda \cdot 1 \notin X$. Indeed, the collection of eigenvalues of all matrices in X is finite, while the set of diagonal matrices $y \in M_n$ satisfying the equations :

$$\mathrm{tr}\,(y) = 0, \quad \mathrm{tr}\,(y^2) = \mathrm{tr}\,(x_0^2),$$

is an algebraic set of dimension $\geq n-2$. By Witt's theorem, there exists $\sigma \in \mathrm{SO}(n^2-1)$ such that $\sigma(x_0) = y_0$. Thus $\sigma(x) = y$, and so $\sigma \notin G_X$. This proves our claim.

If $n = 2$, then $G_X \supset \mathrm{SO}(3)$ because $G = \mathrm{SO}(3)$. On the other hand it is easy to see that $G_X \not\supset \mathrm{SL}(3)$.

If $n = 4$ we claim that $G_X \not\supset \Lambda$. Choose $x = x_0 + \lambda \cdot 1 \in X$, with nonzero $x_0 \in M_4^0(F)$. We can write

$$x_0 = \sum_{i=1}^{k} \lambda(x_i, y_i), \quad 1 \leq k \leq 3,$$

and $x_1, x_2, \ldots, x_k, y_k$ linearly independent elements of $\mathfrak{so}(4)$.

If $k = 1$, then the Λ-orbit of x_0 contains

$$\lambda(\alpha \cdot (e_{12} - e_{21}), e_{34} - e_{43}) = \alpha \cdot \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for all $\alpha \in K^*$. We can choose α such that this matrix is not in X.

If $k=2$, then the Λ-orbit of x_0 contains

$$\lambda(e_{13}-e_{31},e_{14}-e_{41})+\lambda(\alpha\cdot(e_{12}-e_{21}),e_{34}-e_{43}) = \begin{pmatrix} -\alpha & -2 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$$

for all $\alpha \in K^*$. We can choose α such that this matrix is not in X.

If $k=3$ then the Λ-orbit of x_0 has dimension 14. On the other hand, the maximum dimension of a similarity class of matrices in M_4 is 12. Hence Λ cannot preserve X, and our claim is proved.

If $X_F \subset M_n^0(F)$, then $X \subset M_n^0$ and it is easy to see that $G_X \supset GPT_2\langle\tau\rangle$. By Corollary 1, L is reducible. By Theorem A, $L = GPT_1$, and Theorem B implies that $G_X = GPT_2R\langle\tau\rangle$ where $R = T_0 \cap G_X$. As σ is defined over F, we have $\sigma \in (G_X)_F = G_F P_F (T_2)_F R_F\langle\tau\rangle$, and (i) follows.

Now assume that $X_F \not\subset M_n^0(F)$. Choose $x = x_0 + \lambda\cdot 1 \in X_F$ with $x_0 \in M_n^0(F)$ and $\lambda \in F^*$. If $\sigma \in P$ then $\sigma(x) = x_0 + \lambda\sigma(1)$. Since $\sigma(1) = 1+y_0$ where $y_0 \in M_n^0$ may be arbitrary, we conclude that $P \not\subset G_X$. A similar argument shows that $Q \not\subset G_X$.

Let us fix $t \in T_1$. Say t acts on M_n^0 as the scalar α, and on $K\cdot 1$ as α^{1-n^2}. If x is as above, then a simple computation gives :

$$\mathrm{tr}\,(t\circ(a\otimes 1)\circ t^{-1}(x)) = \mathrm{tr}\,(a(\alpha^{-n^2}x_0 + \lambda\cdot 1)).$$

As the latter trace can take infinitely many values for $a \in \mathrm{SL}(n)$, we can choose a so that $t\circ(a\otimes 1)\circ t^{-1}(x) \notin X$. Hence $t(\mathrm{SL}(n)\otimes \mathrm{SL}(n)^{\mathrm{opp}})t^{-1} \not\subset G_X$.

It is obvious that $G_X \not\supset T_1$. Hence, Theorem A implies that $L = G$. By Theorem B, $G_X = GS\langle\tau\rangle$, where $S = T \cap G_X$.

Hence $\sigma \in (G_X)_F = G_F S_F \langle \tau \rangle$, and (ii) follows. $\blacksquare$

When F is algebraically closed, this theorem is also proved, using a different method, in the forthcoming paper [**LP**]. When $x \in M_n^0(F)$ the formulae (2.3)–(2.6) take a simpler form. These simpler formulas were also obtained by R. Guralnick when F is an arbitrary infinite field.

If X_F is the set of zeros in $M_n(F)$ of some polynomial $f \in F[t]$, for example the set of all nilpotent matrices, we obtain the results in [**BP**], [**BPW**], and [**Ho**]. The main result of [**W2**], in which X_F is the similarity class of a regular semisimple element, also follows from Theorem 2.

Now we give the solutions for several new linear preserver problems. Recall that a matrix is called *regular* if its minimal and characteristic polynomials coincide.

Theorem 3. *Let $X_F \subset M_n(F)$ be the set of regular elements and σ a nonsingular linear transformation of $M_n(F)$. If $n > 2$, σ preserves X_F iff σ has the form* (2.1) *or* (2.2). *If $n = 2$, then σ preserves X_F iff σ leaves the subspace $F \cdot 1$ invariant.*

Proof. Let K be an algebraic closure of F and X the set of regular elements of M_n. Since X_F is dense in X, it follows that $\sigma(X) \subset X$. As X is open in M_n, G_X is a group by Lemma 3. It is easy to verify that $G_X \supset GQT\langle \tau \rangle$. If $n = 2$ then all nonzero matrices in M_n^0 are regular and so $G_X \supset \mathrm{GL}(3)$. Since $\mathrm{SL}(n^2)$ is transitive on nonzero matrices in M_n, we have $G_X \not\supset \mathrm{SL}(n^2)$. By Corollary 1, the identity component L of $\mathrm{SL}(n^2) \cap G_X$ is reducible. By Theorem A, $L = \mathrm{SL}(3)QT_1$ if $n = 2$.

Now let $n > 2$. We claim that $\mathrm{SO}(n^2 - 1) \not\subset G_X$. We can choose $x, y \in M_n^0$ such that $x \in X$, $y \notin X$, and $\operatorname{tr}(x^2) = \operatorname{tr}(y^2) = 0$. For instance if $n = 3$ we can take x to be the diagonal matrix with eigenvalues $1, \alpha$, and $-\alpha - 1$, where $\alpha^2 + \alpha - 1 = 0$, and y to be a nonzero matrix such that $y^2 = 0$. By Witt's theorem, there exists $\sigma \in \mathrm{SO}(n^2 - 1)$ such that $\sigma(x) = y$, which proves our claim.

If $n = 4$ we claim that $G_X \not\supset \Lambda$. Consider the matrices $\lambda(x_1, x_2) + \lambda(x_3, x_4)$, where (x_1, x_2, x_3, x_4) are linearly independent elements of $\mathfrak{so}(4)$. We know that Λ is transitive on such matrices. If $x_1 = e_{13} - e_{31}$, $x_2 = e_{14} - e_{41}$, $x_3 = e_{12} - e_{21}$, $x_4 = e_{23} - e_{32}$, then we obtain a nilpotent matrix. On the other hand, the choice $x_1 = e_{12} - e_{21}$, $x_2 = e_{34} - e_{43}$, $x_3 = e_{13} - e_{31} + e_{24} - e_{42}$, $x_4 = e_{12} - e_{21} + e_{14} - e_{41} + e_{23} - e_{32}$, produces a matrix having four distinct eigenvalues. This proves our claim.

By using Theorem A, we can now conclude that, for $n > 2$, $L = GQT_1$. It remains to apply Theorem B and use the fact that σ is defined over F. ∎

Corollary 3. *Let $f(x) = \dim Z(x)$, where $x \in M_n(F)$ and $Z(x)$ is the centralizer of x in $M_n(F)$. If σ is a nonsingular linear transformation of $M_n(F)$, then $f \circ \sigma = f$ iff $\sigma(X_F) \subset X_F$, and so σ has the description as in Theorem* 3.

Proof. If $f \circ \sigma = f$, then $f(x) = n$ implies that $f(\sigma(x)) = n$, i.e., $\sigma(X_F) \subset X_F$. Conversely, if $\sigma(X_F) \subset X_F$, then the theorem implies that $f \circ \sigma = f$. ∎

Theorem 4. *Let $(X_r)_F \subset M_n(F)$, $r < n$, be the set of matrices having at most r distinct eigenvalues in some algebraic closure K of F. Let σ be a nonsingular linear transformation of $M_n(F)$.*

Then $\sigma((X_r)_F) \subset (X_r)_F$ *iff* σ *has the form* (2.1) *or* (2.2).

Proof. Let X_r be the set of matrices $x \in M_n$ having at most r distinct eigenvalues. It is a well known fact that X_r is closed, and so G_{X_r} is an algebraic group. Since we could not find a reference, we shall give a proof of that fact. Let V_r be the set of all points $(\xi_1, \ldots, \xi_n) \in K^n$ such that $\{\xi_1, \ldots, \xi_n\}$ has cardinality $\leq r$. Clearly V_r is closed, since it is a union of finitely many vector subspaces of K^n. Let $\Lambda = K^n/S_n$ be the space of orbits for the standard action of the symmetric group S_n on the affine space K^n. We shall recall some well known facts about Λ, see e.g. [**Fo**, Section I-6]. The space Λ is an affine variety isomorphic to K^n, the natural projection $\pi : K^n \to \Lambda$ is a surjective morphism, its comorphism $\pi^* : K[\Lambda] \to K[x_1, \ldots, x_n]$ is injective, and the image of π^* is the algebra of symmetric polynomials in $x_1, \ldots, x_n$. A subset of Λ is closed iff its inverse image in K^n is closed. If $y_i \in K[\Lambda]$ is chosen so that $\pi^*(y_i)$ is the i-th elementary symmetric function of $x_1, \ldots, x_n$, then $K[\Lambda]$ is the polynomial algebra $K[y_1, \ldots, y_n]$. Since V_r is closed and S_n-invariant, its image, Λ_r, is closed in Λ. Let $\phi : K^n \to \Lambda$ be defined by

$$\phi(\xi_1, \ldots, \xi_n) = \pi(\lambda_1, \ldots, \lambda_n),$$

where $\lambda_1, \ldots, \lambda_n$ are the roots of the polynomial

$$t^n - \xi_1 t^{n-1} + \cdots + (-1)^n \xi_n. \tag{2.7}$$

Then

$$\phi^*(y_i)(\xi_1, \ldots, \xi_n) = \pi^*(y_i)(\lambda_1, \ldots, \lambda_n) = \xi_i,$$

i.e., $\phi^*(y_i) = x_i$. Hence ϕ is an isomorphism. Define $\psi : M_n \to K^n$ by $\psi(x) = (\xi_1, \ldots, \xi_n)$ if (2.7) is the characteristic polynomial of $x \in M_n$. Since ψ is also a morphism and $X_r = \psi^{-1}\phi^{-1}(\Lambda_r)$, we conclude that X_r is closed in M_n.

As X_r is closed, G_{X_r} is an algebraic group. Since every $x \in M_n$ is similar to a triangular matrix, it is easy to see that $(X_r)_F$ is dense in X_r. Consequently $\sigma \in G_{X_r}$. It is easy to check that $G_{X_r} \supset GQT\langle\tau\rangle$. By Corollary 1, G_{X_r} is reducible. By using Witt's theorem, one can show that $G_{X_r} \not\supset \mathrm{SO}(n^2-1)$ if $n > 2$. Clearly, $G_{X_r} \not\supset \mathrm{SL}(3)$ if $n = 2$. As in the proof of Theorem 3, we can show that $G_{X_r} \not\supset \Lambda$ if $n = 4$. By Theorems A and B, $G_{X_r} = GQT\langle\tau\rangle$. It remains to use the fact that σ is defined over F. ∎

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $x \in M_n(F)$, the discriminant of x is defined by

$$D(x) = \prod_{i<j} (\lambda_i - \lambda_j)^2.$$

It is well known that $D : M_n(F) \to F$ is a polynomial function.

Corollary 4. *Let σ be a nonsingular linear transformation of $M_n(F)$. Then $D \circ \sigma = D$ iff σ has the form* (2.1) *or* (2.2) *where $\alpha^{n(n-1)} = 1$.*

Proof. Assume that $D \circ \sigma = D$. Since

$$X_{n-1} = \{x \in M_n : D(x) = 0\},$$

σ has the form (2.1) or (2.2). Now $D \circ \sigma = D$ implies that $\alpha^{n(n-1)} = 1$. The converse is easy to check. ∎

For $n > 5$ this corollary was proved in [**Pi**].

Theorem 5. *Let $(Y_r)_F \subset M_n(F)$ be the set of matrices having exactly r distinct eigenvalues in some algebraic closure K of F. Let σ be a nonsingular linear transformation of $M_n(F)$. Then $\sigma((Y_r)_F) \subset (Y_r)_F$ iff σ has the form* (2.1) *or* (2.2).

Proof. Let $(X_r)_F$ be defined as in Theorem 4. Since $(X_{n-1})_F = M_n(F) \setminus (Y_n)_F$, $\sigma((X_{n-1})_F) \subset (X_{n-1})_F$. Hence the assertion in the case $r = n$ follows from Theorem 4. Now let $r < n$ and let X_r resp. Y_r be the sets of matrices in M_n having at most resp. exactly r distinct eigenvalues. The assertion now follows again from Theorem 4, by using the fact that X_r is the closure of $(Y_r)_F$. Since $(Y_r)_F$ is dense in Y_r, this fact reduces to showing that Y_r is dense in X_r. Due to the lack of reference, we shall sketch the proof of this fact. Note that $Y_r = X_r \setminus X_{r-1}$. Hence, if $x \in X_{r-1}$ we have to show that x belongs to the closure of Y_r. We may assume, without any loss of generality, that $x = (\xi_{ij})$ is an upper triangular matrix with $\xi_{11} = \xi_{22}$. Let Z be the affine space consisting of all upper triangular matrices $y = (\eta_{ij})$ such that $\eta_{ii} = \eta_{jj}$ whenever $1 < i < j \leq n$ and $\xi_{ii} = \xi_{jj}$. Then $Z \cap Y_r$ is a non-empty open subset of Z, and so it is dense in Z. In particular, x belongs to the closure of Y_r. ∎

Theorem 6. *Let $X_F \subset M_n(F)$ be the set of semisimple elements and σ a nonsingular linear transformation of $M_n(F)$. Then $\sigma(X_F) \subset X_F$ iff σ has the form* (2.1) *or* (2.2).

Proof. We claim that $U := (Y_n)_F$, see Theorem 5, is the largest open subset of $M_n(F)$ contained in X_F. It suffices to show that if $x \in X_F \setminus U$, then x belongs to the closure of $V := M_n(F) \setminus X_F$. Without any loss of generality, we may assume that x is in the rational canonical form, i.e.,

$$x = x_1 \oplus \cdots \oplus x_k,$$

where x_i are the companion matrices of certain irreducible polynomials over F. Since $x \notin U$, two of these blocks are the same, say $x_1 = x_2$. Let $x(\lambda)$ be the matrix obtained from x by replacing the (1,2) zero block by $\lambda \cdot 1$. If $\lambda \neq 0$ then $x(\lambda) \in V$, and our claim is proved. Since $\sigma(U) \subset X_F$, it follows now that in fact $\sigma(U) \subset U$. Since $U = (Y_n)_F$, we can apply Theorem 5 to finish the proof. ∎

Theorem 7. *Let $(T_r)_F$, $1 < r \leq n$, be the set of matrices $x \in M_n(F)$ having an eigenvalue of multiplicity at least r, in some algebraic closure K of F. Let σ be a nonsingular linear transformation of $M_n(F)$. Then $\sigma((T_r)_F) \subset (T_r)_F$ iff σ has the form* (2.1) *or* (2.2)*.*

Proof. The Zariski closure, T_r, of $(T_r)_F$ in M_n consists of all matrices $x \in M_n$ having an eigenvalue of multiplicity $\geq r$. The hypothesis implies that $\sigma \in G_{T_r}$. It is clear that $G_{T_r} \supset GQT\langle\tau\rangle$. By the same method as in the proof of Theorem 3, we can show that $\mathrm{SO}(n^2-1) \not\subset G_{T_r}$, $\Lambda \not\subset G_{T_r}$ if $n = 4$, and then that $G_{T_r} = GQT\langle\tau\rangle$. ∎

Theorem 8. *Let X_F be the set of elements of finite order in $\mathrm{GL}(n,F) \subset M_n(F)$. If σ is a nonsingular linear transformation of $M_n(F)$ and $\sigma(X_F) \subset X_F$, then there exists $a \in \mathrm{GL}_n(F)$ and a root of unity $\alpha \in F^*$ such that either $\sigma(x) = \alpha a x a^{-1}$ or $\sigma(x) = \alpha a x' a^{-1}$ for all x.*

Proof. Let (σ_{ij}), $1 \le i, j \le n^2$, be the matrix of σ with respect to the standard basis of $M_n(F)$. Let L be the subfield of F obtained by adjoining to $\mathbf{Q}$ all the entries σ_{ij}. Then σ preserves $M_n(L)$, as well as $X_L = M_n(L) \cap X_F$. Let E be a subfield of L such that L/E is algebraic and $E/\mathbf{Q}$ is purely transcendental. Since L is a finitely generated extension of $\mathbf{Q}$, $m := [L : E]$ is finite. If λ is an eigenvalue of some $x \in X_L$, then λ is a root of unity and $[L(\lambda) : L] \le n$. Let $f \in \mathbf{Q}[t]$ and $g \in E[t]$ be the minimal polynomials of λ over $\mathbf{Q}$ and E, respectively. Since $g|f$, the roots of g and also its coefficients are algebraic over $\mathbf{Q}$. As $E/\mathbf{Q}$ is purely transcendental, it follows that $g \in \mathbf{Q}[t]$. Hence $f = g$ by irreducibility of f. Consequently,

$$[\mathbf{Q}(\lambda) : \mathbf{Q}] = [E(\lambda) : E] \le [L(\lambda) : E] = [L(\lambda) : L]\cdot[L : E] \le mn.$$

Hence there exists a positive integer N such that $x^N = 1$ for all $x \in X_L$. This implies that X_L is a finite union of similarity classes of $M_n(L)$. As $X_L \not\subset L \cdot 1$ and $X_L \not\subset M_n^0(L)$, Theorem 2 implies that σ is given by (2.5) or (2.6).

Assume that (2.5) holds. If $n > 2$, we choose $x_0 = a^{-1}da$ where d is the diagonal matrix with all diagonal entries 1 except the last which is -1. Then we obtain

$$\sigma(x_0) = \alpha d + (\beta - \alpha)\frac{n-2}{n} \cdot 1 \in X_L.$$

Since the diagonal elements of this matrix must be roots of unity, we must have $\alpha = \beta$, and so $\sigma(x) = \alpha a x a^{-1}$ for all x. If $n = 2$, we set $x_0 = a^{-1}da$ where d is the diagonal matrix with diagonal entries α and β. Then

$$\sigma(x_0) = \alpha d + \frac{\beta^2 - \alpha^2}{2} \cdot 1 \in X_L$$

implies that $(\alpha^2+\beta^2)/2$ is a root of unity. Therefore $\alpha^2=\beta^2$, i.e., $\beta=\pm\alpha$. If $\beta=\alpha$, σ has the desired form. If $\beta=-\alpha$ then

$$\sigma(x)=\alpha a(x-\operatorname{tr}(x)\cdot 1)a^{-1}=-\alpha b x' b^{-1},$$

where

$$b=a\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

When σ is given by (2.6), the argument is similar. ∎

We mention that, in the last theorem, the stabilizer subgroup is not an algebraic group, but our method is still applicable. We can obtain a similar result for the set of all potent matrices in $M_n(F)$. Recall that x is a potent matrix if $x^k=x$ for some $k\geq 2$. When F is the complex field such a result was obtained recently in [**BŠ**] (without assuming that σ is nonsingular).

Theorem 9. *Let $P_F\subset M_n(F)$ be the set of potent matrices and σ a nonsingular linear transformation of $M_n(F)$. Then $\sigma(P_F)\subset P_F$ iff there exists a root of unity α and $a\in\mathrm{GL}(n,F)$ such that either $\sigma(x)=\alpha a x a^{-1}$ or $\sigma(x)=\alpha a x' a^{-1}$ for all x.*

Proof. The proof is the same as that of Theorem 8, except that some of the arguments used to show that $\beta=\alpha$ (or $\beta=\pm\alpha$ if $n=2$) have to be modified. ∎

In the next theorem we present a first linear preserver problem in which the group Λ appears as the main component of the stabilizer group. The map λ mentioned in this theorem has been defined in Section 1.

Theorem 10. *Let $n = 4$ and $X \subset M_4^0$ be the set of all matrices $\lambda(x, y)$ with $x, y \in \mathfrak{so}(4)$. Then $G_X = \Lambda PT\langle\tau\rangle$.*

Proof. The inclusion $G_X \supset \Lambda PT\langle\tau\rangle$ follows from the description of Λ. Since Λ is irreducible on M_4^0 and has center of order 3, it is not contained in SO(15). If $G_X \supset$ SO(15), then we would have $G_X \supset$ SL(15), which is obviously impossible. It follows that Λ is the identity component of SL(15) $\cap G_X$, and it remains to apply Theorem B. ∎

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