A SPLITTING PRINCIPLE

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Abstract. A new technical tool for K-theoretical computation is given. It is a "splitting principle". It is applied to compute K-groups of any principal G-homogeneous space X, where G is an arbitrary semisimple simply connected algebraic group over a field F. The answer is given in terms of K-groups of certain central simple F-algebras depending only of G. In particular we prove the isomorphism $K_0(F[X] \otimes \mathcal{D}) \cong K_0(\mathcal{D})$ for any finite dimensional separable F-algebra $\mathcal{D}$.

Introduction

The present paper has two main goals. The first one is to give a new technical tool for certain K-theoretical computations (Theorem 1.1). It is a "splitting principle". The second goal is to apply this "splitting principle" to compute K-groups of any principal G-homogeneous space X, where G is an arbitrary semimisimple and simply connected algebraic group (Theorem 5.1). (Actually G is subject to a condition (see §5), which we omit in the introduction.) For K_0-groups Theorem 5.1 states the isomorphism

$$K_0(\mathcal{D}) \cong K_0(F[X] \otimes_F \mathcal{D}) \tag{*}$$

for any finite-dimensional separable F-algebra $\mathcal{D}$.

Consequences of isomorphisms $(*)$ are the injectivity of the Brauer group homomorphism $\mathrm{Br}(F) \to \mathrm{Br}(F(X))$ and the triviality of the first and the second Chow groups of X (see 5.2).

Actually we compute higher K-groups of mentioned varieties as well. The corresponding result is also stated in 5.1. It expresses K-groups of X in terms of K-groups of certain central simple F-algebras, which are listed in 5.4. Theorem 5.1 generalizes a theorem of Suslin ([Su], 6.3) and a theorem of Levine ([L], 4.3), where K-groups of certain norm varieties are computed.

The model case of Theorem 5.1 is the following result due to Levine ([L], 2.3):

$$\coprod_I K_*(F)[-\operatorname{card}(I)] \cong K_*(G)$$

for every semisimple and simply connected F-split algebraic group G over a field F, where the sum runs over all subsets I of the set $\{1, 2, \ldots, \mathrm{rk}(G)\}$. Our "splitting principle" reduces the general case of Theorem 5.1 to this model case (see 10.1, 10.2).

This paper consists of two parts. The first one is §§1 − −4. It is devoted to the "splitting principle". The second one is §§5 − −14. It is devoted to the K-theory of principal homogeneous spaces. The main result of the first part is Theorem 1.1. The main result of the second part is Theorem 5.1.

To be more specific, in §1 we state the "splitting principle" (Theorem 1.1) and prove a more technical result (see 1.4). In §2, we recall the relations $f_*[\mathcal{O}_{Y\times S}] = [\mathcal{O}_S]$ for a certain class of varieties Y, and prove the triviality of any principal B-homogeneous space for any Borel subgroup B of any semisimple algebraic group. In §3, we introduce the notions of K-module and K-module morphism (see 3.2, 3.4), discuss their properties (3.3, 3.5), and consider examples (3.6, 3.7). In §4, we prove the splitting principle (1.1) using results of §2 and prove Theorem 1.4 using above relations $f_*[\mathcal{O}_{Y\times S}] = [\mathcal{O}_S]$.

In §5, the Main Theorem (see 5.1) is stated. The list of algebras is given in 5.4, and the map from 5.1 is described in 5.3. In §6 we recall the Chevalley theorem about the representation ring of semisimple algebraic group (see 6.1), then we define a notion of fundamental family of representations (see 6.2) and consider examples (see 6.3). In §7, Levine's result is stated (see 7.2) and a modification of it is proved (see 7.6). In §8, we describe Tits' construction of central simple F-algebras arising ¿from representations (see 8.5) and discuss its properties (see 8.7–8.10). In §9 we state variants of Theorem 5.1 using Tits' construction and the fundamental family of representations (see 9.3, 9.5, 9.7). In §10 we prove Theorems 9.5 and 9.7 using 1.1 and 1.4. In §§11, 12, 13 technical lemmas used in §10 are proved. In §14 relations between Clifford algebras and spinor representations are discussed (see 14.8). In §15, we deduce Theorem 5.1 from 9.7 using 8.7 and 13.8.

Throughout the present paper we shall use the following notation and conventions:

F is a field,

$\mathcal{D}$ is a separable F-algebra,

$\mathcal{D}^{\mathrm{op}}$ is the opposite F-algebra.

The word "scheme" means a separate noetherian F-scheme of finite Krull's dimension, and if X is a scheme, then

$\mathcal{P}(X)$ is the category of locally free coherent $\mathcal{O}_X$-modules,

$\mathcal{M}(X)$ is the category of coherent $\mathcal{O}_X$-modules,

$\mathcal{M}(X, \mathcal{D}^{\mathrm{op}})$ is the category of coherent right $\mathcal{D}\otimes_F \mathcal{O}_X$-modules,

$\mathcal{P}(X, \mathcal{D}^{\mathrm{op}})$ is the category of locally projective coherent right $\mathcal{D}\otimes_F \mathcal{O}_X$-modules, or, what is the same, the category of locally free coherent $\mathcal{O}_X$-modules equipped with right $\mathcal{D}$-action and morphisms compatible with this $\mathcal{D}$-action.

Let $K_i(\mathcal{C}) := \pi_{i+1}(BQ\mathcal{C})$ for any exact category $\mathcal{C}$, where $Q\mathcal{C}$ is the category obtained by Quillen's Q-construction ([Q],§2), and $BQ\mathcal{C}$ denotes the classifying space of $Q\mathcal{C}$. As usual, let

$$\begin{aligned} K_i(X) &= K_i(\mathcal{P}(X)), \\ K_i'(X) &= K_i(\mathcal{M}(X)), \\ K_i(X, \mathcal{D}^{\mathrm{op}}) &= K_i(\mathcal{P}(X, \mathcal{D}^{\mathrm{op}})), \\ K_i'(X, \mathcal{D}^{\mathrm{op}}) &= K_i(\mathcal{M}(X, \mathcal{D}^{\mathrm{op}}). \end{aligned}$$

1. Splitting principle I

The main goal of te present sectin is to state our "Splitting principle" in its first form (Theorem 1.1 below). The proof of Theorem 1.1 given in §4 is based on certain properties of functors $K'_A\, K'_X : Sch/F \rightrightarrows$ Gr-Ab given by

$$S \mapsto K'_*(S, A), \S \mapsto K'_*(S \times X)$$

and their transformations $\theta_\alpha : K_A \to K_X$ given by $x \mapsto \alpha \cup_A x$. These properties are discussed in §3, where notions of $\mathbb{K}$-modules and their morphisms are introduced In §4 we restate and prove "Splitting principle" using these notions and Theorem 3.5.

Another goal of the present section is to state a technical result (Theorem 1.4), which also is proved in §4.

Let X be a noetherian F-scheme and let $A_1, A_2, \ldots, A_n$ be finite-dimensional separable F-algebras. Let $\alpha_i \in K_{r_i}(X, A_i^{\mathrm{op}})$ be some elements $(i = 1, 2, \ldots, n)$, where "op" denotes the opposite algebra structure. For any F-scheme S define maps

$$\theta_i(S) : K'_*(S, A_i)[-r_i] \to K'_*(S \times X)$$

by $x \mapsto \alpha_i \cup_{A_i} x$. Set

$$\theta(S) = \sum_{i=1}^{n} \theta_i(S) = \coprod_{i=1}^{n} K'_*(S, A_i)[-r_i] \to K'_*(S \times X)$$

.

1.1. Theorem (Splitting principle I). *If G is a connected semisimple algebraic group over F and $\mathcal{G}$ is a principal G-homogeneous space, then the following statements are equivalent.*

(1) $\theta(S)$ is an isomorphism for every noetherian F-scheme S,

(2) $\theta(E)$ is an isomorphism for any field extension E/F such that $\mathcal{G}(E) \neq \varnothing$.

Proof will be given in § 4 (see 4.2).

1.2. Remark.

1) One should consider not only finite field extensions E/F with the property that $\mathcal{G}(E) \neq \varnothing$.

2) If $\gamma : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to G(F_{\mathrm{sep}})$ is a 1-cocycle corresponding to the principal G-homogeneous space $\mathcal{G}$, then $\mathcal{G}(E) \neq \varnothing$ if and only if γ/E is a coboundary.

1.3. Let B be a finite dimensional separable F-algebra. Define

$$\theta_i^B(S) : K'_*(S, A_i \otimes B)[-r_i] \to K'_*(S, B)$$

by $x \to \alpha_i \cup_{A_i} x$. Set

$$\theta^B(S) = \sum_{i=1}^{n} \theta_i^B(S) : \coprod_{i=1}^{n} K'_*(S, A_i \otimes B)[-r_i] \to K'_*(S \times X, B).$$

1.4. Theorem. *The following statements are equivalent.*

(1) $\theta(S)$ *is an isomorphism for every noetherian* F*-scheme* S*,*
(2) $\theta^D(S)$ *is an isomorphism for any central simple* F*-algebra* D *and any noetherian* F*-scheme* S*.*

Proof will be given in § 4 (see 4.4).

1.5. Remark.

Both of the statements of Theorem 1.4 are equivalent to the following one.

(1) $\theta^D(S)$ is an isomorphism for any finite dimensional separable F-algebra D and any noetherian F-scheme S.

1.6.

This statement arised in discussion with A.Merkurjev. It gives a possibility to deduce Theorem 5.1 from its special case $D = F$, if this special case holds for any scalar extension of F. Before that, the author deduced Theorem 5.1 for any D from the D-variant of Levine's result.

1.7. An example of application of 1.1.

Let $G = \mathrm{PGL}_n$ and let $B_i = End_F(V_n^{\otimes i}) = End_F(V_n)^{\otimes i}$, where V_n is an n-dimensional F-vector space. Let $\mathbb{P}_F^{n-1}$ be the projective space. Consider actions of G on $\mathbb{P}_F^{n-1}$ and on B_i by projective linear transformations and by conjugation, respectively.

Let $\alpha_i = [(\mathcal{O}(-1) \otimes V_n^*)^{\otimes i}]$ be the element of $K_0^G(\mathbb{P}_F^{n-1}, B_i^{\mathrm{op}})$ corresponding to the G-equivariant vector bundle $(\mathcal{O}(-1) \otimes V_n^*)^{\otimes i}$. Let $\gamma : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to G(F_{\mathrm{sep}})$ be a 1-cocycle and let $\mathbb{P}_{F,\gamma}^{n-1}$, $B_{i,\gamma}$, and $\alpha_{i,\gamma} \in K_\circ(\mathbb{P}_{F,\gamma}^{n-1}, B_{i,\gamma}^{\mathrm{op}})$ be twisted forms of $\mathbb{P}_F^{n-1}$, B_i, and $\mathcal{O}(-1) \otimes V_n^*)^{\otimes i}$ respectively.

Then the map

$$\coprod_{i=0}^{n-1} K_*(B_{i,\gamma}) \to K_*(\mathbb{P}_{F,\gamma}^{n-1}) \quad (x_i \mapsto \alpha_{i,\gamma} \otimes_{B_i} x_i)$$

is an isomorphism (see Quillen, [Q], §8, Th. 4.1). Let us prove this using 1.1.

It is sufficient to verify that the map

$$(*) \qquad \coprod_{i=0}^{n-1} K_*(B_{i,\gamma} \otimes_F E) \to K_*(\mathbb{P}_{\gamma,E}^{n-1})$$

is an isomorphism for all field extensions E/F such that $\gamma_E \sim *_E$. If $\gamma_E \sim *_E$, then one can rewrite the last map as the map

$$\coprod_{i=0}^{n-1} K_*(B_i \otimes_F E) \to K_*(\mathbb{P}_E^{n-1}) \quad (y_i \mapsto \alpha_i \otimes_{B_i} y_i)$$

The last map can be changed for an equivalent map

$$\coprod_{i=0}^{n-1} K_*(E) \to K_*(\mathbb{P}_E^{n-1}) \quad (z_i \mapsto \mathcal{O}(-i) \otimes z_i),$$

which is an isomorphism (see [Q], §8, Th. 2.1). Hence (*) is an isomorphism.

This example gives a typical pattern how one can apply "Splitting principle".

2. Three lemmas

2.1. Direct image map. Let S be a noetherian scheme and let $f : T \to S$ be a flat projective morphism. Let $\mathcal{P}(T)$ (respectively $\mathcal{P}(S)$) be the category of finite generated locally free $\mathcal{O}_T$- (respectively, $\mathcal{O}_S$-) modules. Let $\mathcal{P}(f,T)$ be the full subcategory of the category $\mathcal{P}(T)$ consisting of $\mathcal{F}$ such that $R^i f_*(F) = 0$ for $i > 0$. Then the functor $f_* : \mathcal{P}(f,T) \to \mathcal{M}(S)$ actually takes value in $\mathcal{P}(S)$ (see [M], II.5.2) and it determines a map $K(\mathcal{P}(f,T)) \to K(\mathcal{P}(S))$.

Proposition 0.1 of [P1] states that every $\mathcal{F} \in \mathcal{P}(T)$ has a right resolution of finite length

$$0 \to \mathcal{F} \to \mathcal{E}_0 \to \mathcal{E}_1 \to \cdots \to \mathcal{E}_n \to 0,$$

consisting of $\mathcal{E}_i \in \mathcal{P}(f,T)$. Hence the inclusion $\mathcal{P}(f,T) \hookrightarrow \mathcal{P}(T)$ induces an isomorphism $K_*(\mathcal{P}(f,T)) \xrightarrow{\approx} K_*(\mathcal{P}(T)) = K_*(T)$. Composing it with the map $K_*(\mathcal{P}(f,T)) \to K_*(\mathcal{P}(S))$ we obtain a homomorphism

$$f_* : K_*(T) \to K_*(S),$$

which is called the *direct image map*. Three properties of this map are important for this paper:

(1) $f_*([F]) = [f_*(F)]$ for $F \in \mathcal{P}(f,T)$,
(2) $f_*(x \cdot f^*(y)) = f_*(x) \cdot y$ for any $x \in K_0(T), y \in K'_*(S)$, where $K'_*(S) = K_*(calM(S))$,
(3) $g^* \circ f_* = f'_* \circ (g')^* : K_0(T) \to K_0(S')$

for every Cartesian diagram

$$\begin{tikzcd} T' \arrow[r, "g'"] \arrow[d, "f'"'] & T \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

where g is flat and S' is noetherian.

2.2. Let G be a semisimple algebraic group over F and let $P \subset G$ be a closed subgroup such that $\mathcal{F} = G/P$ is a projective variety. Let X be a variety over F such that $X_{\text{sep}} \xrightarrow{\sim} F_{\text{sep}}$.

Lemma. *Let S be an arbitrary Noetherian scheme over F and let*

$$\begin{tikzcd} X \times S \arrow[r, "p_S"] \arrow[d, "p_X"'] & S \arrow[d, "g"] \\ X \arrow[r, "f"] & \operatorname{Spec} F \end{tikzcd}$$

be a Cartesian diagram. Let $f_* : K_\circ(X) \to K_\circ(F)$ be the direct image morphism (see 2.1) and let $(p_S)_* : K_\circ(X \times S) \to K_\circ(S)$ be also the direct image morphism (see 2.1). Then

$$f_*([\mathcal{O}_X]) = [F] \textit{ and } (p_S)_*[\mathcal{O}_{X\times S}] = [\mathcal{O}_S].$$

Proof. First, we have

$$p_X^*[\mathcal{O}_X] = [\mathcal{O}_{X\times S}], \quad g^*[F] = [\mathcal{O}_S] \text{ and } g^* \circ f_* = (p_S)_* \circ (p_X)^*.$$

Hence the second claim follows from the first one. To prove the first claim consider the Cartesian diagrams

$$\begin{array}{ccccccc} X & \xleftarrow{\pi_X} & X_{\mathrm{sep}} & \qquad & \mathcal{F} & \xleftarrow{\pi_{\mathcal{F}}} & \mathcal{F}_{\mathrm{sep}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_{\mathrm{sep}}} & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h\mathrm{sep}} \\ \operatorname{Spec} F & \xleftarrow{\pi} & \operatorname{Spec} F_{\mathrm{sep}}, & & \operatorname{Spec} F & \xleftarrow{\pi} & \operatorname{Spec} F_{\mathrm{sep}}. \end{array}$$

First, note that $\pi^* f_* = (f_{\mathrm{sep}})_* \circ \pi_X^*$ and that $\pi^* : K_0(F) \to K_0(F_{\mathrm{sep}})$ is an isomorphism. Hence to prove the relation $f_*([\mathcal{O}_X]) = [F]$ it is sufficient to verify that $(f_{\mathrm{sep}})_*[\mathcal{O}_{X_{\mathrm{sep}}}] = [F_{\mathrm{sep}}]$. Second, we have $\pi^* \circ h_* = (h_{\mathrm{sep}})_* \circ \pi_{\mathcal{F}}^*$ and so it is sufficient to prove that $h_*[\mathcal{O}_{\mathcal{F}}] = [F]$. It is well known (see [Hu], 28.5) that $\mathcal{F} = G/P$ is an F-rational variety. Therefore

$$H^0(\mathcal{F}, \mathcal{O}_{\mathcal{F}}) = F \quad \text{and} \quad H^i(\mathcal{F}, \mathcal{O}_{\mathcal{F}}) = 0 \quad \text{for } i > 0.$$

These relations show that $\mathcal{O}_{\mathcal{F}} \in \mathcal{P}(h, \mathcal{F})$ (see 2.1) and

$$h_*([\mathcal{O}_{\mathcal{F}}]) = [h_*(\mathcal{O}_{\mathcal{F}})] = [F] \in K_0(F)$$

by 2.1(1). □

2.3. Let $\mathbb{G}_a$ (respectively $\mathbb{G}_m$) be an additive (respectively a multiplicative) group scheme. Let L/F be a finite field extension and let $R_{L/F}(\mathbb{G}_m)$, $R_L(\mathbb{G}_a)$ be the Weil restrictions of $\mathbb{G}_m$ and $\mathbb{G}_a$. One has

$$H^1(F, R_{L/F}(\mathbb{G}_a)) = H^1(L, \mathbb{G}_a) = 0,$$

$$H^1(F, R^{L/F}(\mathbb{G}_m)) = H^1(L, \mathbb{G}_m) = 0.$$

2.4. Lemma. *Assume that $B \subset G$ is a Borel subgroup defined over F. Then $H^1(F, B(F_{\mathrm{sep}})) = *$. In other words, every principal B-homogeneous space is trivial.*

Proof. Using root subgroups of B_{sep} (see [Hu], 28.1) one can find an algebraic group filtration

$$\{1\} = B^0 \subset B^1 \subset \cdots \subset B^m = B$$

of B such that 1) B^i is a closed normal subgroup of B, 2) B^{i+1}/B^i lies in the center of B/B_i, 3) $B^{i+1}/B^i \cong R_{L_i/F}(\mathbb{G}_a)$ or $B^{i+1}/B^i \cong R_{E_i/F}(\mathbb{G}_m)$ for certain finite field extensions L_i/F and E_i/F. Now for any two integers n, i with $n > i$ there exists an exact sequence (see [Se], 5.5)

$$H^1(F, B^{i+1}/B^i) \to H^1(F, B^n/B^i) \to H^1(F, B^n/B^{i+1})$$

An obvious induction on n together with relations from 2.3 show that $H^1(F, B) = *$. □

2.5. Lemma. *Let H be a simply connected semisimple algebraic group over F. Then there exists a simply connected semisimple algebraic group G^q over F and a 1-cocycle $\gamma : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to \overline{G}^q(F_{\mathrm{sep}})$*

taking value in the adjoint group $\overline{G}^q(F_{\mathrm{sep}})$

such that the following holds:

(1) $H \cong G^q(\gamma)$,

(2) *G^q contains a Borel subgroup defined over F, i.e., G^q is F-quasi-split.*

Proof It is known that $H \cong G^d(\varepsilon)$, where G^d is a simply connected F-split semisimple algebraic group over F and

$$\varepsilon : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to \mathrm{Aut}_{F_sep}(G^d_{\mathrm{sep}})$$

is a 1-cocycle. Let $T^d \subset G^d$ be a maximal F-split torus of G^d and let $B^d \subset G^d$ be a Borel subgroup containing the torus T^d. It is known that $\mathrm{Aut}_{F_{\mathrm{sep}}}(G^d_{\mathrm{sep}}) \cong \overline{G}^d(F_{\mathrm{sep}}) \rtimes \mathrm{Aut}(D)$ (see [T2], 1.5.6), where D is the Dynkin diagram of (G^d, T^d).

Set $\varepsilon = (\gamma, \delta)$. Then the cocycle relations $\varepsilon_{\sigma\tau} = \varepsilon_\sigma \cdot^\sigma \varepsilon_\tau$ show that

$$\delta : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to \mathrm{Aut}(D)$$

is a homomorphism and γ is subject to the relations $\gamma^{\sigma\tau} = \gamma_\sigma \cdot (\delta_\sigma \cdot^\sigma \gamma_\tau \cdot \delta_\sigma^{-1})$. This means that γ is a 1-cocycle taking values in $\overline{G}^d(\delta)$, where $\overline{G}^d(\delta)$ is the twisted form of the group $\overline{G}^d$ corresponding to δ. Set $G^q = G^d(\delta)$.

Since $\mathrm{Aut}(D)$ normalizes B^d and δ takes values in $\mathrm{Aut}(D)$, we conclude that $B^d(\delta) \subset G^d(\delta)$. Obviously, $B^d(\delta)$ is a Borel subgroup of $G^d(\delta) = G^q$ defined over F. $\square$

3. K-modules and K-module morphisms

Let NRS $/F$ be the category of noetherian reduced finite dimensional F-schemes and their **flat** morphisms. Let Gr-Ab be the category of graded abelian groups. If $A \in$ Gr-Ab then define $A[r] \in$ Gr-Ab by setting $A[r]_i = A_{i+r}$.

3.1. Let $\Phi :$ NRS $/F \to$ Gr-Ab be a contrvariant functor equipped with the following *extra data*

(1) a morphism $f_* : \Phi(X) \to \Phi(Y)$ is given for every projective morphism $f : X \to Y$ of objects $X, Y \in$ NRS $/F$ (in particular f can be a closed embedding),

(2) for every triple $Z \xrightarrow{i} X \xleftarrow{j} U$ with a closed emberding i and with $U = X \setminus Z$ a morphism

$$\delta : \Phi(U) \to \Phi(Z)[-1]$$

is given,

(3) for every $X \in$ NRS $/F$ a pairing

$$\mu_X : K_0(X) \otimes \Phi(X) \to \Phi(X)$$

is given.

3.2. Definition. The functor $\Phi :$ NRS $/F \to$ Gr-Ab equipped with the extra data 3.1(1)-3.1(3) is called a *K-module*, if the following conditions hold

(a) $\Phi(X_1 \sqcup X_2) = \Phi(X_1) \oplus \Phi(X_2)$,

(b) for every triple $Z \xrightarrow{i} X \xleftarrow{j} U$ ¿from 4.1(2) the sequence

$$\dots \xrightarrow{\delta} \Phi(Z) \xrightarrow{i_*} \Phi(X) \to \Phi(U) \xrightarrow{\delta} \Phi(Z)[-1] \to \dots$$

is exact,

(c) for every $X \in$ NRS $/F$ and every direct sistem U_α of Zariski open subset U_α of X the canonical map

$$\varinjlim \Phi(U_\alpha) \to \Phi(\varprojlim U_\alpha)$$

is an isomorphism;

(d) $(f \circ g)_* = f_* \circ g_*$ for every projective morphisms $f : X \to Y$ and $g : Y \to Z$ of $X, Y, Z \in$ NRS $/F$;

(e) if $1_X = [\mathcal{O}_X] \in K_0(X)$, then $1_X \cdot y = y$ for every $y \in \Phi(X)$,

(f) for every morphism $f : X \to Y$ in the category NRS $/F$ the diagram

$$\begin{array}{ccc} K_0(Y) \otimes \Phi(Y) & \longrightarrow & \Phi(Y) \\ {\scriptstyle f^* \otimes \Phi(f)} \downarrow & & \downarrow {\scriptstyle \Phi(f)} \\ K_0(X) \otimes \Phi(X) & \longrightarrow & \Phi(X) \end{array}$$

is commutative,

(g) for every flat projective $f : X \to Y$ the diagram

$$\begin{array}{ccccc}
K_0(X)\otimes\Phi(Y) & \xrightarrow{\mathrm{id}\otimes\Phi(f)} & K_0(X)\otimes\Phi(X) & \xrightarrow{\mu} & \Phi(X)\\
\Big\downarrow{\scriptstyle f_*\otimes\mathrm{id}} & & \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*}\\
K_0(Y)\otimes\Phi(Y) & \xrightarrow{\mu} & \Phi(Y) & = & \Phi(Y)
\end{array}$$

is commutative, i.e. $f_*(a\cdot\Phi(f)(b))=f_*(a)\cdot b$ for any $a\in K_0(X), b\in\Phi(Y)$, where $f_*: K_0(X)\to K_0(Y)$ is defined in 2.1.

3.3 Remark. A crucial consequence of properties (e) and (g) is the following. Let $f: X\to Y$ be a flat projective morphism of $X,Y\in\mathrm{NRS}/F$ such that $f_*([\mathcal{O}_X])=[\mathcal{O}_Y]$, then the composition

$$\Phi(Y)\overset{\Phi(f)}{\to}\Phi(X)\overset{f_*}{\to}\Phi(Y)$$

coincides with $\mathrm{id}_{\Phi(Y)}$. Indeed,

$$f_*(\Phi(f)(b))=f_*(1_X\cdot\Phi(f)(b))=f_*(1_X)\cdot b=1_Y\cdot b=b$$

for any $b\in\Phi(Y)$.

3.4 Definition. Let $\Phi,\Psi:\mathrm{NRS}/F\to\text{Gr-Ab}$ be two K-modules. A natural transformation $\theta:\Phi\to\Psi$ of the functors is called a *morphism of K-modules*, if $\theta(X_1\amalg X_2)=\theta(X_1)\oplus\theta(X_2)$ and θ is compartible with the extra data above (see 3.1), i.e. the diagrams

$$\begin{array}{ccc}
K_0(X)\otimes\Phi(X) & \longrightarrow & \Phi(X)\\
\Big\downarrow{\scriptstyle id\otimes\theta} & & \Big\downarrow{\scriptstyle\theta}\\
K_0(X)\otimes\Psi(X) & \longrightarrow & \Psi(X)
\end{array}\quad
\begin{array}{ccc}
\Phi(X) & \xrightarrow{f_*} & \Phi(Y)\\
{\scriptstyle\theta}\Big\downarrow & & \Big\downarrow{\scriptstyle\theta}\\
\Psi(X) & \xrightarrow{f_*} & \Psi(Y)
\end{array}\quad
\begin{array}{ccc}
\Phi(U) & \xrightarrow{\partial} & \Phi(Z)[-1]\\
{\scriptstyle\theta}\Big\downarrow & & \Big\downarrow{\scriptstyle\theta}\\
\Psi(U) & \xrightarrow{\partial} & \Psi(Z)[-1]
\end{array}$$

are commutative for all $X\in\mathrm{NRS}/F$, all projective morphisms $f: X\to Y$ of $X,Y\in\mathrm{NRS}/F$, and all triples $Z\overset{i}{\to}X\overset{j}{\leftarrow}U$ from 3.1(2).

3.5. Theorem. *Let G be a semisimple algebraic group over F containing a Borel subgroup B defined over F. Let $\mathcal{G}$ be a principal homogeneons G-space. Then for every morphism $\theta:\Phi\to\Psi$ of K-module the following conditions are equivalent*

(a) $\theta(S)$ is an isomorphism for every $S\in\mathrm{NRS}/F$;

(b) $\theta(E)\overset{def}{=}\theta(\mathrm{Spec}\,E)$ is an isomorphism for every field extension E/F with $\mathcal{G}(E)\neq\varnothing$.

This Theorem is proved below in this section (see 3.8). Let us consider now some examples of K-modules and K-module morphisms.

3.6. Examples of K-modules.

(a) $K':\mathrm{NRS}/F\to\text{Gr-Ab}\quad(X\mapsto\bigoplus_{i\geqslant 0}K'_i(X))$,

(b) $K'_Y:\mathrm{NRS}/F\to\text{Gr-Ab}\quad(X\mapsto\bigoplus_{i\geqslant 0}K'_i(X\times Y))$,

where $Y\in\mathrm{NRS}/F$,

(c) $K'_D : \mathrm{NRS}/F \to \text{Gr-Ab} \quad (X \mapsto \bigoplus_{i\geqslant 0} K'_i(X, D))$,
where D is a finite dimensional separable F-algebra,
(d) $K'_{Y,D} : \mathrm{NRS}/F \to \text{Gr-Ab} \quad (X \mapsto \bigoplus_{i\geqslant 0} K'_i(X \times Y, D))$,
where Y is such as in (b) and D is sush as in (c).

For instance, pairing $K_0 \otimes K'_{Y,D} \to K'_{Y,D}$ is induced by the biexact functors $P(X) \times M(X \times Y, D) \to M(X \times Y, D) \quad (E, F) \mapsto p_X^* E \otimes F$. Directimage maps $f_* : K'_{Y,D}(T) \to K'_{Y,D}(S)$ are defined in ([Q],§7, 2.7). Property 3.2(g) holds by 2.1(2).

3.7. Examples of K-module morphisms.

(a)
$$a\cup ? : K'[-r] \to K'_Y \quad \text{for} \quad a \in K_r(Y),$$
(b)
$$b\cup_D ? : K'_D[-r] \to K'_Y \quad \text{for} \quad b \in K_r(Y, D^{\mathrm{op}}),$$
(c)
$$b\cup_D ? : K'_{D\otimes A}[-r] \to K'_{Y,A} \quad \text{for} \quad b \in K_r(Y, D^{\mathrm{op}}),$$

(d) $\delta : K'_V \to K'_Z[-1]$ for every closed embeding $Z \xrightarrow{i} X$ with $V = X - Z$,

(e) $z_* : K'_X[-r] \to K'_Y$ for $z \in K_r(X \times Y)$, where $X$is a smooth projective over F and $z_*(T) : K'(X \times T)[-r] \to K'(Y \times T)$ is given by $x \mapsto (p_{Y\times T})_*(p_T^*(z) \otimes p_{X\times T}^*(x))$,

(f) $z_* : K'_A[-r] \to K'_B$ for $z \in K_r(A^{\mathrm{op}} \otimes_F B)$, where A, B are finite dimensional separable F-algebras and $z_*(T) : K'_r(T, A) \to K'_r(T, B)$ takes x to $z \otimes_A x$.

Remark also here that there are the following isomorphisms
(i) $K_r(X \times Y) \xrightarrow{\sim} \mathrm{Mor}_{K\text{ -modules}}(K'_X[-r], K'_Y), (z \mapsto z_*)$
(ii) $K_r(A^{\mathrm{op}} \otimes B) \xrightarrow{\sim} \mathrm{Mor}_{K\text{ -modules}}(K'_A[-r], K'_B), (z \mapsto z\cup_A ?)$
(ii) $K_r(Y, A^{\mathrm{op}}) \xrightarrow{\sim} \mathrm{Mor}_{K\text{ -modules}}(K'_A[-r], K'_Y) \quad (z \mapsto z\cup_A ?)$
under the hypotheses of (e) and (f) respectively.

We shall not use the last remark in the present paper.

3.8. Proof of 3.5. Set $\mathcal{F} = G/B$. Then $\mathcal{F}$ is a projective variety (see $[Hu]$,21.3). Let $X = \mathcal{G} \times_G \mathcal{F}$ be the twisted form of variety $\mathcal{F}$ corresponding to principal homogeneons G-space $\mathcal{G}$. We need the following three lemmas, which will be proved in this section below.

3.9. Lemma. *For every $S \in \mathrm{NRS}/F$ and every point $y \in S \times X$ the set $\mathcal{G}(F(y)) \neq \varnothing$, where $F(y)$ is the residue field.*

3.10. Lemma. *Let $Y \in \mathrm{NRS}/F$ be such that maps $\theta(y)$ are isomorphisms for every point $y \in Y$. Then for every closed reduced subscheme $Z \hookrightarrow Y$ map $\theta(Z)$ is also an isomorphism. In particular, $\theta(Y)$ is an isomorphism.*

3.11. Lemma. *Let $S \in \mathrm{NRS}/F$ be such that $\theta(S \times X)$ is an isomorphism, then $\theta(S)$ is also an isomorphism.*

Using these three lemmas we can prove 3.5. Indeed for every $S \in \mathrm{NRS}/F$ and every point $y \in S \times X$ the set $\mathcal{G}(F(y))$ is not empty by 3.9. Therefore $\theta(y)$ is an isomorphism by 3.5(b) and hence $\theta(S \times X)$ is an isomorphism by 3.10. Lemma 3.11 completes the proof of 3.5. $\square$

3.12. Proof of 3.9. Let $G \to G/B = \mathcal{F}$ be the canonical morphism, and let $h : \mathcal{G} = \mathcal{G} \underset{G}{\times} G \to \mathcal{G} \underset{G}{\times} \mathcal{F} = X$ be its twisted from corresponding to principal homogeneos G-space $\mathcal{G}$. If $x \in X$ is a point, then the right B-action on $\mathcal{G}$ (it is the restriction of the right G-action) determines a principal homogeneons $B_{F(x)}$-space structure on the fible $h^{-1}(x)$. Lemma 2.4 shows that it is a trivial pricipal $B_{F(x)}$-homogeneous space. Hence there exists a section $s : x \to h^{-1}(x)$ of the morphism $h^{-1}(x) \to x$.

Let $y \in S \times X$ be a point. If $p_X : S \times X \to X$ is the projection set $x = p_X(y)$. Then the composition

$$y \xrightarrow{p_X} x \xrightarrow{s} h^{-1}(x) \hookrightarrow \mathcal{G}$$

determines an $F(y)$-point of $\mathcal{G}$. Hence $\mathcal{G}(F(y)) \neq \varnothing$.

Lemma 3.9 is proven. □

3.13. Proof of 3.11. Consider the diagram

$$\begin{array}{ccccc}
\Psi(S) & \xrightarrow{\Psi(p_S)} & \Psi(S \times X) & \xrightarrow{(p_S)_*} & \Psi(S) \\
\uparrow{\scriptstyle\theta} & & \uparrow{\scriptstyle\theta} & & \uparrow{\scriptstyle\theta} \\
\Phi(S) & \xrightarrow{\Phi(p_S)} & \Phi(S \times X) & \xrightarrow{(p_S)_*} & \Phi(S).
\end{array}$$

This diagramm is commutative by 3.4. Since $X_{\text{sep}} \xrightarrow{\sim} \mathcal{F}_{\text{sep}}$. Lemma 2.2 and Remark 3.3 shou that the following rebation hold

$$(p_S)_* \circ \Phi(p_S) = \mathrm{id}_{\Phi(S)}, \quad (p_S)_* \circ \Psi(p_S) = \mathrm{id}_{\Psi(S)}.$$

Therefore we have the inclusions

$$\ker(\theta(S)) \hookrightarrow \ker(\theta(S \times X)), \quad \operatorname{coker}(\theta(S)) \hookrightarrow \operatorname{coker}(\theta(S \times X)).$$

Since $\theta(S \times X)$ is an isomorphism, $\theta(S)$ is also an isomorphism. □

3.14. Proof of 3.10. Prove 3.9 by induction on dim Z. If dim $Z = 0$, then Z is a disjoint union of a finite many closed points of Y. Definition 3.4. shows that $\theta(Z) = \oplus \quad \theta(z_i)$, where the sum runs over all points of Z. Since every $\theta(z_i)$ is an isomorphism $\theta(Z)$ is also an isomorphism.

Now assume that $\theta(Z)$ is an isomorphism for every closed reduced subscheme $Z' \overset{i'}{\hookrightarrow} Y$ of dimension $< d$. Let $Z \overset{i}{\hookrightarrow} Y$ be a closed reduced subscheme of dimension d. For every closed subscheme $Z' \overset{i'}{\hookrightarrow} Z$ there is the morphism of long exact sequences of graded abelian group

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \Psi(Z') & \xrightarrow{i'_*} & \Psi(Z) & \xrightarrow{\Psi(j)} & \Psi(Z \backslash Z') & \xrightarrow{\delta} & \Psi(Z')[-1] & \longrightarrow & \ldots \\
 & & {\scriptstyle\theta}\uparrow & & {\scriptstyle\theta}\uparrow & & {\scriptstyle\theta}\uparrow & & {\scriptstyle\theta}\uparrow & & \\
\ldots & \longrightarrow & \Phi(Z') & \xrightarrow{i'_*} & \Phi(Z) & \xrightarrow{\Phi(j)} & \Phi(Z \backslash Z') & \xrightarrow{\delta} & \Phi(Z')[-1] & \longrightarrow & \ldots,
\end{array}$$

because θ is a morphism of K-modules (see 4.3).

Let $x_1, \ldots, x_r$ be all points of Z with $\dim x_i < d$ and let $x_i \neq x_j$ for $i \neq j$. Properties 3.2(d) and 3.2(c) give us the following morphism of exact sequences:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \varinjlim \Psi(Z') & \longrightarrow & \Psi(Z) & \longrightarrow & \Psi(\coprod_{i=1}^{r} x_i) & \xrightarrow{\partial} & \ldots \\
 & & \Big\uparrow{\scriptstyle \varinjlim \theta} & & \Big\uparrow{\scriptstyle \theta} & & \Big\uparrow{\scriptstyle \theta} & & \\
\ldots & \longrightarrow & \varinjlim \Phi(Z') & \longrightarrow & \Phi(Z) & \longrightarrow & \Phi(\coprod_{i=1}^{r} x_i) & \xrightarrow{\partial} & \ldots,
\end{array}$$

with the direct limits taken over all reduced closed subschemes Z' of Z such that $x_i \in Z'$ for every $i = 1, \ldots, r$.

Definition 3.4 shows that $\theta(\amalg x_i) = \oplus \theta(x_i)$. The hypotheses of 3.10 state that $\theta(x_i)$ is an isomorphism ($i = 1, 2, \ldots, r$). Hence $\theta(\amalg x_i)$ also is an isomorphism. The map $\varinjlim \theta$ is an isomorphism by the induction hypotheses. Hence θ is also an isomorphism. Lemma is proven. $\square$

4. Splitting principle II

4.1. Theorem. *(Splitting principle II). Let G be a connected semisimple algebraic group and let $\mathcal{G}$ be a principal G-homogeneous space. Let $\theta : \Phi \to \Psi$ be a K-module morphism. Then the following properties are equivalente*

(1) $\theta(S)$ is an isomorphism for every $S \in \mathrm{NRS}/F$;

(2) $\theta(E)$ is an isomorphism for every E/F such that $\mathcal{G}(E) \neq \varnothing$.

4.2. Corollary. *Theorem 1.1 holds.*

Proof of 4.2. Since $S \to \coprod_{i=1}^{n} K'_*(S, A_i^{\mathrm{op}})[-r_i]$ and $S \to K'_*(S \times X)$ are K-modules (see 3.6(c), 3.6(b)), and $S \to \theta(S) = \sum \theta_i(S)$ is a K-module morphism (see 3.7(b)). Theorem 1.1 follows directly from 4.1. $\square$

4.3. Proof of 4.1. Every semisimple connected algebraic group is an inner from of a semisimple algebraic group containing a Borel subgroup (see $[T2]$, 1.5.6). Let G^q be a semisimple algebraic group containing a Borel subgroup B^q and let $\gamma : \mathrm{Gal}(F_s/F) \to G^q(F_s)$ be a 1-cocycle such that

$$G \cong G^q(\gamma).$$

Evidenly that it is sufficient to prove that (2) follows (1). Consider the following two Claims.

Claim 1. *Let L/F be a field extension such that $\gamma_L \sim *_L$ and let E/L be a field extension such that $\mathcal{G}_L(E) \neq \varnothing$, then $\theta(E)$ is an isomorphism.*

Claim 2. *Let L/F be a field extension such that $\gamma_L \sim *_L$. Then $\theta(L)$ is an isomorphism.*

Till the end of this section we assume 4.1.(2) to hold. Under the conditions of Claim 1 one has $\mathcal{G}_L(E) \neq \varnothing$. Since $\mathcal{G}_L(E) \subset \mathcal{G}(E)$ one has $\mathcal{G}(E) \neq \varnothing$. Now 4.1(2) shows that $\theta(E)$ is an isomorphism. Hence Claim 1 holds.

Now we prove Claim 2. Let L/F be such that $\gamma_L \sim *_L$. Consider the subcategory NRS/L of the category NRS/F and set

$$\Phi_L = \Phi|_{\mathrm{NRS}/L}, \quad \Psi_L = \Psi|_{\mathrm{NRS}/L}, \quad \theta_L = \theta|_{\mathrm{NRS}/L}.$$

Then obviously Φ_L, Ψ_L are K-modules and $\theta_L : \Phi_L \to \Psi_L$ is a K-module morphism with respect to the category NRS/L instead of NRS/F. Since $\gamma_L \sim *_L$, one has L-algebraic group isomorphisms: $G_L \cong G^q(\gamma)_L \cong G^q_L$. Hence G_L has a Borel subgroup defined over L, i.e., G_L is L-quasi-split.

Since Claim 1 holds, Theorem 3.5 applied to $(G_L, \mathcal{G}_L)$ and θ_L shows that $\theta(L)$ is an isomorphism. Claim 2 is proven.

Since Claim 2 holds, Theorem 3.5 applied to (G^q, γ) and $\theta : \Phi \to \Psi$ shows that 4.1(1) holds. Theorem 4.1 is proven. $\square$

4.4. Proof of 1.4. Let Y be the Severi-Brauer variety of D (it is a twisted form of the projective space $\mathbb{P}(V)$, where $\mathrm{dm}_F(V) = \mathrm{dg}_F\, D$). We need the following two lemmas.

4.5. Lemma. *The following statements are equivalent:*
(1) $\theta^D(S \times Y)$ *is an isomorphism,*
(2) $\theta(S \times Y)$ *is an isomorphism.*

4.6. Lemma. *If* $\theta^D(S \times Y)$ *is an isomorphism, then* $\theta^D(S)$ *also is an isomorphism.*

Having these two lemmas complete the proof of 1.4 as follows. Since 1.4(a) the map $\theta(S \times Y)$ is an isomorphism. Hence Lemmas 4.5, 4.6 show that $\theta^D(S)$ is an isomorphism, too. □

4.7. Proof of 4.5. Let J be the Quillen's sheaf on Y (see[Q], §8,1). It is a locally free sheaf of the rank $\mathrm{dg}_F\, D$ such that $\mathrm{End}_Y(J)^{\mathrm{op}} = D$ (see [Q], §8,1). Let $p : Y \to \mathrm{Spec}\, F$ be the structural morphism. The canonical morphism

$$p^*D = D \otimes_F \mathcal{O}_Y \to \mathrm{End}(J)^{\mathrm{op}}$$

is an $\mathcal{O}_Y$-algebra isomorphism (see [Q],§8,1). Since $\mathrm{End}(J)^{\mathrm{op}} \cong J \otimes J^*$, one has

$$(*) \qquad J \otimes_D J^* \cong \mathcal{O}_Y, \quad J \otimes J^* \cong p^*D.$$

For an arbitrary finite dimentional separable F-algebra A, consider two maps ($J_S \stackrel{\mathrm{df}}{=} p_S^* J$):

$$(1) \qquad K_*(S \times Y, A) \stackrel{\cup[J_S]}{\to} K_*(S \times Y, A \otimes D^{\mathrm{op}}),$$

$$(2) \qquad K_*(S \times Y, A \otimes D^{\mathrm{op}} \stackrel{\cup_D[J_S^*]}{\to} K_*(S \times Y, A).$$

Relations (*) show that

$$(x \cup [J_S]) \cup_D [J_S^*] = x \cup ([J_S] \cup_D [J_S^*]) = x \cup [J_S \otimes_D J_S^*] = x,$$

$$(y \cup_D [J_S^*]) \cup [J_S] = y \cup_D ([J_S^*] \cup [J_S]) = y \cup_D ([J_S^* \otimes J_S]) = y.$$

Hence maps (1) and (2) are isomorphisms inverse to each other .

For an arbitrary F-scheme X, the maps

$$K_*(S \times Y \times X) \stackrel{\cup[J_{S \times X}]}{\longrightarrow} K_*(S \times Y \times X, D^{\mathrm{op}})$$

$$K_*(S \times Y \times X, D^{\mathrm{op}}) \stackrel{\cup_D[J_{S \times X}^*]}{\longrightarrow} K_*(S \times Y \times X)$$

also are isomorphisms inverse to each other by the same reason.

For every $\alpha \in K_r(X, A^{\mathrm{op}})$ the diagram

$$\begin{array}{ccc} K_*(S \times Y, A)[-r] & \xrightarrow{\alpha \cup_A ?} & K_*(S \times Y \times X) \\ \Big\downarrow{\scriptstyle \cup[J_S]} & & \Big\downarrow{\scriptstyle \cup[J_{S \times X}]} \\ K_*(S \times Y, A \otimes D^{\mathrm{op}})[-r] & \xrightarrow{\alpha \cup_A ?} & K_*(S \times Y \times X, D^{\mathrm{op}}) \end{array}$$

is commutative, because for any $y \in K_*(S \times Y, A)$ one has

$$(\alpha \cup_A Y) \cup [J_S] = \alpha \cup_A (y \cup [J_{S \times X}]).$$

Since the vertical arrows in the last diagram are isomorphisms, the commutativity of the diagram proves 4.5.

4.8. Proof of 4.6. Once more let $p : Y \to \operatorname{Spec} F$ be the structural morphism. Consider the diagram

$$
(**) \qquad
\begin{array}{ccccc}
\Psi(S) & \xrightarrow{\Psi(p_S)} & \Psi(S \times Y) & \xrightarrow{(p_S)_*} & \Psi(S) \\
\uparrow{\scriptstyle \theta^D(S)} & & \uparrow{\scriptstyle \theta^D(S\times Y)} & & \uparrow{\scriptstyle \theta^D(S)} \\
\Phi(S) & \xrightarrow{\Phi(p_S)} & \Phi(S \times Y) & \xrightarrow{(p_S)_*} & \Phi(S),
\end{array}
$$

where

$$\Phi(T) = \coprod_{i=2}^{n} K'_*(T, A_i \otimes D)[-r_i], \qquad \Psi(T) = K'_*(T \times X, D)$$

for any F-scheme T. Examples 3.6(c) and 3.6(d) show that Φ and Ψ are K-modules. Example 3.7(c) shows that θ^D is a K-module morphism. Hence diagram $(**)$ is commutative (see 3.4).

Since $Y_{\mathrm{sep}} \cong \mathbb{P}(V)_{\mathrm{sep}}$ (see the beginning of 4.4), Lemma 2.2 shows that $(p_S)_*([\mathcal{O}_{Y\times S}]) = [\mathcal{O}_S]$ for any F-scheme S.

Remark 3.3 shows that

$$(p_S)_* \circ \Phi(p_S) = \mathrm{id}_{\Phi(S)}, \quad (p_S)_* \circ \Psi(p_S) = \mathrm{id}_{\Psi(S)}.$$

Therefore we have the inclusions

$$\ker(\theta^D(S)) \hookrightarrow \ker(\theta^D(S \times Y)), \operatorname{coker}(\theta^D(S)) \hookrightarrow \operatorname{coker}(\theta^D(S \times Y)).$$

If $\theta^D(S \times Y)$ is an isomorphism, then $\theta^D(S)$ also is an isomorphism. Lemma is proven. $\square$

5. The Main Result

Theorem 5.1 stated below is the main result of the present paper. To formulate this result consider a connected semisimple algebraic group H over the field F such that the homomorphism (see [T2], 1.5.6)

$$\mathrm{Gal}(F_{\mathrm{sep}}/F) \to \mathrm{Aut}_{F_{\mathrm{sep}}}(H_{\mathrm{sep}})/\,\mathrm{Inn}_{F_{\mathrm{sep}}}(H_{\mathrm{sep}})$$

is trivial. In other words H is an inner form of an F-split semisimple algebraic group, say G.

5.1. Theorem A. *If H is simply connected, then there exist central simple F-algebras $A_1, \dots, A_n$ $(n = \mathrm{rk}(H_{\mathrm{sep}}))$ such that for every principal H-homogeneous space X and finite-dimensional central simple F-algebra $\mathcal{D}$ the following holds:*

$$K_0(\mathcal{D}) \cong K_0(X, \mathcal{D}) \stackrel{\mathrm{df}}{=} K_0(F[X] \otimes_F \mathcal{D}), \tag{1}$$

$$K_1(X, \mathcal{D}) \cong K_0(A_1 \otimes \mathcal{D}) \oplus \cdots \oplus K_0(A_n \otimes \mathcal{D}) \oplus K_1(\mathcal{D}), \tag{2}$$

$$K_*(X, \mathcal{D}) \cong \bigoplus_I K_{*-\mathrm{card}(I)}(A_I \otimes \mathcal{D}), \tag{3}$$

where I runs over all subsets of the set $\{1, 2, \dots, n\}$ and $A_I = A_{i_1} \otimes \cdots \otimes A_{i_r}$ for $I = \{i_1 < i_2 < \cdots < i_r\}$.

This theorem is proved in §14 below using 1.1, 7.2, 9.5 and 9.7. List of algebras $A_1, \dots, A_n$ is given in 5.4 below and explained in §15. Maps $(1-3)$ are discribed in 5.3.

5.2. Corollary (of 5.1(1)). *Let H,$X$and $\mathcal{D}$ be the same as in 5.1, then the following holds*

1. *$K_0(\mathcal{D}) \cong K_0(\mathcal{D}_{F(X)})$ (the map is induced by the scalar extension),*
2. *the map $\mathrm{Br}(F) \to \mathrm{Br}(F(X))$, $\alpha \mapsto \alpha_{F(X)}$, is injective,*
3. *if $\mathcal{D}$ is a skew-field, then $\mathcal{D}_{F(X)}$ is also a skew-field,*
4. *Chow groups $CH^1(X)$ and $CH^2(X)$ are trivial.*

Proof of 5.2. Since X_{sep} is a smooth variety ($X_{\mathrm{sep}} \cong H_{\mathrm{sep}}$), it follows that the variety X is also smooth. Hence $K_0(X, \mathcal{D}) \cong K'_0(X, \mathcal{D})$. Since $K_0(\mathcal{D}) \cong K'_0(\mathcal{D})$ and $K'_0(X, \mathcal{D}) \to K'_0(\mathcal{D}_{F(X)})$ is surjective, the map $K_0(X, \mathcal{D}) \xrightarrow{\alpha} K_0(\mathcal{D}_{F(X)})$ is also surjective. Now 5.1.(1) and isomorphisms $K_0(\mathcal{D}) \cong \mathbb{Z} \cong K_0(\mathcal{D}_{F(X)})$ imply that the natural map $K_0(\mathcal{D}) \to K_0(\mathcal{D}_{F(X)})$ is an isomorphism. Claim (1) is proven. □

If $\mathcal{D}$ is a skew-field, then its class $[\mathcal{D}]$ in $K_0(\mathcal{D})$ generates $K_0(\mathcal{D})$. Since the natural map $K_0(\mathcal{D}) \to K_0(\mathcal{D}_{F(X)})$ is an isomorphism, the generator of $K_0(\mathcal{D}_{F(X)})$ is $[\mathcal{D}_{F(X)}]$. Hence $\mathcal{D}_{F(X)}$ is also a skew-field, which proves (3). Claim (2) follows from (3).

According to 5.1(1), $K_0(X) = \mathbb{Z}$ and the topological filtration is trivial, because $K_0(X) \hookrightarrow K_0(F(X))$. Since the kernel of canonical homomorphism $CH^i(X) \to K(X)^{i/i+1}$ is killed by $(i-1)!$, we see that $CH^1(X) = CH^2(X) = 0$. □

5.3. Remark. To define the isomorphism from 5.1(3) we construct (see §9, 9.1, 9.2, 9.4) certain elements $\alpha_i \in K_1(X, A_i^{\mathrm{op}})$ and define $\alpha_I = \alpha_{i_1} \cup \cdots \cup \alpha_{i_r} \in K_r(X, A_i^{\mathrm{op}})$ for $I = \{i_1 < i_2 < \cdots < i_r\}$. Then the map $K_{*-\mathrm{card}(I)}(A_I \otimes \mathcal{D}) \to K_*(X, \mathcal{D})$ is given by $x \mapsto \alpha_I \cup_{A_I} x$. In other words, this map is the multiplication by α_I over A_I.

5.4 List of algebras. We shall write $C_1 \sim C_2$ for F-algebras C_1 and C_2, if $M_k(C_1) \cong M_l(C_2)$ for some integers k, l. Since the functor

$$C\text{-mod} \to M_K(C)\text{-mod}, V \mapsto V \otimes_F F^k$$

induces an isomorphism $K_*(C) \cong K_*(M_k(C))$, the equivalence relation $\sim$ does not change the K-groups. Here is the list of algebras A_i 5.1.

Cases of simple groups are considered first.

(1). $H = SL_{1,A}$, where A is a central simple F-algebra ($\deg(A) = n+1$). Then

$$A_1 \sim A,\ A_2 \sim A^{\otimes 2},\ \ldots,\ A_n \sim A^{\otimes n}.$$

(2). $H = \mathrm{Spin}(q)$, where q is a nonsingular quadratic form of rank $(2n+1)$. Then

$$A_1 \sim F,\ A_2 \sim F,\ \ldots,\ A_{n-1} \sim F,\ A_n = C_0(q),$$

where $C_0(q)$ is the even Clifford algebra of q.

(3). $H = \mathrm{Sp}(B, \tau)$, where B is a central simple F-algebra of degree $2n$, and $\tau : B \to B$ is an F-algebra involution of symplectic type. Then

$$A_1 \sim B,\ A_2 \sim F,\ A_3 \sim B,\ A_4 \sim F,\ \ldots,\ A_n \sim (B\,\mathrm{or}\,F)$$

(if n is odd, then $A_n \sim B$, otherwise $A_n \sim F$).

(4 a). $H = \mathrm{Spin}(\varphi)$, where φ is a nonsingular quadratic form of rank $2n$ with $(-1)^{n(2n-1)} \cdot \mathrm{disc}(\varphi) \in F^{*2}$. Then

$$A_1 \sim F,\ A_2 \sim F,\ \ldots,\ A_{n-2} \sim F,\ A_{n-1} = C_0^+(\varphi),\ A_n = C_0^-(\varphi),$$

where $C_0^+(\varphi), C_0^-(\varphi)$ are simple components of the even Clifford algebra $C_0(\varphi)$. (Note also that $C_0^+(\varphi) \cong C_0^-(\varphi)$).

(4 b). $H = \mathrm{Spin}(A, \sigma)$, where A is a central simple F-algebra of degree $2n$ and $\sigma : A \to A$ is an F-algebra involution of orthogonal type such that $(-1)^{n(2n-1)}\,\mathrm{disc}(\sigma) \in F^{*2}$. Then

$$A_1 \sim A,\ A_2 \sim F,\ A_3 \sim A,\ A_4 \sim F,\ \ldots,\ A_{n-2} \sim (A\,\mathrm{or}\,F), A_{n-1} = C_0^+(A,\sigma),\ A_n = C_0^-(A,\sigma)$$

where $A_{n-2} \sim A$, if n is odd, $A_{n-2} \sim F$ else, and $C_0^+(A,\sigma)$, $C_0^-(A,\sigma)$ are simple components of the even Clifford algebra $C_0(A,\sigma)$ of the pair (A,σ) (see [J]).

(5). H_{sep} is of G_2-, F_4- or F_8-type. Then $A_i \sim F$ for all i.

(6). H_{sep} is of E_6-type. Then there is an algebra C of degree 27 (its class $[C] \in \mathrm{Br}(F)$ has exponent 1 or 3) such that

$$A_1 = C,\ A_2 \sim F,\ A_3 \sim C^{\mathrm{op}},\ A_4 \sim F,\ A_5 \sim C,\ A_6 \sim C^{\mathrm{op}}.$$

(7). H_{sep} is of E_7-type. Then there is an algebra B of degree 8 (the class $[B] \in \mathrm{Br}(F)$ has exponent 1 or 2) such that

$$A_1 \sim F,\ A_2 \sim B,\ A_3 \sim F,\ A_4 \sim F,\ A_5 \sim B,\ A_6 \sim F,\ A_7 \sim C.$$

(8). $H = \prod_{i=1}^{m} H_i$, where H_i's $(i = 1, 2, \ldots, m)$ are simple. Then H_i's are defined canonically by H. The list of algebras is the disjoint union of lists corresponding to H_i's.

6. Fundamental families of representations

Let G be a semisimple algebraic group over a field F. We shall assume in this sections that G is simply connected and F-split (the last means that there exists a maximal F-split torus). If $\mathrm{Rep}_F(G)$ is the category of finite dimensional F-linear representations of G, then denote by $R(G)$ the Grothendieck group of the category $\mathrm{Rep}_F(G)$ with respect to the usual notion of short exact sequence. The tensor product of representations induces a ring structure on $R(G)$.

If $(V,\rho)\in\mathrm{Rep}_F(G)$, then denote by $[\rho]$ or often $[V]$ the element of $R(G)$ corresponding to (V,ρ). The following classical result can be found in [Ch].

6.1. Theorem (Chevalley). $R_F(G)\cong\mathbb{Z}[X_1,X_2,\dots,X_n]$, *where* $n=\mathrm{rk}(G)$. *More precisely,* $R_F(G)$ *is a polynomial ring over* $\mathbb{Z}$ *freely generated by classes* $[V_1],\dots,[V_n]$ *of irreducible representations* $(V_1,p_1),\dots,(V_n,p_n)$ *corresponding to fundamental weights.*

6.2. Definition. A family $\{(W_i,\rho_i)\}_{i=1}^m$ of F-linear G-representations is called *fundamental*, if $[W_1],\dots,[W_m]$ freely generate $R_F(G)$ as $\mathbb{Z}$-algebra. Note that $m=\mathrm{rk}(G)$ by 6.1.

6.3. Examples.

(a) The family of all fundamental representations (i.e., irreducible representations of G corresponding to fundamental weights) is a fundamental family by 6.1.

(b) The following families are fundamental for groups $SL_{n+1,F}$, $\mathrm{Spin}_{2n+1,F}$, $\mathrm{Sp}_{2n,F}$ and $\mathrm{Spin}_{2n,F}$ respectively (here V_i is the i-dimensional F-vector space F^i):

$$V_{n+1},\ \wedge^2V_{n+1},\ \dots,\ \wedge^nV_{n+1}; \tag{1}$$

$$V_{2n+1},\ \wedge^2V_{2n+1},\ \dots,\ \wedge^{n+1}V_{2n+1},\ \triangle_n; \tag{2}$$

$$V_{2n},\ \wedge^2V_{2n},\ \dots,\ \wedge^nV_{2n}; \tag{3}$$

$$V_{2n},\ \wedge^2V_{2n},\ \dots,\ \wedge^{n-2}V_{2n},\ \triangle_n^+,\ \triangle_n^- \tag{4}$$

(here $\triangle_n$ is the spinor representation of Spin_{2n+1} and $\triangle_n^+,\triangle_n^-$ are two half-spinor representations of Spin_{2n} as described in 13.4 below).

6.4. Lemma. $R_F(G')\otimes_{\mathbb{Z}}R_F(G'')\cong R_F(G'\times G'')$ *(the map is induced by* $[W_1]\otimes[W_2]\mapsto[W_1\otimes W_2]$*).*

6.5. Corollary. *If* $\{(V'_i,\rho'_i)/i=1,\dots,m'\}$ *and* $\{(V''_j,\rho''_j)/j=1,\dots,m''\}$ *are fundamental families of representations for* G' *and* G'' *, then*

$$\{(V'_i,\rho'_i\circ\mathrm{pr}')\}\cup\{(V''_j,\rho''_j\circ\mathrm{pr}'')\}$$

is a fundamental family for the group $G'\times G''$*, where* $\mathrm{pr}':G'\times G''\to G'$ *and* $\mathrm{pr}'':G'\times G''\to G''$ *are two projections.*

7. A result of Levine

Let F be a field and let X be an affine variety over F. Let $F[X]$ be the ring of regular functions on X and let $\mathrm{GL}(F[X]) = \bigcup_{n \geqslant 1} \mathrm{GL}_n(F[X])$ be the general linear group of $F[X]$. If $E(F[X])$ is the subgroup of $\mathrm{GL}(F[X])$ generated by elementary matrices, then

$$K_1(F[X]) = \mathrm{GL}(F[X])/E(F[X])$$

(see [B1],???). For any integer $n \geqslant 1$ identify $\mathrm{Mor}_F(X, \mathrm{GL}_{n,F})$ with $\mathrm{GL}_n(F[X])$ taking f to $(f_{ij})_{i,j=1}^n$ and consider the composition

$$(*) \qquad \mathrm{Mor}_F(X, \mathrm{GL}_{n,F}) = \mathrm{GL}_n(F[X]) \hookrightarrow \mathrm{GL}(F[X]) \to K_1(F[X]).$$

7.1. Notation. For $f \in \mathrm{Mor}_F(X, \mathrm{GL}_{n,F})$ denote by $[f] \in K_1(F[X])$ the image of f under the map (*). If $\{f_i : X \to \mathrm{GL}_{n(i),F}\}_{i=1}^m$ is a family of morphisms, then for every subset $I = \{i_1 < i_2 < \cdots < i_r\}$ of the set $\{1, 2, \ldots, m\}$ denote by

$$[f]_I \in K_r(F[X])$$

the cup product $[f_{i_1}] \cup [f_{i_2}] \cup \cdots \cup [f_{i_r}]$. Denote also by

$$\xi_{I,f} : K_*(F)[-r] \to K_*(F[X])$$

the map $x \mapsto [f]_I \cup x$, and set

$$\xi_f = \sum_I \xi_{I,f} : \coprod_I K_*(F)[-r(I)] \to K_*(F[X]),$$

where I runs over all subset of $\{1, 2, \ldots, m\}$.

7.2. Theorem. (Levine [L], 2.1). *Let G be a connected semisimple F-split algebraic group over F, which is simply connected. Let $\{\rho_i : G \to \mathrm{GL}(V_i)\}_{i=1}^{\mathrm{rk}(G)}$ be a fundamental family of finite dimensional representations of G (see 6.2, 6.3). Then*

$$\xi_\rho : \coprod_I K_*(F)[-r(I)] \to K_*(F[G])$$

is an isomorphism of graded abelian groups.

7.3. Corollary.

$$K_0(G) \cong \mathbb{Z}, \quad K_1(G) \cong K_1(F) \oplus \mathbb{Z}^{\mathrm{rk}(G)}.$$

7.4. Remark. The cases $G = \mathrm{SL}_{n,F}$, $\mathrm{Sp}_{2n,F}$ were proved also by Suslin in [Su], 2.1, 2.9. A similar result for the complex topological K-groups was proved by L. Hodgkin in [H], Th.A.

The group $K_*^{\mathrm{tp}}(SU_n(\mathbb{C}))$ was computed by Atiyah in [A], 2.7.17.

7.5. We shall need a slight modification of Theorem 7.2. Namely, let $v_i : \mathrm{GL}_{n(i),F} \to \mathrm{GL}_{n(i),F}$ be the left multiplication by $x_i \in \mathrm{GL}_{n(i)}(F)$. Set

$$g_i = v_i \circ \rho_i : G \to \mathrm{GL}_{n(i),F} .$$

Then the following claim holds.

7.6. Corollary. Under the notation of 7.1 the map

$$\xi_{\rho,v} := \xi_g : \coprod_I K_*(F)[-r(I)] \xrightarrow{\approx} K_*(F[G]).$$

is an isomorphism.

Proof of 7.6. Remark first that $[g_i] = [\rho_i] + [x_i]$ in $K_1(F[G])$, because above the map (*) is a group homomorphism. Further define $K_*(F)$-linear filtrations on $\Phi = \coprod_I K_*(F)[-r(I)]$ and $K_*(F[G])$ as follows:

$$\Phi^r = \coprod_{r(I) \leqslant r} K_*(F)[-r(I)], \quad K_*^r(F[G]) = \xi_\rho(\Phi^r).$$

Relations $[g_i] = [\rho_i] + [x_i] \quad (i = 1, \dots, \mathrm{rk}(G))$ show that

$$[g]_I = [\rho]_I \mod K_*^{r(I)-1}(F[G])$$

for every subset I of $\{1, 2, \dots, \mathrm{rk}(G)\}$. The last relations also show that the map $\xi_g : \Phi \to K_*(F[G])$ is a morphism of filtrated $K_*(F)$-modules, and the morphism $\mathrm{Gr}\,\xi_g : \mathrm{Gr}\,\Phi \to \mathrm{Gr}\,K_*(F[G])$ of the adjoint modules is an isomorphism. Hence ξ_g is also an isomorphism. $\square$

8. A construction of Tits

The main goal of the present section is to recall Tits' construction of certain central simple F-algebra arising from representation of algebraic group. (see [T1], §3).

8.1. Let G be a semisimple connected F-split algebraic group over F which is simply connected. Let $T \subset G$ be a maximal F-split torus of G. Let $Z \hookrightarrow G$ be the center of G and set $\overline{G} = G/Z$. It is well known that Z is not necessarily reduced closed F-group subscheme of T and if $\overline{T} = T/Z$, then

$$0 \to \mathrm{Hom}(\overline{T}, \mathbb{G}_m) \to \mathrm{Hom}(T, \mathbb{G}_m) \to \mathrm{Hom}(Z, \mathbb{G}_m) \to 0$$

is an exact sequence.

8.2. Notation. Set $\mathrm{Ch} = \mathrm{Hom}(Z, \mathbb{G}_m)$. It is the character group of Z. The group Ch is finite and does not change in passing to a field extension of F.

8.3. Category $\mathrm{Rep}^{\chi}_F(G)$. Let $\chi \in \mathrm{Ch}$. Denote by $\mathrm{Rep}^{\chi}_F(G)$ the full subcategory of the category $\mathrm{Rep}_F(G)$ (see §6) consisting of the representations (V, ρ) such that Z acts on V by the character χ (this means that $zv = \chi(z)v$ for any z from Z and v from V).

8.4. Definition. A representation (V, ρ) of G is called Ch-homogeneous, if $(V, \rho) \in \mathrm{Rep}^{\chi}_F(G)$ for some character $\chi \in \mathrm{Ch}$.

8.5. Let $\chi \in \mathrm{Ch}$. Choose a non-trivial representation $V_{(\chi,\rho)} \in \mathrm{Rep}^{\chi}_F(G)$ and set $A_\chi = \mathrm{End}_F(V_\chi)$. Consider the G-action on A_χ $((g, \alpha) \mapsto g\alpha g^{-1})$. It is factorized through a $\overline{G}$-action, because Z acts trivially. Therefore

$$A_\chi = \mathrm{End}_F(V_\chi)$$

is an algebra equipped with a $\overline{G}$-action by F-algebra automorphisms.

8.6. Tits' construction. Let $\gamma : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to \overline{G}(F_{\mathrm{sep}})$ be a 1-cocycle. Using this cocycle one obtains a new $\mathrm{Gal}(F_{\mathrm{sep}}/F)$-action on $A_\chi \otimes_F F_{\mathrm{sep}}$:

$$\sigma \circ x = \overline{\rho}(\gamma(\sigma)) \circ^{\sigma} x \circ \overline{\rho}(\gamma(\sigma))^{-1}.$$

Set

$$A_\chi(\gamma) = [A_\chi \otimes_F F_{\mathrm{sep}}]^{\mathrm{Gal}(F_{\mathrm{sep}}/F)}.$$

The algebra $A_\chi(\gamma)$ is a twisted form of the algebra A_χ and corresponds to the 1-cocycle

$$\mathrm{Gal}(F_{\mathrm{sep}}/F) \xrightarrow{\gamma} \overline{G}(F_{\mathrm{sep}}) \xrightarrow{\overline{\rho}} \mathrm{PGL}(V_\chi \otimes_F F_{\mathrm{sep}}) = \mathrm{Aut}_{F_{\mathrm{sep}}-\mathrm{alg}}(A_\chi \otimes_F F_{\mathrm{sep}}).$$

where $\overline{\rho} : \overline{G} \to \mathrm{PGL}(V_\chi)$ makes the following diagram commutative

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & Z & \longrightarrow & G & \longrightarrow & \overline{G} & \longrightarrow & \{1\} \\
 & & \downarrow & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \overline{\rho}} & & \\
\{1\} & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathrm{GL}(V_\chi) & \longrightarrow & \mathrm{PGL}(V_\chi) & \longrightarrow & \{1\}.
\end{array}$$

8.7. Lemma. ([T1], 3.5).
1) if $V_\chi, V'_\chi \in \mathrm{Rep}_F^\chi(G)$, *then* $A_\chi(\gamma) \sim A'_\chi(\gamma)$ *in* $\mathrm{Br}(F)$;
2) if $V \in \mathrm{Rep}_F(\overline{G})$, *then* $A_\chi(\gamma) \sim F$;
3) $A_\chi(\gamma) \otimes_F A_{\chi'}(\gamma) \sim A_{\chi\cdot\chi'}(\gamma)$,
4) $A_{\chi^{-1}}(\gamma) \sim A_\chi^{\mathrm{op}}(\gamma)$.
In other words the correspondence $\chi \mapsto [A_\chi(\gamma)]$, $\mathrm{Ch} \to \mathrm{Br}(F)$ *is a well-defined group homomorphism.*

Proof is given in 8.10.

8.8 Centers of simple groups. If G is one of the groups: SL_{n+1}, Spin_{2n+1}, Sp_{2n}, Spin_{2n}, G_2, F_4, E_6, E_7 or E_8, then its center Z is the F-group scheme

$$\mu_{n+2},\ \mu_2,\ \mu_2,\ (\mu_2 \times \mu_2 \,\mathrm{or}\, \mu_4),\ \{1\},\ \{1\},\ \{1\},\ \mu_2 \text{ or } \mu_3$$

respectively. The character groups Ch are

$$\mathbb{Z}/(n+1)\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} \text{ or}$$
$$\mathbb{Z}/4\mathbb{Z},\ \{1\},\ \{1\},\ \{1\},\ \mathbb{Z}/2\mathbb{Z} \text{ or } \mathbb{Z}/3\mathbb{Z}$$

respectively (see [Hu], Appendix).

8.9. If V_1 is the first representation from examples 6.3 for groups

$$SL_{n+1},\ \mathrm{Spin}_{2n},\ \mathrm{Sp}_{2n} \text{ and } \mathrm{Spin}_{2n}, \text{ then } V_1 \in \mathrm{Rep}_F^{\chi_1}(G)$$

for

$$\chi_1 = \overline{1} \in \mathbb{Z}/(n+1)\mathbb{Z},\ \chi_1 = \overline{0} \in \mathbb{Z}/2\mathbb{Z},\ \chi_1 = \overline{1} \in \mathbb{Z}/2\mathbb{Z},\ \chi_1 = (\overline{1},\overline{1}) \text{ or } \overline{2}.$$

8.10. Proof of 8.7. Let $V \in \mathrm{Rep}_F(\overline{G})$. Then the pairing $\mathrm{End}(V)\times V \to V$ is compatible with the $\overline{G}$-action. Twist it with γ. The result is a pairing $\mathrm{End}(V)(\gamma)\otimes V(\gamma) \to V(\gamma)$. It induces an F-algebra homomorphism $\mathrm{End}(V)(\gamma) \to \mathrm{End}(V(\gamma))$. Since the both F-algebras are simple, it is an isomorphism. Hence

$$\mathrm{End}(V)(\gamma) \sim F$$

in the sense of 5.4.

If $W \in \mathrm{Rep}_F(\overline{G})$, then $\mathrm{End}(V\otimes W) \cong \mathrm{End}(V)\otimes\mathrm{End}(W)$ and this F-algebra isomorphism is compatible with the $\overline{G}$-actions. Hence

$$\mathrm{End}(V\otimes W)(\gamma) \cong \mathrm{End}(V)(\gamma)\otimes\mathrm{End}(W)(\gamma) \cong \mathrm{End}(V)(\gamma)\otimes\mathrm{End}(W(\gamma))$$

and

$$\mathrm{End}(V\otimes W)(\gamma) \sim \mathrm{End}(V)(\gamma).$$

Apply this remark to prove (1). Set $W_1 = V\otimes V^*$ and $W_2 = V^*\otimes V'$. Then $W_1, W_2 \in \mathrm{Rep}_F(\overline{G})$ and one have

$$\mathrm{End}(V)(\gamma) \sim \mathrm{End}(V\otimes W_2)(\gamma) \cong \mathrm{End}(W_1\otimes V')(\gamma) \sim \mathrm{End}(V')(\gamma).$$

To prove Claim (4) take $W = V\otimes V^*$. Then $W \in \mathrm{Rep}_F(\overline{G})$ and

$$\mathrm{End}(V)(\gamma)\otimes\mathrm{End}(V^*)(\gamma) \cong \mathrm{End}(W)(\gamma) \sim F.$$

Since $V \in \mathrm{Rep}_F^\chi(G)$ one has $V^* \in \mathrm{Rep}_F^{\chi^{-1}}(G)$. This proves (4).

To prove (3) note that the canonical map $\mathrm{End}(V)\otimes\mathrm{End}(V') \to \mathrm{End}(V\otimes V')$ is an F-algebra isomorphism compatible with the $\overline{G}$-actions. Hence

$$\mathrm{End}(V)(\gamma)\otimes\mathrm{End}(V')(\gamma) \cong \mathrm{End}(V\otimes V')(\gamma).$$

Since $V\otimes V' \in \mathrm{Rep}_F^{\chi\cdot\chi'}(G)$, the last isomorphism and 8.7(1) prove 8.7(3). □

9. Theorems B and C

Let B be a central simple F-algebra and let $e_1, \ldots, e_m$ be a base of B as a F-vector space. Let $x_i : \mathrm{GL}_{1,B} \to \mathbb{A}^1_F$ be the i-th coordinate function with respect to the base $e_1, e_2, \ldots, e_m$. For every affine F-variety X set $B[X] = B \otimes_F F[X]$ (comp. §7). Let $\mathrm{GL}(B[X]) = \bigcup_{i \geqslant 1} \mathrm{GL}_i(B[X])$ and let $E(B[X])$ be the subgroup of $\mathrm{GL}(B[X])$ generated by elementary matrixes. Recall that

$$K_1(B[X]) = \mathrm{GL}(B[X])/E(B[X])$$

(see [B1]). Identify $\mathrm{Mor}_F(X, \mathrm{GL}_{1,B})$ with $\mathrm{GL}_1(B[X])$ taking α to $\{\alpha_i\}_{i=1}^m \in B[X]$, where $\alpha_i = x_i \circ \alpha$, and consider the composition

$$\begin{aligned} \mathrm{Mor}_F(X, \mathrm{GL}_{1,B}) &= \mathrm{GL}_1(B[X]) \\ (*) \qquad\qquad &= \mathrm{GL}_1(B^{\mathrm{op}}[X]) \hookrightarrow \mathrm{GL}(B^{\mathrm{op}}[X]) \to K_1(B^{\mathrm{op}}[X]). \end{aligned}$$

9.1. Notation. For $g \in \mathrm{Mor}_F(X, \mathrm{GL}_{1,B})$ denote by

$$\{g\} \in K_1(B^{\mathrm{op}}[X]) = K_1(X, B^{\mathrm{op}})$$

the image of g under the map $(*)$. If $\{g_i : X \to \mathrm{GL}_{1,B_i}\}_{i=1}^l$ is a family of morphisms, then for every subset $I = \{i_1 < i_2 < \cdots < i_r\}$ of $\{1, 2, \ldots, l\}$ denote by

$$\{g\}_I \in K_r(X, B_1^{\mathrm{op}} \otimes \cdots \otimes B_r^{\mathrm{op}})$$

the cup product $\{g_{i_1}\} \cup \{g_{i_2}\} \cup \cdots \cup \{g_{i_r}\}$. Denote by

$$\Theta_{I,g} : K_*(B_I)[-r(I)] \to K_*(F[X]) = K_*(X)$$

the map $x \mapsto \{g\}_I \cup_{B_I} x$, where $B_I = B_{i_1} \otimes \cdots \otimes B_{i_r}$. Finally set

$$\Theta_g = \sum_I \Theta_{I,g} : \coprod_I K_*(B_I)[-r(I)] \to K_*(X),$$

where the sum is taken over all subsets I of $\{1, 2, \ldots, l\}$.

9.2. Let G, Z and $\overline{G}$ be the same as in §8. Let Ch be again the character group of Z (see 8.2). Let $(V, \rho) \in \mathrm{Rep}_F^{\chi}(G)$ be a non-zero representation for some character $\chi \in \mathrm{Ch}$ (see 8.3). If

$$A = \mathrm{End}(V),$$

then $\mathrm{GL}(V) = \mathrm{GL}_{1,A}$ as algebraic groups. Consider $\overline{G}$-actions on G and on $\mathrm{GL}(V) = \mathrm{GL}_{1,A}$ given by $(g, \alpha) \mapsto g\alpha g^{-1}$ and $(g, \beta) \mapsto \rho(g)\beta\rho(g)^{-1}$ respectively. Then $\rho : G \to \mathrm{GL}(V)$ is a $\overline{G}$-equivariant morphism.

If $\gamma : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to \overline{G}(F_{\mathrm{sep}})$ be a 1-cocycle, then twist $G, \mathrm{GL}(V) = \mathrm{GL}_{1,A}$, and ρ with γ. One obtains

$$G(\gamma), \;\; \mathrm{GL}(V)(\gamma) = \mathrm{GL}_{1,A(\gamma)} \;\text{ and }\; \rho(\gamma) : G(\gamma) \to \mathrm{GL}_{1,A(\gamma)},$$

and $\rho(\gamma)$ is an algebraic group morphism.

9.3. Theorem. *Let G be a simply connected semisimple F-split algebraic group over F. Let $\rho_1, \dots, \rho_{\mathrm{rk}(G)}$ be a fundamental family of* Ch*-homogeneous representations of G (see 6.2, 6.3 and 8.4). Let $\gamma : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to \overline{G}(\gamma_{\mathrm{sep}})$ be the 1-cocycle from 9.2. Then the family of algebras*

$$\{A_i(\gamma) = \mathrm{End}(V_i)(\gamma) | i = 1, \dots, \mathrm{rk}(G)\}$$

(see 9.2) and the family of morphisms

$$\{\rho_i(\gamma) : G(\gamma) \to \mathrm{GL}_{1,A_i(\gamma)} | i = 1, \dots, \mathrm{rk}(G)\}$$

(see 9.2) determines a map (see 9.1)

$$\Theta_{\rho(\gamma)} : \coprod_I K_*(A_I(\gamma))[-r(I)] \to K_*(G(\gamma))$$

which is an isomorphism of graded abelian groups.

This Theorem is a partial case of Theorem 9.5.

9.4. In this subsection we use notations and agreements of 9.2. Let

$$\delta : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to G(\gamma)(F_{\mathrm{sep}}) = G(F_{\mathrm{sep}})$$

be a 1-cocycle. This means that $\delta_{\sigma\tau} = \delta_\sigma \cdot ({}^{\sigma'}\delta_\tau)$, where ${}^{\sigma'}\delta_\tau = \gamma_\sigma \cdot^\sigma (\delta_\tau) \cdot \gamma_\sigma^{-1}$. Twist $G(\gamma)$, $\mathrm{GL}(V)(\gamma) = \mathrm{GL}_{1,A(\gamma)}$ and $\rho(\gamma)$ from 9.2 with δ. One obtains

$${}_\delta G(\gamma),\ {}_\delta \mathrm{GL}_{1,A(\gamma)} \text{ and } {}_\delta\rho(\gamma) : {}_\delta G(\gamma) \to {}_\delta \mathrm{GL}_{1,A(\gamma)},$$

where ${}_\delta G(\gamma)$ is a principal $G(\gamma)$-homogeneous space, ${}_\delta \mathrm{GL}_{1,A(\gamma)}$ is a principal $\mathrm{GL}_{1,A(\gamma)}$-homogeneous space. Below in this paper we write $\rho(\gamma,\delta)$ for ${}_\delta\rho(\gamma)$ and $G(\gamma,\delta)$ for ${}_\delta G(\gamma)$.

Since $A(\gamma)$ is a central simple F-algebra (see 8.6), one has

$$H^1(F, \mathrm{GL}_{1,A(\gamma)}(F_{\mathrm{sep}})) = *$$

(see [Se], III, §1, L.1). Hence there exists an isomorphism

$$u : {}_\delta \mathrm{GL}_{1,A(\gamma)} \to \mathrm{GL}_{1,A(\gamma)}$$

of principal $\mathrm{GL}_{1,A(\gamma)}$-homogeneous spaces. Set

$$f = u \circ \rho(\gamma,\delta) : G(\gamma,\delta) \to \mathrm{GL}_{1,A(\gamma)}.$$

9.5. Theorem B. *Let $G, \rho_1, \dots, \rho_{\mathrm{rk}(G)}$ and γ be the same as in 9.3. Let $\delta :$ $\mathrm{Gal}(F_{\mathrm{sep}}/F) \to G(\gamma)(F_{\mathrm{sep}})$ be the 1-cocycle from 9.4. Then the family of algebras*

$$\{A_i(\gamma) | i = 1, 2, \dots, \mathrm{rk}(G)\}$$

¿from 9.3 and the family of morphisms

$$\{f_i = u_i \circ \rho(\gamma,\delta) : G(\gamma,\delta) \to \mathrm{GL}_{1,A_i(\gamma)} | i = 1, 2, \dots, \mathrm{rk}(G)\}$$

(see 9.4) determine the map (see 9.1)

$$\Theta_f : \coprod_I K_*(A_I(\gamma))[-r(I)] \to K_*(G(\gamma,\delta)),$$

which is an isomorphism of graded abelian groups.

Proof will be given in §10.

9.6. Let $\mathcal{D}$ be an arbitrary central simple F-algebra. Let $\{A_i(\gamma)|i=1,\dots,\mathrm{rk}(G)\}$ be the family of algebras from 9.5 and let

$$\{f_i : G(\gamma,\delta) \to \mathrm{GL}_{1,A_i(\gamma)}\,|i=1,\dots,\mathrm{rk}(G)\}$$

be the family of morphisms from 9.5. Let

$$\{f\}_i \in K_1(G(\gamma,\delta),A_i(\gamma)^{\mathrm{op}}),\ \{f\}_I \in K_{r(I)}(G(\gamma,\delta),A_I(\gamma)^{\mathrm{op}})$$

be the corresponding elements (see 9.1). Denote by

$$\Theta^{\mathcal{D}}_{I,f} : K_*(A_I(\gamma))[-r(I)] \to K_*(G(\gamma,\delta))$$

the map taking x to $\{f\}_I \cup_{A_I(\gamma)} x$. Set

$$\Theta^{\mathcal{D}}_f = \sum_I \Theta^{\mathcal{D}}_{I,f} : \coprod_I K_*(A_I(\gamma)\otimes\mathcal{D})[-r(I)] \to K_*(G(\gamma,\delta),\mathcal{D}),$$

where I runs over all subset of the set $\{1,2,\dots,\mathrm{rk}(G)\}$.

9.7. Theorem C. *Under the notation and the assumptions of 9.5 and 9.6 the map $\Theta^{\mathcal{D}}_f$ is an isomorphism of graded abelian groups.*

Proof will be given in §10.

10. Proofs of Theorems B and C

The family of algebras $\{A_i(\gamma) | i = 1, \ldots, \mathrm{rk}(G)\}$ and the family of morphisms $\{f_i : G(\gamma, \delta) \to \mathrm{GL}_{A_i(\gamma)} | i = 1, \ldots, \mathrm{rk}(G)\}$ from 9.5 determine elements

$$\{f_i\} \in K_1(G(\gamma, \delta), A_i(\gamma)^{\mathrm{op}}), \ \{f\}_I \in K_{r(I)}(G(\gamma, \delta), A_I(\gamma)^{\mathrm{op}})$$

(see 9.1 and 9.6).

For every F-scheme S we define the map (c. 1.1)

$$\Theta_f(S) : \coprod_I K'_*(S, A_I(\gamma))[-r(I)] \to K'_*(S \times G(\gamma, \delta))$$

as the sum of maps

$$\Theta_{I,f}(S) : K'_*(S, A_I(\gamma))[-r(I)] \to K'_*(S \times G(\gamma, \delta))$$

given by $x \mapsto \{f\}_I \cup_{A_I(\gamma)} x$. Similarly define the map

$$\Theta_f^{\mathcal{D}}(S) : \coprod_I K'_*(S, A_I(\gamma) \otimes \mathcal{D})[-r(I)] \to K'_*(S \times G(\gamma, \delta), \mathcal{D})$$

for every central simple F-algebra $\mathcal{D}$

To prove Theorem B (see 9.5) state the following result (see 10.1), which also proves Theorem C (see 9.7) by 1.4.

10.1. Theorem. *$\Theta_f(S)$ is an isomorphism for every noetherian F-scheme S.*

Proof. of 10.1. Consider the semisimple algebraic group $\overline{G} \times G(\gamma)$ and the 1-cocycle

$$\gamma \times \delta : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to \overline{G}(F_{\mathrm{sep}}) \times G(\gamma)(F_{\mathrm{sep}}).$$

Theorem 1.1 and Remark 1.2(2) show that to prove 10.1 it is sufficient to prove the following.

10.2. Proposition. *If L/F is a field extension such that $\gamma|_L \sim *_L$ and $\delta|_L \sim *L$, then map $\Theta_f(L)$ is an isomorphism.*

Proof. of 10.2. Roughly speaking, $G_L = G(\gamma, \delta)_L$, $\quad A_i(\gamma)_L = \mathrm{End}(V_i)_L$ under the assumptions of 10.2. Moreover, the map $f_{i,L} : G(\gamma, \delta)_L \to \mathrm{GL}_{1,A_i(\gamma)}$ coincides with the composition

$$G_L \xrightarrow{\rho_{i,L}} \mathrm{GL}(V_i)_L \xrightarrow{l_i} \mathrm{GL}(V_i)_L,$$

where ρ_i is the representation of G, and l_i is the left translation by an element $x_i \in \mathrm{GL}(V_i)(L)$. Therefore one can reduce 10.2 to 7.6.

This is the idea of the proof of 10.2. To give a complete proof of 10.2 we need the following three lemmas, which are proved below in §§11, 12 and 13.

10.3. Lemma. *Set $G(\gamma,\delta) = {}_\delta G(\gamma)$. Then under the hypotheses of 10.2 there exist commutative diagrams*

$$\begin{array}{ccc} G(\gamma,\delta)_L & \xrightarrow{f_{i,L}} & \mathrm{GL}(V_i)(\gamma)_L = \mathrm{GL}_{1,A_i(\gamma)_L} \\ {\scriptstyle s}\uparrow & & {\scriptstyle r_i}\uparrow \\ G_L & \xrightarrow{g_i} & \mathrm{GL}(V_i)_L = \mathrm{GL}_{1,\mathrm{End}(V_i)_L}, \end{array}$$

where s and r_i $(i = 1, 2, \ldots, \mathrm{rk}(G))$ are isomorphisms, and every r_i is induced by an L-algebra isomorphism

$$\mathrm{End}(V_i) \otimes_F L \to \mathrm{End}(V_i)(\gamma) \otimes_F L = A_i(\gamma) \otimes_F L,$$

and $g_i = v_i \circ \rho_{i,L}$, where $v_i : \mathrm{GL}(V_i)_L \to \mathrm{GL}(V_i)_L$ is given by $y \mapsto x_i \cdot y$ for certain $x_i \in \mathrm{GL}(V_i)(L)$ $(i = 1, \ldots, \mathrm{rk}(G))$.

10.4. Lemma. *Let $X' \xrightarrow{t} X$ be an isomorphism of affine L-varieties, and let $\{T_i : \mathcal{D}'_i \to \mathcal{D}_i\}_{i=1}^m$ be a family of central simple L-algebras isomorphisms. Let $\{f_i : X \to \mathrm{GL}_{1,\mathcal{D}_i}\}_{i=1}^m$ and $\{f'_i : X' \to \mathrm{GL}_{1,\mathcal{D}'_i}\}_{i=1}^m$ be two families such that diagrams*

$$\begin{array}{ccc} X & \xrightarrow{f_i} & \mathrm{GL}_{1,\mathcal{D}_i} \\ {\scriptstyle t}\uparrow & & \uparrow{\scriptstyle T_i} \\ X' & \xrightarrow{f'_i} & \mathrm{GL}_{1,\mathcal{D}'_i} \end{array}$$

are commutative $(i = 1, 2, \ldots, m)$. Let Θ_f and $\Theta_{f'}$ be the maps from 9.1. Then the following properties are equivalent:

(1) *Θ_f is an isomorphism,*
(2) *$\Theta_{f'}$ is an isomorphism.*

10.5. Lemma. *Let $\{h_i : X \to \mathrm{GL}(V_i) = \mathrm{GL}_{1,\mathrm{End}(V_i)}\}_{i=1}^m$ be a family of morphisms. Let ξ_h and Θ_h be graded abelian group morphisms from 7.1 and 9.1 respectively. Then the following properties are equivalent:*

(1) *Θ_h is an isomorphism,*
(2) *ξ_h is an isomorphism.*

10.6.The end of the proof of 10.2.

Having these three lemmas, complete the proof of 10.2. To do this, consider the families $\{f_{i,L}\}$ and $\{g_i = v_i \circ \rho_{i,L}\}$ from 10.3. Then there exist the homomorphisms (see 9.1 and 7.1 for definitions):

$$\Theta_{f_L} : \coprod_I K_*(A_I(\gamma) \otimes_F L)[-r(I)] \to K_*(G(\gamma,\delta)_L),$$

$$\Theta_g : \coprod_I K_*(\mathrm{End}(V_I) \otimes_F L)[-r(I)] \to K_*(G_L),\ (V_I = V_{i_1} \otimes \cdots \otimes V_{i_r})$$

$$\xi_g : \coprod_I K_*(L)[-r(I)] \to K_*(G_L).$$

Observe now that Θ_{f_L} coincides with $\Theta_f(L)$ and ξ_g coincides with $\xi_{\rho_L,v}$ of 7.5 and 7.6. The map $\xi_{\rho_L,g}$ is an isomorphism by 7.6. Lemmas 10.3-10.5 show that Θ_{f_L} is also an isomorphism. Hence $\Theta_f(L)$ is an isomorphism. Proposition 10.2 is proven, Theorem 10.1 is proven. Hence Theorem B is proven and Theorem C is also proven by 1.4. $\square$

11. Proof of Lemma 10.3

Proof of 10.3. Let $L_{\rm sep}$ be a separable closure of L and let $F_{\rm sep} \hookrightarrow L_{\rm sep}$ be the separable closure of F in $L_{\rm sep}$. For every $\sigma \in \mathrm{Gal}(L_{\rm sep}/L)$ set $\overline{\sigma} = \sigma|_{F_{\rm sep}}$. Since (see [Se], Ch.I, 5.1) $\gamma|_L \sim *_L$, there exists $b \in \overline{G}(L_{\rm sep})$ such that $\gamma_{\overline{\sigma}} = b^{-1} \cdot^{\sigma} b$ for every $\sigma \in \mathrm{Gal}(L_{\rm sep}/L)$. Since $\delta|_L \sim *_L$, there exists $c \in G(L_{\rm sep})$ such that

$$\delta_{\overline{\sigma}} = c^{-1} \cdot^{\sigma'} (c) = c^{-1} \cdot (\gamma_{\overline{\sigma}} \cdot^{\sigma} c \cdot \gamma_{\overline{\sigma}}^{-1}) = c^{-1} \cdot (b^{-1} \cdot^{\sigma} b) \cdot^{\sigma} c \cdot ({}^{\sigma} b^{-1} \cdot b).$$

For every affine F-variety X and every 1-cocycle $\varepsilon : \mathrm{Gal}(F_{\rm sep}/F) \to \mathrm{Aut}_{F_{\rm sep}}(X_{\rm sep})$ the variety $({}_{\varepsilon}X)_{\rm sep}$ coincides with the variety $X_{\rm sep}$, but the $\mathrm{Gal}(F_{\rm sep}/F)$-action is given by $x \mapsto \varepsilon_{\sigma} \cdot^{\sigma} x$ (see [Se], Ch.I, 5.2). Hence the variety $({}_{\varepsilon}X)_{L_{\rm sep}}$ coincides with the variety $X_{L_{\rm sep}}$, but the $\mathrm{Gal}(L_{\rm sep}/L)$-action is given by $y \mapsto \varepsilon_{\overline{\sigma}} \cdot \sigma_y$. Therefore the varieties $G(\gamma)_{L_{\rm sep}}$, $G(\gamma,\delta)_{L_{\rm sep}}$, $\mathrm{GL}(V_i)(\gamma)_{L_{\rm sep}}$ and $\mathrm{GL}(V_i)(\gamma,\delta)_{L_{\rm sep}}$ coincide with $G_{L_{\rm sep}}$, $G_{L_{\rm sep}}$, $\mathrm{GL}(V_i)_{L_{\rm sep}}$ and $\mathrm{GL}(V_i)_{L_{\rm sep}}$ respectively, but the $\mathrm{Gal}(L_{\rm sep}/L)$-actions are given by the rules

(1) $x \overset{\sigma}{\mapsto} \gamma_{\overline{\sigma}} \cdot^{\sigma} x \cdot \gamma_{\overline{\sigma}}^{-1}$ ($\overline{G}$ acts on G by the conjugation),
(2) $y \overset{\sigma}{\mapsto} \delta_{\overline{\sigma}} \cdot (\gamma_{\overline{\sigma}} \cdot^{\sigma} x \cdot \gamma_{\overline{\sigma}}^{-1})$,
(3) $z \mapsto \rho_i(\gamma_{\overline{\sigma}}) \cdot^{\sigma} z \cdot \rho_i(\gamma_{\overline{\sigma}})^{-1}$,
(4) $t \mapsto \rho_i(\delta_{\overline{\sigma}}) \cdot (\rho_i(\gamma_{\overline{\sigma}}) \cdot^{\sigma} t \cdot \rho_i(\gamma_{\overline{\sigma}})^{-1})$
respectively.

Consider the maps

$$\begin{aligned} s_{\rm sep} : G_{L_{\rm sep}} &\to G(\gamma,\delta)_{L_{\rm sep}} && (g \mapsto c^{-1}(b^{-1}gb)),\\ q_{i,\rm sep} : \mathrm{GL}(V_i)_{L_{\rm sep}} &\to \mathrm{GL}(V_i)(\gamma,\delta)_{L_{\rm sep}} && (h \mapsto \rho_i(c)^{-1} \cdot (\rho_i(b)^{-1} \cdot h \cdot \rho_i(b)),\\ r_{i,\rm sep} : \mathrm{GL}(V_i)_{L_{\rm sep}} &\to \mathrm{GL}(V_i)(\gamma)_{L_{\rm sep}} && (h \mapsto \rho_i(b^{-1}) \cdot h \cdot \rho_i(b)). \end{aligned}$$

These maps are compatible with the $\mathrm{Gal}(L_{\rm sep}/L)$-actions (2-4).

Therefore they determine certain maps

$$\begin{aligned} s : G_L &\to G(\gamma,\delta)_L,\\ q_i : \mathrm{GL}(V_i)_L &\to \mathrm{GL}(V_i)(\gamma,\delta)_L,\\ r_i : \mathrm{GL}(V_i)_L &\to \mathrm{GL}(V_i)(\gamma)_L. \end{aligned}$$

Since $q_{i,\rm sep} \circ \rho_{i,\rm sep} = \rho_i(\gamma,\delta)_{\rm sep} \circ s_{\rm sep}$, the diagram

$$\begin{tikzcd} G(\gamma,\delta)_L \arrow[r, "\rho_i(\gamma,\delta)_L"] & \mathrm{GL}(V_i)(\gamma,\delta)_L \\ G_L \arrow[u, "s"] \arrow[r, "\rho_{i,L}"] & \mathrm{GL}(V_i)_L \arrow[u, "q_i"'] \end{tikzcd}$$

is commutative.

Let $u_i : \mathrm{GL}(V_i)(\gamma,\delta) \to \mathrm{GL}(V_i)(\gamma)$ be the isomorphism from 9.5. Set $v_i = r_i^{-1} \circ u_{i,L} \circ q_i$, and $g_i = v_i \circ \rho_{i,L}$. Then the diagram

$$\begin{tikzcd} G(\gamma,\delta)_L \arrow[r, "f_{i,L}"] & \mathrm{GL}(V_i)(\gamma)_L \\ G_L \arrow[u, "s"] \arrow[r, "g_i"] & \mathrm{GL}(V_{i,L}) \arrow[u, "r_i"'] \end{tikzcd}$$

is commutative, because $f_{i,L} = u_{i,L} \circ \rho_i(\gamma, \delta)_L$ (see 6.5).

Prove now that $v_i : \mathrm{GL}(V_i)_L \to \mathrm{GL}(V_i)_L$ is the left multiplication by an $x_i \in \mathrm{GL}(V_i)(L)$. Since

$$u_{i,F_{\mathrm{sep}}} : \mathrm{GL}(V_i)_{F_{\mathrm{sep}}} = \mathrm{GL}(V_i)(\gamma, \delta)_{F_{\mathrm{sep}}} \to \mathrm{GL}(V_i)(\gamma)_{F_{\mathrm{sep}}} = \mathrm{GL}(V_i)_{F_{\mathrm{sep}}}$$

is a morphism of the right principal $\mathrm{GL}(V_i)$-homogeneous spaces, it is the left multiplication by a certain $d_i \in \mathrm{GL}(V_i)(F_{\mathrm{sep}})$. More precisely, $d_i = u_{i,\mathrm{sep}}(1)$. Hence

$$v_{i,\mathrm{sep}} : \mathrm{GL}(V_i)_{L_{\mathrm{sep}}} \to \mathrm{GL}(V_i)_{L_{\mathrm{sep}}}$$

is the left multiplication by $x_i = \rho_i(b) \cdot (d_i \cdot \rho_i(c)^{-1} \cdot \rho_i(b)^{-1}$. Since $x_i = v_{i,\mathrm{sep}}(1) = v_i(1)$, one has $x_i \in \mathrm{GL}(V_i)(L)$, and

$$v_i : \mathrm{GL}(V_i)_L \to \mathrm{GL}(V_i)_L$$

is the left multiplication by x_i.

To finish the proof of 10.3 it is sufficient to verify that the above map r_i is induced by an L-algebra isomorphism $\mathrm{End}(V_i)_L \to A_i(\gamma)_L$.

The algebra $A_i(\gamma)$ is the $\mathrm{Gal}(F_{\mathrm{sep}}/F)$-invariants of the algebra $\mathrm{End}(V_i) \otimes_F F_{\mathrm{sep}}$ under the action given by

$$a \overset{\sigma}{\mapsto} \rho_i(\gamma_\sigma) \cdot {}^{\sigma} a \cdot \rho_i(\gamma_\sigma)^{-1}.$$

Hence $A_i(\gamma)_L$ is the $\mathrm{Gal}(L_{\mathrm{sep}}/L)$-invariants of the algebra $\mathrm{End}(V_i) \otimes_F L_{\mathrm{sep}}$ under the action given by

$$a \overset{\sigma}{\mapsto} \rho_i(\gamma_{\overline{\sigma}}) \cdot {}^{\overline{\sigma}} a \cdot \rho_i(\gamma_{\overline{\sigma}})^{-1},$$

where $\overline{\sigma} = \sigma|_{F_{\mathrm{sep}}}$. The map

$$h \mapsto \rho_i(b^{-1}) \cdot h \cdot \rho_i(b)$$

determines an isomorphism

$$T_{i,\mathrm{sep}} : \mathrm{End}(V_i) \otimes_F L_{\mathrm{sep}} \to \mathrm{End}(V_i) \otimes_F L_{\mathrm{sep}},$$

which induces an L-algebra isomorphism

$$T_i : \mathrm{End}(V_i)_L \to A_i(\gamma)_L.$$

It is obvious that $T_{i,\mathrm{sep}}$ induces $r_{i,\mathrm{sep}}$ above. Hence T_i induces $r_i : \mathrm{GL}(V_i)_L \to \mathrm{GL}_{1,A_i(\gamma)_L} = \mathrm{GL}(V_i)(\gamma)_L$. Lemma is proven. $\square$

12. Proof of Lemma 10.4

For an arbitrary ring homomorphism $u : A \to B$ let $u_* : K_*(A) \to K_*(B)$ denote the homomorphism induced by the exact functor $P \mapsto B \otimes_A P$, $\mathcal{P}(A) \to \mathcal{P}(B)$, where $\mathcal{P}(A)$, $\mathcal{P}(B)$ are the categories of left funitely generated projective A and B-modules respectively. It is known (see [Gr]) that K_* defined by Quillen ([Q], §2) coincides with K_* defined by the Quillen plus construction. Moreover, above the map $u_* : K_*(A) \to K_*(B)$ coincides with the map induced by the group homomorphism $\mathrm{GL}(u) : \mathrm{GL}(A) \to \mathrm{GL}(B)$.

12.1. Consider ring homomorphisms

$$\begin{aligned} \mathrm{id} = \mathrm{id}_{F[X]} &: F[X] \to F[X], \\ T_i &: \mathcal{D}'_i \to \mathcal{D}_i, \\ \mathrm{id}_i = \mathrm{id}_{\mathcal{D}_i} &: \mathcal{D}_i \to \mathcal{D}_i, \\ u = t^* &: F[X] \to F[X'], \\ \mathrm{id}' = \mathrm{id} &: F[X'] \to F[X']. \end{aligned}$$

The commutativity of diagrams from 10.4 shows that the following relation holds in $K_1(\mathcal{D}_i^{\mathrm{op}} \otimes_F F[X'])$:

$$(T_i \otimes \mathrm{id}')_*\{f'_i\} = \{T_i \circ f'_i\} = \{f_i \cdot t\} = (\mathrm{id}_i \otimes u)_*\{f_i\}. \tag{12.1(1)}$$

(see 9.1 for the notation).

12.2. Lemma. *Set*

$$\begin{aligned} \mathcal{D}_I = \mathcal{D}_{i_1} \otimes \cdots \otimes \mathcal{D}_{i_r},\ & \mathcal{D}'_I = \mathcal{D}'_{i_1} \otimes \cdots \otimes \mathcal{D}'_{i_r}, \\ T_I = T_{i_1} \otimes \cdots \otimes T_{i_r},\ & \mathrm{id}_I = \mathrm{id}_{i_1} \otimes \cdots \otimes \mathrm{id}_{i_r} \end{aligned}$$

for $I = \{i_1 < i_2 < \cdots < i_r\}$. *Then*

$$(T_I \otimes \mathrm{id}')_*\{f'\}_I = (\mathrm{id}_I \otimes u)_*\{f\}_I$$

(see 9.1 for the definitions of $\{\ldots\}_I$*??).*

Proof. The relation follows from 12.1(1) and the fact that

$$(T_I \otimes \mathrm{id})_*(a_1 \cup \cdots \cup a_r) = (T_{i_1} \otimes \mathrm{id})_*(a_1) \cup \cdots \cup (T_{i_r} \otimes \mathrm{id})_*(a_r),$$

$$(\mathrm{id}_I \otimes u)_*(b_1 \cup \cdots \cup b_r) = (\mathrm{id}_{i_1} \otimes u)_*(b_1) \cup \cdots \cup (\mathrm{id}_{i_r} \otimes u)_*(b_r)$$

for any

$$a_j \in K_*(\mathcal{D}_{i_j}^{\mathrm{op}} \otimes F[X]),\ b_j \in K_*(D_{i_j}^{\mathrm{op}} \otimes F[X']) \quad (j = 1, \ldots, r).$$

12.3. Lemma. *If* $\mathcal{D} = \mathcal{D}_I$, $\mathcal{D}' = \mathcal{D}'_I$, $T = T_I$, $\mathrm{id} = \mathrm{id}_I$, *then the diagram*

$$\begin{array}{ccc}
K_*(\mathcal{D}'^{\mathrm{op}} \otimes F[X']) \otimes K_*(\mathcal{D}') & \longrightarrow & K_*(F[X']) \\
\Big\downarrow{\scriptstyle (T\otimes \mathrm{id}')_* \otimes T_*} & & \Big\downarrow{\scriptstyle \mathrm{id}'_*} \\
K_*(\mathcal{D}^{\mathrm{op}} \otimes F[X']) \otimes K_*(\mathcal{D}) & \longrightarrow & K_*(F[X']) \\
\Big\uparrow{\scriptstyle (\mathrm{id}\otimes u)_* \otimes \mathrm{id}_*} & & \Big\uparrow{\scriptstyle u_*} \\
K_*(\mathcal{D}^{\mathrm{op}} \otimes F[X]) \otimes K_*(\mathcal{D}) & \longrightarrow & K_*(F[X])
\end{array}$$

is commutative, where horisontal arrows are induced by biexact functors $(P', Q') \mapsto P' \otimes_{\mathcal{D}'} Q'$, $(P'', Q') \mapsto P'' \otimes_{\mathcal{D}} Q'$ *and* $(P, Q) \mapsto P \otimes_{\mathcal{D}} Q$ *respectively.*

Proof. The commutativity of the top square follows from the fact that $F[X']$-modules

$$(\mathcal{D}^{\mathrm{op}}[X'] \otimes_{\mathcal{D}'[X']} P') \otimes_{\mathcal{D}} (\mathcal{D} \otimes_{\mathcal{D}'} Q') \text{ and } P' \otimes_{\mathcal{D}'} Q'$$

are canonically isomorphic. Similary the commutativity of the bottom square is verified.

12.4. Corollary. *The diagram*

$$\begin{array}{ccc}
K_*(\mathcal{D}'_I) & \xrightarrow{\{f'\}_I \cup ?} & K_*(F[X']) \\
\Big\downarrow{\scriptstyle T_*} & & \Big\uparrow{\scriptstyle u_*} \\
K_*(\mathcal{D}_I) & \xrightarrow{\{f\}_I \cup ?} & K_*(F[X])
\end{array}$$

is commutative.

Proof. This follows from 12.2 and 12.3.

12.5. Corollary. *Lemma 10.4 holds.*

Proof. The maps $(T_I)_* : K_*(\mathcal{D}_I'^{\mathrm{op}}) \to K_*(\mathcal{D}_I^{\mathrm{op}})$ and u_* are isomorphisms. Hence 10.4 follows from 12.4.

13. Proof of Lemma 10.5

Let X be an affine variety and $\mathcal{P}(X)$ be the category of finitely generated projective $F[X]$-modules. Let V be a finite dimensional F-vector space and V^* be its dual vector space. Let

$$\eta_V : \mathcal{P}(X) \to \mathcal{P}(X, \mathrm{End}(V)^{\mathrm{op}})$$

be the functor $P \mapsto V^* \otimes_F P$.

13.1. Lemma. *The diagram (see 7.1 and 9.1 for notation)*

$$\begin{array}{ccc} \mathrm{Mor}(X, \mathrm{GL}(V)) & \xrightarrow{\mathrm{id}} & \mathrm{Mor}(X, \mathrm{GL}_{1,\mathrm{End}(V)}) \\ \Big\downarrow{\scriptstyle [??]} & & \Big\downarrow{\scriptstyle \{\}} \\ K_1(X) & \xrightarrow{\eta_V} & K_1(X, \mathrm{End}(V)^{\mathrm{op}}) \end{array}$$

is commutative.

Proof. This is obvious.

13.2. If $V_1, \ldots, V_r$ are finite dimensional F-vector spaces, then the diagram

$$\begin{array}{ccc} K_*(X) \otimes \cdots \otimes K_*(X) & \xrightarrow{\eta_{V_1} \otimes \cdots \otimes \eta_{V_r}} & K_*(X, \mathrm{End}(V_1^{\mathrm{op}})) \otimes \cdots \otimes K_*(X, \mathrm{End}(V_r^{\mathrm{op}})) \\ \Big\downarrow{\scriptstyle \cup} & & \Big\downarrow{\scriptstyle \cup} \\ K_*(X) & \xrightarrow{\eta_{V_1 \otimes \cdots \otimes V_r}} & K_*(X, \mathrm{End}(V_1 \otimes \cdots \otimes V_r)^{\mathrm{op}}). \end{array}$$

Indeed, this follows from the fact that $\mathrm{End}(V_1 \otimes \cdots \otimes V_r)^{\mathrm{op}} \otimes_F F[X]$-modules $(P_1 \otimes \cdots \otimes P_r) \otimes V_1^* \otimes \cdots \otimes V_r^*$ and $(V_1^* \otimes P_1) \otimes (V_2^* \otimes P_2) \otimes \cdots \otimes (V_r^* \otimes P_r)$ are canonically isomorphic, and from the definition of the cup-product in K-theory (see [Ji], Appendix).

13.3. Lemma. *For any family $\{f_i : X \to \mathrm{GL}(V_i)\}_{i=1}^r$ the following relation holds*

$$\eta_{V_1 \otimes \cdots \otimes V_r}([f_1] \cup \cdots \cup [f_r]) = \{f_1\} \cup \cdots \cup \{f_r\}.$$

Proof. See 13.2 and 13.1.

13.4. Lemma. *The diagram*

$$\begin{array}{ccc} K_*(X) \otimes K_*(F) & \xrightarrow{\cup} & K_*(X) \\ \Big\downarrow{\scriptstyle \eta_V \otimes \eta_{V^*}} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ K_*(X, \mathrm{End}(V)^{\mathrm{op}}) \otimes K_*(\mathrm{End}(V)) & \longrightarrow & K_*(X) \end{array}$$

is commutative, where the unnamed arrow is induced by the biexact functor

$$(P, Q) \mapsto P \otimes_{\mathrm{End}(V)} Q.$$

Proof. This follows from the fact that $F[X]$-modules $(V^* \otimes U) \otimes_{\mathrm{End}(V)} (V \otimes W)$ and $U \otimes_F W$ are canonically isomorphic.

13.5. Corollary. *Under the hypotheses of 10.5 the diagram*

$$\begin{array}{ccc}
\coprod_I K_*(F)[-r(I)] & \xrightarrow{\xi_f} & K_*(X) \\
\Big\downarrow \coprod_I \eta_{V_I} & & \Big\downarrow \mathrm{id} \\
\coprod_I K_*(\mathrm{End}(V_I))[-r(I)] & \xrightarrow{\Theta_f} & K_*(X)
\end{array}$$

is commutative (see 7.1 and 9.1 for notation).

Proof. The claim follows from 13.4 and 13.3.

13.6. Corollary. *Lemma 10.5 holds.*

Proof. It is sufficient to prove that η_{V_I} is an isomorphism for every I. To prove this, note that for the vector space V the functors

$$\begin{aligned}
\eta_V &: \mathcal{P}(X) \to \mathcal{P}(X, \mathrm{End}(V)^{\mathrm{op}}) && (P \mapsto V^* \otimes_F P), \\
\eta_V^{\mathrm{op}} &: \mathcal{P}(X, \mathrm{End}(V)^{\mathrm{op}}) \to \mathcal{P}(X) && (Q \mapsto V \otimes_{\mathrm{End}(V)^{\mathrm{op}}} Q)
\end{aligned}$$

are category equivalences inverse to each other. Indeed,

$$(V \otimes V^*) \otimes_{\mathrm{End}(V)^{\mathrm{op}}} P \cong P \quad \text{and} \quad (V^* \otimes V) \otimes_{\mathrm{End}(V)^{\mathrm{op}}} Q \cong Q.$$

$\square$

14. Clifford algebras and spinors

Let V be a vector space over F and let $q \in S^2V^*$ be a nonsingular quadratic form on V. Let $C(q)$ be the Clifford algebra of (V, q) and let $C(q) = C_0(q) \oplus C_1(q)$ be the decomposition of $C(q)$ into the even and odd parts. Then $C_0(q)$ is a subalgebra of $C(q)$ and $C_1(q)$ is a $C_0(q)$-module.

14.1 Notation. Set

$$C(2n) = C(x_1y_1 + \cdots + x_ny_n)$$

and

$$C(2n+1) = C(x_1y_1 + \cdots + x_ny_n + z^2).$$

14.2. Lemma. *(See [Sw], 4.3). There exist algebra isomorphisms*

$$1)\, C_0(2n+1) \cong M_{2^n}(F), \quad 2)\, C_0(2n) \cong M_{2^{n-1}}(F) \times M_{2^{n-1}}(F).$$

They induce algebraic group isomorphisms

$$\varphi_{2n+1} : \mathrm{GL}_{1,C_0(2n+1)} \xrightarrow{\sim} \mathrm{GL}_{2^n,F};$$
$$\varphi_{2n} = \varphi_{2n}^+ \times \varphi_{2n}^- : \mathrm{GL}_{1,C_0(2n)} \xrightarrow{\sim} \mathrm{GL}_{2^{n-1}} \times \mathrm{GL}_{2^{n-1}}\,.$$

14.3. Lemma. *There exists a closed embedding of algebraic groups*

$$\mathrm{Spin}(q) \overset{\mathrm{id}}{\hookrightarrow} \mathrm{GL}_{1,C_0(q)}\,.$$

Proof. See [B2].

14.4. Notation. Set

$$\delta_n = \varphi_{2n+1} \circ i_{2n+1} : \mathrm{Spin}_{2n+1,F} \to \mathrm{GL}_{2^n,F},$$
$$\delta_n^{\pm} = \varphi_{2n}^{\pm} \circ i_{2n} : \mathrm{Spin}_{2n,F} \to \mathrm{GL}_{2^{n-1},F}\,.$$

Let $\triangle_n$ denote the F-vector space F^{2^n} endowed with the Spin_{2n+1}-action $(g, x) \mapsto \delta_n(g)x$, and let $\triangle_n^+$ (respectively $\triangle_n^-$), denote the F-vector space $F^{2^{n-1}}$ endowed with the Spin_{2n}-action given by $(g, x) \mapsto \delta_n^+(g)\cdot x$ (respectively $(g, x) \mapsto \delta_n^-(g)\cdot x$).

14.5.Agreements. Below we assume that the group Spin_{2n+1} acts on $C_0(2n+1)$ by the conjugation, i.e.,

$$(x, \alpha) \mapsto i_{2n+1}(x) \cdot \alpha \cdot i_{2n+1}(x)^{-1}.$$

Also Spin_{2n+1} acts on $\mathrm{End}_F(\triangle_n)$ by the conjugation

$$(x, \beta) \mapsto \delta_n(x) \cdot \beta \cdot \delta_n(x)^{-1}$$

(here $\delta_n(x)$ is considered as an element of $\mathrm{GL}_{1,\mathrm{End}(\triangle_n)}$). Similarly define Spin_{2n}-action on the algebras $C_0(2n)$ and $\mathrm{End}_F(\triangle_n^+) \times \mathrm{End}_F(\triangle_n^-)$, using i_{2n} and $\delta_n^{\pm}$.

14.6. Remark. Isomorphisms (1) and (2) of 14.2 rewritten as isomorphisms

$$\psi_{2n+1} : C_0(2n+1) \xrightarrow{\sim} \mathrm{End}_F(\triangle_n),$$
$$\psi^+_{2n} \times \psi^-_{2n} : C_0(2n) \xrightarrow{\sim} \mathrm{End}_F(\triangle^+_n) \times \mathrm{End}_F(\triangle^-_n)$$

are compatible with the $\mathrm{Spin}_{2n+1,F}$-and the $\mathrm{Spin}_{2n,F}$-actions described in 14.5. Moreover these actions are factorized through the groups $\mathrm{SO}_{2n+1,F}$ and $\mathrm{PSO}_{2n,F}$ respectively.

14.7. Corollary. *Let $\gamma : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to \mathrm{SO}_{2n+1}(F_{\mathrm{sep}})$ and $\varepsilon : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to \mathrm{SO}_{2n}(F_{\mathrm{sep}})$ be 1-cocycles. Then*

$$\psi_{2n+1}(\gamma) : C_0(2n+1)(\gamma) \xrightarrow{\sim} \mathrm{End}_F(\triangle_n)(\gamma),$$
$$\psi_{2n}(\varepsilon) : C_0(2n)(\varepsilon) \xrightarrow{\sim} \mathrm{End}(\triangle^+_n)(\varepsilon) \times \mathrm{End}(\triangle^-_n)(\varepsilon)$$

are F-algebra isomorphisms.

14.8. Proposition. *Let $q(\gamma)$ and $q(\varepsilon)$ be the quadratic forms corresponding to γ and ε respectively. Then there exist F-algebra isomorphisms*

$$C_0(q(\gamma)) \xrightarrow{\sim} \mathrm{End}(\triangle_n)(\gamma),$$
$$C_0(q(\varepsilon)) \xrightarrow{\sim} \mathrm{End}(\triangle^+_n)(\varepsilon) \times \mathrm{End}(\triangle^-_n)(\varepsilon).$$

Proof. Since $q \mapsto C_0(q)$ is a functor, one has an F-algebra isomorphism $C_0(q(\gamma)) \xrightarrow{\sim} C_0(2n+1)(\gamma)$. Similary, $C_0(q(\varepsilon)) \xrightarrow{\sim} C_0(2n)(\varepsilon)$. Now 14.8 follows from 14.7.

14.9. Let $\xi : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to \mathrm{PSO}_{2n}(F_{\mathrm{sep}})$ be a 1-cocycle and let (A_ξ, σ_ξ) be the corresponding algebra with involution of the orthogonal type. (Here A_ξ corresponds to the 1-cocycle

$$\mathrm{Gal}(F_{\mathrm{sep}}/F) \xrightarrow{\xi} \mathrm{PSO}_{2n}(F_{\mathrm{sep}}) \hookrightarrow \mathrm{PGL}_{2n}(F_{\mathrm{sep}}).)$$

Then the following holds.

14.10. Proposition. *Let $C_0(A_\xi, \sigma_\xi) = C_0(2n)(\xi)$ be the even Clifford algebra of (A_ξ, σ_ξ) (see [J]). Then there exists an F-algebra isomorphism*

$$\psi_{2n}(\xi) : C_0(A_\xi, \sigma_\xi) \xrightarrow{\sim} \mathrm{End}_F(\triangle^+_n)(\xi) \times \mathrm{End}_F(\triangle^-_n)(\xi).$$

Proof. This follows from 14.6.

15. Proof of Theorem A

In this section Theorem A (see 5.1) is deduced from Theorem C (see 9.7). Let G be a connected semisimple F-split algebraic group over F such that $H_{\mathrm{sep}} \xrightarrow{\sim} G_{\mathrm{sep}}$ as algebraic groups over F_{sep}. Then under the hypotheses of §5

$$H = G(\gamma)$$

for a certain 1-cocycle $\gamma : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to \overline{G}(F_{\mathrm{sep}})$, where $\overline{G}$ the factor of G modulo its center Z.

15.1. Proof of Theorem A. First note that $X = {}_{\delta}H$ for a 1-cocycle $\delta : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to H(F_{\mathrm{sep}})$ and hence $X = {}_{\delta}G(\gamma) = G(\gamma,\delta)$ under the notation of 9.4. Theorem C (see 9.7) shows that

$$K_*(X,\mathcal{D}) = K_*(G(\gamma,\delta),\mathcal{D}) \cong \coprod_I K_*(A_I(\gamma)\otimes\mathcal{D})[-r(I)],$$

where $A_I(\gamma) = A_{i_1}(\gamma)\otimes\cdots\otimes A_{i_r}(\gamma)$. This proves the first part of Theorem A.

To compute algebras $A_i(\gamma)$ choose a fundamental family of Ch-homogeneous representation of G (see 6.2, 8.4) and first consider the cases when G is simple. Begin with cases $G = \mathrm{SL}_{n+1}$, Spin_{2n+1}, Sp_{2n} and Spin_{2n} . Take the following table of representations (here we consider $Z' = \ker(\mathrm{Spin}_{2n} \to \mathrm{SO}_{2n})$ and $Z = \ker(\mathrm{Spin}_{2n} \to \mathrm{PSO}_{2n})$):

Table of representations

(G,Z)	V_1	V_2	$\dots$	V_{n-2}	V_{n-1}	V_n
$(\mathrm{SL}_{n+1},\mu_{n+1})$	V_{n+1}	$\wedge^2 V_{n+1}$	$\dots$	$\wedge^{n-2}V_{n+1}$	$\wedge^{n-1}V_{n+1}$	$\wedge^n V_{n+1}$
$(\mathrm{Spin}_{2n+1},\mu_2)$	V_{2n+1}	$\wedge^2 V_{2n+1}$	$\dots$	$\wedge^{n-2}V_{2n+1}$	$\wedge^{n-1}V_{2n+1}$	$\triangle_n$
(Sp_{2n},μ_2)	V_{2n}	$\wedge^2 V_{2n}$	$\dots$	$\wedge^{n-2}V_{2n}$	$\wedge^{n-1}V_{2n}$	$\wedge^n V_{2n}$
(Spin_{2n},Z')	V_{2n}	$\wedge^2 V_{2n}$	$\dots$	$\wedge^{n-2}V_{2n}$	$\triangle_n^+$	$\triangle_n^-$
(Spin_{2n},Z)	V_{2n}	$\wedge^2 V_{2n}$	$\dots$	$\wedge^{n-2}V_{2n}$	$\triangle_n^+$	$\triangle_n^-$

In all other cases of simple group G take the family of all fundamental representations (see 6.3(a)) as the desired fundamental family of Ch-homogeneous representations of G.

Let $\chi_1,\chi_2,\dots,\chi_n$ be the characters of Z corresponding to representations V_1, $V_2,\dots,V_n$ from every line of Table of representations. These characters can be listed in the following table (actually, in the case $G = \mathrm{Spin}_{2n}$, we consider the characters of $Z' = \ker(\mathrm{Spin}_{2n} \to \mathrm{SO}_{2n})$ and of $Z = \ker(\mathrm{Spin}_{2n} \to \mathrm{PSO}_{2n})$):

Table of characters						
(G,Z)	$\chi_1,$	$\chi_2,$	$\dots$ $\chi_{n-2},$	$\chi_{n-1},$	χ_n	Ch
$(\mathrm{SL}_{n+1},\mu_{n+1})$	$\overline{1},$	$\overline{2},$	$\dots$ $\overline{n}-\overline{2},$	$\overline{n}-\overline{1},$	$\overline{n}$	$\mathbb{Z}/(n+1)\mathbb{Z}$
$(\mathrm{Spin}_{2n+1},\mu_2)$	$\overline{0},$	$\overline{0},$	$\dots$ $\overline{0},$	$\overline{0},$	$\overline{1}$	$\mathbb{Z}/2\mathbb{Z}$
(Sp_{2n},μ_2)	$\overline{1},$	$\overline{2},$	$\dots$ $\overline{n}-\overline{2},$	$\overline{n}-\overline{1},$	$\overline{n}$	$\mathbb{Z}/2\mathbb{Z}$
(Spin_{2n},Z')	$\overline{0},$	$\overline{0},$	$\dots$ $\overline{0},$	$\overline{0},$	$\overline{1}$	$\mathbb{Z}/2\mathbb{Z}$
(Spin_{2n},Z)	$\chi_1,$	$2\chi_1,$	$\dots$ $(n-2)\chi_1,$	$\chi_{n-1},$	χ_n	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$

where $\chi_1 = (\overline{1},\overline{1})$ in the case Ch $= \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and $\chi_1 = \overline{2}$ in the case Ch $= \mathbb{Z}/4\mathbb{Z}$.

Let V_1 be the first representation in every line of Table of Representations. Let $A_1 = \mathrm{End}_F(V_1)$ be the corresponding algebra (see 8.5), and let $A_1(\gamma) = \mathrm{End}(V_1)(\gamma)$ (see 8.6). We consider two possibilities for the cocycle γ in the case $G = \mathrm{Spin}_{2n}$: first, γ takes values in $\mathrm{SO}_{2n}(F_{\mathrm{sep}})$, second, γ takes values in $\mathrm{PSO}_{2n}(F_{\mathrm{sep}})$. Lemma 8.7 and Table of characters show that the algebras $A_i(\gamma) \overset{\mathrm{df}}{=} \mathrm{End}(V_i)(\gamma)$ of 9.5 are listed in the table

Table of algebras							
A_1	A_2	A_3	A_4	$\dots$	A_{n-2}	A_{n-1}	A_n
$A_1(\gamma)$	$A_1(\gamma)^{\otimes 2}$	$A_1(\gamma)^{\otimes 3}$	$A_1(\gamma)^{\otimes 4}$	$\dots$	$A_1(\gamma)^{\otimes n-2}$	$A_1(\gamma)^{\otimes n-1}$	$A_1(\gamma)^{\otimes n}$
F	F	F	F	$\dots$	F	F	$\mathrm{End}(\triangle_n)(\gamma)$
$A_1(\gamma)$	F	$A_1(\gamma)$	F	$\dots$	$\dots$	$\dots$	$\dots$
F	F	F	F	$\dots$	F	$\mathrm{End}(\triangle_n^+)(\gamma)$	$\mathrm{End}(\triangle_n^-)(\gamma)$
$A_1(\gamma)$	F	$A_1(\gamma)$	F	$\dots$	F or $A_1(\gamma)$	$\mathrm{End}(\triangle_n^+)(\gamma)$	$\mathrm{End}(\triangle_n^-)(\gamma)$

In the third line of this Table the algebra F stands in every even position, and $A_1(\gamma)$ stands in every odd position. The same holds for the fifth line of this Table except of the two last positions. If

$$G = \mathrm{SL}_{n+1}, \quad \text{then } G(\gamma) = \mathrm{SL}_{n+1}(\gamma) = \mathrm{SL}_{1,A_1(\gamma)}\,.$$

Set $\mathcal{D} = A_1(\gamma)$, then first line of Table of Algebras coincides with the line of algebras from 5.4(1). If $G = \mathrm{Spin}_{2n+1}$, then $G(\gamma) = \mathrm{Spin}_{2n+1}(\gamma) = \mathrm{Spin}(q)$, where q is the quadratic rank $(2n+1)$-form corresponding to γ. Proposition 14.8 proves now that the second line of Table of algebras coincides with the line of algebras from 5.4(2).

If $G = \mathrm{Sp}_{2n}$, then $G(\gamma) = \mathrm{Sp}_{2n}(\gamma) = \mathrm{Sp}(A_1(\gamma),\tau_1(\gamma))$ where $\tau_1 : A_1 \to A_1$ is the involution on $A_1 = \mathrm{End}(V_{2n})$ corresponding to the standart skew-symmetric form on V_{2n}.

Set $(B,\tau) = (A_1(\gamma),\tau_1(\gamma))$. Then the third line of Table of Algebras coincides with the line of algebras from 5.4(3).

If $G = \mathrm{Spin}_{2n}$ and γ takes values in $\mathrm{SO}_{2n}(F_{\mathrm{sep}})$ rather than in $\mathrm{PSO}_{2n}(F_{\mathrm{sep}})$, then $G(\gamma) \cong \mathrm{Spin}_{2n}(\gamma) = \mathrm{Spin}(\varphi)$, where φ is the quadratic rank $2n$ form corresponing to γ. Since $\mathrm{End}(\triangle_n^{\pm})(\gamma)$ are simple F-algebras, the isomorphism from 14.8 shows that $\mathrm{End}(\triangle_n^{\pm})(\gamma)$ are two different simple compfonents of $C_0(\varphi)$. This proves that the fourth line of Table of Algebras coincides with the line of algebras from 5.4.(4a).

If $G = \mathrm{Spin}_{2n}$ and γ takes values in $\mathrm{PSO}_{2n}(F_{\mathrm{sep}})$, then $G(\gamma) = \mathrm{Spin}_{2n}(\gamma) = \mathrm{Spin}(A_1(\gamma), \sigma_1(\gamma))$, where $\sigma_1 : A_1 \to A_1$ is the F-algebra involution corresponding to the quadratic form $x_1y_1 + \cdots + x_ny_n$ on the F-space V_1, where $V_1 = V_{2n}$ now. Set $(A, \sigma) = (A_1(\gamma), \sigma_1(\gamma))$. Since F-algebras $\mathrm{End}(\triangle_n^{\pm})(\gamma)$ are again simple F-algebras, the isomorphism from 14.10 shows that $\mathrm{End}(\triangle_n^{\pm})(\gamma)$ are two different simple components of $C_0(A, \sigma)$. This proves that the fifth line of Table of Algebras coincides with the line of algebras from 5.4.(4b).

Now let H_{sep} be of the type G_2, F_4 or E_8. For every of these cases consider the family of fundamental representations of G (see 6.3(a)). Center of G is trivial (see 8.8). Hence all algebras $\mathrm{End}(V_i)(\gamma)$ are equivalent to the field F (see 8.7(2)). This explains the lists of algebras from 5.4.(5).

Now let H_{sep} be of the type E_6. Take again the family of fundamental representations of G (see 6.3(a). The highest weights of these representations are described in ([Bo], Table 5). Their restrictions to characters $\mathrm{Ch} = \mathrm{Hom}(Z, \mathbb{G}_m) = \mathbb{Z}/3\mathbb{Z}$ of the center Z of G (see 8.1) are the following:

$$\overline{w}_1 = \overline{1},\ \ \overline{w}_2 = \overline{0},\ \ \overline{w}_3 = \overline{2},\ \ \overline{w}_4 = \overline{0},\ \ \overline{w}_5 = \overline{1},\ \ \overline{w}_6 = 2.$$

Set $C = \mathrm{End}(V_1)(\gamma)$. Then $\deg_F(C) = \dim_F(V_1) = 27$. Lemma 8.7 explains the list of algebras from 5.4.(6).

Now let H_{sep} be of the type E_7. Take again the family of fundamental representations of G (see 6.3(a). Their highest weights are described in ([Bo], Table VI). Their restrictions to characters $\mathrm{Ch} = \mathrm{Hom}(Z, \mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z}$ of the center Z of G (see 8.1) are the following:

$$\overline{w}_1 = \overline{0},\ \overline{w}_2 = \overline{1},\ \overline{w}_3 = \overline{0},\ \overline{w}_4 = \overline{0},\ \overline{w}_5 = \overline{1},\ \overline{w}_6 = \overline{0},\ w_7 = \overline{1}.$$

If $B = \mathrm{End}(V_2)(\gamma)$, then $\deg_F B = \dim_F(V_2) = 8$. Lemma 8.7 explains the list of algebras from 5.4.(7).

Now let H be semisimple. Let G and $\gamma : \mathrm{Gal}(F_{\mathrm{sep}}/F) \to G(F_{\mathrm{sep}})$ be the same as in the beginning of this section. Decompose G into a direct product of F-split simply connected simple algebraic group over F:

$$G \cong G_1 \times \cdots \times G_m.$$

Then $\gamma = \gamma_1 \times \cdots \times \gamma_m$ and each of γ_i is a 1-cocycle. If $\{V_j^i | j = 1, \ldots, \mathrm{rk}(G_i)\} = \mathbb{V}^i_{is}$ the fundamental family of G_i-representations considered above in this section, then $\mathbb{V}^1 \cup \cdots \cup \mathbb{V}^m$ is a fundamental family of G-representations by 6.2. The corresponding list of F-algebras is $\coprod_{i=1}^{m} \{\mathrm{End}(V_j^i)(\gamma_i) | j = 1, \ldots, \mathrm{rk}(G_i)\}$. This proves 5.4(8). Theorem A is proven. $\square$

References

[A] Atiyah, M. F., *K-Theory*, Harvard University, Cambridge, Mass., 1965.

[B1] Bass, H., *Algebraic K-theory*, Benjamin, 1968.

[B2] Bass, H., Algebra.

[Bo] Bourbaki, N., *Groupes et algébras de Lie, Chap. IV-VI*, Hermann, Paris, 1969.

[Ch] Chevalley, C., *Séminaire sur la classification des groupes de Lie algébriques,* Vol.2, Paris, 1958.

[Gr] Grayson,D., *Higher K-theory II (after d.Quillen)*, Lect.Notes Math., **551** (1958), Springer-Verlag, New York.

[H] Hodgkin, L., *The K-theory of Lie groups*, Topology **6** (1967), 1–36.

[Hu] Humphreys, J. E., *Linear algebraic groups*, Springer, New York, 1972.

[J] Jacobson, N., *Clifford algebras for algebras with involution of type D*, J. Algebra **1** (1964), 288–300.

[Ji] Jillet, ., *Riemann-Roch Theorems for higher algebraic K-theory,*.

[L] Levine, M., *The algebraic K-theory of the classical groups and some twisted forms*, Duke Math. J. **20** (1993), 405–443.

[M] Mumford, D., *Abelian varieties*, Oxford Univ. Press, Oxford, 1970.

[P1] Panin, I. A., *On algebraic K-theory of generalized flag fibre bundles and some of their twisted forms;* in Algebraic K-theory, Advances in Soviet Math., Vol. 4 (A. A. Suslin, ed.), AMS, Providence, R. I., 1991, pp. 21–46.

[P2] Panin, I. A., *On algebraic K-theory of principal homogeneous spaces;* in Algebraic K-theory, Advances in Soviet Math., Vol. 4 (A. A. Suslin, ed.), AMS, Providence, R. I., 1991, pp. 47–51.

[P3] Panin, I.A., *On the algebraic K-theory of twisted flag varieties*, to appear.

[P4] Panin, I.A., *A splitting principle and the algebraic K-theory of some homogeneous varieties.*

[Q] Quillen, D., *Higher algebraic K-theory I,*, Lect.Notes Math., **341** (1972), Springer-Verlag, Berlin, 85–147.

[Se] Serre, J.-P., Lect.Notes Math., (1964), Springer-Verlag, Berlin.

[Su] Suslin, A.A., *K-theory and K-cohomology of certain group varieties*, in Algebraic K-theory, Advances in Soviet Math., Vol. 4 (A. A. Suslin, ed.), AMS, Providence, R. I., 1991, pp. 53–74.

[Sw] Swan, R.G., *K-theory of quadratic hyperfaces,*, Ann. Math **121** (1985), 113–153.

[T1] Tits, J., *Representations linéaires irréductibles d'un groupe réductif sur un corps quelconque,*, Reine Angew. Math **247** (1971), 196–220.

[T2] Tits, J., *Classification of algebraic semisimple groups,*, Proc.Symp Pure Math. **IX** (1966), 33–62.