Symmetric chain decompositions of linear lattices

Frank Vogt
TH Darmstadt

Bernd Voigt
Universität Bielefeld

Introduction

Symmetric chain decompositions were introduced by de Bruin, Tengbergen, and Kruyswijk [1] as a tool for an alternative proof of Sperner's Theorem on the maximal size of an antichain in a finite power set [7]. Symmetric chain decompositions turned out to be useful for several problems, therefore the existence and construction of symmetric chain decompositions in different partially ordered sets has been a standing question in combinatorics (e. g. see [3]). Griggs proved in [6] that the lattice of subspaces of a finite vector space has a symmetric chain decompostion. Since his proof was indirect, Greene and Kleitman formulated the problem of giving an explicit construction for a symmetric chain decomposition of these linear lattices [5]. A partial solution (with restrictions for the underlying field of the vector space) was given by Fleischer [4], a student of Leeb. In this paper we introduce an inductive construction for a symmetric chain decompostion of the lattice of subspaces of a finite dimensional vector space over an arbitrary field. The construction is, in a certain sense, an extension of the well known construction for boolean lattices [5].

Notation

Let us recall first that a *symmetric chain* of a ranked partially ordered set P of finite length n is a chain $x_k \prec x_{k+1} \prec \ldots \prec x_{n-k}$, where the rank of each x_i is i. Then a *symmetric chain decomposition* of P is a partition of P into symmetric chains.

For the following, let $\mathcal{F}$ be a field. All matrices are matrices with entries in $\mathcal{F}$. Then E_t denotes the $(t \times t)$–unit–matrix, whereas 0 denotes the zero–matrix, which, by definition, is a matrix whose entries are just zeros; by abuse of language the matrix 0 always has the right dimensions.

An $(n \times k)$–matrix A with rows $a_1, \ldots, a_n$ is in *Schubert–normal–form* (or short: is an $(n \times k)$–Schubert–normal–form) if

(1) there are $i_1 < i_2 < \ldots < i_k$ such that a_{i_j} equals the j-th k–unit–vector e_j, and

(2) each a_i is a linear combination of the $e_1, \ldots, e_l$ with $i_l \leq i$.

$$\begin{pmatrix}
0 & \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} & 0 \\
\vdots & & & & & \vdots \\
0 & \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} & 0 \\
1 & 0 & \multicolumn{3}{c}{\cdots\cdots\cdots} & 0 \\
* & 0 & \multicolumn{3}{c}{\cdots\cdots\cdots} & 0 \\
\vdots & \vdots & & & & \vdots \\
* & 0 & \multicolumn{3}{c}{\cdots\cdots\cdots} & 0 \\
0 & 1 & 0 & \multicolumn{2}{c}{\cdots\cdots} & 0 \\
* & * & 0 & \multicolumn{2}{c}{\cdots\cdots} & 0 \\
\vdots & \vdots & \vdots & & & \vdots \\
* & * & 0 & \multicolumn{2}{c}{\cdots\cdots} & 0 \\
0 & 0 & 1 & 0 & \cdot\cdot & 0 \\
 & & & & \ddots & \\
* & \multicolumn{3}{c}{\cdots\cdots\cdots} & * & 0 \\
0 & \multicolumn{3}{c}{\cdots\cdots\cdots} & 0 & 1 \\
* & \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} & * \\
\vdots & & & & & \vdots \\
* & \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} & *
\end{pmatrix}$$

Figure 1: A matrix in Schubert–normal–form

Observe that the columns of each $(n \times k)$–Schubert–normal–form A generate a k–dimensional subspace U of $\mathcal{F}^n$, the n–dimensional vector space over $\mathcal{F}$. More precisely, A describes the embedding of $\mathcal{F}^k$ into $\mathcal{F}^n$ which maps $\mathcal{F}^k$ onto U. It is easy to see that for each k–dimensional subspace U of $\mathcal{F}^n$ there is exactly one Schubert–normal–form whose colums generate U. Hence there is a bijection between the subspaces of $\mathcal{F}^n$ and the Schubert–normal–forms over $\mathcal{F}$ with n rows. The inclusion of subspaces of $\mathcal{F}^n$ can be described as follows: Let A be an $(n \times k)$– and B be an $(n \times m)$–Schubert–normal–form. Then the subspace generated by A is a subspace of the subspace generated by B if and only if there is an $(m \times k)$–Schubert–normal–form C with $A = B \cdot C$. For more information on Schubert–normal–forms, see [2].

$\mathcal{L}\binom{n}{k}$, where $k \leq n$, denotes the set of $(n \times k)$–Schubert–normal–forms, i. e., this is the set of k–dimensional subspaces of $\mathcal{F}^n$. Then $\mathcal{L}(n) = \bigcup_{k=0}^{n} \mathcal{L}\binom{n}{k}$ is the *n–dimensional linear lattice* over $\mathcal{F}$, i. e., the lattice of subspaces of $\mathcal{F}^n$. The lattice structure of $\mathcal{L}(n)$ can be described just by matrix–multiplication as explained above.

In the sequel, vectors $a \in \mathcal{F}^n$ are viewed as *column–vectors*, a' denotes the transposed vector (tranposed matrix, resp.), i. e., a' is a row–vector with n columns.

The crucial definition is the definition of the sets $\mathcal{L}_t\binom{n}{k}$ and $\mathcal{L}_t(n)$. These are defined as follows:

Definition 1 Let $t \geq 0$ and let

$$\begin{aligned}
\mathcal{L}_t\binom{n}{k} &= \left\{ \left(\begin{array}{c|c} E_t & 0 \\ \hline A & B \end{array} \right) \in \mathcal{L}\binom{t+n}{t+k} \;\middle|\; \operatorname{rank}(A) = t \right\} , \\
\mathcal{L}_t(n) &= \bigcup_{k=0}^{n} \mathcal{L}_t\binom{n}{k} .
\end{aligned}$$

Remark 2 Obviously, $\mathcal{L}_t(n) = \emptyset$ for $n < t$. If C is in $\mathcal{L}_t\binom{n}{k}$, then C is a (t+k)–dimensional subspace of $\mathcal{F}^{t+n}$.

A Decomposition

First we introduce some additional definitions.

Definition 3 Let a' be a row–vector with t columns, let $b \in \mathcal{L}\binom{t}{1}$ be a one–dimensional linear subspace of the t–dimensional space over $\mathcal{F}$ and let c be a column–vector with t rows. Then let

$$\begin{aligned}
\mathcal{L}_t^{a,-}(n) &= \left\{ \left(\begin{array}{c|c} E_t & \\ \cline{1-1} a' & 0 \\ \hline A & B \end{array} \right) \in \mathcal{L}_t(n) \;\middle|\; \mathrm{rank}(A) = t \right\} , \\
\mathcal{L}_{t-1}^{a,b}(n) &= \left\{ \left(\begin{array}{c|c} E_t & \\ \cline{1-1} a' & 0 \\ \hline A & B \end{array} \right) \in \mathcal{L}_t(n) \;\middle|\; \mathrm{rank}(A) = t-1,\ A \cdot b = 0,\ a' \cdot b \neq 0 \right\} , \\
\mathcal{L}_t^{c,+}(n) &= \left\{ \left(\begin{array}{c|c|c} \multicolumn{2}{c|}{E_{t+1}} & 0 \\ \hline A & A \cdot c & B \end{array} \right) \in \mathcal{L}_t(n) \;\middle|\; \mathrm{rank}(A) = t \right\} .
\end{aligned}$$

We continue with some more or less obvious observations:

Observation 4 $\mathcal{L}_t^{a,-}(n) \cong \mathcal{L}_t(n-1)$.

Proof Consider the mapping $\Psi_a : \mathcal{L}_t^{a,-}(n) \longrightarrow \mathcal{L}_t(n-1)$ which is defined by

$$\Psi_a \left(\left(\begin{array}{c|c} E_t & \\ \cline{1-1} a' & 0 \\ \hline A & B \end{array} \right) \right) = \left(\begin{array}{c|c} E_t & 0 \\ \hline A & B \end{array} \right) .$$

Obviously, Ψ_a is onto. Furthermore,

$$\left(\begin{array}{c|c} E_t & \\ \cline{1-1} a' & 0 \\ \hline A_1 & B_1 \end{array} \right) = \left(\begin{array}{c|c} E_t & \\ \cline{1-1} a' & 0 \\ \hline A_2 & B_2 \end{array} \right) \cdot \left(\begin{array}{c|c} E_t & \\ \hline X & Y \end{array} \right)$$

holds if and only if

$$\left(\begin{array}{c|c} E_t & 0 \\ \hline A_1 & B_1 \end{array} \right) = \left(\begin{array}{c|c} E_t & 0 \\ \hline A_2 & B_2 \end{array} \right) \cdot \left(\begin{array}{c|c} E_t & \\ \hline X & Y \end{array} \right)$$

is satisfied. Hence, Ψ_a is an order–embedding and therefore an order–isomorphism. $\square$

Observation 5 $\mathcal{L}^{a,b}_{t-1}(n) \cong \mathcal{L}_{t-1}(n-1)$.

Proof Consider the order–isomorphism $\Theta_{a,b} : \mathcal{L}^{a,b}_{t-1}(n) \longrightarrow \mathcal{L}_{t-1}(n-1)$ which is defined by

$$\Theta_{a,b}\left(\left(\begin{array}{c|c} \begin{array}{c} E_t \\ \hline a' \end{array} & 0 \\ \hline A & B \end{array}\right)\right) = \left(\begin{array}{c|c} E_{t-1} & 0 \\ \hline \tilde{A} & B \end{array}\right) ,$$

where $\tilde{A}$ is derived from A by deleting the column of A which corresponds to the leading 1 of b. Observe that, for obtaining an isomorphism, it is sufficient to delete one column of A which is linearly dependent from the others. Such a column does always exist because we have $Ab = 0$. We choose the first column from the left which is possible in order to make the definition of $\Theta_{a,b}$ unique. □

Observation 6 $\mathcal{L}^{c,+}_{t}(n) \cong \mathcal{L}_{t}(n-1)$.

Proof Consider the order–isomorphism $\Phi_c : \mathcal{L}^{c,+}_{t}(n) \longrightarrow \mathcal{L}_{t}(n-1)$ which is defined by

$$\Phi_c\left(\left(\begin{array}{c|c} E_{t+1} & 0 \\ \hline A \mid A\cdot c & B \end{array}\right)\right) = \left(\begin{array}{c|c} E_t & 0 \\ \hline A & B \end{array}\right) .$$

□

Convention 7 For completeness we also define the identity

$$\Lambda : \left\{ \left(\begin{array}{c|c} E_{t+1} & 0 \\ \hline A & B \end{array}\right) \in \mathcal{L}_t(n) \;\middle|\; \mathrm{rank}(A) = t+1 \right\} \longrightarrow \mathcal{L}_{t+1}(n-1) .$$

□

Now we are able to formulate the following almost obvious decomposition theorem which will lead to a symmetric chain decomposition of $\mathcal{L}(n)$ in the next section.

Theorem 8 (Decomposition Theorem) For all $n \geq 1$ and $t \geq 0$,

$$\begin{aligned} \mathcal{L}_t(n) &= \bigcup \{\Psi_a^{-1}(\mathcal{L}_t(n-1)) \mid a \in \mathcal{F}^t\} \\ &\cup \bigcup \{\Theta_{a,b}^{-1}(\mathcal{L}_{t-1}(n-1)) \mid a \in \mathcal{F}^t,\, b \in \mathcal{L}\tbinom{t}{1}\} \\ &\cup \bigcup \{\Phi_c^{-1}(\mathcal{L}_t(n-1)) \mid c \in \mathcal{F}^t\} \\ &\cup \Lambda^{-1}(\mathcal{L}_{t+1}(n-1)) , \end{aligned}$$

and this union is disjoint. □

Symmetric Chain Decompositions

As a consequence of the decomposition theorem we obtain:

Theorem 9 For every choice of t and n, the partially ordered set $\mathcal{L}_t(n)$ has a symmetric chain decomposition.

Proof We prove the theorem by induction on n, following the decomposition described in the decomposition theorem. Obviously, $\mathcal{L}_t(0)$ is either a singleton ($t = 0$) or the empty set ($t > 0$). In either case, $\mathcal{L}_t(0)$ has a symmetric chain decomposition. Now, let $n > 0$. Observe that the subsets $\mathcal{L}_{t-1}^{a,b}(n)$ as well as $\mathcal{L}_{t+1}(n-1)$ lie symmetrically in $\mathcal{L}_t(n)$. Hence, the decompositions of these parts into symmetric chains (which exist according to the inductive assumption and the order–isomorphisms introduced in the previous section) extend to $\mathcal{L}_t(n)$.

Therefore it remains to care about the other parts of the decomposition, which are the subspaces in $\mathcal{L}_t^{a,-}(n)$ as well as in $\mathcal{L}_t^{c,+}(n)$. We finish the proof of Theorem 9 by the following three lemmata. Observe that

Lemma 10

$$\begin{aligned} \mathcal{L}_t^{a,-}(n) &\subseteq \bigcup_{k=0}^{n-t-1} \mathcal{L}_t\binom{n}{k} \quad \text{and} \\ \mathcal{L}_t^{c,+}(n) &\subseteq \bigcup_{k=1}^{n-t} \mathcal{L}_t\binom{n}{k} . \end{aligned}$$

□

Essentially, this lemma says that the subspaces in $\mathcal{L}_t^{a,-}(n)$ have dimensions from t to $n-1$ and those in $\mathcal{L}_t^{c,+}(n)$ have dimensions from $t+1$ to n.

To each column–vector $c \in \mathcal{F}^t$, we associate an appropriate row–vector $a_c \in \mathcal{F}^t$. We then consider corresponding chains in $\mathcal{L}_t^{c,+}(n)$ and in $\mathcal{L}_t^{a_c,-}(n)$. We chop of the head (maximal element) from each chain in $\mathcal{L}_t^{c,+}(n)$ and declare this to be the new head of the corresponding chain in $\mathcal{L}_t^{a_c,-}(n)$. This way we get two new symmetric chains in $\mathcal{L}_t(n)$ for each corresponding pair of chains. The following lemma contains the necessary technical details.

Lemma 11 Let T be a regular $(t \times t)$–matrix. The mapping $\Gamma_T : \mathcal{L}_t(n-1) \longrightarrow \mathcal{L}_t(n-1)$ defined by

$$\Gamma_T \left(\begin{array}{c|c} E_t & 0 \\ \hline A & B \end{array} \right) = \left(\begin{array}{c|c} E_t & 0 \\ \hline A \cdot T & B \end{array} \right)$$

is an order–isomorphism. If

$$T = T_{a,c} = E_t + c \cdot a' = \begin{pmatrix} 1 + a_1c_1 & a_2c_1 & \dots & a_tc_1 \\ a_1c_2 & 1 + a_2c_2 & \dots & a_tc_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1c_t & a_2c_t & \dots & 1 + a_tc_t \end{pmatrix}$$

with $a = (a_1, \ldots, a_t)'$ and $c = (c_1, \ldots, c_t)'$, then for every subspace $C \in \mathcal{L}_t^{c,+}(n)$ it follows that

$$\Psi_a^{-1}(\Gamma_T(\Phi_c(C))) \subseteq C \ .$$

Proof The assumption that T is regular is necessary because otherwise $\Gamma_T(D)$ would not be in $\mathcal{L}_t(n-1)$ for any $D \in \mathcal{L}_t(n-1)$. Observe now that we have

$$\left(\begin{array}{c|c} E_t & 0 \\ \hline A_1 & B_1 \end{array}\right) = \left(\begin{array}{c|c} E_t & 0 \\ \hline A_2 & B_2 \end{array}\right) \cdot \left(\begin{array}{c|c} E_t & 0 \\ \hline X & Y \end{array}\right)$$

if and only if

$$\left(\begin{array}{c|c} E_t & 0 \\ \hline A_1 \cdot T & B_1 \end{array}\right) = \left(\begin{array}{c|c} E_t & 0 \\ \hline A_2 \cdot T & B_2 \end{array}\right) \cdot \left(\begin{array}{c|c} E_t & 0 \\ \hline X \cdot T & Y \end{array}\right)$$

holds. Thus Γ_T is an order–isomorphism. The last assertion we conclude from

$$\left(\begin{array}{c|c} \begin{array}{c} E_t \\ \hline a' \end{array} & 0 \\ \hline A \cdot T & B \end{array}\right) = \left(\begin{array}{c|c} E_{t+1} & 0 \\ \hline A \,|\, A \cdot c & B \end{array}\right) \cdot \left(\begin{array}{c|c} \begin{array}{c} E_t \\ \hline a' \end{array} & 0 \\ \hline 0 & B \end{array}\right) ,$$

where $C = \left(\begin{array}{c|c} E_{t+1} & 0 \\ \hline A \,|\, A \cdot c & B \end{array}\right)$. □

Remark 12 Observe that $\dim C = \dim \Psi_a^{-1}(\Gamma_T(\Phi_c(C))) + 1$, i. e., C is an upper cover of $\Psi_a^{-1}(\Gamma_T(\Phi_c(C)))$. Hence Γ_T yields the desired correspondence between $\mathcal{L}_t^{c,+}(n)$ and $\mathcal{L}_t^{a_c,-}(n)$ if $T_{a_c,c}$ is regular.

Lemma 13 For every t, there exists a bijection $\varphi_t : \mathcal{F}^t \longrightarrow \mathcal{F}^t$, $c \longmapsto a_c$, such that the matrices $T_{a_c,c}$ are regular for every choice of c.

Proof Observe that $T_{a_c,c}$ is regular if and only if $a_c' \cdot c \neq -1$. Observe also that for every $d \in \mathcal{F}$, $d \neq 0$, the mapping $\mu_d : \mathcal{F} \longrightarrow \mathcal{F}$ defined by

$$\mu_d(x) = \begin{cases} 1 & \text{if } x = 0 \\ \dfrac{d}{x} + 1 & \text{if } x \neq 0 \end{cases}$$

is a bijection such that $\mu_d(x) \cdot x \neq d$ for every $x \in \mathcal{F}$. Thus, we can define $\varphi_1 = \mu_{-1}$ and

$$\varphi_{t+1}\left(\frac{v}{x}\right) = \left(\frac{\varphi_t(v)}{\mu_\alpha(x)}\right)$$

for all $v \in \mathcal{F}^t$ and $x \in \mathcal{F}$, where $\alpha = -\varphi_t(v)' \cdot v - 1$. Obviously, φ_t is a bijection for all t and $a_c' \cdot c = \varphi_t(c)' \cdot c \neq -1$ □

Lemma 13 finishes the proof of Theorem 9. Since this proof is constructive we have

Corollary 14 (Inductive Symmetric Chain Decomposition)
The decomposition theorem can be used to construct inductively symmetric chain decompositions of $\mathcal{L}_t(n)$ for every t and n. □

An Explicit Symmetric Chain Decomposition

In this section, we show how a variation of the bracketing algorithm of Greene and Kleitman [5] can be used to obtain an explicit symmetric chain decomposition of $\mathcal{L}_t(n)$.

Given a $(k+t)$–dimensional subspace $D \in \mathcal{L}_t\binom{n}{k}$ we may use the decomposition theorem to decompose D until we reach a space in $\mathcal{L}_0\binom{0}{0}$. Such a decomposition is determined once we know which mappings Ψ_a, $\Theta_{a,b}$, Φ_c and Λ are involved in it. Thus we could start from the (uniquely determined) space in $\mathcal{L}_0\binom{0}{0}$, use the appropriate embeddings and eventually reach D.

Next we present some notation for writing down such a complete decomposition. Each space in $\mathcal{L}_t(n)$ is decomposed using one of the mappings Ψ, Θ, Φ or Λ, see Figure 2 for a typical situation. Thus we can represent each space in $\mathcal{L}_t(n)$ by a *decomposition sequence* containing the symbols Ψ, Θ, Φ, or Λ, together with certain subscripts.

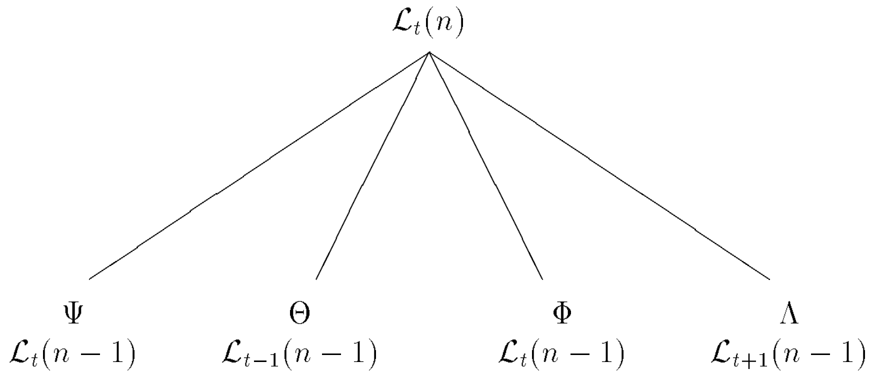

Figure 2: The decomposition of $\mathcal{L}_t(n)$

Definition 15 Let $(\Omega_i \mid i < n)$ be a $\{\Psi, \Theta, \Phi, \Lambda\}$–sequence. We say that the occurrences of Λ are matched by those of Θ, if there exists a matching associating to each Λ precisely one Θ to the right of it.

We may think of Λ as a left (opening) bracket '[' and, accordingly, of Θ as a right (closing) bracket ']'. Then the definition requires that the subsequence of $(\Omega_i \mid i < n)$ consisting of all '[' and of those ']' they match forms a mathematically correct bracketing.

Theorem 16 A sequence $(\Omega_i \mid i < n)$ is a decomposition sequence of some (in fact, uniquely determined, then) subspace $D \in \mathcal{L}_t(n)$, i. e., $D = \Omega_0^{-1}(\Omega_1^{-1}(\ldots(\Omega_{n-1}^{-1}(\omega))\ldots))$, where ω denotes the unique subspace in $\mathcal{L}_0(0)$, if and only if the following hold:

(1) All Λ are matched by Θ.

(2) The number of unmatched Θ is precisely t.

(3) If $\Omega_j \in \{\Psi_a, \Theta_{a,b}, \Phi_c\}$, where $a, c \in \mathcal{F}^k$ and $b \in \mathcal{L}\binom{k}{1}$, then the number of Λ that are unmatched in $(\Omega_i \mid i < j)$ is precisely $k - t$.

If the properties (1) – (3) are satisfied, then the dimension of the subspace represented by $(\Omega_i \mid i < n)$ is the total number of occurrences of Φ and Θ.

Proof Observe that Λ is a map from a subset of $\mathcal{L}_t(n)$ onto $\mathcal{L}_{t+1}(n-1)$, i. e., the subscript is increased from t to $t+1$. Similarly, application of Θ decreases the subscript by 1, and Ψ and Φ leave it unchanged. Since we start with $D \in \mathcal{L}_t(n)$ and end with $\omega \in \mathcal{L}_0(0)$, this proves (1). The subspace $\Omega_j^{-1}(\Omega_{j+1}^{-1}(\dots(\Omega_{n-1}^{-1}(\omega))\dots))$ is in $\mathcal{L}_m(n-j)$, where m equals the number of Θ that are unmatched in $(\Omega_i \mid j \le i < n)$. For $j = 0$ we get (2). If $a, c \in \mathcal{F}^k$, $b \in \mathcal{L}\binom{k}{1}$, and $\Omega_j \in \{\Psi_a, \Theta_{a,b}, \Phi_c\}$, then we have $k = m$, since Ω_j only can be applied to a subspace in $\mathcal{L}_k(n-j)$. By (1) and (2), this implies (3). Hence the decomposition sequence of any subspace D must satisfy (1) – (3).

Conversely, let the sequence $(\Omega_i \mid i < n)$ be such that (1) – (3) are satisfied. We have to show that, for $j = 0, \dots, n-1$, the isomorphism Ω_j^{-1} can be applied to the space $D_{j+1} = \Omega_{j+1}^{-1}(\dots(\Omega_{n-1}^{-1}(\omega))\dots)$. This is obvious if $\Omega_j = \Lambda$. Otherwise, there is $k \ge 0$ such that $\Omega_j \in \{\Psi_a, \Theta_{a,b}, \Phi_c\}$ for appropriate $a, c \in \mathcal{F}^k$ and $b \in \mathcal{L}\binom{k}{1}$. From (1) – (3), we conclude that D_{j+1} is in $\mathcal{L}_k(n-j-1)$ if $\Omega_j \in \{\Psi_a, \Phi_c\}$, and that D_{j+1} is in $\mathcal{L}_{k-1}(n-j-1)$ if $\Omega_j = \Theta_{a,b}$. In either case, Ω_j^{-1} can be applied to D_{j+1}, and we have $D = D_0 \in \mathcal{L}_t(n)$ by (2).

The latter assertion of the theorem holds because the isomorphisms Φ and Θ reduce the number of columns of the Schubert–normal–form, whereas Ψ and Λ leave it unchanged. Thus, the number of occurrences of Φ and Θ equals the number of columns of D, i. e., the dimension of the subspace. □

We may think of the occurrences of Φ and Ψ as left and right angles, i. e., Φ corresponds to '$\langle$' and Ψ corresponds to '$\rangle$'. But here we do not require that these angles match. Observe that, in Theorem 16, for $t = 0$ the number of Λ equals the number of Θ. Therefore the dimension of the subspace is exactly the number of opening brackets and angles ('[' and '$\langle$'). Let us recall the correspondence between Ψ, Θ, Φ, or Λ and the brackets and angles, as explained above:

Convention 17

1. For Ψ_a we write $\rangle_a$.
2. For $\Theta_{a,b}$ we write $]_{a,b}$.
3. For Φ_c we write $\langle_c$.
4. For Λ we write $[$.

Let $(\Omega_i \mid i < n)$ be a decomposition sequence representing a subspace $D \in \mathcal{L}_t(n)$. Let $(\Omega_{l_i} \mid i < m)$ be the subsequence of $(\Omega_i \mid i < n)$ consisting of those Φ– and Ψ–entries, which are not matched. So, typically

$$\begin{array}{rcl} (\Omega_{l_i} \mid i < m) & = & (\Psi_{a_{l_0}}, \ \ldots, \ \Psi_{a_{l_{p-1}}}, \ \Phi_{c_{l_p}}, \ \ldots, \ \Phi_{c_{l_{m-1}}}) \\ & = & (\rangle_{a_{l_0}}, \ \ldots, \ \rangle_{a_{l_{p-1}}}, \ \langle_{c_{l_p}}, \ \ldots, \ \langle_{c_{l_{m-1}}}) \,. \end{array}$$

In this situation, we define $p(D) = l_p$ if $p < m$. Otherwise we say that $p(D)$ does not exist. Similarly, let $s(D) = l_{p-1}$ if $p > 0$. If $p = 0$ then $s(D)$ does not exist.

Now the chain, which D belongs to, consists of subspaces that have the same matched brackets and angles as D in their decomposition sequence and differ only in the angles which are not matched (In this consideration we do not care about the subscripts of the brackets and angles). We then conclude that this symmetric chain has length $m + 1$, and D itself is the $(p + 1)$-st element of this chain. The immediate predecessor of D in the chain is obtained by changing the left angle '$\langle$' at the $p(D)$-th position to a right angle '$\rangle$'. Conversely, we get the immediate successor by changing the '$\rangle$' at the $s(D)$-th position to '$\langle$'. For obtaining the predecessor and the successor of D, resp., it remains to calculate the appropriate subscripts of the new angles.

Observe that, due to the construction presented in the previous section, we have to change the left angle '$\langle_{c_{p(D)}}$' in the decompostion sequence of D to the right angle '$\rangle_{\varphi_t(c_{p(D)})}$' for obtaining the decomposition sequence of the predecessor $D^{\downarrow}$ of D in its chain. But this construction changes the shape of the Schubert–normal–form, therefore the entries in the decomposition sequence of $D^{\downarrow}$ right of the position $p(D)$ will differ from those of the decomposition sequence of D. More precisely, we can use the transformation matrix T given by Lemma 11 to formulate how the entry at position $p(D) + 1$ in the decompostion sequence of $D^{\downarrow}$ can be derived from the corresponding entry in the decomposition sequence of D. If we are able to derive a new transformation matrix from T which can be used for the computation of position $p(D) + 2$ then we can proceed by induction up to the last position $n - 1$. The similar works for the successor $D^{\uparrow}$ of D.

For the calculation of the transformation matrices, we need some more notation. Let $\mathcal{I}_t(n) = \{\Psi_a, \Theta_{a,b}, \Phi_c, \Lambda \mid a, c \in \mathcal{F}^t, b \in \mathcal{L}\binom{t}{1}\}$ be the set of all isomorphisms which may be applied to some elements of $\mathcal{L}_t(n)$. For any $x \in \mathcal{L}\binom{t}{1}$, let $\lambda(x)$ be the position of the leading '1' in x. Furthermore, we denote by y^* the $(t \times 1)$–Schubert–normal–form of the one–dimensional subspace of $\mathcal{F}^t$ that is generated by $y \in \mathcal{F}^t \setminus \{0\}$.

$C_{t,i}$ denotes the unit matrix E_t where the i-th column has been deleted, whereas $R_{t,i} = C'_{t,i}$ denotes the unit matrix E_t where the i-th row has been deleted. Again, e_i denotes the i-th unit vector.

Definition 18 For every regular $(t \times t)$–matrix T mappings ι_T and μ_T which map every element of $\mathcal{I}_t(n)$ to another element of $\mathcal{I}_t(n)$ and a certain matrix, resp., are defined by

$$\begin{array}{rlcl} \iota_T: & \Psi_a & \longmapsto & \Psi_{T'a} \\ & \Theta_{a,b} & \longmapsto & \Theta_{T'a,(T^{-1}b)^*} \\ & \Phi_c & \longmapsto & \Phi_{T^{-1}c} \\ & \Lambda & \longmapsto & \Lambda \end{array} \qquad \begin{array}{rlcl} \mu_T: & \Psi_a & \longmapsto & T \\ & \Theta_{a,b} & \longmapsto & R_{t,\lambda(b)} \cdot (E_t - b \cdot e'_{\lambda(b)}) \cdot T \cdot C_{t,\lambda((T^{-1}b)^*)} \\ & \Phi_c & \longmapsto & T \\ & \Lambda & \longmapsto & \left(\begin{array}{c|c} T & 0 \\ \hline 0 & 1 \end{array}\right) \end{array}$$

Lemma 19 $\mu_T(\Omega)$ is regular for all $\Omega \in \mathcal{I}_t(n)$. Furthermore,

$$\iota_{T^{-1}}(\iota_T(\Omega)) = \Omega \quad \text{and} \quad \mu_{T^{-1}}(\iota_T(\Omega)) = (\mu_T(\Omega))^{-1} .$$

Proof The assertions are obvious for $\Omega \neq \Theta_{a,b}$, and $\iota_{T^{-1}}(\iota_T(\Theta_{a,b})) = \Theta_{a,b}$ follows from $(T((T^{-1}b)^*))^* = b$. Hence it remains to prove that $\mu_{T^{-1}}(\iota_T(\Theta_{a,b})) = (\mu_T(\Theta_{a,b}))^{-1}$.

For this, observe that

$$R_{t,i} \cdot C_{t,i} = E_{t-1} \quad \text{and} \quad C_{t,i} \cdot R_{t,i} = E_t - e_i \cdot e_i'. \tag{1}$$

Furthermore, for all $x \in \mathcal{L}\binom{t}{1}$ we have

$$\begin{aligned}
&(E_t - e_{\lambda(x)} \cdot e'_{\lambda(x)}) \cdot (E_t - x \cdot e'_{\lambda(x)}) \\
&\quad = E_t - x \cdot e'_{\lambda(x)} - e_{\lambda(x)} \cdot e'_{\lambda(x)} + e_{\lambda(x)} \cdot e'_{\lambda(x)} \cdot x \cdot e'_{\lambda(x)} \\
&\quad = E_t - x \cdot e'_{\lambda(x)}
\end{aligned} \tag{2}$$

since $e'_{\lambda(x)} \cdot x = 1$. We calculate

$$\begin{aligned}
&\mu_T(\Theta_{a,b}) \cdot \mu_{T^{-1}}(\iota_T(\Theta_{a,b})) \\
&\quad = R_{t,\lambda(b)} \cdot (E_t - b \cdot e'_{\lambda(b)}) \cdot T \cdot C_{t,\lambda((T^{-1}b)^*)} \cdot R_{t,\lambda((T^{-1}b)^*)} \\
&\qquad \cdot (E_t - (T^{-1}b)^* \cdot e'_{\lambda((T^{-1}b)^*)}) \cdot T^{-1} \cdot C_{t,\lambda(b)} \\
&\quad = R_{t,\lambda(b)} \cdot (E_t - b \cdot e'_{\lambda(b)}) \cdot T \cdot (E_t - (T^{-1}b)^* \cdot e'_{\lambda((T^{-1}b)^*)}) \cdot T^{-1} \cdot C_{t,\lambda(b)}
\end{aligned} \tag{3}$$

using (1) and (2) with $x = (T^{-1}b)^*$. Now observe that there is $\alpha \in \mathcal{F}$ such that $\alpha \cdot b = T \cdot (T^{-1}b)^*$. Hence

$$\begin{aligned}
&(E_t - b \cdot e'_{\lambda(b)}) \cdot T \cdot (E_t - (T^{-1}b)^* \cdot e'_{\lambda((T^{-1}b)^*)}) \cdot T^{-1} \\
&\quad = E_t - b \cdot e'_{\lambda(b)} - T \cdot (T^{-1}b)^* \cdot e'_{\lambda((T^{-1}b)^*)} \cdot T^{-1} \\
&\qquad + b \cdot e'_{\lambda(b)} \cdot T \cdot (T^{-1}b)^* \cdot e'_{\lambda((T^{-1}b)^*)} \cdot T^{-1} \\
&\quad = E_t - b \cdot e'_{\lambda(b)} - \alpha \cdot b \cdot e'_{\lambda((T^{-1}b)^*)} \cdot T^{-1} + b \cdot e'_{\lambda(b)} \cdot \alpha \cdot b \cdot e'_{\lambda((T^{-1}b)^*)} \cdot T^{-1} \\
&\quad = E_t - b \cdot e'_{\lambda(b)}
\end{aligned}$$

because of $e'_{\lambda(b)} \cdot b = 1$. Thus, equation (3) becomes

$$\mu_T(\Theta_{a,b}) \cdot \mu_{T^{-1}}(\iota_T(\Theta_{a,b})) \quad = \quad R_{t,\lambda(b)} \cdot C_{t,\lambda(b)} - R_{t,\lambda(b)} \cdot b \cdot e'_{\lambda(b)} \cdot C_{t,\lambda(b)} .$$

If we use $e'_{\lambda(b)} \cdot C_{t,\lambda(b)} = 0$ and again (1) we obtain

$$\mu_T(\Theta_{a,b}) \cdot \mu_{T^{-1}}(\iota_T(\Theta_{a,b})) \quad = \quad E_{t-1} .$$

Therefore $\mu_{T^{-1}}(\iota_T(\Theta_{a,b}))$ is regular, and its inverse is $\mu_T(\Theta_{a,b})$. □

The decomposition theorem (Theorem 8) says that for every $D \in \mathcal{L}_t(n)$ there is only one isomorphism $\Omega \in \mathcal{I}_t(n)$ that can be applied to D. We say then that Ω is the *corresponding isomorphism* to D.

Lemma 20 Let $\Omega \in \mathcal{I}_t(n)$ be the corresponding isomorphism to $D \in \mathcal{L}_t(n)$. Then $\iota_T(\Omega)$ is the corresponding isomorphism to $\Gamma_T(D)$. Additionally, $\Gamma_{\mu_T(\Omega)} \circ \Omega = \iota_T(\Omega) \circ \Gamma_T$.

Proof Simple calculations with Schubert–normal–forms show that $\iota_T(\Omega)$ is the corresponding isomorphism to $\Gamma_T(D)$, and that $\Gamma_{\mu_T(\Omega)} \circ \Omega = \iota_T(\Omega) \circ \Gamma_T$ for $\Omega \neq \Theta_{a,b}$. Only the proof of $\Gamma_{\mu_T(\Theta_{a,b})} \circ \Theta_{a,b} = \iota_T(\Theta_{a,b}) \circ \Gamma_T$ requires more complicated calculations. Hence let

$$C = \left(\begin{array}{c|c} \begin{array}{c} E_t \\ \hline a' \end{array} & 0 \\ \hline A & B \end{array} \right) \in \mathcal{L}_{t-1}^{a,b}(n) \,.$$

Then we have

$$\Theta_{a,b}\left(\left(\begin{array}{c|c} \begin{array}{c} E_t \\ \hline a' \end{array} & 0 \\ \hline A & B \end{array} \right) \right) = \left(\begin{array}{c|c} E_{t-1} & 0 \\ \hline A \cdot C_{\lambda(b)} & B \end{array} \right) ,$$

and application of $\Gamma_{\mu_T(\Theta_{a,b})}$ yields

$$\left(\begin{array}{c|c} E_{t-1} & 0 \\ \hline A \cdot C_{\lambda(b)} \cdot R_{t,\lambda(b)} \cdot (E_t - b \cdot e'_{\lambda(b)}) \cdot T \cdot C_{t,\lambda((T^{-1}b)^*)} & B \end{array} \right) .$$

But we have

$$\begin{aligned} A \cdot C_{\lambda(b)} \cdot R_{t,\lambda(b)} \cdot (E_t - b \cdot e'_{\lambda(b)}) &= A \cdot (E_t - e_{\lambda(b)} \cdot e'_{\lambda(b)}) \cdot (E_t - b \cdot e'_{\lambda(b)}) \\ &= A \cdot (E_t - b \cdot e'_{\lambda(b)}) \\ &= A - A \cdot b \cdot e'_{\lambda(b)} = A \,, \end{aligned}$$

since $A \cdot b = 0$. Now it is easy to see that the resulting Schubert–normal–form is exactly $\iota_T(\Theta_{a,b})(\Gamma_T(C)))$. □

From now on we assume that a family (φ_t) of bijections $\varphi_t : \mathcal{F}^t \longrightarrow \mathcal{F}^t$ with $\varphi_t(c)' \cdot c \neq -1$ is fixed (e. g. the one introduced in the proof of Lemma 13). Next we use the mappings ι_T and μ_T to describe the decomposition sequences which arise from the construction mentioned in Remark 12.

Lemma 21 Let $(\Omega_i \mid i < n)$ be the decomposition sequence of a $D \in \mathcal{L}_t(n)$.

1. If $D \in \mathcal{L}_t^{c,+}(n)$ for some $c \in \mathcal{F}^t$ then the decomposition sequence $(\Omega_i^{\Downarrow} \mid i < n)$ of the subspace $\Psi^{-1}_{\varphi_t(c)}(\Gamma_{T_{\varphi_t(c),c}}(\Phi_c(D)))$ is obtained by

$$\Omega_i^{\Downarrow} = \begin{cases} \Psi_{\varphi_t(c)} & \text{if } i = 0 \\ \iota_{T_{i-1}^{\Downarrow}}(\Omega_i) & \text{if } i > 0 \end{cases}$$

   where $T_0^{\Downarrow} = T_{\varphi_t(c),c} = E_t + c \cdot \varphi_t(c)'$ and $T_i^{\Downarrow} = \mu_{T_{i-1}^{\Downarrow}}(\Omega_i)$ for all $i > 0$.

2. If $D \in \mathcal{L}_t^{a,-}(n)$ for some $a \in \mathcal{F}^t$ then the decomposition sequence $(\Omega_i^{\Uparrow} \mid i < n)$ of the subspace $\Phi^{-1}_{\varphi_t^{-1}(a)}(\Gamma^{-1}_{T_{a,\varphi_t^{-1}(a)}}(\Psi_a(D)))$ is obtained by

$$\Omega_i^{\Uparrow} = \begin{cases} \Phi_{\varphi_t^{-1}(a)} & \text{if } i = 0 \\ \iota_{T_{i-1}^{\Uparrow}}(\Omega_i) & \text{if } i > 0 \end{cases}$$

   where $T_0^{\Uparrow} = T^{-1}_{a,\varphi_t^{-1}(a)} = (E_t + \varphi_t^{-1}(a) \cdot a')^{-1}$ and $T_i^{\Uparrow} = \mu_{T_{i-1}^{\Uparrow}}(\Omega_i)$ for all $i > 0$.

Proof Let $(\Omega_i^{\Downarrow} \mid i < n)$ be the decomposition sequence of $\Psi_{\varphi t(c)}^{-1}(\Gamma_{T_{\varphi t(c),c}}(\Phi_c(D)))$. By definition we have $\Omega_0^{\Downarrow} = \Psi_{\varphi t(c)}$. Furthermore, $(\Omega_i \mid 0 < i < n)$ is the decomposition sequence of $D_0 = \Phi_c(D)$ whereas $(\Omega_i^{\Downarrow} \mid 0 < i < n)$ is the decomposition sequence of $D_0^{\Downarrow} = \Gamma_{T_{\varphi t(c),c}}(\Phi_c(D)) = \Gamma_{T_0^{\Downarrow}}(D_0)$.

By the first part of Lemma 20, we know that $\Omega_1^{\Downarrow} = \iota_{T_0^{\Downarrow}}(\Omega_1)$. Now, let $D_1 = \Omega_1(D_0)$ and $D_1^{\Downarrow} = \Omega_1^{\Downarrow}(D_0^{\Downarrow})$. Then we have

$$D_1^{\Downarrow} \quad = \quad \Omega_1^{\Downarrow}(\Gamma_{T_0^{\Downarrow}}(D_0)) \quad = \quad \Gamma_{\mu_{T_0^{\Downarrow}}(\Omega_1)}(D_1) \quad = \quad \Gamma_{T_1^{\Downarrow}}(D_1)$$

by the second part of Lemma 20. Hence inductive applications of Lemma 20 prove the assertion. The assertion concerning the $\Omega_i^{\Uparrow}$ can be shown dually. □

From now on, we understand $^{\Downarrow}$ and $^{\Uparrow}$ as operators that can be applied to an arbitrary decomposition sequence $(\Omega_i \mid i < n)$, i. e., we can choose $j < n$ such that $\Omega_j = \Phi_c$ and apply $^{\Downarrow}$ to the subsequence $(\Omega_i \mid j \leq i < n)$, and similarly for $^{\Uparrow}$. Using $^{\Downarrow}$ and $^{\Uparrow}$ this way, we can formulate the algorithm that computes for each subspace in $\mathcal{L}_t(n)$ the decomposition sequence of its predecessor and its successor, resp.

Theorem 22 A symmetric chain decomposition of $\mathcal{L}_t(n)$ can be obtained by the following construction, where $(\Omega_i \mid i < n)$ denotes the decomposition sequence of a $D \in \mathcal{L}_t(n)$.

1. If $p(D)$ does not exist, then D is the minimal element of a chain. Otherwise the decomposition sequence $(\Omega_i^{\downarrow} \mid i < n)$ of the predecessor $D^{\downarrow}$ of D in its chain is obtained by
$$\Omega_i^{\downarrow} \quad = \quad \begin{cases} \Omega_i & \text{if } i < p(D) \\ \Omega_i^{\Downarrow} & \text{if } i \geq p(D) \end{cases}$$
2. If $s(D)$ does not exist, then D is the maximal element of a chain. Otherwise the decomposition sequence $(\Omega_i^{\uparrow} \mid i < n)$ of the successor $D^{\uparrow}$ of D in its chain is obtained by
$$\Omega_i^{\uparrow} \quad = \quad \begin{cases} \Omega_i & \text{if } i < s(D) \\ \Omega_i^{\Uparrow} & \text{if } i \geq s(D) \end{cases}$$

Proof We prove the theorem by induction on n, again following the structure described in the decomposition theorem. If $D \in \mathcal{L}_t^{a,-}(n)$, $D \in \mathcal{L}_{t-1}^{a,b}(n)$, or $D \in \mathcal{L}_{t+1}(n-1)$ then the immediate predecessor of D in its chain is in the same set (if it exists). Therefore we have $\Omega_0 = \Omega_0^{\downarrow}$. Furthermore, in all these cases, $p(D) > 0$ by definition, because Ω_0 cannot be Φ_c for any $c \in \mathcal{F}^t$. Since, by assumption, the assertion holds for $n-1$, we can compute the decomposition sequence of $D^{\downarrow}$ as described, and $D^{\downarrow}$ exists if and only if $p(D)$ exists.

Now we consider the case $D \in \mathcal{L}_t^{c,+}(n)$. By the inductive assumption we know that $p(D) = 0$ if and only if D is the maximal element of a symmetric chain in $\mathcal{L}_t^{c,+}(n)$. Thus, if D is not the maximal element of a symmetric chain of $\mathcal{L}_t^{c,+}(n)$, then $D^{\downarrow}$ is also in $\mathcal{L}_t^{c,+}(n)$, and similar to the cases above the assertion holds for n. It remains to consider the case that D is the maximal element of a symmetric chain in $\mathcal{L}_t^{c,+}(n)$, i. e., $p(D) = 0$. By the construction of the symmetric chains in the previous section, we have

$D^{\downarrow} = \Psi^{-1}_{\varphi_t(c)}(\Gamma_{T_{\varphi_t(c),c}}(\Phi_c(D)))$. Hence, a simple application of Lemma 21 proves this case, too.

So far we have proved that we get the symmetric chain decomposition of $\mathcal{L}_t(n)$ which was introduced in the previous section via computing the decomposition sequences $(\Omega_i^{\downarrow} \mid i < n)$. It remains to show that the immediate successor of a subspace can be obtained from the calculation of $(\Omega_i^{\uparrow} \mid i < n)$ as stated above. So let

$$\Omega_i^{\dagger} \quad = \quad \begin{cases} \Omega_i & \text{if } i < s(C) \\ \Omega_i^{\Uparrow} & \text{if } i \geq s(C) \end{cases}$$

for all $C \in \mathcal{L}_t(n)$ where $s(C)$ exists. Obviously, $s(D^{\downarrow}) = p(D)$ and $\Omega_i^{\downarrow\dagger} = \Omega_i$ for all $i < p(D)$. By construction, we have

$$\Omega^{\downarrow\dagger}_{p(D)} \quad = \quad \Omega^{\downarrow\Uparrow}_{p(D)} \quad = \quad \Omega^{\Downarrow\Uparrow}_{p(D)} \quad = \quad \Psi^{\Uparrow}_{\varphi_t(c)} \quad = \quad \Phi_{\varphi_t^{-1}(\varphi_t(c))} \quad = \quad \Phi_c \quad = \quad \Omega_{p(D)}$$

and $T^{\Uparrow}_{p(D)} = (T^{\Downarrow}_{p(D)})^{-1}$. Induction on i and application of Lemma 19 leads to

$$T_i^{\Uparrow} \quad = \quad \mu_{(T^{\Uparrow}_{i-1})}(\iota_{T^{\Downarrow}_{i-1}}(\Omega_i)) \quad = \quad \mu_{(T^{\Downarrow}_{i-1})^{-1}}(\iota_{T^{\Downarrow}_{i-1}}(\Omega_i)) \quad = \quad (\mu_{T^{\Downarrow}_{i-1}}(\Omega_i))^{-1} \quad = \quad (T_i^{\Downarrow})^{-1}$$

for all $i > p(D)$. Again using Lemma 19, we get

$$\Omega_i^{\downarrow\dagger} \quad = \quad \Omega_i^{\downarrow\Uparrow} \quad = \quad \Omega_i^{\Downarrow\Uparrow} \quad = \quad \iota_{(T^{\Uparrow}_{i-1})}(\iota_{T^{\Downarrow}_{i-1}}(\Omega_i)) \quad = \quad \iota_{(T^{\Downarrow}_{i-1})^{-1}}(\iota_{T^{\Downarrow}_{i-1}}(\Omega_i)) \quad = \quad \Omega_i$$

4	$\langle\langle\langle\langle$				
3	$\rangle\langle\langle\langle$	$\langle\rangle\langle\langle$	$\langle\langle\rangle\langle$	$\langle\langle\langle\rangle$	$[\,]_{1,1}\langle\langle$
2	$\rangle\rangle\langle\langle$	$\langle\rangle\rangle\langle$	$\rangle\langle\rangle\langle$	$\rangle\langle\langle\rangle$	$[\,]_{1,1}\rangle\langle$
1	$\rangle\rangle\rangle\langle$	$\langle\rangle\rangle\rangle$	$\rangle\langle\rangle\rangle$	$\rangle\rangle\langle\rangle$	$[\,]_{1,1}\rangle\rangle$
0	$\rangle\rangle\rangle\rangle$				

3	$\langle[\,]_{1,1}\langle$	$\langle\langle[\,]_{1,1}$	$[\langle_0]_{1,1}\langle$	$[\langle_1]_{1,1}\langle$	$\langle[\langle_0]_{1,1}$
2	$\rangle[\,]_{1,1}\langle$	$\rangle\langle[\,]_{1,1}$	$[\rangle_1]_{1,1}\langle$	$[\rangle_0]_{1,1}\langle$	$\rangle[\langle_0]_{1,1}$
1	$\rangle[\,]_{1,1}\rangle$	$\rangle\rangle[\,]_{1,1}$	$[\rangle_1]_{1,1}\rangle$	$[\rangle_0]_{1,1}\rangle$	$\rangle[\rangle_1]_{1,1}$

3	$\langle[\langle_1]_{1,1}$	$[\langle_0\langle_0]_{1,1}$	$[\langle_0\langle_1]_{1,1}$	$[\langle_1\langle_0]_{1,1}$	$[\langle_1\langle_1]_{1,1}$
2	$\rangle[\langle_1]_{1,1}$	$[\rangle_1\langle_0]_{1,1}$	$[\rangle_1\langle_1]_{1,1}$	$[\rangle_0\langle_0]_{1,1}$	$[\rangle_0\langle_1]_{1,1}$
1	$\rangle[\rangle_0]_{1,1}$	$[\rangle_1\rangle_1]_{1,1}$	$[\rangle_1\rangle_0]_{1,1}$	$[\rangle_0\rangle_1]_{1,1}$	$[\rangle_0\rangle_0]_{1,1}$

2	$\langle\langle\rangle\rangle$	$\langle\rangle\langle\rangle$	$[\,]_{1,1}\langle\rangle$	$\langle[\,]_{1,1}\rangle$	$\langle\rangle[\,]_{1,1}$

2	$[\langle_0]_{1,1}\rangle$	$[\langle_1]_{1,1}\rangle$	$\langle[\rangle_0]_{1,1}$	$\langle[\rangle_1]_{1,1}$	$[\langle_0\rangle_0]_{1,1}$

2	$[\langle_0\rangle_1]_{1,1}$	$[\langle_1\rangle_0]_{1,1}$	$[\langle_1\rangle_1]_{1,1}$	$[\,]_{1,1}[\,]_{1,1}$	$[[\,]_{\binom{0}{1},\binom{0}{1}}]_{1,1}$

2	$[[\,]_{\binom{0}{1},\binom{1}{1}}]_{1,1}$	$[[\,]_{\binom{1}{0},\binom{1}{0}}]_{1,1}$	$[[\,]_{\binom{1}{0},\binom{1}{1}}]_{1,1}$	$[[\,]_{\binom{1}{1},\binom{0}{1}}]_{1,1}$	$[[\,]_{\binom{1}{1},\binom{1}{0}}]_{1,1}$

Figure 3: A symmetric chain decomposition of $\mathcal{L}(4)$ over GF(2)

for all $i > p(D)$. This proves $D^{\downarrow\dagger} = D = D^{\downarrow\uparrow}$, and $D^{\uparrow\downarrow} = D^{\uparrow\downarrow}$ follows dually. Hence we have $C^{\dagger} = C^{\uparrow}$ whenever $C^{\uparrow}$ exists, what finishes the proof. $\square$

Let us close our paper with the example $\mathrm{GF}(2)^4$. In this case we only have to consider the mapping $\varphi_1 : \mathrm{GF}(2) \longrightarrow \mathrm{GF}(2)$ which maps 0 to 1 and 1 to 0. Then the tables in Figure 3 show the resulting symmetric chains of $\mathcal{L}(4)$, represented by their decomposition sequences. Each column represents a symmetric chain, the numbers in the leftmost column indicate the dimension of the subspace. For keeping notations shorter we omit the parentheses of the one–dimensional vectors in the subscripts.

Acknowledgements The results of this work were obtained while the first author was at the *Universität Bielefeld* as a guest of the *Graduiertenkolleg Mathematik*.

References

[1] N. G. de Bruin, D. Kruyswijk, and C. A. v. Tengbergen. On the set of divisors of a number. *Nieuw Arch. Wiskunde* **23**(2) (1951), 191–193.

[2] W. Deuber and B. Voigt. Partitionseigenschaften endlicher affiner und projektiver Räume. *Europ. J. Combinatorics* **3** (1982), 329–340.

[3] K. Engel and H.–D. O. F. Gronau. *Sperner theory in partially ordered sets.* B. G. Teubner Verlagsgesellschaft, Leipzig, 1985.

[4] K. Fleischer. *Explizite Zerlegungen des Vektorraum–Verbandes.* Master's thesis, Universität Erlangen–Nürnberg, 1983.

[5] C. Greene and D. J. Kleitman. Proof techniques in the theory of finite sets. In: G.–C. Rota (ed.), *Studies in combinatorics*, vol. 17 of MAA Studies in Mathematics, 1978.

[6] J. R. Griggs. Sufficient conditions for a symmetric chain order. *SIAM J. Appl. Math.* **32** (1977), 807–809.

[7] E. Sperner. Ein Satz über Untermengen einer endlichen Menge. *Math. Z.* **27** (1928), 544–548.