ON THE RATE OF NORMAL APPROXIMATION IN $D[0,1]$

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Abstract.
The paper gives estimates of the rate of convergence in the Central Limit Theorems for stochastically continuous cadlag processes proved recently by Bézandry and Fernique (*Ann. Inst H. Poincare*, **28**) and Bloznelis and Paulauskas (to appear in *Stoch. Proc. Appl.*).

1. Introduction

Let $X = \{X(t),\ t \in [0,1]\}$ be a centered stochastically continuous random process with cadlag (right continuous and having left limits) sample paths. Let $D[0,1]$ denote the Skorohod space of cadlag functions on $[0,1]$ endowed with the usual topology. The process X is said to satisfy the Central Limit Theorem (CLT) in $D[0,1]$ if the distributions $\mathcal{L}(S_n)$ of the normalized sums $S_n = n^{-1/2}(X_1 + \cdots + X_n)$ converge weakly to the distribution of a Gaussian process, say Y, with sample paths in $D[0,1]$. Here $X_1, X_2, \ldots$ are independent copies of X. The aim of the paper is to estimate the rate of convergence in the following limit theorem proved recently by Bloznelis and Paulauskas (1993 b) and Fernique (1993).

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Theorem 1. *Let $p,\ q \geq 2$. Let $f,\ g$ be nonnegative increasing functions on $[0, +\infty)$. Let X be a r. process with mean 0, finite second moments, and sample paths in $D[0,1]$ satisfying*

$$\mathbf{E}\big(|X(s) - X(t)| \wedge |X(t) - X(u)|\big)^p \leq f(u - s), \tag{1.1}$$

$$\mathbf{E}|X(s) - X(t)|^q \leq g(t - s), \tag{1.2}$$

for all $s \leq t \leq u$, and

$$\int_0 f^{1/p}(u) \cdot u^{-1-1/p} du < \infty, \quad \int_0 g^{1/q}(u) \cdot u^{-1-1/(2q)} du < \infty. \tag{1.3}$$

Then the process X satisfies the CLT in $D[0,1]$.

We write $x \wedge y = min\{x; y\}$. For $x \in D[0,1]$ we shall denote $\|x\| = \sup_{t\in[0,1]} |x(t)|$. In the paper we estimate the uniform (Kolmogorov) distance between the distributions $\mathcal{L}(\|S_n\|)$ and $\mathcal{L}(\|Y\|)$.

Theorem 2. *Let conditions of Theorem 1 hold with the functions $f(u) = c \cdot u^{1+\alpha}$ and $g(u) = c \cdot u^{1/2+\alpha}$, $\alpha > 0$. Then there exists a constant $C_1 = C_1(\mathcal{L}(Y), p, q, \alpha)$ such that*

$$\sup_{r\geq 0} |\mathbf{P}(\|S_n\| \leq r) - \mathbf{P}(\|Y\| \leq r)| \leq C_1(1 + \mathbf{E}\|X\|^3)n^{-1/6}\log^{2/3}(n). \tag{1.4}$$

Here and in what follows letters C and c with indices or without denote constants which depend only on the quantities indicated in the brackets. Theorem 2 is a particular case of more general results formulated in section 2 below. Denote

$$\varDelta(r, S_n, Y) = |\mathbf{P}(\|S_n\| \leq r) - \mathbf{P}(\|Y\| \leq r)|.$$

Theorem 2 gives the bound for $\varDelta(r, S_n, Y)$ which is uniform with respect to $r \geq 0$. Next theorem gives the so called non – uniform bound.

Theorem 3. *Let $p > 3$ and $q \geq 2$. Let conditions of Theorem 1 hold with the functions $f(u) = cu^{1+\alpha}$, $g(u) = cu^{1/2+\alpha}$, $\alpha > 0$. Assume that $\mathbf{E}\|X\|^\beta < \infty$ for some $\beta > 3$. Then there exists a constant $C_2 = C_2(\mathcal{L}(Y), p, q, \alpha, \beta)$ such that for $r \geq 0$*

$$|\mathbf{P}(\|S_n\| \leq r) - \mathbf{P}(\|Y\| \leq r)|(1 + r^3) \leq C_2(1 + \mathbf{E}\|X\|^\beta)n^{-1/6}\log^2(n). \tag{1.5}$$

Remarks. Let F and G be increasing continuous functions defined on $[0,1]$. Theorems $1, 2$ and 3 remain true if one replaces $f(u-s)$ and $g(t-s)$ in the right hand sides of (1.1) and (1.2) by $f(F(u) - F(s))$ and $g(G(t) - G(s))$ respectively.

The rate of order $\mathcal{O}(n^{-1/6})$ appears usually when one considers the CLT for a Banach function space valued random elements, see, e.g., Paulauskas and Račkauskas (1989) and Bentkus et al (1990), moreover this rate is optimal in a sense, see ibidem. Earlier, the rate of convergence in the CLT in $D[0,1]$ was considered by Paulauskas and Juknevičiene (1989) and Paulauskas and Stieve (1990).

2. Main Results and Proofs

Here we formulate and prove main results. In proofs we combine the finite-dimensional approximations with estimates of the rate of convergence in the CLT for l_∞^N valued random vectors. Such estimates of order $n^{-1/6}$ (here n denotes the number of random vectors in the sum) which depend weakly on the growing dimension ($N \to \infty$) were obtained in Bentkus (1982) and Račkauskas (1984).

Theorem 4. *Assume the process X satisfies conditions of Theorem 1 with $p = 2, q \geq 2$. Then there exists a constant $C_3 = C_3(\mathcal{L}(Y), q)$ such that for any sequence of natural numbers $\{N_n \ , n \geq 1\}$, satisfying $N_n \geq n^{1/3}$, the following bound holds*

$$\sup_{r \geq 0} \Delta(r, S_n, Y) \leq$$

$$C_3\big((1 + \mathbf{E}\|X\|^3)n^{-1/6}\log^{2/3} n + [T_1(N_n^{-1})]^{1/2} + [N_n^{-1}\log^{1/2}\log(N_n)]^{1/2}\big), \quad (2.1)$$

where

$$T_1(s) = \int_0^s u^{-3/2} f^{1/2}(u)du + \int_0^s u^{-1-1/(2q)} g^{1/q}(u)\log^{1/2-1/(2q)}(1 + 1/u)du.$$

Note that the integrand of the first integral in the definition of $T_1(\cdot)$ is precisely the same as that in condition (1.3), with $p = 2$. It seems that the logarithmic factor in the second integral is superfluous.

The following theorem 5 gives the bound for the rate of convergence in the CLT due to Bézandry and Fernique (1992).

Theorem 5. *Let $X, X_1, X_2, X_3, \ldots$ be i.i.d. stochastically continuous centered cadlag processes defined on the probability space $(\Omega_1, \mathcal{F}_1, P_1)$. Let f, g, θ be increasing functions such that $f(0) = g(0) = \theta(0) = 0$, and let θ be concave. Assume that for all $s \leq t \leq u$, and $A \in \mathcal{F}_1$,*

$$\mathbf{E}|X(s) - X(t)|^2 \wedge |X(t) - X(u)|^2 \mathbb{I}_A \leq f(u - s)\theta(P(A)), \quad (2.2)$$

$$\mathbf{E}|X(s) - X(t)|^q \leq g(t - s). \quad (2.3)$$

Then there exists a constant $C_4 = C_4(\mathcal{L}(Y), \theta, q)$ such that for any sequence of natural numbers $\{N_n, n \geq 1\}$ such that $N_n \geq n^{1/3}$, the bound (2.1) holds with the constant C_4 instead of C_3 and with $T_1(\cdot)$ replaced by $T_2(\cdot)$,
where

$$T_2(s) = \int_0^s u^{-3/2}\theta^{1/2}\big(u\log(1 + 1/u)/\log 2\big)f^{1/2}(u)du +$$

$$\int_0^s u^{-1-1/(2q)}\log^{1/2-1/(2q)}(1 + 1/u)g^{1/q}(u)du. \quad (2.4)$$

For $x \in D[0,1]$ define $\Delta_x(s,t,u) = |x(s)-x(t)| \wedge |x(t)-x(u)|$. Let $Z = \{Z(t),\ t \in [0,1]\}$ be a separable cadlag random process defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $h_1, h_2, \theta_1, \theta_2$ denote increasing functions, $h_1(0) = h_2(0) = \theta_1(0) = \theta_2(0) = 0$ and let θ_1, θ_2 be concave. The following result is proved in Bézandry and Fernique (1992) (see Lemma 1.1.3 and Theorem 1.2 ibidem).

Theorem 6. *Assume the process $Z = \{Z(t), t \in [0,1]\}$ is right-continuous in L_1 at each $t \in [0,1]$. Assume for each $0 \le s \le t \le u \le 1$ and each $A \in \mathcal{F}$*

$$\mathbf{E}|Z(s) - Z(t)| \wedge |Z(t) - Z(u)|\mathbb{I}_A \le h_1(u-s)\theta_1(\mathbf{P}(A)) + h_2(u-s)\theta_2(\mathbf{P}(A)).$$

Then there exists a sequence of random functions $\{f_k,\ k \ge 1\}$,
$f_k : \Omega \times [0,1] \to \mathbf{S}_k := \{j \cdot 2^{-k},\quad 0 \le j \le 2^k\}$ satisfying:
i) For each $k \ge 1$ and each $\omega \in \Omega$ the function $f_k(\omega, \cdot) \to \mathbf{S}_k$ is non decreasing, $f_k(\omega, t) = t$ for $t \in \mathbf{S}_k$;
ii) For each $k \ge 1$

$$\sup_{t\in[0,1]} |Z(\omega,t) - Z(\omega, f_k(\omega,t))| \le \sum_{m \ge k} \sum_{t \in \mathbf{S}_{m+1} \setminus \mathbf{S}_m} \Delta_{Z(\omega,\cdot)}(t - 2^{-k-1}, t, t + 2^{-k-1}) \tag{2.5}$$

and

$$\mathbf{E} \sup_{t\in[0,1]} |Z(\omega,t) - Z(\omega, f_k(\omega,t))| \le 2 \int_0^{2^{-k}} u^{-2}(h_1(u)\theta_1(u) + h_2(u)\theta_2(u))du. \tag{2.6}$$

For a cadlag process Z satisfying conditions of Theorem 4 define

$$Z^{\{k\}} = \{Z(\omega, f_k(\omega,t)),\ t \in [0,1], \omega \in \Omega\}, k \ge 1.$$

One may define the other approximating process

$$Z^{[k]}(\omega,t) = \sum_{i=0}^{2^k-1} Z(\omega, i \cdot 2^{-k})\mathbb{I}\{i \cdot 2^{-k} \le t < (i+1) \cdot 2^{-k}\} + Z(\omega,1)\mathbb{I}\{t = 1\}.$$

Remark, that $\|Z^{\{k\}}\| = \max\{|Z(i \cdot 2^{-k})|,\ 0 \le i \le 2^k\} = \|Z^{[k]}\|$ and hence $\mathit{\Delta}(\cdot, Z^{\{k\}}, Y) = \mathit{\Delta}(\cdot, Z^{[k]}, Y)$.

Lemma 1. *For $k \ge 1$ and $r, \varepsilon > 0$,*

$$|\mathbf{P}(\|S_n\| < r) - \mathbf{P}(\|Y\| < r)| \le$$

$$I_1(S_n, \varepsilon) + 3I_1(Y, \varepsilon) + 2I_2(Y, \varepsilon) + 2 \cdot \mathit{\Delta}(r, S_n^{\{k\}}, Y^{[k]}) + \mathit{\Delta}(r - \varepsilon, S_n^{\{k\}}, Y^{[k]}),$$

where

$$I_1(S_n,\varepsilon)=\mathbf{P}(\|S_n^{\{k\}}-S_n\|>\varepsilon), I_1(Y,\varepsilon)=\mathbf{P}(\|Y^{[k]}-Y\|>\varepsilon),$$

$$I_2(Y,\varepsilon)=\mathbf{P}(\|Y^{[k]}\|\in[r-\varepsilon,r+\varepsilon]).$$

Proof of Lemma 1. The proof is standard, see e.g., Bentkus et al.(1990).

Proof of Theorem 4. Let k be the integer satisfying $2^{k-1}<N_n\le 2^k$. Fix $r,\varepsilon>0$ and apply lemma 1 to the processes S_n and Y. It suffices to estimate quantities $I_1(S_n,\varepsilon)$, $I_1(Y,\varepsilon)$, $I_2(Y,\varepsilon)$, $\Delta(\cdot,S_n^{[k]},Y^{[k]})$, see also remark before the lemma 1. It follows from (2.6) and lemma 3 below that

$$I_1(S_n,\varepsilon)=\mathbf{P}(\|S_n^{\{k\}}-S_n\|\ge\varepsilon)\le\varepsilon^{-1}\mathbf{E}\|S_n^{\{k\}}-S_n\|\le 2c_2\varepsilon^{-1}T_1(2^{-k}). \qquad (2.7)$$

Define the pseudometric $d(s,t)=\big(\mathbf{E}(Y(s)-Y(t))^2\big)^{1/2}$, $s,t\in[0,1]$. Let $N(\delta)$, $\delta>0$ be the minimal number of balls $B(u,\delta)=\{v\in[0,1]:\ d(u,v)<\delta\}\subset[0,1]$, $u\in[0,1]$ that cover the interval $[0,1]$. By Theorem 5.5 of Jain and Marcus (1978), there exists an absolute constant c such that for each $\varepsilon,\delta>0$

$$\mathbf{P}(\sup_{|u-v|<\delta}|Y(u)-Y(v)|>\varepsilon)\le c\varepsilon^{-1}(\int_0^\delta H^{1/2}(s)ds+\delta\log^{1/2}\log(1/\delta)), \qquad (2.8)$$

where $H(\delta)=\log N(\delta)$. A simple calculation shows that condition (1.3) implies $g(u)\le cu^{1/2}$. Hence,

$$\big(\mathbf{E}(Y(s)-Y(t))^2\big)^{1/2}=\big(\mathbf{E}(X(s)-X(t))^2\big)^{1/2}\le$$

$$(\mathbf{E}|X(s)-X(t)|^q)^{1/q}\le c|t-s|^{1/(2q)},$$

and $N(\delta)\le 1+\delta^{-2q}$. We have

$$I_1(Y,\varepsilon)=\mathbf{P}(\|Y-Y^{[k]}\|>\varepsilon)\le c\varepsilon^{-1}2^{-k}k^{1/2}. \qquad (2.9)$$

Theorem 4 of Lifshits (1986) applies to the Gaussian process Y and yields

$$\sup_{\varepsilon>0}\sup_{r\ge0}\varepsilon^{-1}\mathbf{P}(\|Y\|\in[r,r+\varepsilon])\le C_5, \qquad (2.10)$$

where $C_5=C_5(Y)$. Hence, $I_2(Y,\varepsilon)\le C_5\varepsilon$. It remains to estimate the quantity

$$\Delta(S_n^{[k]},Y^{[k]}):=\sup_{r\ge0}\Delta(r,S_n^{[k]},Y^{[k]}).$$

The problem is equivalent to the estimation of the rate of convergence in the CLT uniformly over the class of all balls with the center at the origin in the finitedimensional Banach space $l_\infty^{2^k}$, where $2^{k-1} < N_n \le 2^k$. Using the standard technique due to Bentkus (1982) and Račkauskas (1984), see, e.g., Paulauskas and Račkauskas (1989), one may obtain the following bound

$$\Delta(S_n^{[k]}, Y^{[k]}) \le C(Y) n^{-1/6} \log^{2/3}(2^k)(1 + \mathbf{E}\|X\|^3), \tag{2.11}$$

if $2^k \ge N_n \ge n^{1/3}$, $n \ge 1$.

It follows from Lemma 1 and the estimates of $I_1(S_n, \varepsilon)$, $I_1(Y, \varepsilon)$, $I_2(Y, \varepsilon)$, $\Delta(\cdot, S_n^{[k]}, Y^{[k]})$ that

$$\Delta(r, S_n, Y) \le C(Y) n^{-1/6} \log^{2/3}(N_n)(1 + \mathbf{E}\|X\|^3) +$$
$$C_5\varepsilon + c_2\varepsilon^{-1} T_1(N_n^{-1}) + c\varepsilon^{-1} N_n^{-1} \log^{1/2}\log(N_n).$$

Choosing $\varepsilon = \big(T_1(N_n^{-1}) + N_n^{-1}\log^{1/2}\log(N_n)\big)^{1/2}$ gives the desired bound (2.1). Theorem is proved.

Proof of Theorem 5. The proof is almost the same as that of Theorem 4. Only now we use Lemma 4 instead of Lemma 3 when estimating

$$I_1(S_n, \varepsilon) \le \varepsilon^{-1}\mathbf{E}\|S_n^{\{k\}} - S_n\| \le 2c_3\varepsilon^{-1} T_2(2^{-k}), \tag{2.12}$$

by means of Theorem 6. Theorem is proved.

Proof of Theorem 2. If $p = 2$, $q \ge 2$ we apply Theorem 4. It is easy to see that if $f(u) = cu^{1+\alpha}$ and $g(u) = cu^{1/2+\alpha}$ then

$$T_1(u) \le \int_0^u s^{-1+\alpha/2} ds + \int_0^u s^{-1+\alpha/q} \log^{1/2-1/(2q)}(1 + 1/s) ds \le c(\alpha, q) u^{\alpha/(2+q)},$$

$u \in [0,1]$. Choosing $N_n = n^\gamma$ with $\gamma \ge 2 + [2(2+q)/\alpha]$ gives

$$[T_1(N_n^{-1})]^{1/2} + [N_n^{-1}\log^{1/2}\log(N_n)]^{1/2} \le c(\alpha, q) n^{-1}.$$

Now (1.4) follows from (2.1).

If $p > 2, q \ge 2$ we apply Theorem 5 with the functions

$$f(u) = cu^{2(1+\alpha)/p},\ \theta(u) = u^{1-2/p},\ q(u) = u^{1/2+\alpha}.$$

We have

$$T_2(u) \le \int_0^u s^{-1+\alpha/p} \log^{1/2-1/(2p)}(1 + 1/s) ds +$$
$$\int_0^u s^{-1+\alpha/q} \log^{1/2-1/(2q)}(1 + 1/s) ds \le c(\alpha, p, q) u^{\alpha/(p+q)}, u \in [0,1].$$

The bound (1.4) follows from Theorem 5 with $N_n = n^\gamma$ and $\gamma \ge 2 + [2(p+q)/\alpha]$ in (2.4). Theorem is proved.

Rest of the section is devoted to the non-uniform bound, i.e., the bound for $(1 + r)^3 \Delta(r, S_n, Y)$.

Lemma 2. *(see, e.g., Lemma 4.1 in Bentkus et al. (1990)). For all* $n, k \geq 1$ *and* $r > \varepsilon > 0$ *the following inequality holds*

$$\Delta(r, S_n, Y)) \leq$$

$$\Delta(r-\varepsilon, S_n^{\{k\}}, Y^{[k]}) + \Delta(r, S_n^{\{k\}}, Y^{[k]}) + I_3(S_n, \varepsilon, r) + I_3(Y, \varepsilon, r) + I_2(Y, \varepsilon),$$

where $I_3(S_n, \varepsilon, r) = \mathbf{P}(\|S_n^{\{k\}} - S_n\| > \varepsilon, \|S_n\| > r)$, $I_3(Y, \varepsilon, r) = \mathbf{P}(\|Y^{[k]} - Y\| > \varepsilon, \|Y\| > r)$.

Proof of Theorem 3. Denote $d = \min(p, \beta)$. Assume without loss of generality that $r > 10$. If $r \leq 10$ then theorem follows from Theorem 2. Fix integer k satisfying $2^k > n^{1/3}$ and $\varepsilon = n^{-1/6}$. By Lemma 2, it suffices to estimate $I_2(Y, \varepsilon)$, $I_3(S_n, \varepsilon, r)$, $I_3(Y, \varepsilon, r)$ and $\Delta(r, S_n^{[k]}, Y^{[k]})$.

Combining Hölder's inequality, (2.12) and Lemma 5, we get

$$I_3(S_n, \varepsilon, r) = \mathbf{E}\mathbb{I}\{\|S_n^{\{k\}} - S_n\| > \varepsilon\}\mathbb{I}\{\|S_n\| > r\} \leq$$

$$[\mathbf{P}(\|S_n^{\{k\}} - S_n\| > \varepsilon)]^{1-3/d}\mathbf{P}^{3/d}(\|S_n\| > r) \leq$$

$$c(\beta, p, q)(1+r)^{-3}(1 + \mathbf{E}\|X\|^\beta)[\varepsilon^{-1}T_1(2^{-k})]^{1-3/d}.$$

Similarly, by (2.9) and Chebyshev inequality,

$$I_3(Y, \varepsilon, r) \leq [c\varepsilon^{-1}2^{-k}k^{1/2}]^{1/2}r^{-3}(\mathbf{E}\|Y\|^6)^{1/2}.$$

Here we use the fact that sample continuous Gaussian process Y has finite moments $\mathbf{E}\|Y\|^p < \infty$ for each $p > 0$.

Combining (2.10) with the estimates for the density function of the distribution of $\|Y\|$ due to Tsirel'son (1975), we get

$$I_2(Y, \varepsilon) \leq C_6(1+r)^{-3}\varepsilon, \quad r > 0, \varepsilon \leq 1,$$

where $C_6 = C_6(Y)$ is an absolute constant. It remains to estimate $\Delta(r, S_n^{[k]}, Y^{[k]})$. In what follows we apply Theorem 4.5 from Bentkus et al. (1990). Define

$$M_3 = n^{-1/6}(1 \vee (\mathbf{E}\|X\|^3 + \mathbf{E}\|Y\|^3)^{1/3}); D_n(2^{-k}) = M_3 \vee W(2^{-k}, M_3).$$

$$W(2^{-k}, t) = \sup_{r \geq 0}(1+r)^3\mathbf{P}(\sup_{|u-v| \leq 2^{-k}}|Y(u) - Y(v)| > t,\ \|Y\| > r);$$

We write $x \vee y = \max\{x; y\}$. By Theorem 4.5 in Bentkus et al. (1990), there exists an absolute constant $C_7 = C_7(Y)$ such that for each $r > \varepsilon > 0$

$$\Delta(r, S_n^{[k]}, Y^{[k]}) \leq C_7(1+r)^{-3}\log^2(2^k)[D_n(2^{-k}) \vee D_n^3(2^{-k})].$$

Using Hölder and Chebyshev inequalities and (2.8), we estimate

$$W(2^{-k},t) \le \sup_{r\ge 0}(1+r)^{-3}\mathbf{P}^{1/2}(\sup_{|u-v|<2^{-k}}|Y(u)-Y(v)|>t)\mathbf{P}^{1/2}(\|Y\|>r) \le$$

$$\sup_{r\ge 0}(1+r)^{3}[ct^{-1}2^{-k}k^{1/2}]^{1/2}[1\wedge(r^{-3}\left(\mathbf{E}\|Y\|^{6}\right)^{1/2})] \le C(Y)t^{-1/2}2^{-k/2}k^{1/4}.$$

A simple calculation gives

$$D_n(2^{-k})\vee D_n^3(2^{-k}) \le n^{-1/6}(1+\mathbf{E}\|X\|^3+\mathbf{E}\|Y\|^3)+C(Y)n^{1/12}2^{-k/2}k^{1/4},$$

since $2^k \ge n^{1/3}$. Hence

$$\varDelta(r,S_n^{[k]},Y^{[k]}) \le C(Y)(1+r)^{-3}k^2(n^{-1/6}(1+\mathbf{E}\|X\|^3+\mathbf{E}\|Y\|^3)+n^{1/12}2^{-k/2}k^{1/4}).$$

Combining the estimates of $I_2(Y,\varepsilon)$, $I_3(S_n,\varepsilon,r)$, $I_3(Y,\varepsilon,r)$ and $\varDelta(r,S_n^{[k]},Y^{[k]})$ and Lemma 2, we get

$$\begin{aligned} \varDelta(r,S_n,Y)(1+r)^3 \le \\ C(Y,p,q,\beta)\big(k^2[n^{-1/6}(1+\mathbf{E}\|X\|^3+\mathbf{E}\|Y\|^3)+n^{1/12}2^{-k/2}k^{1/4}]+ \\ (1+\mathbf{E}\|X\|^{\beta})[\varepsilon^{-1}T_2(2^{-k})]^{1-3/d}+[\varepsilon^{-1}2^{-k}k^{1/2}]^{1/2}+\varepsilon\big). \end{aligned} \tag{2.13}$$

Recall, that $T_2(2^{-k}) \le c(\alpha,p,q)(2^{-k})^{\alpha/(p+q)}$, see proof of Theorem 2. Choose $2^{-k} \le n^{-\gamma}$ where γ satisfies

$$\gamma > 1/3;\ [\gamma(\alpha/(p+q))-1/6]\cdot(1-3/d) > 1/6;, \gamma/2-1/12 > 1/6; \gamma-1/6 > 1/6.$$

Then the sum in the right hand side of (2.13) does not exceed

$$C(Y,p,q,\beta)k^2n^{-1/6}(1+\mathbf{E}\|X\|^{\beta}+\mathbf{E}\|Y\|^3).$$

Theorem is proved.

The argument used in proof of Theorem 3 allows to get more general results. For instance the non uniform estimates (but with a slower speed of convergence with respect to n) can be obtained if instead of condition $f(u) \le cu^{1+\alpha}$, $g(u) \le cu^{1/2+\alpha}$ one requires the finiteness of the integrals that appear in the proof of Lemma 5.

It was mentioned in Bloznelis and Paulauskas (1993a) that a transformation of the parameter set by means of increasing continuous function does not influence the quantity $\varDelta(\cdot,S_n,Y)$. It is possible to construct a sample continuous process X which satisfies the CLT in $C[0,1]$ and for which the rate of order $n^{-1/6}$ can not be improved, cf. Bentkus et al (1990).

3. Auxiliary results

Lemma 3. *Assume X satisfies conditions of Theorem 4. Then there exists a constant $c_2 = c_2(q)$ such that for all $A \in \mathcal{F}_1$ and each $0 \le s \le t \le u \le 1$*

$$\mathbf{E}|S_n(s) - S_n(t)| \wedge |S_n(t) - S_n(u)| \mathbb{I}_A \le$$

$$c_2 f^{1/2}(u-s)\mathbf{P}^{1/2}(A) + c_2 g^{1/q}(u-s)[\mathbf{P}(A)]^{1-1/(2q)} \log^{1/2-1/(2q)}(1+1/\mathbf{P}(A)).$$

Lemma 4. *Assume X satisfies conditions of Theorem 5. Then there exists a constant $c_3 = c_3(q)$ such that for all $A \in \mathcal{F}_1$ and all $0 \le s \le t \le u \le 1$,*

$$\mathbf{E}|S_n(s) - S_n(t)| \wedge |S_n(t) - S_n(u)| \mathbb{I}_A \le$$

$$c_3 f^{1/2}(u-s)\mathbf{P}^{1/2}(A)\theta^{1/2}(\mathbf{P}(A)\log(1+1/\mathbf{P}(A))/\log 2) +$$
$$c_3 g^{1/q}(u-s)\mathbf{P}^{1-1/(2q)}(A)\log^{1/2-1/(2q)}(1+1/\mathbf{P}(A)). \tag{3.1}$$

The proof of Lemmas 3 and 4 goes along the lines of the proof of Lemmas 2.4.1, 2.4.2 and 2.4.4 of Bézandry and Fernique (1992), see also proof of Theorem 2 in Bloznelis and Paulauskas (1993b).

Lemma 5. *Assume that X satisfies conditions of Theorem 3. Then there exists a constant $c_4 = c_4(p,q,\beta)$ such that for all $r \ge 1, n \ge 1$, $\mathbf{P}(\|S_n\| \ge r) \le c_4(1 + \mathbf{E}\|X\|^\beta) r^{-d}$, where $d = \min(\beta, p)$.*

Proof of Lemma 5. Put $\lambda_m = a/(m\log^2(m)), m \ge 1$ and let the constant a be such that
$\sum_{m\ge1} \lambda_m = 1$. By (2.5),

$$\|S_n^{\{1\}} - S_n\| \le \sum_{m\ge1} \sum_{t \in \mathbf{S}_{m+1}\setminus\mathbf{S}_m} \Delta_{S_n}(t - 2^{-m-1}, t, t+2^{-m-1}).$$

Hence, for each $r \ge 0$

$$\mathbf{P}(\|S_n\| \ge 2r) \le \mathbf{P}(\|S_n^{\{1\}}\| \ge r) +$$

$$\mathbf{P}(\sum_{m\ge1} \sum_{t \in \mathbf{S}_{m+1}\setminus\mathbf{S}_m} \Delta_{S_n}(t - 2^{-m-1}, t, t+2^{-m-1}) \ge r \sum_{m\ge1} \lambda_m). \tag{3.2}$$

The standard argument yields

$$\mathbf{P}(\|S_n^{\{1\}}\| \ge r) = \mathbf{P}(\max(|S_n(0)|, |S_n(1/2)|, S_n(1)|) \ge r) \le c(\beta)\mathbf{E}\|X\|^\beta r^{-\beta},$$

since the process X is centered. The second summand in (3.2) does not exceed

$$\mathbf{I} := \sum_{m\geq 1} \sum_{t\in \mathbf{S}_{m+1}\setminus \mathbf{S}_m} \mathbf{P}(\Delta_{S_n}(t-2^{-m-1}, t, t+2^{-m-1}) \geq r\lambda_m).$$

Below we will prove that for all $s \leq t \leq u$ and each $\lambda > 0$

$$\mathbf{P}(\Delta_{S_n}(s,t,u) \geq \lambda) \leq$$

$$c(p,q)[\lambda^{-p} f(u-s) + \lambda^{-2q} g^2(u-s) + \lambda^{-2v}(f^{2v/p}(u-s) + g^{2v/q}(u-s))], \quad (3.3)$$

where $v = \max(p,q)$. Then

$$\mathbf{I} \leq \sum_{m\geq 1} 2^m c(p,q)[r^{-p}(m\log^2(m))^p f(2^{-m}) + r^{-2q}(m\log^2(m))^{2q} g^2(2^{-m})] +$$

$$\sum_{m\geq 1} 2^m c(p,q)[r^{-2v}(m\log^2(m))^{2v}(f^{2v/p}(2^{-m}) + g^{2v/q}(2^{-m}))] \leq$$

$$c'(p,q)\Big(r^{-p}\int_0^1 u^{-2}h^p(u)f(u)du + r^{-2q}\int_0^1 u^{-2}h^{2q}(u)g^2(u)du +$$

$$r^{-2v}\int_0^1 u^{-2}h^{2v}(u)\big(f^{2v/p}(u) + g^{2v/q}(u)\big)du\Big),$$

where $h(u) = \big(u\log(1+u^{-1})\big)$. These integrals are finite, since $f(u) \leq cu^{1+\alpha}$, $g(u) \leq cu^{1/2+\alpha}$. Hence $\mathbf{I} \leq c''(p,q)(r^{-p} + r^{-2q} + r^{-2v})$ and the lemma follows.

It remains to prove (3.3). Let $X_1, X_2, ..., X_n, X_1', X_2', ..., X_n'$ be independent copies of X. Let $\varepsilon, \varepsilon_1, \varepsilon_2, ...$ be i.i.d. Bernoulli random variables $\big(\mathbf{P}(\varepsilon = +1) = \mathbf{P}(\varepsilon = -1) = 1/2\big)$. Assume the sequences $\{X_n\}$, $\{X_n'\}$, $\{\varepsilon_n\}$ are independent. Define $X^s = X' - X$ and $S_n^s = S_n - S_n'$, where $S_n' = n^{-1/2}(X_1' + \cdots + X_n')$.

Let $\overline{a} = S_n(s) - S_n(t), \overline{b} = S_n(t) - S_n(u)$, $\overline{a}' = S_n'(s) - S_n'(t), \overline{b}' = S_n'(t) - S_n'(u)$. A simple inequality

$$|\overline{a}| \wedge |\overline{b}| \leq |\overline{a} - \overline{a}'| \wedge |\overline{b} - \overline{b}'| + |\overline{a}'| \wedge |\overline{b}| + |\overline{a}| \wedge |\overline{b}'| \quad (3.4)$$

yields

$$\mathbf{P}(|\overline{a}| \wedge |\overline{b}| > 3\lambda) \leq$$

$$\mathbf{P}(|\overline{a} - \overline{a}'| \wedge |\overline{b} - \overline{b}'| > \lambda) + \mathbf{P}(|\overline{a}'| \wedge |\overline{b}| > \lambda) + \mathbf{P}(|\overline{a}| \wedge |\overline{b}'| > \lambda) = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3.$$

The second and the third probabilities are estimated in the same way. Let us estimate the second one,

$$P_2 \leq \lambda^{-2q}\mathbf{E}|S_n(s) - S_n(t)|^q |S_n'(t) - S_n'(u)|^q =$$

$$\lambda^{-2q}\mathbf{E}|S_n(s)-S_n(t)|^q\mathbf{E}|S_n'(t)-S_n'(u)|^q \le$$
$$c(q)\lambda^{-2q}\mathbf{E}|X(s)-X(t)|^q\mathbf{E}|X(t)-X(u)|^q \le c(q)\lambda^{-2q}g^2(u-s).$$

Analogously $\mathbf{P}_3 \le c(q)\lambda^{-2q}g^2(u-s)$. Let us estimate $\mathbf{P}_1$. Denote

$$x = n^{-1/2}(X^s(s)-X^s(t)), y = n^{-1/2}(X^s(t)-X^s(u)), m = |x|\wedge|y|,$$

$$\alpha = sign(x), \beta = sign(y), u = |x|-m, v = |y|-m.$$

Then

$$\mathbf{P}_1 = \mathbf{P}(|\sum_{i=1}^n \varepsilon_i\alpha_i x_i|\wedge|\sum_{i=1}^n \varepsilon_i\beta_i y_i| \ge \lambda) \le \mathbf{P}(|\sum_{i=1}^n \varepsilon_i\alpha_i m_i| \ge \lambda/3)+$$

$$\mathbf{P}(|\sum_{i=1}^n \varepsilon_i\beta_i m_i| \ge \lambda/3) + \mathbf{P}(|\sum_{i=1}^n \varepsilon_i\alpha_i u_i|\wedge|\sum_{i=1}^n \varepsilon_i\beta_i v_i| \ge \lambda/3) \le \mathbf{P}_4+\mathbf{P}_5+\mathbf{P}_6.$$

The last probability $\mathbf{P}_6$ does dot exceed

$$(\lambda/3)^{-2q}\mathbf{E}(|\sum_{i=1}^n \varepsilon_i\alpha_i u_i|\wedge|\sum_{i=1}^n \varepsilon_i\beta_i v_i|)^{2q} \le$$

$$c(q)(\lambda/3)^{-2q}\mathbf{E}|X^s(s)-X^s(t)|^q\mathbf{E}|X^s(t)-X^s(u)|^q,$$

see proof of Theorem 2 in Bloznelis and Paulauskas (1993 b). It follows from (1.2) that $\mathbf{P}_6 \le c(q)(\lambda)^{-2q}g^2(u-s)$. It remains to estimate probabilities $\mathbf{P}_4$ and $\mathbf{P}_5$. In what follows we apply the iterated Hoffmann-Jorgensen inequality. For each $v \ge 1$ there exist constants $c_1(v),\ c_2(v),\ c_3(v)$ such that for each $\lambda > 0$ and $n \ge 1$

$$\mathbf{P}(|\sum_{i=1}^n \varepsilon_i\alpha_i m_i| \ge c_1(v)\lambda) \le c_2(v)\sum_{i=1}^n \mathbf{P}(|m_i| \ge \lambda) + c_3(v)\mathbf{P}^v(|\sum_{i=1}^n \varepsilon_i\alpha_i m_i| \ge \lambda). \tag{3.5}$$

Denote $a = n^{-1/2}(X(s)-X(t)),\ b = n^{-1/2}(X(t)-X(u)),\ a' = n^{-1/2}(X'(s)-X'(t)),\ b' = n^{-1/2}(X'(t)-X'(u))$. Inequality (3.4) with $a,\ b,\ a',\ b'$ instead of $\overline{a},\ \overline{b},\ \overline{a}',\ \overline{b}'$ yields

$$\mathbf{P}(|m| > 4\lambda) = \mathbf{P}(|a-a'|\wedge|b-b'| > 4\lambda) \le$$

$$\lambda^{-p}(\mathbf{E}(|a|\wedge|b|)^p + \mathbf{E}(|a'|\wedge|b'|)^p) + \lambda^{-2q}(\mathbf{E}(|a'|\wedge|b|)^{2q} + \mathbf{E}(|a|\wedge|b'|)^{2q}) \le$$
$$n^{-p/2}\lambda^{-p}2f(u-s) + n^{-q}\lambda^{-2q}2g^2(u-s). \tag{3.6}$$

On the other hand

$$\mathbf{P}(|\sum_{i=1}^{n}\varepsilon_i\alpha_i m_i|\geq\lambda)\leq\lambda^{-2}\sum_{i=1}^{n}\mathbf{E}|m_i|^2\leq\lambda^{-2}c(f^{2/p}(u-s)+g^{2/q}(u-s)).\quad(3.7)$$

In the last inequality we estimate m by means of (3.4) but with $a,\ b,\ a',\ b'$ instead of $\overline{a},\ \overline{b},\ \overline{a}',\ \overline{b}'$ and apply Hölder's inequality. Combining (3.6), (3.7) and (3.5) with $v=\max(p,q)$ we get

$$\mathbf{P}_4\leq c\big(\lambda^{-p}f(u-s)+\lambda^{-2q}g^2(u-s)+\lambda^{-2v}(f^{2v/p}(u-s)+g^{2v/q}(u-s))\big).$$

Analogous estimate holds also for $\mathbf{P}_5$. Combining estimates of $\mathbf{P}_1\leq\mathbf{P}_4+\mathbf{P}_5+\mathbf{P}_6$ and $\mathbf{P}_2,\ \mathbf{P}_3$ gives (3.3). Lemma is proved

4. Appendix

Here we prove Lemmas 3 and 4. In what follows we assume that all the random variables considered below are defined on the common probability space $(\Omega,\mathcal{F},P)$.

Lemma 6. *Assume X satisfies conditions of Theorem 4. Then there exists a constant c depending only on q such that for each $A\in\mathcal{F}$ and all $0\leq s\leq t\leq u\leq 1$*

$$\mathbf{E}|S_n^s(s)-S_n^s(t)|\wedge|S_n^s(t)-S_n^s(u)|\mathbb{I}_A\leq$$

$$cf^{1/2}(u-s)+cg^{1/q}(u-s)[\mathbf{P}(A)]^{1-1/(2q)}\log^{1/2-1/(2q)}(1+1/\mathbf{P}(a)).\quad(4.1)$$

Proof of Lemma 6. Denote

$$x=n^{-1/2}(X^s(s)-X^s(t)),y=n^{-1/2}(X^s(t)-X^s(u)),m=|x|\wedge|y|,$$

$$\alpha=sign(x),\beta=sign(y),u=|x|-m,v=|y|-m.$$

It suffices to estimate

$$\mathbf{E}\{|\sum_{i=1}^{n}\varepsilon_i\alpha_i x_i|\wedge|\sum_{i=1}^{n}\varepsilon_i\beta_i y_i|\mathbb{I}_A\}\leq$$

$$\mathbf{E}|\sum_{i=1}^{n}\varepsilon_i\alpha_i m_i|\mathbb{I}_A+\mathbf{E}|\sum_{i=1}^{n}\varepsilon_i\beta_i m_i|\mathbb{I}_A+\mathbf{E}\{|\sum_{i=1}^{n}\varepsilon_i\alpha_i u_i|\wedge|\sum_{i=1}^{n}\varepsilon_i\beta_i v_i|\mathbb{I}_A\}.\quad(4.2)$$

The last expectation is estimated in Bloznelis and Paulauskas (1993 b). We have

$$\mathbf{E}\{|\sum_{i=1}^{n}\varepsilon_i\alpha_i u_i|\wedge|\sum_{i=1}^{n}\varepsilon_i\beta_i v_i|\mathbb{I}_A\}\leq cg^{1/q}(u-s)[\mathbf{P}(A)]^{1-1/(2q)}.\quad(4.3)$$

The first and the second expectations are estimated in the same way. Let us estimate the first one,

$$\mathbf{E}|\sum_{i=1}^{n}\varepsilon_i\alpha_i m_i|\mathbb{I}_A \le \big(\mathbf{E}|\sum_{i=1}^{n}\varepsilon_i\alpha_i m_i|^2\mathbb{I}_A\big)^{1/2}[\mathbf{P}(A)]^{1/2} =$$

$$\big(\mathbf{E}(\mathbf{E}_2|\sum_{i=1}^{n}\varepsilon_i\alpha_i m_i|^2\mathbb{I}_A)\big)^{1/2}[\mathbf{P}(A)]^{1/2}. \tag{4.4}$$

Here $\mathbf{E}_2$ denotes the expectation taken with respect to $\mathbf{P}_2$. Using the standard Orlicz space technique, see Bézandry and Fernique (1992), one obtains

$$E_2|\sum_{i=1}^{n}\varepsilon_i\alpha_i m_i|^2\mathbb{I}_A \le c\sum_{i=1}^{n} m_i^2\mathbf{P}_2(A)\log(1+1/\mathbf{P}_2(A)). \tag{4.5}$$

Denote $\xi(u) = u\log(1+1/u)/\log 2$. Put

$$a = n^{-1/2}(X(s)-X(t)), a' = n^{-1/2}(X'(s)-X'(t)), b = n^{-1/2}(X(t)-X(u)),$$

$$b' = n^{-1/2}(X'(t)-X'(u)), d = |a|\wedge|b|, d' = |a'|\wedge|b'|.$$

A simple inequality

$$|m| = |a-a'|\wedge|b-b'| \le d+d'+|a'|\wedge|b|+|a|\wedge|b'|$$

yields

$$\mathbf{E}|m|^2\xi(\mathbf{P}_2(A)) \le 4\mathbf{E}d^2\xi(\mathbf{P}_2(A)) + 4\mathbf{E}|d'|^2\xi(\mathbf{P}_2(A)) +$$

$$4\mathbf{E}|a'|^2\wedge|b|^2\xi(\mathbf{P}_2(A)) + 4\mathbf{E}|a|^2\wedge|b'|^2\xi(\mathbf{P}_2(A)). \tag{4.6}$$

We have

$$\mathbf{E}d^2\xi(\mathbf{P}_2(A)) \le \mathbf{E}d^2 \le n^{-1}f(u-s), \mathbf{E}|d'|^2\xi(\mathbf{P}_2(A)) \le \mathbf{E}|d'|^2 \le n^{-1}f(u-s). \tag{4.7}$$

By Lemma 1.1.3 of Fernique (1981),

$$\mathbf{E}|a'|^2\wedge|b|^2\xi(\mathbf{P}_2(A)) \le n^{-1}g^{2/q}(u-s)[\mathbf{E}\xi(\mathbf{P}_2(A))]^{1-1/q}$$

provided

$$\mathbf{E}|a'|^2\wedge|b|^2\mathbb{I}\{|a'|^2\wedge|b|^2 \ge M\} \le n^{-1}g^{2/q}(u-s)[\mathbf{P}(|a'|^2\wedge|b|^2 \ge M)]^{1-1/q}, \forall M \ge 0.$$

Observe, that the last inequality follows from the independence of $|a'|$ and $|b|$, Hölder's inequality and condition (2.1) of Theorem 4. We have

$$\mathbf{E}|a'|^2 \wedge |b|^2 \xi(\mathbf{P}_2(A)) \leq n^{-1} g^{2/q}(u-s)[\mathbf{E}\xi(\mathbf{P}_2(A))]^{1-1/q} \leq$$
$$n^{-1} g^{2/q}(u-s)[\xi(\mathbf{P}(A))]^{1-1/q}, \tag{4.9}$$

since, by Jensen inequality, $\mathbf{E}\xi(\mathbf{P}_2(A)) \leq \xi(\mathbf{P}(A))$. Analogously,

$$\mathbf{E}|a|^2 \wedge |b'|^2 \xi(\mathbf{P}_2(A)) \leq n^{-1} g^{2/q}(u-s)[\xi(\mathbf{P}(A))]^{1-1/q}. \tag{4.10}$$

Combining inequalities (4.2,4,5,6) and estimates (4.3,7,9,10) we get (4.1). Lemma is proved.

Proof of Lemma 3. Let

$$x = S_n(s) - S_n(t), y = S_n(t) - S_n(u), x' = S'_n(s) - S'_n(t), y' = S'_n(t) - S'_n(u).$$

A simple inequality

$$|x| \wedge |y| \leq |x - x'| \wedge |y - y'| + |x'| \wedge |y| + |x| \wedge |y'|$$

yields

$$\mathbf{E}|x| \wedge |y| \mathbb{I}_A \leq \mathbf{E}|x - x'| \wedge |y - y'| \mathbb{I}_A + \mathbf{E}|x'| \wedge |y| \mathbb{I}_A + \mathbf{E}|x| \wedge |y'| \mathbb{I}_A. \tag{4.11}$$

The first summand is estimated in Lemma 6. Let us estimate the last two expectations. We have

$$\mathbf{E}|x'| \wedge |y| \mathbb{I}_A \leq \mathbf{E}|x'| \mathbb{I}_A = \mathbf{E}|x'| \mathbf{E}\mathbb{I}_A = \mathbf{E}|S'_n(s) - S'_n(t)| \mathbf{P}(A),$$

since $\mathbb{I}_A$ and S'_n are independent. Further,

$$\mathbf{E}|S'_n(s) - S'_n(t)| \leq \left(\mathbf{E}|S'_n(s) - S'_n(t)|^2\right)^{1/2} =$$
$$\left(\mathbf{E}(X(s) - X(t))^2\right)^{1/2} \leq \left(\mathbf{E}|X(s) - X(t)|^q\right)^{1/q} \leq g^{1/q}(u-s).$$

Hence $\mathbf{E}|x'| \wedge |y| \mathbb{I}_A \leq g^{1/q}(u-s)\mathbf{P}(A)$. Analogously, $\mathbf{E}|x| \wedge |y'| \mathbb{I}_A \leq g^{1/q}(u-s)\mathbf{P}(A)$. Substitution of these estimates and (4.1) in (4.11) gives the desired bound. Lemma is proved.

Proof of Lemma 4. The proof goes along the lines of the argument we used in proof of Lemmas 6 and 3. The only difference is that now expectations $\mathbf{E}|d|^2 \xi(\mathbf{P}_2(A))$ and $\mathbf{E}|d'|^2 \xi(\mathbf{P}_2(A))$ are estimated as in proof of Lemma 2.4.1 in Bézandry and Fernique (1992),

$$\mathbf{E}|d|^2 \xi(\mathbf{P}_2(A)) \leq cf(u-s)\theta(\xi[\mathbf{P}(A)\log(1 + 1/\mathbf{P}(A))/\log 2]),$$
$$\mathbf{E}|d'|^2 \xi(\mathbf{P}_2(A)) \leq cf(u-s)\theta(\xi[\mathbf{P}(A)\log(1 + 1/\mathbf{P}(A))/\log 2]).$$

The rest of the proof of the lemma coincides with that of Lemmas 6 and 3. Lemma is proved.

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