Generalized Weyl algebras

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1 Introduction

The purpose of this paper is to study the simple modules over one class of generalized Weyl algebras (briefly, GWA, the definition is in sec. 3), containing the classical Weyl algebras A_n. In sec. 4 the classification (up to irreducible elements of some Euclidian ring) of simple modules over GWA of degree 1 is considered. As particular case the classification of simple $sl(2)$-modules is obtained. The present method of approach adopt R. Block's idea to use some localization functor for the classification of the irreducible representations (of differential rings). For GWA of an arbitrary degree some natural classes of simple modules are classified, namely, finite dimensional and strong D-torsion (sec. 5), polynomial (sec. 5), which are analogous of the weight and Whittaker's modules over simple Lie algebras, respectively.

In the spirit of I. Bernstein [5,7] the theory of the dimension and the multiplicity for finitely generated modules is developed (sec. 6–8). In contrast to Bernstein theory the multiplicity of a module is a fractional number, not necessarily an integer, and the growth of the modules is described not only by one (Hilbert) polynomial, but by several. Nevertheless, Bernstein's inequality is taking place for considered GWA (sec. 7). In sec. 9 the modules over the Weyl algebras A_n which have the (minimal possible Gelfand-Kirillov) dimension n and the multiplicity 1 are classified. They are the "smallest" simple A_n-modules.

Considered generalized Weyl algebras of degree n are the tensor product of n generalized Weyl algebras of degree 1 for which the simple modules are classified (for example, $A_n = A_1 \otimes \ldots \otimes A_1$, n times). Then the tensor product of simple modules over GWA of degree 1 is a simple module over the tensor product of algebras (sec. 10). In particular, $\hat{A}_1 \otimes \ldots \otimes \hat{A}_1 \subset \hat{A}_n, sl(2)^\wedge \otimes \ldots \otimes sl(2)^\wedge \subset (sl(2) \times \ldots \times sl(2))^\wedge$, (corollary 10.1), where $\hat{A}_1$ is the set of isomorphic classes of simple A_1-modules.

For considered generalized Weyl algebras the global homological dimension, the Krull and Gelfand-Kirillov dimension are computed (sec. 5), the criterion for simplicity of algebras is established (sec. 5), Stafford's Theorem [6] are generalized (sec. 13).

2 Definition of generalized Weyl algebras [1–3]

Let D be a ring, $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a set of commuting automorphisms of $D, (\sigma_i\sigma_j = \sigma_j\sigma_i$, for all $i, j), a = (a_1, \ldots, a_n)$ be a set of (non-zero) elements of the centre $Z(D)$ of D, such that $\sigma_i(a_j) = a_j$ for all $i \neq j$. The *generalized Weyl algebra* $A = D(\sigma, a)$ (briefly, GWA) of degree n with *basic* ring D is the ring generated by D and $2n$ indeterminates $X_1^+, \ldots, X_n^+, X_1^-, \ldots, X_n^-$, subject to the relations:

$$X_i^- X_i^+ = a_i, X_i^+ X_i^- = \sigma_i(a_i), X_i^{\pm}\alpha = \sigma_i^{\pm 1}(\alpha)X_i^{\pm}, \forall \alpha \in D\,,$$

$$[X_i^-, X_j^-] = [X_i^+, X_j^+] = [X_i^+, X_j^-] = 0\,\forall i \neq j\,,$$

where $[x, y] = xy - yx$. We say that a and σ are the sets of *defining* elements and automorphisms of A, respectively. If D is a Noetherian ring (resp., without zero divisors), then A is a Noetherian ring (resp., without zero divisors) [1,3].

The category of generalized Weyl algebras is closed under the tensor product (over a basic field) of algebras, under joining to the algebra a set $C = \{C_i | i \in I\}$ of independent commuting variables. An opposite algebra to GWA is also GWA. More precisely,

$$A \otimes A' = D \otimes D'(\sigma \cup \sigma', a \cup a'),$$

$$A^0 = D^0(\sigma^{-1}, \sigma(a)), \text{ where } \sigma^{-1} = (\sigma_i^{-1}), \sigma(a) = (\sigma_i(a_i)),$$

$$A[C] = D[C](\sigma, a).$$

In the latter case we can make a more complicated construction by putting an action of σ_i on C_i in many nontrivial ways.

For any vector $k = (k_1, \ldots, k_n) \in \mathbf{Z}^n$ we put $v_k = v_{k1}(1) \ldots v_{kn}(n)$, where for any $1 \leq i \leq n$ and $m \geq 0 v_{\pm m}(i) = (X_i^{\pm})^m, v_0(i) = 1$. In the case where $n = 1$ we write v_m for $v_m(1)$. It follows from the definition of the GWA that

$$A = \bigoplus_{k \in \mathbf{Z}^n} A_k$$

is a $\mathbb{Z}^n$-graded algebra $(A_k A_e \subset A_{k+e}, \forall k, e \in \mathbb{Z}^n)$, where $A_k = Dv_k$.

3 The principal object

Further, unless it is indicated otherwise, A is GWA of degree n with basic polynomial ring $D = K[H_1, \ldots, H_n]$ in n variables with coefficients in the algebraically closed field K of characteristic 0, $\sigma_i(H_j) = H_j - \delta_{ij}(1 \leq i, j \leq n$, where $\delta_{ij} = 1$, if $i = j$ and 0, otherwise). Every defining element $a_i \in K[H_i]$ of A is a polynomial of positive degree. GWA A is Noetherian without zero divisors, hence, by Goldie's theorem A has a quotient field.

The *Weyl algebra* $A_n = A_n(K)$ of degree n over a field K is an associative K-algebra generated by $2n$ indeterminates $X_1, \ldots, X_n, \partial_1, \ldots, \partial_n$, subject to the relations:

$$[X_i, X_j] = [\partial_i, \partial_j] = [\partial_i, X_j] = 0 \text{ for } i \neq j, [\partial_i, X_i] = 1\,.$$

A_n is GWA of considered type with the set $\{a_i = H_i, 1 \leq i \leq n\}$ of defining elements.

An algebra A (for the specific defining elements) and its finite dimensional irreducible representations are also considered by E.K. Sklyanin [11]. GWA A of degree 1 and simple A-modules are studied thoroughly in [3].

4 The classification (up to irreducible elements of Euclidian ring) of simple A-modules

The results of this section are evident and non-difficult generalization of [3], their proofs are in [10]. In this section denote by $\mathcal{A}$ the GWA of degree 1 with non-zero defining elements $a \neq 0$ and basic *Dedekind* ring D. Let $k = S^{-1}\mathcal{D}$ be the quotient field of $\mathcal{D}, \mathcal{S} = \mathcal{D} - \{0\}$. The defining automorphism σ of $\mathcal{A}$ has the property

$$(C) \quad \sigma^n(\mathbf{p}) \neq \mathbf{p}, \forall \mathbf{n} \neq \mathbf{0} \in \mathbf{Z}, \mathbf{p} \neq \mathbf{0} \subset \mathcal{D}, \text{ prime ideal.}$$

For simplicity put $X = X_1^+$ and $Y = X_1^-$. The localization $B := S^{-1}\mathcal{A}$ of the ring $\mathcal{A}$ at S is a *skew Laurant* polynomial ring $k[X, X^{-1}; \sigma]$ which is an Euclidian ring with length function $l : l(\alpha X^m + \ldots + \beta X^n) = n - m$, where $\alpha, \beta \in k$ are non-zero, $m < \ldots < n$. Remind that every simple module over Euclidian ring B has the form B/Bb for some irreducible element $b \in B$. $B/Bb \simeq B/Bc$ if and only if b and c are *similar*, that is, there exists $d \in B$ such that 1 is a greatest common right divisor of c and d and bd is a least common left multiple of c and d [15].

For every $\mathcal{A}$-module M the set $\mathrm{tor}(M) = \{m \in M : sm = 0 \text{ for some } s \neq 0 \in \mathcal{D}\}$ is the ($\mathcal{D}$-*torsion*) submodule. Hence,

- every simple $\mathcal{A}$-module M is either $\mathcal{D}$-torsionfree ($\mathrm{tor}(M) = 0$) or $\mathcal{D}$-torsion ($\mathrm{tor}(M) = M$).

By (C) the group $= \mathbf{Z} < \sigma >$ acts freely on the set $\mathcal{P} = \mathrm{Specm}(\mathcal{D})$ of maximal ideals of $\mathcal{D}, (\sigma^i, \mathbf{p}) \rightarrow \sigma^i(\mathbf{p})$, i.e. by the map $\sigma^i(\mathbf{p}) \rightarrow \mathbf{i}$ every orbit $\mathcal{O} = \{\sigma^i(\mathbf{p}), \mathbf{i} \in \mathbf{Z}\}$ can be identified with $\mathbf{Z}$. So, for every orbit we use, without mentioning it explicitly, all the definitions and notations which are suitable for $\mathbf{Z}$ (the order, the segment, etc.). For example, $\sigma^i(\mathbf{p}) \leq \sigma^j(\mathbf{p})$ if and only if $i \leq j$. An orbit $\mathcal{O}$ is *degenerated* if it contains an ideal $\mathbf{p}$ such that $a \in \mathbf{p}$ (we call such ideals *marked*. Since $a \neq 0$ and $\mathcal{D}$ is Dedekind, there is only finite number of marked ideals and degenerated orbits). Marked ideals $\mathbf{p_1} < \ldots < \mathbf{p_s}$ of a generated orbit $\mathcal{O}$ divide it into $s + 1$ parts:

$$\Gamma_1 = (-\infty, \mathbf{p}_1], \Gamma_2 = (\mathbf{p_1}, \mathbf{p_2}], \ldots, \Gamma_{s+1} = (\mathbf{p_s}, \infty).$$

Ideals $\mathbf{p}, \mathbf{q} \in \mathcal{P}$ are *equivalent* ($\mathbf{p} \sim \mathbf{q}$) if they belong either to a non-degenerated orbit or to some Γ_i. There are only finite number of degenerated orbits and finite equivalent classes.

Proposition 4.1 [10]. *For every left (right) ideal $J \neq 0$ the $\mathcal{A}$-module $\mathcal{A}/\mathcal{J}$ has the finite length. So, the Krull dimension of $\mathcal{A}$ is K-dim $(\mathcal{A}) = 1$.*

Let A be a ring and Q be an isomorphism-invariant property of the simple A-modules. Denote by the $\hat{A}(Q)$ the set of isomorphism classes of simple A-modules which have the property Q.

Theorem 4.1 (Classification of the simple $\mathcal{D}$-torsion $\mathcal{A}$-modules). *The map Specm $(\mathcal{D})/\sim \rightarrow \hat{\mathcal{A}}(\mathcal{D}$-torsion$), \Gamma \rightarrow [L(\Gamma)]$, is a bijection, where*

1. *if Γ is non-degenerated orbit, then $L(\Gamma) = \mathcal{A}/\mathcal{A}\mathbf{p}, \mathbf{p} \in \Gamma$;*
2. *if $\Gamma = (-\infty, \mathbf{p}]$, then $L(\Gamma) = \mathcal{A}/\mathcal{A}(\mathbf{p}, X)$;*
3. *if $\Gamma = (\sigma^{-n}(\mathbf{p}), \mathbf{p}], n \in \mathbf{N}$, then $L(\Gamma) = \mathcal{A}/\mathcal{A}(Y^n, \mathbf{p}, X)$. This is precisely all the simple $\mathcal{A}$-modules which have the finite length $l_{\mathcal{D}}(L(\Gamma)) = n < \infty$ as $\mathcal{D}$-modules (there are only finitely many of these modules. Their number is equal to the total number of the finite classes of equivalence);*
4. *if $\Gamma = (\mathbf{p}, \infty)$, then $L(\Gamma) = \mathcal{A}/\mathcal{A}(\sigma(\mathbf{p}), Y)$.*

For $\alpha, \beta \in \mathcal{D}$ we write $\alpha < \beta$ if for each two maximal ideals $\mathbf{p} \ni \alpha$ and $\mathbf{q} \ni \beta$ which belongs to the same orbit the following is true: $\mathbf{p} < \mathbf{q}$. An element $b = v_{-m}\beta_{-m} + \ldots + \beta_0 \in \mathcal{A}, \beta_i \in \mathcal{D}$, is *normal* if $\beta_0 < \beta_{-m}$ and $\beta_0 < a$.

Lemma 4.1 *For any $b = v_{-m}\beta_{-m} + \ldots + \beta_0 \in \mathcal{A}, \beta_i \in \mathcal{D}$, of positive length $m > 0$ there exist $\alpha, \beta \in \mathcal{D}$ such that $\beta b \alpha^{-1} \in \mathcal{A}$ is normal. For this purpose it is sufficient to put*

$$\alpha = \sqcap \left\{ \sigma^j(\beta_0) | -s \leq j \leq 0 \right\}, \ \beta = \sqcap \left\{ \sigma^j(\beta_0) | -m-s \leq j \leq 1 \right\},$$

where s is a sufficiently large (easy calculated) integer.

Theorem 4.2 (Classfication of the simple $\mathcal{D}$-torsionfree $\mathcal{A}$-modules).

1. *The canonical map $S^{-1} : \hat{\mathcal{A}}(\mathcal{D}$-torsionfree$) \rightarrow \hat{B}, [M] \rightarrow [S^{-1}M]$, is a bijection, with an inverse $Soc : [N] \rightarrow [Soc_{\mathcal{A}}(N)]$, where $Soc_{\mathcal{A}}(N)$ is the socle of the module ${}_{\mathcal{A}}N$ and $S^{-1}M = B \otimes {}_{\mathcal{A}}M$ is a localization of M at S.*
2. *An $\mathcal{A}$-module M is simple $\mathcal{D}$-torsionfree if and only if $M \simeq \mathcal{A}/\mathcal{A} \cap Bb$, where $b \in \mathcal{A}$ is a normal irreducible in B (up to similarity uniquely determined) element.*

Remark. Multiplying by a convenient v_i each irreducible element $b \in B$ can be turned into normal element $c = \beta v_i b \alpha^{-1}$ by Lemma 4.1. By Theorem 4.2 $\mathcal{A}/\mathcal{A} \cap Bc \simeq (\mathcal{A}\alpha + Bb)/Bb$ is the simple $\mathcal{D}$-torsionfree $\mathcal{A}$-module.

Examples of algebras $\mathcal{A}$.

1. $\mathcal{A} = K[H](\sigma, a \neq 0), \sigma(H) = H - 1$ is an automorphism of the basic polynomial ring $K[H]$ over a field K of characteristic 0. In particular, for $a = H$ we get the 1-st Weyl algebra A_1.

2. Let U= $Usl(2)$ be the universal enveloping algebra of the Lie algebra $sl(2) =< X, Y, H \,|\, [H, X] = X, [H, Y] = -Y, [X, Y] = 2H >$ over a field K of characteristic 0, $C = H(H+1) + YX$ be Casimir element. Evidently,

$$U \simeq K[H, C](\sigma, a = C - H(H+1)),$$

where $\sigma(H) = H - 1, \sigma(C) = C$. Then for any $\lambda \in K$ the factoralgebra $U(\lambda) := U/U(C - \lambda)$ is the GWA from Example 1

$$U(\lambda) \simeq K[H](\sigma, a = \lambda - H(H+1)), \quad [9].$$

3. Consider the K-algebra $\Lambda(b)$ [2,13], the deformation of $Usl(2)$. It is generated by X, Y, H subject to the relations:

$$[H, X] = X, [H, Y] = -Y, [X, Y] = b \neq 0 \in K[H].$$

Then [6]

$$\Lambda(b) \simeq K[H, C](\sigma, a = X - \alpha)$$

where σ from Example 2 and $\alpha \in K[H]$ is a solution of the equation $\alpha - \sigma(\alpha) = b$. The centre of $\Lambda(b)$ is $K[C]$ and for any $\lambda \in K$ the factoralgebra

$$\Lambda(b, \lambda) := \Lambda(b)/\Lambda(b)(C - \lambda)$$

is isomorphic to the GWA from Example 1 with the defining element $\lambda - \alpha$.

4. $\mathcal{A}$ has $\mathcal{D} = K[H, (H - \mu/(1-\lambda))^{-1}], \sigma(H) = \lambda H + \mu$, where $\lambda \neq 0 \in K$ is not a root of $1, \mu \in K$.

5 Classification of the simple strong D-torsion A-modules, the simplicity criterion, the global homological dimension and the Krull dimension of the algebra A

Let $\mathcal{A}_i := \mathcal{D}_i(\sigma_i, a_i)$ be the GWA of degree 1 from Example 1, sec. 3 with the basic ring $\mathcal{D}_i = K[H_i], 1 \leq i \leq n$. Then the considered algebra A of degree n is the tensor product

$$A = \bigotimes_{i=1}^{n} \mathcal{A}_i$$

of the algebras $\mathcal{A}_i$. The group $\mathbf{Z}^n = \sqcap_{i=1}^n < \sigma_i >$ acts on the set of maximal ideals Specm $D \simeq \sqcap_{i=1}^n$ Specm $\mathcal{D}_i \simeq K^n$ of the polynomial ring D. Define an equivalence $\sim$ on Specm D as the direct product of the equivalences on Specm $\mathcal{D}_i$. An A-module M is *strong* D-torsion if for each $i = 1, \dots n$ M is a $\mathcal{D}_i$-torsion module. Then strong D-torsion A-module M is a direct sum of D-submodules (prime components):

$$M = \oplus_{\chi \in \mathcal{H}^*} M^\chi, M^\chi = \{m \in M : (h - \chi(h)^n)m = 0 \text{ for all } h \in \mathcal{H}, n = n(m)\} ,$$

where $\mathcal{H}^*$ is dual to the vector space $\mathcal{H}$ generated by the elements H_i, and vice versa. $\mathrm{Supp}(M) = \{\chi \in \mathcal{H}^* : M^\chi \neq 0\}$ is the *support* of M. The module M is the direct sum of A-submodules

$$M = \oplus M^{\mathcal{O}}, \mathcal{O} \in \mathrm{Specm}(D)/ \sim, M^{\mathcal{O}} = \bigoplus_{\chi \in \mathcal{O}} M^\chi .$$

So, M is indecomposable if and only if $M = M^{\mathcal{O}}$ for some orbit $\mathcal{O}$. A module M is *weight*, if

$$M = \oplus M_\chi$$

where $M_\chi = \{m \in M : (h - \chi(h))m = 0\}$ is the weight subspace of M of the weight χ. Evidently, a module is weight if and only if it is a semisimple D-module. The category of strong D-torsion modules is a full subcategory of the category of A-modules.

Theorem 5.1 (Classification of the simple strong D-torsion A-modules). *The map Specm $(D)/ \sim \to \hat{A}$ (strong D-torsion), $\Gamma = \sqcap_{i=1}^n \Gamma_i \to [L(\Gamma) := \otimes_{i=1}^n L_i(\Gamma_i)]$, is a bijection, where $L_i(\Gamma_i)$ from Theorem 4.1. This map establishes $1-1$ correspondence between the finite equivalence classes and the simple finite dimensional A-modules;* $\dim L(\Gamma) = \sqcap_{i=1}^n |\Gamma_i|$.

Proof follows from Theorem 4.1, since each simple strong D-torsion A-module is a factormodule of $A/A(H_1 - \lambda_1, \dots, H_n - \lambda_n) \simeq \bigotimes_{i=1}^n \mathcal{A}_i/\mathcal{A}_i(H_i - \lambda_i)$ for some $\lambda_i \in K$.

Theorem 5.2 *1. There are only finitely many two-sided ideals of A, each of them is homogeneous up to the grading of A.*

2. An algebra A is simple if and only if for each pair of distinct roots of the polynomials a_i $(i = 1, \dots, n)$ their difference is not a whole number.

3. If an algebra A is simple and each root of the polynomial a_i $(i = 1, \dots, n)$ has the multiplicity 1, then the global dimension gl.dim $A = n$.

4. The Krull dimension of A is K-dim $A = n$.

6 Hilbert functions, the Poincaré series

In this section $A = \bigoplus_{n\geq 0} A_n$ is a graded commutative Noetherian ring. Then the ring A_0 is Noetherian and the A_0-algebra A is generated by some elements $x_1, \ldots, x_n$ of degree $k_1, \ldots, k_n$, respectively. Let $M = \bigoplus_{n\geq 0} M_n$ be a finitely generated graded A-module ($A_n M_m \subset M_{n+m}$ for all n, m), then the A_0-module M_n is also finitely generated for each $n \geq 0$. Let λ be some *additive* function from the set of finitely generated A_0-modules to $\mathbf{Z}$ (if $0 \to N \to M \to L \to 0$ is an exact sequence of finitely generated A_0-modules, then $\lambda(M) = \lambda(N) + \lambda(L)$). The function λ is *non-negative* if its value is non-negative for every module. In applications the following (non-negative) additive functions are come cross:

1. A_0 is a field, $\lambda(N) = \dim(N)$ is the dimension of the vector space N;
2. A_0 is an Artinian ring, $\lambda(N) = l_{A_0}(N)$ is the length of the A_0-module N.

The *Poincaré series* of M (relatively to λ) is

$$P(M,t) = \sum_{n=0}^{\infty} \lambda(M_n) t^n \in \mathbf{Z}[[t]]\,.$$

The functions

$$\lambda_M(n) := \lambda(M_n),\ \chi_M(n) := \sum_{i=0}^{n} \lambda(M_i),\ n \geq 0\,,$$

are called a *Hilbert function* and an *integral Hilbert function* of M respectively.

Theorem 6.1 (Hilbert-Serre).

1. *$P(M,t)$ is a rational function of the type $f(t)/\prod_{i=0}^{n}(1-t^{k_i})$, where $f(t) \in \mathbf{Z}[t]$.*
2. *If $k_1 = \ldots = k_s = 1$, then for all sufficiently large n the Hilbert functions $\lambda_M(n)$ and $\chi_M(n)$ are polynomials of the degree $d-1$ and d respectively with coefficients which are rational number of the form $[(d-1)!]^{-1}\mathbf{Z}$ and $[d!]^{-1}\mathbf{Z}$, where $d = d(M)$ is an order of the pole $t = 1$ of $P(M,t)$.*

The main result of sec. 5 is the following generalization of Theorem 5.1.

Theorem 6.2 *Let $k = l.c.m.(k_1, \ldots, k_s)$ be the least common multiple of the k_i-th, then*

1. *there exist k polynomials $\psi_0, \ldots, \psi_{k-1} \in \mathbf{Q}[t]$ of the degree $\leq d-1$ with coefficients in $[k^{d-1}(d-1)!]^{-1}\mathbf{Z}$ such that*

$$\lambda_M(n) = \psi_j(n),\ \textit{for all}\ n \gg 0\ \textit{such that}\ n \equiv j (\mathrm{mod} k)\,;$$

2. *there exist k polynomials $\gamma_0, \dots, \gamma_{k-1} \in \mathbf{Q}[t]$ of the degree $\le d$ with coefficients in $[k^d d!]^{-1}\mathbf{Z}$ such that*

$$\chi_M(n) = \gamma_j(n), \text{ for all } n \gg 0 \text{ such that } n \equiv j (\mathrm{mod} k);$$

3. *if λ is non-negative, then all polynomials γ_j have the same degree $d(=$ an order of the pole $t = 1$ of $P(M,t))$ and the same highest coefficients.*

For the integral Hilbert function χ_M define its *degree*

$$\deg(\chi_M) := \max \deg(\gamma_j);$$

highest coefficient

$$\mathrm{hc}(\chi_M) := \sum \{\mathrm{hc}(\gamma_j) | \deg(\gamma_j) = \deg(\chi_M)\} ,$$

where $\mathrm{hc}(\gamma_j)$ is the highest coefficient of the polynomial γ_j;

$$\textit{multiplicity:} \qquad e(\chi_M) := (\deg(\chi_M))!\, \mathrm{hc}(\chi_M).$$

All the same notions can be introduced for the Hilbert function λ_M.

Corollary 6.1 1. *The degree $\deg(\chi_M)$ and the multiplicity $e(\chi_M)$ of the integral Hilbert function of M (relatively to λ) equal to the order $d(M)$ of the pole $t = 1$ of $P(M,t)$ and to $e(\chi_M) = k((1-t)^{d(M)}P(M,t))|_{t=1}$ respectively.*

2. *The degree and the multiplicity of the Hilbert function of M equal to $\deg(\lambda_M) = d(M) - 1$ and to $e(\lambda_M) = e(\chi_M)$ respectively.*

The results of sec. 6 are proved in [16].

7 An analogue of Bernstein inequality, the Hilbert function, filtrations

The GWA A of degree n (sec. 2) has the following filtrations: the filtration

$$B : B_m = \sum \{D_i v_j \mid 2|i| + s_1|j_1| + \dots + s_n|j_n| \le m\} ,$$

where D_i is the usual filtration on the polynomial ring D, s_k is the degree of the (defining) polynomial $a_k, 1 \le k \le n, j = (j_1, \dots, j_n) \in \mathbf{Z}^n$. Evidently, the space B_m is finite dimensional for each $m \ge 0$.

The filtration Σ:

$$\Sigma_m = \sum \{Dv_j \,|\, |j_1| + \dots + |j_n| \le m\} .$$

The associated graded algebra $\mathrm{gr}_B(A)$ is commutative and generated by $3n$ variables x_i, y_i, h_i subject to the relations:

$$h_i^{s^i} = x_i y_i, \ 1 \leq i \leq n \, .$$

We say that function $\chi : \mathbf{Z}_+ \to \mathbf{R}$ has the *polynomial growth*, if for some integer d there exists a non-zero limit

$$\lambda = \lim \chi(t)/t^d, \ t \to \infty \, .$$

For the filtration B analogously to the case of the Weyl algebras [4,5] a good filtration $\Gamma = \{\Gamma_i\}$ exists on every finitely generated A-module M. Then by Theorem 6.2.3 (for $\mathrm{gr}_B(A)$), there exists a function of the polynomial growth:

$$\chi(M, \Gamma, t) = (e/d!)t^d + \ldots, \ ek^d, \ d \in \mathbf{Z}_+, \ k = l.c.m.(s_1, \ldots, s_n, 2) \, ,$$

such that $\dim \Gamma_j = \chi(M, \Gamma, j)$ for all $j \gg 0$, where three dots denote a term of smaller order.

Definition. We call $d = d(M)$ and $e = e(M)$ the *dimension* and the *multiplicity* of the module M.

These two constants depend on the module M only and not on the particular choice of the good filtration Γ. They have the same properties as in the case of the Weyl algebras [4]. In particular, for every exact sequence $0 \to M \to N \to L \to 0$, $d(M) = \max\{d(M), d(L)\}$; if $d(M) = d(L)$, then $e(N) = e(M) + e(L)$. It is clear that $d(M)$ is the *Gelfand-Kirillov* dimension of M.

The Gelfand-Kirillov dimension of GWA A equals $GK\text{-}dim\ A = 2n$, since

$$\dim B_j = j^{2n}/(2n)! s_1 \ldots s_n + \ldots \, .$$

As in the case of Weyl algebras [7] *Bernstein inequality* holds for GWA A.

Theorem 7.1 *,[16]. Let A be a simple algebra and $M \neq 0$ be a finitely generated A-module. Then $d(M) \geq n$.*

If A is not a simple algebra and $1 \leq t \leq n$ is the total number of non-simple tensor multiples $\mathcal{A}_i$ of A, then the following inequality is true in Theorem 7.1

$$d(M) \geq n - t \, [16].$$

8 The Bernstein class of A-modules

Lemma 8.1 *Let M and N be finitely generated modules over generalized Weyl algebras A and A' respectively. Then their tensor product $M \otimes N$ is the module over the tensor product of algebras $A \otimes A'$, which has the dimension $d(M \otimes N) = d(M) + d(N)$ and the mulitplicity $e(M \otimes N) = e(M)e(N)$.*

Let us introduce the *Bernstein class* $\mathcal{B} = \mathcal{B}(A)$ of A-modules which consists of all non-zero finitely generated A-modules which have the minimal possible dimension (n for the Weyl algebra A_n). Evidently, the Bernstein class is a full subcategory of the category of A-modules.

- Each module from $\mathcal{B}(A)$ has the finite length.

It follows from the results of the previous section. Let $\mathcal{B}', \mathcal{B}''$ and $\mathcal{B}$ be the Bernstein classes of modules for algebras A', A'' and $A = A' \otimes A''$ respectively, then by Lemma 8.1

$$\mathcal{B}' \otimes \mathcal{B}'' \subset \mathcal{B}$$

It follows from the inclusion $\hat{A}_1 \subset \mathcal{B}(A_1)$ that

$$\hat{A}_1 \otimes \ldots \otimes \hat{A}_1 \subset \mathcal{B}(A_n)\,,$$

i.e. the tensor product of n simple A_1-modules is a A_n-module of the finite length which has the dimension n (moreover, it is a simple A_n-module, Corollary 10.1).

Theorem 8.1 *If $M \in \mathcal{B}(A), A' := \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_{n-1}$ is a subalgebra of A. Then for each $u \neq 0 \in \mathcal{A}_n$ the kernel $Ker(u_M)$ and the cokernel $Coker(u_M)$ belong to the Bernstein class $\mathcal{B}(A')$, where $u_M : M \to M, m \to um$.*

9 Classification of small A_n-modules

Definition. A module from the Bernstein class will be called *small* if it has a minimal possible multiplicity.

- *The small module is simple (since the multiplicity is additive and restricted) and, in some sense is the "smallest" simple module.*

It follows from the definition that A_n-module M is small iff $d(M) = n$ and $e(M) = 1$. Denote by $\mathcal{K}_n$ the set of all small A_n-modules, then by Lemma 8.1

$$\mathcal{K}_n \otimes \mathcal{K}_m \subset \mathcal{K}_{n+m}$$

for all $n, m \in \mathbf{N}$. A_n-module

$$W_n := A_n/A_n(\partial_1, \ldots, \partial_n) = (K[X_1, \ldots, X_n], \partial_j(F) = \partial F/\partial X_j, X_j(F) = X_j F)$$

and

$$V_n := A_n/A_n(X_1, \ldots, X_n)$$

are small.

Let $\mathrm{Aut}_B(A_n)$ be the group of automorphisms which preserve the *Bernstein* filtration on $A_n(\tau(B)_n = B_n, \forall n \in \mathbf{N})$. It is a subgroup of $GL(V)$ all elements of which preserve the bilinear antisymmetric form $[u,v] = uv - vu$ of the vector space $V =< 1, X_1, \ldots, X_n, \partial_1, \ldots, \partial_n >$.

Theorem 9.1 *,[17]. M is a small A_n-module iff $M \simeq {}^{\tau}W_n$ (a twisted module) for some algebra automorphism $\tau \in \mathrm{Aut}_B(A_n)$, where $W_n = A_n/A_n\ (\partial_1, \ldots, \partial_n)$.*

10 The tensor product of simple modules

Let K be an algebraically closed uncountable field. In the present section

$$R_i = \mathcal{D}_i(\sigma_i, a_i), \ (1 \leq i \leq n)$$

is GWA of degree 1, $\mathcal{D}_i$ is a commutative finitely generated domain which has the *restricted minimum condition* (i.e. each proper factormodule $\mathcal{D}_i$ has a finite length), σ_i is a K-automorphism of $\mathcal{D}_i, a_i \neq 0$. We assume that for each simple $\mathcal{D}_i$-torsion R_i-module M there exists a commutative finitely generated subalgebra L of R_i such that L-module $M = \oplus M_\nu$ is semisimple with simple non-isomorphic summands M_ν. This condition holds if additionally assume that for each maximal ideal $\mathbf{p}$ of $\mathcal{D}_i$ the ideals $\sigma_i^j(\mathbf{p}), j \in \mathbf{Z}$, are all different. Then it is sufficient to put $L = \mathcal{D}_i$ [10] (see sec. 4). Examples 1,4 (for example, the Weyl algebra A_1) and the factoralgebras $U(\lambda), \Lambda(b, \lambda)$ from Examples 2,3, sec. 4 are algebras of this kind (other examples are in [3]).

Lemma 10.1 *Let M be a simple $\mathcal{D}_i$-torsionfree R_i-module. Then for each $m \neq 0 \in M$ there exists an element $p \in R_i$ which has the kernel $Ker(p_m) = Km$ under the natural action on M.*

Proof. Let B be the localization of $R = R_i$ at $\mathcal{D}_i - \{0\}$. It is clear, that $M = Rm \simeq M_p := R/R \cap Bp$ for some irreducible (in B) element $p \in R, pm = 0$. Then

$$0 \to V_p := R \cap Bp/Rp \to R/Rp \to M_p \to 0 \qquad (*)$$

is an exact sequence of R-modules, V_p is a $\mathcal{D}_i$-torsion module, hence $\mathrm{Hom}_R(V_p, M) = 0$. Applying the functor $\mathrm{Hom}_R(-, M)$ to $(*)$ we have $K \simeq \mathrm{Hom}_R(M_p, M) \simeq \mathrm{Hom}_R(R/Rp, M) \simeq \mathrm{Ker}(p_M)$.

Theorem 10.1

$$\hat{R}_1 \otimes \ldots \otimes \hat{R}_n \subset (R_1 \otimes \ldots \otimes R_n)^\wedge,$$

i.e. the tensor product of simple R_i-modules $M_i, (i = 1, \ldots, n)$ is a simple module over the tensor product $R = \otimes R_i$ of algebras R_i. Two such modules $\otimes M_i$ and $\otimes N_i$ are isomorphic iff the R_i-modules M_i and N_i are isomorphic for all i.

Proof. A module M is simple iff it is generated by any of its element $u \neq 0$. So, it is sufficient to show that $Ru \ni z = \otimes z_i, z_i \neq 0 \in M_i$ (then $Ru \supseteq Rz = \otimes R_i z_i = M$). Prove it.

Let M_1 be a $\mathcal{D}_1$-torsion module. Then we can write u as a sum $u = \sum u_j \otimes w_j, u_j \in M_1, w_j \in N = \bigotimes_{i \geq 2} M_i$, of say m, non-zero elements such that

$$\text{(i)} \quad Ku_i \cap Ku_j = 0, \quad \text{if } i \neq j;$$

$$\text{(ii)} \quad \{w_j\} \text{ are } K\text{-linearly independent.}$$

By Lemma 10.1 there exist $p \in R_1$ such that $\mathrm{Ker}(p_{M_1}) = Ku_1$, then by (i), (ii) $u' := pu = \sum_2^m pu_j \otimes w_j \neq 0$ and we complete the proof by using induction on m and n.

Let M_1 be a $\mathcal{D}_1$-torsionfree module. Then choose a commutative finitely generated subalgebra L of R_1 such that $M_1 = \sum (M)_k$ is the sum of non-isomorphic (1-dimensional) simple L-modules. Hence, $u = \sum u_j \otimes w_j, u_j \in (M_1)_j$, where $\{w_j \in N\}$ are linearly independent. Then $du = u_1 \otimes w_1 \neq 0$ for some $d \in L$. Induction now completes the argument.

It is evident that the R_i-module M is semisimple and each simple summand of M is isomorphic to M_i. Hence, the R-modules $\otimes M_i$ and $\otimes N_i$ are isomorphic iff $M_i \simeq N_i$, for all $1 \leq i \leq n$.

Corollary 10.1 *1.* $\hat{A}_1 \otimes \ldots \otimes \hat{A}_1 \subset \hat{A}_n$.

2. $sl(2)^\wedge \otimes \ldots \otimes sl(2)^\wedge \subset (sl(2) \times \ldots \times sl(2))^\wedge$.

11 The multiplicity of the simple A_n-modules which has the dimension n

Theorem 11.1 *For each integer $m \in \mathbf{N}$ there exists a simple A_n-module M which has the dimension $d(M) = n$ and the multiplicity $e(M) = n$.*

Proof. By Lemma 8.1. and Corollary 10.1 it is enough to prove this Theorem for A_1. Consider the automorphism

$$\tau : \partial_1 \longrightarrow \partial_1 - \alpha, X_1 \longrightarrow X_1\,,$$

of the Weyl algebra A_1, where the polynomial $\alpha \in K[X_1]$ has the degree $d \geq 1$. Then the module

$$M := A_1/A_1(\partial_1 + \alpha) \simeq {}^{\tau}W_1$$

is simple (W_1 is from sec. 9) and has a vector space it equals to $K[X_1]$. Using the Bernstein filtration B_i of A_1 we build the good filtration on $M = \cup \left\{ M_i = \sum_{j=0}^{d} B_i X_1^i = \sum_{s=0}^{(i+1)d} K X_1^s \right\}$, hence $e(M) = d$.

12 The polynomial modules

The category $\mathbf{C}$ of modules (without isomorphic object) is called *normal* if the following conditions hold:

C1) each module M of $\mathbf{C}$ has a finite length and each submodule of M is essential (that is, it intersects non-trivially each proper submodule of M);

C2) each non-zero morphism is injective;

C3) if $M \in \mathbf{C}$, then each simple submodule of M belongs to $\mathbf{C}$;

C4) for given simple module $L \in \mathbf{C}$ there are only finitely many modules holding L.

On $\mathbf{C}$ define a partial order $<$ and an equivalence $\sim$: $M < N (M \sim N)$, if $\mathrm{Hom}_{\mathbf{C}}(M, N) \neq 0$ (resp., in $\mathbf{C}$ there exists a chain of modules $M = M_1, \ldots, M_s = N$ such that for each i either $M_i > M_{i+1}$ or $M_i < M_{i+1}$).

Lemma 12.1 *1. Each class $E \subset \mathbf{C}$ of equivalence is finite and holds a least element.*

2. Each module $M \in \mathbf{C}$ is indecomposable, its endomorphism ring is a division ring, M has finitely many submodules and there are finitely many modules of $\mathbf{C}$ which hold M.

Proof. By C1,2 choose a simple module $M = M(E) \in E$. By C4 the set $F = \{N \in E : M < N\}$ is finite. Show that $E = F$. Suppose contrary, then there exist a module $L \in E - F$ such that $L < N$ for some $N \in F$. By C1,2 L is the submodule of N, hence L holds M (a contradiction). The rest is clear.

The map $\mathbf{C}/\sim \to \hat{\mathbf{C}}, E \to M(E)$, is a bijection between the classes of equivalence and the simple modules of $\mathbf{C}$. If M and N are non-isomorphic and $\mathrm{Hom}_{\mathbf{C}}(M, N) \neq 0$, then $\mathrm{Hom}_{\mathbf{C}}(N, M) = 0$.

Definition. A module over GWA is *polynomial* if it is a free module of rank 1 over the basic ring of GWA.

Let $\mathcal{A} = \mathcal{D}(\sigma, a)$ be the GWA of degree 1, $\mathcal{D}$ be a principal commutative domain, σ has the property (C), $a \neq 0$. For simplicity we assume that the set $\mathcal{D}^*$ of inverse elements of $\mathcal{D}$ coincides with the set $\mathcal{D}^\sigma = \{d \neq 0 \in \mathcal{D} : \sigma(d) = d\}$ of fixed points of σ (for example, $\mathcal{A}$ from Example 1, sec. 3). Then Proposition 11 follows from Lemma 11.2 and the defining relations of $\mathcal{A}$.

Proposition 12.1 *1. Each polynomial $\mathcal{A}$-module is isomorphic to $W_f = \mathcal{A}/\mathcal{A}(X - f, Y - a/\sigma^{-1}(f))$ for some divisor $f \neq 0 \in \mathcal{D}$ of $\sigma(a)$ $(f|\sigma(a))$, and vice versa. $W_f \simeq W_g$ iff $f = g$.*

2. Each submodule $N \neq 0$ of W_f is polynomial $N = W_\alpha$ and $N = \mathcal{D}h$ for some $h \in W_f = \mathcal{D}$ such that $\sigma(h)f = h\alpha$.

3. The category of the polynomial $\mathcal{A}$-modules is normal.

The set $\mathcal{M}$ of irreducible elements of $\mathcal{D}$ is *marked*, if each irreducible element of $\mathcal{D}$ divide some element of $\mathcal{M}$ and if $p = \lambda\sigma^i(q)$ for some $p, q \in \mathcal{M}, i \in \mathbf{Z}, \lambda \in \mathcal{D}^*$, then $\lambda = 1$. Fix the marked set $\mathcal{M}$ (for example, for $\mathcal{D} = K[H]$ $\mathcal{M}$ is the set of irreducible polynomials with the highest coefficients 1), then the group $\mathbf{Z} =< \sigma >$ acts on $\mathcal{M}$. For any $\alpha \neq 0 \in k$ (the quotient field of $\mathcal{D}$) denoted by $\mathcal{O}(\alpha)$ the set of orbits of $\mathcal{M}/\mathbf{Z}$ holding the irreducible divisors of $f, g \in \mathcal{D}$, where $\alpha = f/g$ is the irreducible quotient. Then

$$\alpha = \alpha^* \sqcap \{\alpha_{\mathcal{O}} \,|\, \mathcal{O} \in \mathcal{O}(\alpha)\} \; ,$$

where $\alpha_{\mathcal{O}} \in k$ contains all the irreducible divisors from $\mathcal{O}, \alpha^* \in \mathcal{D}^*$ is a uniquely determined element. Denote by $\nu(p, \alpha)$ the multiplicity of $p \in \mathcal{M}$ in α.

Lemma 12.2 *If $\alpha \neq 0 \in k$, then the solution $h \in k$ of the equation*

$$\sigma(h) = \alpha h$$

exists iff there exists the solution $h^* \in \mathcal{D}^*$ for $\alpha = \alpha^*$ and

$$< \alpha, \mathcal{O} >:= \sum_{i \in \mathbf{Z}} \nu(\sigma^i(p), \alpha) = 0$$

for each orbit $\mathcal{O} \in \mathcal{O}(\alpha)$. The solution is

$$h = h^* \sqcap_{\mathcal{O} \in \mathcal{O}(\alpha)} \sqcap_{p \in \mathcal{O}} p^{\Sigma(p)},$$

where $\Sigma(p) = -\sum_{i \geq 0} \nu(\sigma^{-i}(p), \alpha)$. If the solution h exists, then $h \in \mathcal{D}$ iff $\Sigma(p) \geq 0$ for each $\mathcal{O} \in \mathcal{O}(\alpha)$ and $p \in \mathcal{O}(\alpha)$.

Corollary 12.1 (The simplicity criterion for W_f). *The polynomial $\mathcal{A}$-module W_f is simple if and only if for every proper divisor $\alpha' \neq 0 \in \mathcal{D}$ of $\sigma(a)$ there is no solution of the equation $\sigma(h) = \alpha h$ for $\alpha = \alpha'/f$.*

Theorem 12.1 *Suppose $A = \otimes(\mathcal{A}_i = K[H_i](\sigma_i, a_i))$ is* GWA *of degree n from sec. 3, $i = 1, \ldots, n$. Then*

1. *each polynomial A-module is the tensor product $W_f := \otimes W_{f_i}$ of the polynomial $\mathcal{A}$-modules $W_{f_i}, f = (f_1, \ldots, f_n), f_i \in K[H_i]$. Moreover, $W_f \simeq W_g$ iff $f = g$.*

2. *W_f is a simple A-module iff W_{f_i} is a simple $\mathcal{A}_i$-module for all $i = 1, \ldots, n$.*

3. *the category of the polynomial A-modules is normal.*

The following modules are the first examples of the polynomial modules: the A_n-modules $W_\lambda \simeq A_n/A_n(\partial_1 - \lambda_1, \ldots, \partial_n - \lambda_n), \lambda = (\lambda_1 \neq 0, \ldots, \lambda_n \neq 0) \in K^n$, the simple Whittaker $sl(2)$-module $W_\lambda(\mu) = K[h]$, $\mu \neq 0, \lambda \in K$ [12], where

$$Hh^n = h^{n+1}, Xh^n = \mu(h-1)^n, Yh^n = -\mu^{-1}(h-\lambda)(h+\lambda+1), n \geq 0.$$

For another example of the polynomial examples $sl(2)$-modules (more precisely, $U(\lambda)$-modules) see [8]. E.K. Sklyanin [11] considered a simple finite dimensional A-module as a factormodule of polynomial (if we use the definitions of the present paper).

13 The module structure of A

In the paper [6] J.T. Stafford put the following question: "find a more general class of simple rings for which the results of section 3 (in [6]) still hold". Simple generalized Weyl algebras A (sec. 3) is the example of such class of simple rings. Before the formulation of the results give one definition.

Let R be a Noetherian domain with quotient ring Q and let M be a finitely generated R-module. Define rank $M = s : Q \otimes\ _R M \simeq Q^{(s)}$.

Theorem 13.1 *Suppose that generalized Weyl algebra A is simple, then*

1. *any right ideal of A can be generated by two elements. Moreover, if a right ideal $I = aA + bA + cA$ and $d \neq 0 \in A$, then there exist $f, g \in A$ such that $I = (a + cfd)A + (b + cgd)A$;*
2. *any finitely generated A-module M is isomorphic to $N \oplus A^{(s)}, s \in \mathbf{N}$, where the module N has the rank ≤ 1;*
3. *let M be a finitely generated A-module with rank $M \geq 2$. Suppose that $M \oplus A \simeq N \oplus A$ for some module N. Then $M \simeq N$.*

Let $\mathcal{A}_1 := K[H_1](\sigma_1, a_1)$ be GWA of degree 1 from Example 1, sec. 3 and B_1 be a localization of $\mathcal{A}_1$ (sec. 4). For any field $L \supset K$ we write $\mathcal{A}_1(L) = L \otimes \mathcal{A}_1$.

Lemma 13.1 *Suppose that the algebra $\mathcal{A}_1$ is simple and $b_1, \dots, b_r \in \mathcal{A}_1$ are linearly independent over K. Let $S = \mathcal{A}$ or B_1 and choose $t \neq 0 \in S$. Then $\mathcal{A}_1(b_1, \dots, b_r)tS = S^{(r)}$.*

Proof. It is enough to consider the case $S = \mathcal{A}_1$ and $t = 1$. For sufficiently large integer $s : b_i X^s \in K < H, X >$ for all i, so without lose of generality put $s = 0$. If the polynomial $\alpha \in K[H]$ has the degree n, then $\deg(\sigma(\alpha) - \alpha) = n - 1$. Hence there exists an integer m such that

$$(\mathrm{ad}X)^m (b_1, \dots, b_r) = (\lambda_1 X^{n_1}, \dots, \alpha_r X^{n_r}) \qquad (**)$$

where $\lambda_i \in K$ and $(\mathrm{ad}X)u = Xu - uX, u \in \mathcal{A}_1$. Since the elements b_i are linearly independent, replacing some b_j by $\Sigma \mu_i b_i$, with $\mu_i \in K, \mu_j = 1$ (this operation commutes with $\mathrm{ad}X$), we can assume that $n_1 > n_2 \geq \dots$. Multiplying $(**)$ by Y^{n_1} on the left side and using $\mathrm{ad}Y$ sufficiently many times, say n, we have $(\lambda Y^n, 0, \dots, 0)$, for some $\lambda \neq 0 \in K$. Since $\mathcal{A}_1$ is simple, then $(\mathcal{A}_1, 0, \dots, 0) \subset \mathcal{A}_1(b_1, \dots, b_r)S$. We complete the proof by induction.

Now, using Lemma 13.1, we can repeat the argument of the paper [6] to prove Theorem 13.1. For this purpose we should change the rings $R, S, A_T, A_{(r)}$ from [6] on the following rings.

Case 1. $R = A_T \supset A$ is a localization of A at the multiplicative closed subset generated by $T = \{H_2, \dots, H_m\}$ for some m. Put

$$S = \mathcal{A}_1\,(K(H_2, \dots, H_m)) \supset U = S \cap A \supset \mathcal{A}_1\,.$$

Case 2. $R = A_t \supset A$, where $T = \{H_1, \dots, H_n, X_1, \dots, X_{r-1}\}$ for some $r \leq n$. Denote by Q the division ring of the subalgebra of A, generated by the set T, then we can write $R = Q < X_r, \dots, X_n, Y_r, \dots, Y_n >$. If $S = Q < X_r, Y_r >= Q[X_r, X_r^{-1}; \sigma_r]$ is a skew Laurant polynomial ring, then $S = T_{r-1}(H_r, \dots, H_n)[X_r, X_r^{-1}; \sigma_r]$, where T_r is a division ring of the algebra $A_{(r)} := \bigotimes_1^r \mathcal{A}_i, U = S \cap A = A_{(r)}[H_{r+1}, \dots, H_n] \supset \mathcal{A}_1$.

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