Tensor homological minimal algebras, global dimension of the tensor product of algebras and of generalized Weyl algebras

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Abstract

Let d be the left homological dimension or weak dimension. In the present paper we give for a large class of algebras the answer to the question:

- when the dimension of the tensor product of algebras is the sum of dimensions of the multiples,

$$\mathrm{d}(\Lambda_1 \otimes \ldots \otimes \Lambda_n) = \mathrm{d}\Lambda_1 + \ldots + \mathrm{d}\Lambda_n.$$

Tensor d-minimal algebras are introduced and it is proved that a large class of algebras are tensor d-minimal algebras.

1 Introduction, main results

Denote by Alg (respectively, $\mathcal{LN}$; $\mathcal{LFN}$; $\mathcal{SC}$) the class of all (respectively, left Noetherian; left Noetherian and finitely generated; somewhat commutative, see sec. 3) algebras over a fixed field K. In this paper the tensor product $\otimes$ means $\otimes_K$ and a module means a left module. Denote by d one of the following dimensions of rings: wd, the *weak* dimension; lgd, the *left homological* dimension; kd, the *Krull* dimension (in the sense of Gabriel-Rentschler [GR]).

In the present paper we give for a large class of (infinite dimensional non-commutative) algebras the answer to the question:

- when the dimension of the tensor product of algebras is the sum of dimensions of the multiples,

$$\mathrm{d}(\Lambda_1 \otimes \ldots \otimes \Lambda_n) = \mathrm{d}\Lambda_1 + \ldots + \mathrm{d}\Lambda_n. \qquad (1.1)$$

Usually, (1.1) is not true. For example, if L_n is the division ring of the polynomial ring $P_n = K[X_1, \ldots, X_n]$ in n variables, then lgd $L_n \otimes L_n = n \neq 0 =$ lgd L_n+ lgd L_n ([MR], Proposition 5.10, p. 251). Therefore, we should put on the algebras Λ_i some restricted conditions for (1.1) to be satisfied. It is well known, that for all above dimensions d=wd, kd (respectively, lgd) ([Aus],[MR])

$$\mathrm{d}(A \otimes B) \geq \mathrm{d}A + \mathrm{d}B, \text{ for all (respectively, left Noetherian) algebras } A, B. \qquad (1.2)$$

So it is natural to give the following

Definition. An algebra A is a *tensor* d-*minimal* with respect to some class of algebras Ω, if

$$\mathrm{d}(A\otimes B)=\mathrm{d}A+\mathrm{d}B,\ \text{for each}\ B\in\Omega. \tag{1.3}$$

In the case wd (lgd) we say that the algebra A is *tensor weak minimal* (*tensor homological minimal*), briefly, TWM (THM). The polynomial ring P_n is tensor homological minimal with respect to Alg, since for each algebra B

$$\mathrm{lgd}(P_n\otimes B)=n+\mathrm{lgd}\,B=\mathrm{lgd}\,P_n+\mathrm{lgd}\,B\ (Hilbert's\, syzygy\, theorem).$$

The global dimension of the tensor product (over a commutative ring) of algebras was studied in [ERZ]. At the beginning of the article the authors write that "*the questions concerning the dimension of the tensor product of two algebras have turned out to be surprisingly difficult.*" Theorem 8 of [ERZ] states that the algebra $T_n(K)$ of all $n\times n$ matrices (k_{ij}) with entries in (commutative ring) K satisfying $k_{ij}=0$ for $i<j$ is THM and TWM with respect to Alg. In [Ha] the tensor homological (and weak) minimality was proved for matrix ring $M_n(K)$ with entries in (commutative ring) K. If the algebras A, B and $A\otimes B$ are semi-primary and the global dimension gl.dim $A\otimes B$ $<\infty$, then gl.dim $A\otimes B$ = gl.dim $A+$ gl.dim B ([Aus 1], Theorem 5).

We shall assume that all considered classes Ω contain the field K.

One of the aims of this paper is to give sufficient conditions for an algebra to be tensor d-minimal and, as a consequence, prove this property for many well-known algebras and their tensor products (for example, the Weyl algebra A_n of degree n over an algebraically closed resp. and uncountable field is THM with respect to $\mathcal{SC}$ resp. $\mathcal{LFN}$). Since many "good" algebras are the tensor product of tensor d-minimal ones (for example, the Weyl algebra $A_n=A_1\otimes\ldots\otimes A_1, n$ times), so their dimension can be computed via (1.1), using Lemma 1.1 (which is an easy corollary of the definition above).

Lemma 1.1 *Let Λ_i $(i=1,\ldots,n)$ be tensor d-minimal algebras with respect to Ω such that $\Lambda_i\otimes\Omega\subseteq\Omega$ for all i. Then the tensor product $\otimes_{i=1}^n\Lambda_i$ is a tensor d-minimal algebra with respect to Ω and d$(\otimes_{i=1}^n\Lambda_i\otimes B)=\sum_{i=1}^n\mathrm{d}\Lambda_i+\mathrm{d}B$ for each $B\in\Omega$.* ∎

The structure of the paper is as follows. In sec. 2 the definitions and examples of generalized Weyl algebras are presented; generalized Weyl algebras include many well-known algebras: the Weyl algebra A_1; the universal enveloping algebra $Usl(2)$ of the Lie algebra $sl(2)$; their quantum analogs and natural deformations $A_1(q), U_qsl(2), \Lambda(b)$; the quantum plane Λ; the quantum Heisenberg algebra $\mathcal{H}_q$.

For some of these algebras we prove their tensor d-minimality, where d=wd or lgd (see Corollary 1.5, below) and using (1.1), in many cases calculate their homological dimension.

In sec. 3 *filter-, finite-, factor-* and *subsplit* algebras and modules are introduced (for the purpose of convenience all the definitions are given at the begining of sec. 3) and their relations are examined.

If a ring R is commutative, then wd $R=\sup\{\,\mathrm{wd}\,R_{\mathbf{m}}\mid \mathbf{m}\in$ Specm R, the set of maximal ideals of $R\}$, where $R_{\mathbf{m}}$ is the localization of R at m. J.-E. Roos [R] generalized this result for some non-commutative rings. Namely, if for some ring A there exist "sufficiently many" localizations $\{A_\alpha \,|\, \alpha \in I\}$, i.e. rings A_α such that an A-module M equals to 0 if and only if $A_\alpha \otimes_A M = 0$ for all $\alpha \in I$, then wd $A = \sup\{\mathrm{wd}\, A_\alpha \mid \alpha \in I\}$ (see Theorem 4.7). Using this result J.-E. Roos [R] calculated the global dimension of the Weyl algebra A_n over a field of characteristic zero, gl.dim $A_n = n$ (recall that for a left Noetherian ring its left homological dimension equals to the weak dimension). On the contrary, the generalized Weyl algebras have no "much" natural localizations, so we can not apply Roos results. The next Theorem show how to calculate the weak dimension for this situation.

Theorem 1.2 *Suppose that $\{A_\alpha \to A \,|\, \alpha \in I\}$ is a set of flat extensions of a ring A such that wd $A < \infty$, $\nu = \sup\{\mathrm{wd}\, A_\alpha \,|\, \alpha \in I\} < \infty$, $\mu = \sup\{\mathrm{fd}_A M \,|\, M$ is cyclic A-module such that $A_\alpha \otimes_A M = 0$ for all $\alpha \in I\}$; the set I either finite or the ring A is right coherent. Then*

1. *either* wd $A = \mu$ *or* $\mu < \mathrm{wd}\, A \leq \nu$.
2. *If, additionally,* $\nu \leq \mathrm{wd}\, A$ *(for example, all the rings A_α are two-sided localizations of A), then* wd $A = \max\{\mu, \nu\}$. ■

The proof of Theorem 1.2 is given in sec. 4 where there are many helpful propositions which are used to compute the global dimension of generalized Weyl algebras.

In Sec. 5 the following Proposition is proved.

Proposition 1.3 *Suppose that d=wd (respectively, d=lgd). If an algebra (respectively, a left Noetherian algebra) A is finite-split (respectively, filter-split) with respect to some class of algebras Ω (respectively, $\Omega \subseteq \mathcal{LN}$), then A is tensor d-minimal with respect to Ω.* ■

In sec. 5 we give sufficient conditions for tensor d-minimality of algebras (Corollaries 5.3–5.6). In particular, we prove that the universal enveloping algebra $U(\mathcal{G})$ of the finite dimensional Lie algebra $\mathcal{G}$ over an algebraically closed field, the Weyl algebra A_n over a field of finite characteristic and the quantum Weyl algebra $A_n(q_1, \ldots, q_n)$ over an arbitrary field are tensor homological minimal with respect to $\mathcal{LN}$.

In sec. 6 the tensor homological minimality with respect to $\mathcal{LFN}$ of considered generalized Weyl algebras is proved by using the following Theorem.

Theorem 1.4 *Suppose that d=wd (respectively, d=lgd) and a tensor d-minimal with respect to Ω (respectively, $\Omega \subseteq \mathcal{LN}$) algebra $B = S^{-1}A$ is a localization of an algebra (respectively, a left Noetherian algebra) A (respectively, such that $A \otimes \mathcal{LN} \subseteq \mathcal{LN}$) at a multiplicatively closed subset S of A, $dB < \infty$, $\mathbf{a}$ is the set of all finitely generated S-torsion A-modules M, i.e. $S^{-1}M := B \otimes_A M = 0$. Then*

1. *if each cyclic module from $\mathbf{a}$ is finite-split with respect to Ω, then A is tensor d-minimal with respect to Ω;*

2. *in the case d=lgd suppose that each module from* $\mathbf{a}$ *is Artinian and any of the conditions of Proposition 3.2 hold, for example, A is Schurian, then A is tensor d-minimal with respect to* Ω. ∎

Using Theorem 1.4 and results of sec. 5 one can easily prove Corollary 1.5.

Corollary 1.5 *Let K be an algebraically closed uncountable field. Consider the following generalized Weyl algebras of degree 1 with non-zero defining element a.*

1. *char* $K = 0$, $K[H](\sigma, a)$, $\sigma(H) = H - \mu$, $\mu \neq 0 \in K$ *(if* $\sigma(H) = H - 1$ *we have the Weyl algebra* $A_1 \simeq K[H](\sigma, a = H)$ *and all prime quotients of* $Usl(2) : U(\lambda) = Usl(2)/(C - \lambda) \simeq K[H](\sigma, a = \lambda - H(H+1))$, *where C is the Casimir element);*

2. $K[H, (H - \mu/(1-\lambda))^{-1}](\sigma, a)$, $\sigma(H) = \lambda H + \mu$, $\lambda \neq 0, 1 \in K$ *is not a root of 1,* $\mu \in K$ *(if* $\sigma(H) = \lambda H$, $\lambda \neq 0, 1$, *then* $K[H, H^{-1}](\sigma, a = H)$ *is a localization of the quantum plane* $\Lambda = K < X, Y | XY = \lambda YX >$ *at the multiplicatively closed set* $S = \{(YX)^n, n = 0, 1, \ldots\}$*).*

Each tensor product $\Lambda = \otimes_{i=1}^n \Lambda_i$ *of these algebras is a tensor homological minimal algebra with respect to* $\mathcal{LFN}$ *, therefore*

$\mathrm{lgd}(\otimes_{i=1}^n \Lambda_i \otimes B) = \sum_{i=1}^n \mathrm{lgd}\Lambda_i + \mathrm{lgd}\, B$, for any left Noetherian finitely generated algebra B.

If K in not necessarily uncountable, then Λ, *where all algebras* Λ_i *from 1, is a tensor homological minimal algebra with respect to* $\mathcal{SC}$.

In many cases the considered algebras are the generalized Weyl algebras $A = D(\sigma, a)$ of degree 1 with Dedekind ring D (see definition in sec. 2). By Theorem 1.6 we can compute their global homological dimension.

Theorem 1.6 *Let* $A = D(\sigma, a)$ *be the generalized Weyl algebra of degree 1 with basic Dedekind ring D,* $Da = \mathbf{p}_1^{n_1} \ldots \mathbf{p}_n^{n_s}$ *is the product of the distinct maximal ideals of D. Then the global dimension of A is*

$$\mathrm{gl.dim}\, A = \begin{cases} \infty & \text{, if } a = 0 \text{ or } n_i \geq 2 \text{ for some } i; \\ 2 & \text{, if } a \neq 0,\ n_1 = \ldots = n_s = 1,\ s \geq 1 \text{ or } a \text{ is invertible and there exists} \\ & \quad \text{an integer } k \geq 1 \text{ such that either } \sigma^k(\mathbf{p}_i) = \mathbf{p}_j \text{ for some } i, j \\ & \quad \text{or } \sigma^k(\mathbf{q}) = \mathbf{q} \text{ for some maximal ideal } \mathbf{q} \text{ of } D; \\ 1 & \text{, otherwise.} \end{cases}$$ ∎

The proof of Theorem 1.6 is given in sec. 7. Using Corollary 1.5 and Theorem 1.6 the left homological dimension of the tensor product $\otimes_{i=1}^n \Lambda_i$ of algebras can be easily computed.

When $D = K[H]$ is the polynomial ring over an algebraically closed field K, char $K = 0$, and the automorphism $\sigma \in \mathrm{Aut}(D)$ acts as follows $\sigma(H) = H - 1$, Theorem 1.6 was proved in ([Hod 1] and [Bav 1,4]). The primitive factoralgebras $U(\lambda)$, $\lambda \in K$ (see Example

3 in sec. 2) of the universal enveloping algebra $Usl(2)$ of the Lie algebra $sl(2)$ are of this kind, where $a = \lambda - H(H+1)$. Theorem 1.6 for $U(\lambda)$ was proved in [St].

Since the Weyl algebra A_1 is the generalized Weyl algebra $A_1 = K[H](\sigma, a = H)$, $\sigma(H) = H - 1$, of degree 1 (see sec. 2, Example 1), then by Theorem 1.6 we get gl.dim $A_1 = 1$, if char $K = 0$; 2, if char $K \neq 0$, [Ri]. If the field K is algebraically closed, then by Corollary 1.5 A_1 is the tensor homological minimal algebra with respect to somewhat commutative algebras, hence, gl.dim$(A_n = \otimes_1^n A_1) = \sum_1^n$ gl.dim $A_1 = n$, if char $K = 0$, [Ro]; $2n$, if char $K \neq 0$, [Ri].

2 Generalized Weyl algebras, examples

Definition of the generalized Weyl algebras ([Bav 1-3]). Let D be a ring, $\sigma = (\sigma_1, ..., \sigma_n)$ a set of commuting automorphisms of D, ($\sigma_i\sigma_j = \sigma_j\sigma_i$, for all i, j), $a = (a_1, ..., a_n)$ be a set of (non-zero) elements of the centre $Z(D)$ of D, such that $\sigma_i(a_j) = a_j$ for all $i \neq j$

The *generalized Weyl algebra* $A = D(\sigma, a)$ (briefly GWA) of degree n with a *basic* ring D is a ring generated by D and the $2n$ indeterminates $X_1^+, ..., X_n^+, X_1^-, ..., X_n^-$ subject to the relations:

$$X_i^- X_i^+ = a_i, \qquad X_i^+ X_i^- = \sigma_i(a_i),$$

$$X_i^{\pm}\alpha = \sigma_i^{\pm 1}(\alpha)X_i^{\pm}, \forall\ \alpha \in D,$$

$$[X_i^-, X_j^-] = [X_i^+, X_j^+] = [X_i^+, X_j^-] = 0,\ \forall\ i \neq j,$$

where $[x, y] = xy - yx$. We say respectively that a and σ are the sets of *defining* elements and automorphisms of A. For any vector $k = (k_1, ..., k_n) \in \mathbf{Z}^n$ we put $v_k = v_{k_1}(1)...v_{k_n}(n)$ where for any $1 \leq i \leq n$ and $m \geq 0$ $v_{\pm m}(i) = (X_i^{\pm})^m$, $v_0(i) = 1$. In the case $n = 1$ we write v_m for $v_m(1)$. It follows from the definition of the GWA that

$$A = \oplus_{k \in \mathbf{Z}^n} A_k$$

is a $\mathbf{Z}^n$-graded algebra ($A_k A_e \subset A_{k+e}$, for every $k, e \in \mathbf{Z}^n$), where $A_k = Dv_k$.

The category of generalized Weyl algebras is closed under the tensor product (over a basic field) of algebras, $A \otimes A' = D \otimes D'(\sigma \cup \sigma', a \cup a')$.

Proposition 2.1 *([Bav 3,4]).Let A be a generalized Weyl algebra with basic ring D. Then*

1. *if D is left (right) Noetherian, then A is left (right) Noetherian;*
2. *if D has no zero divisors and $a_i \neq 0$, $\forall\ i = 1, ..., n$, then A has no zero divisors.* ∎

The Weyl algebra A_n. Define the n-th *Weyl algebra*, $A_n = A_n(K)$, over a field (a ring) K to be the associative K-algebra with identity generated by the $2n$ indeterminates X_1, ..., X_n, $\partial_1, ..., \partial_n$, subject to the relations:

$$[X_i, X_j] = [\partial_i, \partial_j] = [\partial_i, X_j] = 0 \text{ for } i \neq j,\ [\partial_i, X_i] = 1 \text{ for all } i.$$

The Weyl algebra A_n is the generalized Weyl algebra $A = D(\sigma; a)$ of degree n with a basic polynomial ring $D = K[H_1, ..., H_n]$ in n variables with coefficients in K. The sets of defining elements and automorphisms of A are $\{a_i = H_i | 1 \leq i \leq n\}$ and $\{\sigma_i | \sigma_i(H_j) = H_j - \delta_{ij}\}$, respectively, where $\delta_{ij} = 1$, if $i = j$; and 0, otherwise. Moreover, the map

$$A_n \to A,\ X_i \to X_i^+,\ \partial_i \to X_i^-,\ \partial_i X_i \to H_i,\ i = 1, \ldots, n,$$

is an isomorphism of the algebras.

Examples of the generalized Weyl algebras of degree 1. Let $A = D(\sigma, a)$ be a generalized Weyl algebra of degree 1, $a \in Z(D)$, $\sigma \in \mathrm{Aut}(D)$.

The ring A is generated by D and the 2 indeterminates $X = X_1^+$ and $Y = X_1^-$ subject to the relations:

$$X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y,\ \forall \alpha \in D,\ YX = a \text{ and } XY = \sigma(a) \qquad (2.1)$$

The algebra $A = \oplus_{n \in \mathbf{Z}} A_n$ is $\mathbf{Z}$-graded, where $A_n = Dv_n$, $v_n = X^n$ $(n > 0)$, $v_n = Y^{-n}$ $(n < 0)$, $v_0 = 1$. It follows from the above relations that

$$v_n v_m = (n, m) v_{n+m} = v_{n+m} < n, m >$$

for some $(n, m) \in D$. If $n > 0$ and $m > 0$ then

$$n \geq m : \ (n, -m) = \sigma^n(a) ... \sigma^{n-m+1}(a),\ (-n, m) = \sigma^{-n+1}(a) ... \sigma^{-n+m}(a),$$

$$n \leq m : \quad (n, -m) = \sigma^n(a) ... \sigma(a),\ \ (-n, m) = \sigma^{-n+1}(a) ... a,$$

in other cases $(n, m) = 1$.

Let $A_{(i)} = D_i(\sigma_i, a_i)\,(i = 1, ..., n)$ be a GWA of degree 1 over some field K and assume that each σ_i is a K-automorphism, then their tensor product

$$\otimes_1^n A_{(i)} = (\otimes_1^n D_i)((\sigma_i), (a_i))$$

is a GWA of degree n over K.

This construction allows us to build a great deal of examples of generalized Weyl algebras of degree n. For example, the n-th Weyl algebra $A_n(K)$ can be written in this way as $A_n(K) = A_1 \otimes ... \otimes A_1$ (n times).

Example 1. $A = K[H](\sigma, a)$, where σ is an arbitrary automorphism of $K[H]$, i.e. $\sigma(H) = \lambda H + \mu$, $\lambda \neq 0, \mu \in K$, $a \in K[H]$. When $\sigma(H) = H - 1$, these algebras are studied in [Hod 1] and [Bav 4]. By putting $a = H$ we get the 1-st Weyl algebra $A_1 \simeq K[H](\sigma, a = H)$.

Let $F_m = A_1^G$ be the fixed ring where G is the cyclic group of order m, acting by multiplication $\partial \to \omega\partial$, $X \to \omega^{-1}X$, where ω is a primitive m'th root of unity. Then

$$F_m = K < \partial^m, \partial X, X^m > \simeq K[H](\sigma, a = m^m H(H - 1/m) \ldots (H - (m-1)/m),$$

$$\partial^m \leftrightarrow Y, X^m \leftrightarrow X, \partial X/m \leftrightarrow H,$$

is a GWA of degree 1, where $\sigma(H) = H - 1$, char $K = 0$, [Hod 1].

Case, $\mu = 0$. Let $\Lambda = K < X, Y | XY = \lambda YX >$ (*the quantum plane*), then $\Lambda \simeq K[H](\sigma, a = H)$.

Let $A(S^2_\lambda) = K < X, Y, H \mid XH = \lambda HX,\ YH = \lambda^{-1}HY,\ \lambda^{1/2}YX = -(c-H)(d+H),\ \lambda^{-1/2}XY = -(c-\lambda H)(d+\lambda H) >$ be the algebra of functions on the quantum 2-dimensional sphere, then $A(S^2_\lambda) \simeq K[H](\sigma, a = -\lambda^{-1/2}(c-H)(d+H))$.

The *quantum Weyl* algebra $A_1(q) =< x, \partial | \partial x - qx\partial = 1 >$ of degree 1 over K ($q \neq 0 \in K$) is the GWA $K[H](\sigma, a = H)$ of degree 1 where $\sigma(H) = q^{-1}(H-1)$.

Example 2. $A = D(\sigma, a)$, where $D = K[H, (H - \mu/(1-\lambda))^{-1}], \sigma(H) = \lambda H + \mu, \lambda \neq 0, 1, \mu \in K$, $a \neq 0 \in D$. In particular, when $\mu = 0$ we have $A = K[H, H^{-1}](\sigma, a)$, $\sigma(H) = \lambda H$.

Example 3. Consider the K-algebra $\Lambda(b)$, deformation of $Usl(2)$, ([Sm], [Bav 1]). It is generated by X, Y, Z subject to the relations: $[H, X] = X$, $[H, Y] = -Y$, $[X, Y] = b \neq 0 \in K[H]$. Then $\Lambda(b) \cong K[H, C](\sigma, a = C - \alpha)$ where $\sigma : H \to H - 1, C \to C$ and $\alpha \in K[H]$ is a solution of the equation $\alpha - \sigma(\alpha) = b$, [Bav 1]. For $b = 2H$, $\Lambda(b) = Usl(2)$. If K is a field of characteristic zero, then the center of $\Lambda(b)$ is $K[C]$. For any $\lambda \in K$ the factor algebra $\Lambda(b, \lambda) := \Lambda(b)/\Lambda(b)(C-\lambda)$ is isomorphic to the GWA from Example 1 with the defining element $\lambda - \alpha$. Let $U(\lambda) = Usl(2)/Usl(2)(C-\lambda) \simeq K[H](\sigma, \lambda - H(H+1))$ be the infinite dimensional primitive factor of $Usl(2)$ (see [Bav 4]).

Example 4. The *quantum Heisenberg* algebra [KS], [Ma], [Ros]:

$$\mathcal{H}_q = K < X, Y, H \mid XH = q^2HX,\ YH = q^{-2}HY,\ XY - q^{-2}YX = q^{-1}H >,\ q \in K,\ q^4 \neq 1$$

is isomorphic to the GWA of degree 1:

$$\mathcal{H}_q \simeq k[H, C](\sigma; a = \rho^{-1}C - \mu H = q^2(C - H/q(1-q^4)),\ \sigma(H) = q^2H,\ \sigma(C) = q^{-2}C.$$

The element $\Omega = HC$ belongs to the centre of $\mathcal{H}_q$, [AD]. For each $\lambda \neq 0 \in K$ the factoralgebra $\mathcal{H}_q(\lambda) := \mathcal{H}_q/(\Omega - \lambda)$ is the GWA of degree 1:

$$\mathcal{H}_q(\lambda) \simeq K[H, H^{-1}](\sigma; a = q^2(\lambda H^{-1} - H/q(1-q^4)),\ \sigma(H) = q^2H.$$

For other examples the reader is referred to [Bav 1–4].

3 Factor-,sub- and filter-split algebras and modules

Let Ω be some class of algebras, usually, $\Omega = \mathcal{LN}$, $\mathcal{LFN}$, $\mathcal{SC}$ or *Alg*. Let $(I, \leq)$ be a well-ordered set and A be an algebra, an A-module M is *I-ordered* if $M = \cup_{i \in I} M_i$ is a union of A-submodules such that $M_i \subseteq M_j$, if $i \subseteq j$. Put $M_{<i} := \cup_{j<i} M_j$, then $M_{<i} \subseteq M_i$ for each i. Denote by $\hat{A}$ the set of isoclasses of simple A-modules.

Definitions.

- An A-module M is *filter-split* (respectively, *finite-split*) with respect to Ω iff for each $C \in \Omega$ every factormodule L of an $A \otimes C$-module $M \otimes C$ is I-ordered $L = \cup_{i \in I} L_i$ for some (respectively, finite) well-ordered set $I = I(L)$ and for each $i \in I$ the $A \otimes C$-module $L_i/L_{<i} \simeq M_i \otimes N_i$ for some ${}_AM_i$ and ${}_CN_i$.

- An algebra A is *filter-split* (respectively, *finite-split*) with respect to Ω iff the A-module ${}_AA$ is so.

- A simple A-module M is *factorsplit* (respectively, *subsplit*) with respect to Ω iff for each $C \in \Omega$ every factormodule (respectively, submodule) of the $A \otimes C$-module $M \otimes C$ is isomorphic (respectively, equals) to $M \otimes N$ for some C-module N (respectively, for some left ideal N of C).

- An algebra A is *factorsplit* (respectively, *subsplit*) with respect to Ω iff each simple A-module is factorsplit (respectively, subsplit) with respect to Ω.

- An algebra A is *Schurian* iff each simple A-module M is *Schurian*, i.e. $\mathrm{End}_A(M) = K$.

- An algebra A is *tensor-simple* iff the tensor product $M \otimes N$ of simple $A-$ and B-modules M and N respectively, is a simple $A \otimes B$-module for each algebra B, each simple B-module N and each simple A-module M.

- An A-module M is A-*kernel* (briefly, *kernel*) if for each $m \neq 0 \in M$ exists a subset J of A such that $\cap_{j \in J} \mathrm{Ker}\, j_M = Km$, where j_M is the K-linear map $M \to M, x \to jx$.

  A semisimple module is *strong semisimple* if it is a sum of non-isomorphic simple submodules (then this sum is direct and unique).

- An A-module M is *semikernel* iff there exists a subalgebra D such that ${}_DM = \oplus_{i \in I} M_i$ is a strong semisimple module with D-kernel (non-isomorphic) simple components ${}_DM_i$.

Theorem 3.0 [Bav 7]. *Let A be an algebra. Then A is Schurian implies subsplit with respect to Alg implies factorsplit with respect to Alg implies tensor-simple.* ∎

Lemma 3.1 [Bav 7]. *1. Each subsplit module or algebra with respect to Ω is factorsplit with respect to Ω.*

2. If M is a non-zero semikernel A-module, then for any algebra C and any non-zero C-module N each non-zero $A \otimes C$-submodule L of $M \otimes N$ contains a non-zero element $m \otimes n$ for some $m \in M$, $n \in N$ (hence, $M \otimes N$ contains a non-zero submodule $M_1 \otimes N_1$, $M_1 = Am$, $N_1 = Cn$.)

3. Each simple semikernel A-module M is subsplit (hence, factorsplit) with respect to Alg.

4. A simple module is Schurian iff it is semikernel. So, an algebra A is Schurian iff each simple A-module is semikernel. ∎

We say that a set $\mathbf{a}$ of A-modules is *closed* under submodules and factormodules if for any exact sequence of A-modules $0 \to N \to M \to L \to 0$, $M \in \mathbf{a}$ if and only if $N, L \in \mathbf{a}$.

Proposition 3.2 *Let $\mathbf{a}$ be a set of Artinian A-modules closed under sub- and factormodules. Consider the following statements:*

1. *each A-module $M \in \mathbf{a}$ is filter-split with respect to Ω;*

2. *each simple A-module from* $\mathbf{a}$ *is factorsplit with respect to* Ω*;*

3. *each simple A-module from* $\mathbf{a}$ *is subsplit with respect to* Ω*;*

4. *each simple A-module from* $\mathbf{a}$ *is semikernel (i.e. Schurian);*

5. *the algebra A is Schurian.*

Then $1 \Leftarrow 2 \Leftarrow 3 \Leftarrow 4 \Leftarrow 5$.

Proof. $1 \Leftarrow 2$. Let M be as in 1. Since the module M is Artinian and the set $\mathbf{a}$ is closed under sub- and factormodules, $M = \cup_{i\in I} M_i$ is I-ordered for some well-ordered set I such that the A-module $\bar{M}_i = M_i/M_{<i} \in \mathbf{a}$ is simple for all $i \in I$. If $C \in \Omega$, then $A\otimes C$-module $L = M \otimes C$ is I-ordered, $L = \cup_{i\in I}(L_i = M_i \otimes C)$ and $L_i/L_{<i} \simeq \bar{M}_i \otimes C$ for all $i \in I$. Let $\tilde{L} = L/J$ be a factormodule of L for some submodule J of L, then $\tilde{L} = \cup_{i\in I}(\tilde{L}_i = L_i + J)$ is I-ordered and for each $i \in I$ $A\otimes C$-module $\tilde{L}_i/\tilde{L}_{<i}$ is a factormodule of $L_i/L_{<i}$. Since A-module $\bar{M}_i$ is factorsplit with respect to Ω and $L_i/L_{<i} \simeq \bar{M}_i \otimes C$, hence $\tilde{L}_i/\tilde{L}_{<i} \simeq \bar{M}_i \otimes N_i$ for some ${}_C N_i$.

$2 \Leftarrow 3 \Leftarrow 4 \Leftarrow 5$ follow from Lemma 3.1. ∎

Corollary 3.3 *Suppose that an algebra A is Artinian and Ω is a class of algebras. Then A is Schurian implies A is subsplit with respect to Ω which implies A is factorsplit with respect to Ω which implies A is filter-split with respect to Ω.*

Proof. This follows from Proposition 3.2 where $\mathbf{a}$ is the set of all factors of ${}_A A$, i.e. the set of all factormodules M/N where M and N are submodules of ${}_A A$ such that $N \subseteq M$. ∎

Corollary 3.4 *Let A be an algebra and* $\mathbf{a}$ *be a set of A-modules closed under sub- and factormodules. Consider the following statements*

1. *each cyclic A-module $M \in$*$\mathbf{a}$ *is finite-split with respect to* Ω;

2. *each cyclic A-module $M \in$*$\mathbf{a}$ *has finite length and each simple A-module from* $\mathbf{a}$ *is factorsplit (or subsplit) with respect to* Ω;

3. *each cyclic A-module $M \in$*$\mathbf{a}$ *has finite length and the algebra A is Schurian.*

Then $1 \Leftarrow 2 \Leftarrow 3$.

The proof is the same as the proof of "$1 \Leftarrow 2$" in Proposition 3.2 but in our case the set I is finite. ∎

Let A be an associative K-algebra endowed with an increasing filtration $A_0 \subset A_1 \subset \dots$ by K-subspaces such that $1 \in A_0$, $A_n A_m \subset A_{n+m}$, $A = \cup A_n$, $gr\, A = \oplus A_n/A_{n-1}$ is a finitely generated commutative K-algebra. Then the algebra A is called *somewhat commutative* and the set of all such algebras we denote by $\mathcal{SC}$, [MR].

Theorem 3.5 *If K is algebraically closed (and uncountable), then each somewhat commutative (respectively, finitely generated) algebra is Schurian, [Qu] ([Di]).* ∎

4 The generalization of Roos Theorem, the weak dimension of rings

One of the goals of the present paper is to compute the left homological dimension of generalized Weyl algebras (see sec. 7, 8) and to prove their tensor d-minimality. All the considered algebras being Noetherian their left (right) and weak homological dimensions are equal. So, in this section we deal with weak dimension. The central result of this section is Theorem 1.2, which generalized Roos Theorem 4.7 [R].

At the begining of sec. 4 some helpfull elementary results are given, most of them are well known. But for the reason of completeness we give their proofs.

Let A be a ring, denote by $\mathrm{pd}_A\, M$ and $\mathrm{fd}_A\, M$ the projective and flat dimension of a module M, respectively. If $\mathbf{a}$ is a set of A-modules, then $\mathrm{pd}_A\, \mathbf{a} = \sup\{\mathrm{pd}_A\, M \mid M \in \mathbf{a}\}$ and $\mathrm{fd}_A\, \mathbf{a} = \sup\{\mathrm{fd}_A\, M \mid M \in \mathbf{a}\}$ are called the *projective* and *flat* dimension of $\mathbf{a}$. Stress that $\mathrm{lgd}\ A = \sup\{\mathrm{pd}_A\, M \mid {}_AM \text{ is cyclic}\}$ and $\mathrm{wd}\ A = \sup\{\mathrm{fd}_A\, M \mid {}_AM \text{ is cyclic }\}$.

Proposition 4.1 *([Aus]).Let M be a left A-module, I a nonempty well-ordered set and $\{M_i\}_{i\in I}$ a family of submodules of M such that if $i, j \in I$ and $i \leq j$, then $M_i \leq M_j$. If $M = \cup_{i\in I} M_i$ and $pd_A(M_i/M_j) \leq n$ for all $i \in I$ where $M_{<i} = \cup_{j<i} M_j$, then $pd_A M \leq n$.* ∎

We say that a set of A-modules $\mathbf{a}$ is *closed* under submodules and factormodules if for any exact sequence of A-modules $0 \to N \to M \to L \to 0$, $M \in \mathbf{a}$ if and only if $N, L \in \mathbf{a}$.

Lemma 4.2 *If a set of A-modules* $\mathbf{a}$ *is closed under sub- and factormodules then* $\mathrm{pd}_A\, \mathbf{a} = \sup\{\mathrm{pd}_A\, M \mid {}_AM \text{ is cyclic}\}$ *and* $\mathrm{fd}_A\, \mathbf{a} = \sup\{\mathrm{fd}_A\, M \mid {}_AM \text{ is cyclic}\}$.

Proof. In the case of projective dimension it is a consequence of Proposition 4.1. Since every module M is the direct limit of its finitely generated submodules, $M = \lim_{\to} M_i$, and $\mathrm{fd}_A(M = \lim_{\to} M_i) \leq \sup\ \mathrm{fd}_A\, M_i$, so the rest is clear, since $\mathbf{a}$ is closed under submodules. ∎

Any ring homomorphism $A \to B$ is called an *extension* of A. An extension $A \to B$ is *left (right) flat* , if ${}_AB$ (B_A) is flat. An extension is flat if it is both left and right flat. In the same way one can define a (left, right) projective extension. Every (left, right) projective extension is (left, right) flat.

Lemma 4.3 *1. If $A \to B$ is a right flat extension, then* $\mathrm{fd}_B(B \otimes {}_AM) \leq \mathrm{fd}_A\, M$ *for each ${}_AM$.*

2. If $A \to B$ is a left flat extension, then $\mathrm{fd}_A N \leq \mathrm{fd}_B N$ *for each ${}_BN$.*

3. If $A \to B$ is a flat extension, then

a) $\mathrm{fd}_B(B \otimes {}_AN) \leq \mathrm{fd}_A N \leq \mathrm{fd}_B N$ for every ${}_BN$;

b) $\mathrm{fd}_A(B \otimes {}_AM) \leq \mathrm{fd}_B(B \otimes {}_AM) \leq \mathrm{fd}_A M$ for every ${}_AM$.

Proof. 1. Let $\ldots \to P_1 \to P_0 \to {}_AM \to 0$ be a flat resolution of ${}_AM$. Since B_A is flat, then $\ldots \to B \otimes_A P_1 \to B \otimes_A P_0 \to B \otimes_A M \to 0$ is a flat resolution of ${}_B(B \otimes_A M)$, hence $\mathrm{fd}_B(B \otimes_A M) \leq \mathrm{fd}_A M$.

2. Let L be a right A-module and $\ldots \to Q_1 \to Q_0 \to L_A \to 0$ its flat resolution. Since ${}_AB$ is flat we get the flat resolution $\ldots \to Q_1 \otimes_A B \to Q_0 \otimes_A B \to L \otimes_A B \to 0$ of the right B-module $L \otimes_A B$. The abelian groups $(Q_i \otimes_A B) \otimes_B N \simeq Q_i \otimes_A N$ are isomorphic , so $\mathrm{Tor}_i^A(L, N) \simeq \mathrm{Tor}_i^B(L \otimes_A B, N)$ and $\mathrm{fd}_A N \leq \mathrm{fd}_B N$.

3 follows from 1 and 2. ∎

Definition. An extension $A \to B$ is called *cochosen* (respectively, *chosen*) iff for any ${}_BM$ (respectively, ${}_AM$) exists ${}_BN$ (respectively, ${}_AN$) such that the B-modules (respectively, A-modules) $B \otimes_A M$ and $M \oplus N$ are isomorphic.

If a subring A of B is a direct summand of B as a A-bimodule, i.e. ${}_AB_A = A \oplus C$ for some C, then the extension $A \to B$ is chosen.

Corollary 4.4 *1. If $A \to B$ is a right flat cochosen extension, then for each ${}_BM$ $\mathrm{fd}_B M \leq \mathrm{fd}_A M$ and, hence, wd $B \leq$wd A.*

2. If $A \to B$ is a left flat chosen extension, then for each ${}_AM$, $\mathrm{fd}_A M \leq \mathrm{fd}_B(B \otimes_A M)$ and, hence, wd $A \leq$wd B.

3. If $A \to B$ is a flat cochosen (respectively, chosen) extension, then all the inequalities of Lemma 4.3 (a) (respectively, (b)) become the equalities.

4. If $A \to B$ is flat chosen and cochosen extension, then wd $A =$wd B.

Proof. 1. Since the extension $A \to B$ is cochosen , then the B-modules $B \otimes_A M$ and $M \oplus N$ are isomorphic for some ${}_BN$, hence, $\mathrm{fd}_B M \leq \mathrm{fd}_B(B \otimes_A M)$. Since B_A is flat, then by Lemma 1.3.(1) $\mathrm{fd}_B(B \otimes_A M) \leq \mathrm{fd}_A M$ and we get $\mathrm{fd}_B M \leq \mathrm{fd}_A M$.

2. Since the extension $A \to B$ is chosen, then the A-modules $B \otimes_A M$ and $M \oplus N$ are isomorphic for some ${}_AN$, hence, $\mathrm{fd}_A M \leq \mathrm{fd}_A(B \otimes_A M)$. Since ${}_AB$ is flat, then by Lemma 1.3.(2) $\mathrm{fd}_A(B \otimes_A M) \leq \mathrm{fd}_B(B \otimes_A M)$, so $\mathrm{fd}_A M \leq \mathrm{fd}_B(B \otimes_A M)$.

3. The proof follows from the above inequalities (in 1 and 2).

4 is clear (see 1, 2). ∎

Let a ring $B = S^{-1}A$ be a (two-sided) localization (in the sense of Ore) of a ring A at the multiplicatively closed subset S of A. Then $A \to B$ is a flat cochosen extension, hence, by Corollary 4.4.(1) wd $B \leq$ wd A (a well known fact).

Let A and B be algebras over a field K, then the extension $A \to A \otimes B$, $a \to a \otimes 1$, is flat chosen, hence, by Corollary 4.4.(2) wd $A \leq$wd $(A \otimes B)$.

Let A be a ring and $F(A)$ be one of the following rings: a ring of skew polynomials $A[X;\sigma]$, a skew Laurent polynomial ring $A[X, X^{-1};\sigma]$, a skew power series $A[[X;\sigma]]$, a skew Laurent power series $A[[X, X^{-1};\sigma]]$, a generalized Weyl algebra $A(\sigma, a)$ of any degree (see [MR], sec. 2 for definitions), then the extension $A \to F(A)$ is flat chosen, hence, by Corollary 4.4.(2) wd $A \leq$wd $F(A)$.

Let $M_n(A)$ be a matrix $n \times n$ ring with coefficients in A. Then the extension $A \to M_n(A)$ is flat chosen and cochosen, hence, by Corollary 4.4.(4) wd $M_n(A)$ =wd A.

A ring A is *right coherent* if every direct product of left flat A-modules is also a left flat A-module. Every left Noetherian ring is right coherent (see [KE]).

Proof of Theorem 1.2 (see the Introduction).

1. It is evident that $n :=$wd $A \geq \mu$.

Given A-module M for each $\alpha \in I$ put $M_\alpha := A_\alpha \otimes {}_A M$. Denote by $\mathbf{a}(M)$, $\mathbf{b}(M)$ and $\mathbf{c}(M)$, respectively, the kernel, image and cokernel of the canonical A-homomorphism $M \to \prod_{\alpha\in I} M_\alpha$, then following sequences of A-modules are exact

$$0 \to \mathbf{a}(M) \to M \to \mathbf{b}(M) \to 0, \tag{4.1}$$

$$0 \to \mathbf{b}(M) \to \prod_{\alpha\in I} M_\alpha \to \mathbf{c}(M) \to 0. \tag{4.2}$$

Let $\mathbf{A} := \{{}_A M$ such that $M_\alpha = 0$ for all $\alpha \in I\}$ and $\mathbf{B} := \{{}_A M$ such that the map $M \to \prod_{\alpha\in I} M_\alpha$ is injective $\}$. Then $\mathbf{a}(M) \in \mathbf{A}$.

Let us prove that $\mathbf{b}(M) \in \mathbf{B}$. Tensoring (4.1) by the right flat A-module A_α we get the exact sequence of A_α-modules $0 \to \mathbf{a}(M)_\alpha \to M_\alpha \to \mathbf{b}(M)_\alpha \to 0$. Since $\mathbf{a}(M)_\alpha = 0$, the A_α-modules M_α and $\mathbf{b}(M)_\alpha$ are isomorphic for each $\alpha \in I$. Suppose that an element $\bar{m} = m + \mathbf{a}(M) \in \mathbf{b}(M)$ belongs to the kernel of the homomorphism $\mathbf{b}(M) \to \prod_{\alpha\in I} \mathbf{b}(M)_\alpha \simeq \prod_{\alpha\in I} M_\alpha$, then m belongs to the kernel of the homomorphism $M \to \prod_{\alpha\in I} M_\alpha$, i.e. $m \in \mathbf{a}(M)$ and $\bar{m} = 0$, it is equivalent to the inclusion $\mathbf{b}(M) \in \mathbf{B}$.

$\mathbf{A}$ is closed under submodules and factormodules. In fact, if $0 \to M \to N \to L \to 0$ is an exact sequence of A-modules, then tensoring it by the flat module $(A_\alpha)_A$ we have the exact sequence $0 \to M_\alpha \to N_\alpha \to L_\alpha \to 0$. If $N \in \mathbf{A}$, then $N_\alpha = 0$, hence, $M_\alpha = N_\alpha = 0$ for each $\alpha \in I$, i.e. $M, N \in \mathbf{A}$.

Then by Lemma 4.2, $\mathrm{pd}_A \mathbf{A} = \mu$. It follows from (4.1) that $\mathrm{fd}_A M \leq \max\{\mathrm{fd}_A \mathbf{a}(M), \mathrm{fd}_A \mathbf{b}(M)\}$, then

$$n = \max\{\mu, \lambda\}, \tag{4.3}$$

where $\lambda = \sup\{\mathrm{fd}_A M \mid M \in \mathbf{B}\}$. Since ${}_A(A_\alpha)$ is flat for each $\alpha \in I$, either the set I is finite or the ring A is right coherent, then By Lemma 4.3.(2) for any A-module M exists a chain of the following inequalities

$$\mathrm{fd}_A(\prod_{\alpha\in I} M_\alpha) \leq \sup\{\mathrm{fd}_A(M_\alpha) \mid \alpha \in I\} \leq \sup\{\mathrm{fd}_{A_\alpha}(M_\alpha)\} \leq \nu. \tag{4.4}$$

If $n = \mu$, there is nothing to prove. Consider the remaining case, i.e. $n > \mu$, then by (4.3) we can choose a module $M \in \mathbf{B}$ such that $n = \mathrm{fd}_A M$. Prove that $n \leq \nu$. Suppose the contrary, $n > \nu$. Since $M \in \mathbf{B}$, i.e. $M = \mathbf{b}(M)$, then we have the exact sequence of A-modules (4.2): $0 \to M \to \prod_{\alpha\in I} M_\alpha \to \mathbf{c}(M) \to 0$. By (4.4) $\mathrm{fd}_A \mathbf{c}(M) = \mathrm{fd}_A M + 1 = n + 1$, but $\mathrm{fd}_A \mathbf{c}(M) \leq$wd $A = n < \infty$, a contradiction, hence, $n \leq \nu$.

2. It is a consequence of 1. ∎

Corollary 4.5 *Let a ring $A_\alpha = S_\alpha^{-1} A$, $\alpha \in I$, be a localization of the ring A and*

1. *wd $A < \infty$;*
2. *either the set I is finite or the ring A is right coherent (for example, A is left Noetherian);*

then wd $A = \max\{\nu, \mu\}$, where ν and μ are as in proof of Theorem 1.2.

Proof. Since the ring A_α is a localization of A, then wd $A_\alpha \leq$ wd A for each $\alpha \in I$, hence, $\nu \leq$ wd A. Now the proof follows from Theorem 1.2.(2). ∎

A set of extensions $\{A \to A_\alpha | \alpha \in I\}$ is *left exact* if ${}_AM = 0 \Leftrightarrow A_\alpha \otimes {}_AM = 0, \forall\, \alpha \in I$. Let R be a commutative ring and $R_{\mathbf{m}}$ is a localization of R at $\mathbf{m}$, where $\mathbf{m} \in$Specm (R) is a maximal ideal of R. Then $\{R \to R_{\mathbf{m}} | \mathbf{m} \in \text{Specm}\, R\}$ is the left exact set of extensions.

Corollary 4.6 *Suppose that conditions of Theorem 1.2.(2) hold and the set of extensions $\{A \to A_\alpha | \alpha \in I\}$ is left exact, then wd $A = \sup\{$ wd $A_\alpha | \alpha \in I\}$.* ∎

Theorem 1.2 is the generalization of Roos Theorem 4.7 [R], which for the readers convenience we formulate in the terminology of the present paper and for left modules.

Theorem 4.7 *([R]). Let A be a ring and $A_\alpha = S_\alpha^{-1}A$, $\alpha \in I$, is a left localization of the ring A. If*

1. *the set of extensions $\{A \to A_\alpha | \alpha \in I\}$ is a left exact;*
2. *a left A-module A_α is flat for each $\alpha \in I$;*
3. *wd $A < \infty$;*
4. *the set I is finite or the ring A is right coherent;*

then wd $A = \sup\{$ wd $A_\alpha \,|\, \alpha \in I\}$. ∎

For a given ring A usually it is difficult to say either wd $A < \infty$ or wd $A = \infty$. So the next Corollary is very usefull for applications since in its formulation there is no the finite condition wd $A < \infty$.

Corollary 4.8 *Let $B = S^{-1}A$ be a localization of a ring A at a multiplicatively closed subset S of A, $\nu =$ wd $B < \infty$. Then wd $A = \sup\{\mu, \nu\}$ where $\mu = \sup\{\mathrm{fd}_A N \,|\, {}_AN\, \text{is cyclic}, B \otimes {}_AN = 0\}$.*

Proof. Since wd $A \geq \mu$, it is enough to consider the case $\mu < \infty$. It is clear that B is the left and right flat A-module and except wd $A < \infty$, all the conditions of Theorem 1.2.(2) hold in the case of one extension $A \to B$.

Let us prove that wd $A < \infty$. Let M be an A-module and A-modules $\mathbf{a}(M)$, $\mathbf{b}(M)$, and $\mathbf{c}(M)$ be the same as in the proof of Theorem 1.2 in the case $A \to B$. Then $\mathbf{a}(M)$ and $\mathbf{c}(M)$ belong to $\mathbf{A} := \{{}_AN\, \text{such that}\, B \otimes {}_AN = 0\}$. Since ${}_AB$ is flat, by Lemma 4.3

$$\mathrm{fd}_A(B \otimes {}_AM) \leq \mathrm{fd}_B(B \otimes {}_AM) \leq \nu < \infty.$$

It follows from the sequences (4.1), (4.2) that $\mathrm{fd}_A M \leq \sup\{\mathrm{fd}_A(B \otimes {}_AM), \mathrm{fd}_A \mathbf{c}(M)\}$. Hence, wd $A \leq \max\{\mu, \nu\} < \infty$ and we complete the proof by using Theorem 1.2.(2). ∎

5 Constructions F of algebras, (d, F)-extremal algebras and their connections with tensor d-minimal algebras

Proof of Proposition 1.3.

For $C \in \Omega$ denote by Λ the tensor product of algebras $A \otimes C$. Then every cyclic Λ-module L is the factormodule of ${}_\Lambda\Lambda$.

Let d=wd (respectively, lgd). Since A is finite-split (respectively, filter-split) with respect to Ω, $L = \cup_{i\in I} L_i$ is an I-ordered Λ-module for some well-ordered set I such that $L_i/L_{<i} \simeq M_i \otimes N_i$, the tensor product of modules ${}_AM_i$, ${}_CN_i$. In the case d=wd the set I is finite, therefore

$$\mathrm{fd}_\Lambda L \leq \sup\{\mathrm{fd}_\Lambda(L_i/L_{<i}) = \mathrm{fd}_A M_i + \mathrm{fd}_C N_i,\, i \in I\} \leq \mathrm{wd}\, A + \mathrm{wd}\, C.$$

Since wd $\Lambda = \sup\{\mathrm{fd}_\Lambda L \mid L$ is cyclic Λ-module $\}$ and wd $\Lambda \geq$wd $A+$ wd C, we have wd $\Lambda =$wd $A+$ wd C.

If d=lgd, then by Proposition 4.1

$$\mathrm{pd}_\Lambda L \leq \sup\{\mathrm{pd}_\Lambda(L_i/L_{<i}) \leq \mathrm{pd}_A M_i + \mathrm{pd}_C N_i,\, i \in I\} \leq \mathrm{lgd}\, A + \mathrm{lgd}\, C.$$

Since lgd $\Lambda = \sup\{\mathrm{pd}_\Lambda L \mid L$ is a cyclic Λ-module $\}$ and lgd $\Lambda \geq$ lgd $A+$ lgd C, we have lgd $\Lambda =$lgd $A+$ lgd C. ∎

Definition. Suppose that d=wd (respectively, lgd, kd). An A-module M is *tensor* d-*restricted* with respect to Ω iff for each algebra $C \in \Omega$ any factormodule L of the tensor product $M \otimes C$ of the modules ${}_AM$ and ${}_CC$ has flat (respectively, projective, Krull) dimension less or equals to dA+dC, i.e.

$$\mathrm{fd}_{A\otimes C} L \leq \mathrm{wd}\, A + \mathrm{wd}\, C \;\; \text{(respectively, } \mathrm{pd}_{A\otimes C} L \leq \mathrm{lgd}\, A + \mathrm{lgd}\, C, \ldots).$$

It follows from the proof of Proposition 1.3 that for d=wd (respectively, d=lgd) *each finite-split (respectively, filter-split) with respect to Ω module is tensor* d-*restricted with respect to* Ω.

Given a set U denote, as usually, by 2^U the set of "all" subsets of U.

Definition. A map F from *Alg* (or from some subset of *Alg*) to 2^U, $A \to F[A]$, is called a *construction* (of algebras), i.e. the construction F for each algebra A takes into correspondence the set of algebras $F[A]$.

Given a construction F, denote by $F(A)$ a fixed algebra from the set $F[A]$, i.e. $F(A) \in F[A]$.

Examples of constructions (see [MR] for definitions).

1. $F_{sp}(A) = A[X; \sigma, \partial]$, a skew polynomial ring. In special cases we get $F_s(A) = A[X; \sigma, \partial = 0] = A[X; \sigma]$, and $F_{dr}(A) = A[X; \sigma = 1, \partial] = A[X; \partial]$, a differential ring.

2. Given a filtered algebra $A = \cup_{i\geq 0} A_i$ ($A_iA_j \subseteq A_{i+j}$ for all $i, j \geq 0$), $F_{gr}(A) =$gr $A := \oplus_{i\geq 0} A_i/A_{i-1}$, the associated graded algebra (we put $A_{-1} = 0$).

3. $F_{GWA}(A) = A(\sigma, a)$, a generalized Weyl algebra of an arbitrary degree.

4. $F_{sl}(A) = A[X, X^{-1}; \sigma]$, a ring of skew Laurent polynomials.
5. $F_{mat}(A) = M_n(A)$, a matrix ring with coefficients in A.
6. $F_{lps}(A) = A[[X, X^{-1}; \sigma]]$, a skew Laurent power series ring.
7. $F_{sps}(A) = A[[X; \sigma]]$, a skew power series ring.

We say that a construction F *commutes* with the tensor product $\otimes$ iff for any two algebras A and B the algebra $F(A) \otimes B$ is isomorphic to some algebra $F(A \otimes B)$ or equivalently, $F[A] \otimes B \subseteq F[A \otimes B]$.

A construction F *preserves* a property Q of algebras iff an algebra A has the property Q it follows that all the algebras $F(A)$ have so.

Lemma 5.1 *All the constructions above commute with the tensor product and preserve a left (right) Noetherianity.*

Proof. For F_{GWA} see Proposition 2.1. For other constructions see ([MR]). ∎

In this section we give sufficient conditions when a construction preserve the tensor d-minimality of algebras (Theorem 5.2) and we derive some usufull corollaries and apply them to the universal enveloping algebras, the (quantum) Weyl algebras and to other algebras.

Let the dimension d be one of the considered above: wd, lgd, kd (or any other). Then for a given construction F define the *upper* and *lower* d-indices:

$$\text{d.ind}\, F \;= \text{d.ind}^+\, F \;:= \inf\{i \in \mathbf{Z} \,|\, dF(A) \leq dA + i, \text{ for any algebra } A\},$$

$$\text{d.ind}^-\, F \;:= \sup\{i \in \mathbf{Z} \,|\, dF(A) \geq dA + i, \text{ for any algebra } A\}.$$

Mainly we deal with the upper d-index, so the word "upper" we usually don't write, i.e. the d-index means the upper one. It is evident that $\text{d.ind}^+ F \geq \text{d.ind}^- F$ for any construction F. Given two constructions F and G, then

$$\text{d.ind}\, F \circ G \leq \text{d.ind}\, F \;+\; \text{d.ind}\, G,$$

where their composition $F \circ G$ is defined in the following way

$$(F \circ G)(A) = F(G(A)), \text{ or equivalently, } (F \circ G)[A] = \cup_{G(A) \in G[A]} F[G(A)].$$

The difference $\triangle dF = \text{d.ind}^+ F - \text{d.ind}^- F$ is called the *d-defect* of the construction F. A construction F is *d-exact* if $\triangle dF = 0$, i.e. $dF(A) = dA + \text{d.ind}\, F$, for each algebra A and $F(A) \in F[A]$. The composition of d-exact constructions is also d-exact. The constructions $F_s(A) = A[X; \sigma]$ and $F_{sl}(A) = A[X, X^{-1}; \sigma = 1]$ are lgd-exact and their lgd-index is 1 ([MR], Theorem 5.3, p. 247). The matrix ring construction $F_{mat}(A) = M_n(A)$ is d-exact and $\text{d.ind}\, F_{mat} = 0$ where d=wd or d=lgd. An algebra Λ is TWM iff the construction $F_\Lambda(A) = \Lambda \otimes A$ is wd-exact, then $\text{wd.ind}\, F_\Lambda = \text{wd}\, \Lambda$.

Definition. An algebra A is (d^+, F)-*extremal* (respectively, *proper* (d^+, F)-*extremal*) iff

$$dF(A) = dA + \text{d.ind}^+\, F \text{ for some (respectively, for all) } F(A) \in F[A].$$

An algebra $F(A)$ is called the *partner* of A if the above equality holds, the set of all partners is denoted by $Par^{+}(A)$. In the same way, but changing $+$ to $-$, one can define the *(proper)* (d^{-}, F)*-extremal* algebras and $Par^{-}(A)$.

Remark. If we consider a class of algebras Ω "smaller" than Alg (for example, left Noetherian algebras $\mathcal{LN}$), then we can give the same definitions of d-indecies, but all the algebras belong to Ω. In this case we will stress which class is considered. For example, if $\Omega = \mathcal{LN}$, then the construction $F_{sps}(A) = A[[X;\sigma]]$ of a skew power ring is lgd-exact ([MR], Theorem 5.3, p. 247). An algebra Λ is THM with respect to $\mathcal{LN}$ iff the construction $F_{\Lambda}(A) = \Lambda \otimes A$ is lgd-exact in $\mathcal{LN}$, then lgd.ind $F_{\Lambda} =$ lgd Λ.

Theorem 5.2 *Suppose that d=wd, kd (respectively, lgd) and let a construction F commuting with tensor product. Then*

1. *If* d.ind$^{+} = F < \infty$ *and an algebra* A *is* (d^{+}, F)*-extremal and tensor* d*-minimal with respect to* Ω *(respectively,* $\Omega \subseteq \mathcal{LN}$*), then every* $F(A) \in Pat^{+}(A)$ *is a tensor* d*-minimal algebra with respect to* Ω.

2. *If* d.ind$^{-} = F < \infty$*, an algebra* A *is* (d^{-}, F)*-extremal and an algebra* $F(A) \in Pat^{-}(A)$ *is tensor* d*-minimal with respect to* Ω *(respectively,* $\Omega \subseteq \mathcal{LN}$*), then* A *is tensor* d*-minimal with respect to* Ω.

Proof. 1. Since F commutes with $\otimes$, for any algebra $B \in \Omega$ we have $dF(A) + dB \leq dF(A) \otimes B = dF(A \otimes B)$. Since A is a tensor d-minimal algebra with respect to Ω and $F(A) \in Par^{+}(A)$ we get $dF(A \otimes B) \leq d(A \otimes B) + \mathrm{d.ind}\, F = dA + dB + \mathrm{d.ind}\, F = dF(A) + dB$, so instead of all the inequalities we can write the equalities, hence, $F(A)$ is the tensor d-minimal algebra with respect to Ω.

2. Given an algebra $B \in \Omega$, since F commutes with $\otimes$ we can choose algebras $F(A) \otimes B$ and $F(A \otimes B)$ which are isomorphic. Then $dA + dB \leq d(A \otimes B) \leq dF(A \otimes B) - \mathrm{d.ind}^{-} F = dF(A) \otimes B - \mathrm{d.ind}^{-} F = dF(A) + dB - \mathrm{d.ind}^{-} F$ (since the algebra $F(A)$ is tensor d-minimal) $= dA + dB$ (since A is (d^{-}, F)-extremal, $F(A) \in Par^{-}(A)$), hence, $d(A \otimes B) = dA + dB$ for any $B \in \Omega$, i.e. A is tensor d-minimal. ∎

Corollary 5.3 *Let* d *and* F *be as in Theorem 5.2. If the construction* F *is* d*-exact and an algebra* A *is tensor* d*-minimal with respect to* Ω *(*$\Omega \subseteq \mathcal{LN}$ *when d=lgd), then every algebra* $F(A)$ *is so.*

Proof follows from Theorem 5.2 since for the d-exact construction F all the algebras are (d^{+}, F)-extremal. ∎

Denote by $ExCon(d)$ the monoid generated by all d-exact constructions. If A is a tensor d-minimal algebra with respect to Ω then $F(A)$ is so for any $F \in ExCon(d)$. Using this procedure one can build sophisticated tensor d-minimal algebras. For example, the algebra $B = M_n(K[X, X^{-1}])[Y;\sigma]$ is THM with respect to $\mathcal{LN}$ since $B = F_s \circ F_{mat} \circ F_{sl}(K)$, $F_s, F_{mat}, F_{sl} \in ExCon(\mathrm{lgd})$ and K is THM.

Corollary 5.4 *Suppose that d=wd (respectively, lgd) and let an algebra $B = S^{-1}A$ be a localization of an (respectively, left Noetherian) algebra A and assume that dA =dB. If A is the tensor d-minimal algebra with respect to Ω ($\Omega \subseteq \mathcal{LN}$), then B is so.*

Proof. Let C be an algebra and $F_{loc}(C) = T^{-1}C$ is the localization of C at some multiplicatively closed subset T of C. Then the construction F_{loc} commutes with $\otimes$ since $T^{-1}C \otimes D \simeq T^{-1}(C \otimes D)$ for any algebra D. For both considered dimensions d.ind$^+$ $F_{loc} = 0$. Since dA =dB the algebra A is (d^+, F_{loc})-extremal and B is the partner of A, so we complete the proof by using Theorem 5.2.(1). ∎

Corollary 5.5 *Let $A = \cup_{i \geq 0} A_i$ be a filtered left Noetherian algebra such that the associated graded algebra gr A is a tensor homological minimal with respect to $\Omega \subseteq \mathcal{LN}$, lgd A =lgd gr A. Then A is the tensor homological minimal with respect to Ω. Particularly, the universal enveloping algebra $U(\mathcal{G})$ of the finite dimensional Lie algebra $\mathcal{G}$ over an arbitrary field and the Weyl algebra A_n of the degree n over a field of finite characteristic are tensor homological minimal with respect to $\mathcal{LN}$.*

Proof. For any algebra B the tensor product $A \otimes B = \cup_{i \geq 0} A_i \otimes B$ is a filtered algebra such that associated graded algebra equals to gr$(A \otimes B) = \cup_{i \geq 0} A_i \otimes B / A_{i-1} \otimes B \simeq \oplus_{i \geq 0}(A_i/A_{i-1}) \otimes B =$ gr $A \otimes B$. It means that the construction $F_{gr}(A) =$ gr A commutes with the tensor product. By Corollary 6.18 ([MR], p. 259), lgd $A \leq$ lgd gr A for each filtered algebra A, i.e. lgd.ind$^- F_{gr} = 0$. Since lgd $A =$ lgd gr A the algebra A is (d^-, F)-extremal and $F_{gr}(A) =$ gr $A \in Par^-(A)$, so the proof follows from Theorem 5.2.(2).

The universal enveloping algebra $U(\mathcal{G})$ has the filtration such that gr $U(\mathcal{G}) = P_n$ is the polynomial ring in n =dim$_K\mathcal{G}$ variables. Since the polynomial ring P_n is THM and lgd $U(\mathcal{G}) = n$ =lgd P_n =lgd gr $U(\mathcal{G})$ the proof is clear.

The same sort of arguments we can apply to A_n. The Weyl algebra A_n of degree n (see definition in sec. 2) has the filtration such that gr $A_n = P_{2n}$. If char $K > 0$, then lgd $A_n = 2n$ (see [Ri])= lgd P_{2n} =lgd gr A_n. ∎

Remark. If char $K = 0$, then lgd $A_n = n < 2n$ =lgd gr A_n ([Ro]), so in this case Corollary 5.4 does not work, but the result is true for an algebraically closed uncountable field and for $\mathcal{LFN}$ (Corollary 1.5, the proof is in sec. 6).

If K is algebraically closed field of characteristic zero and $\mathcal{G}$ is a solvable Lie algebra, then the universal enveloping algebra $U(\mathcal{G})$ is THM with respect to *Alg* ([MR], Theorem 5.8, p. 250).

Let A be a ring. An element $x \in A$ is *regular* (respectively, *normal*) if it is not both a left and right zero divisor (respectively, $Ax = xA$.)

Corollary 5.6 *If a left Noetherian algebra A is tensor homological minimal with respect to $\Omega \subseteq \mathcal{LN}$ and $x \in A$ is normal regular non-unit element such that lgd A =lgd $A/Ax + 1$, then the factor-algebra A/Ax is tensor homological minimal with respect to Ω.*

Proof. If lgd $A/Ax = \infty$, then A/Ax is THM. Suppose that lgd $A/Ax < \infty$. Let $\mathcal{M}$ be the set of algebras with a fixed normal regular non-unit element $x \in A$ such that lgd

$A/Ax < \infty$. By Theorem 3.5 ([MR], p. 238) lgd $A/Ax \leq$lgd $A - 1$, so the construction $F_{nr} : \mathcal{M} \to 2^{Alg}$, $F_{nr}(A) = A/Ax$, has lgd.ind$^+ F_{nr} = -1$ and commutes with $\otimes$, since $A/Ax \otimes B \simeq A \otimes B/(A \otimes B)(x \otimes 1)$, $x \otimes 1 \in A \otimes B$ is normal regular non-unit. Since lgd $A =$lgd $A/Ax + 1$ the algebra A is $(lgd^+, F_{nr})-$extremal, then by Theorem 5.2.(1) the factor algebra A/Ax is tensor homological minimal with respect to Ω. ∎

Let K be a field, $A_1(q) = K < X, Y \mid XY - qYX = 1 >$, the *quantum Weyl algebra* of degree 1, $q \neq 0, 1 \in K$. By definition, the *the quantum Weyl algebra* $A_n(\bar{q} = (q_1, \ldots, q_n))$ of degree n is the tensor product $\otimes_{i=1}^n A_1(q_i)$ of the quantum Weyl algebra of degree 1. Evidently, the quantum Weyl algebra $A_1(q)$ is the skew polynomial ring $K[Y][X; \sigma, \partial]$ where $\sigma(Y) = qY$ and $\partial(Y) = 1$, thus $A_1(q)$ is the Noetherian ring.

Proposition 5.7 *The quantum Weyl algebra $A_n(\bar{q})$ is the tensor homological minimal algebra with respect to left Noetherian algebras and lgd $A_n(\bar{q}) = 2n$.*

Proof. Let us show that lgd $A_1(q) = 2$. Let R be any ring such that lgd $R < \infty$, then

$$\text{lgd}\, R \,\leq\, \text{lgd}\, R[X; \sigma, \partial] \,\leq\, \text{lgd}\, R + 1, \ ([MR], \text{Theorem } 5.3, \text{ p.} \, 247). \tag{5.1}$$

Thus lgd $A_1(q) =$ lgd $K[Y][X; \sigma, \partial] \leq$lgd $K[Y] + 1 = 2$. A straightforward calculation shows that $H = XY + (q-1)^{-1}$ is the normal regular non-unit element of $A_1(q)$ and the factoralgebra $A_1(q)/(H)$ is isomorphic to the Laurent polynomial ring $K[X, X^{-1}]$. Since lgd $K[X, X^{-1}] = 1 < \infty$ then by Theorem 3.5 ([MR], p. 235) $2 \leq \text{lgd}\, A_1(q)/(H) + 1 \leq$ lgd $A_1(q)$, thus lgd $A_1(q) = 2$.

Prove that $A_1(q)$ is THM with respect to $\mathcal{LN}$. Since $A_1(q) = K[Y][X; \sigma, \partial] = F_{sp}(K[Y]))$ and lgd $A_1(q) = 2$, by (5.1) we have lgd.ind $F_{sp} = 1$. So, the algebra $K[Y]$ is (lgd$^+, F_{sp}$)-extremal. The construction F_{sp} commutes with $\otimes$ and the polynomial ring $K[Y]$ is THM with respect to $\mathcal{LN}$, then by Theorem 5.2.(1) $F_{sp}(K[Y]) = A_1(q)$ is THM with respect to $\mathcal{LN}$.

The construction F_{sp} preserves the left Noetherianity, so $A_1(q) \otimes \mathcal{LN} \subseteq \mathcal{LN}$ and by Lemma 1.1, the n'th quantum Weyl algebra $A_n(\bar{q})$ is THM with respect to $\mathcal{LN}$ and lgd $A_n(\bar{q}) = \sum_{i=1}^n$ lgd $A_i(q_i) = 2n$. ∎

Lemma 5.8 *1. Let K be an algebraically closed uncountable (respectively, algebraically closed) field. Then the field of rational functions $K(H)$ in one variable is a tensor homological minimal algebra with respect to $\mathcal{LFN}$ (respectively, $\mathcal{SC}$).*

2. Suppose that a field L is a tensor homological minimal algebra with respect to $\mathcal{LFN}$ resp. to $\mathcal{SC}$ (for example, $L = K(H)$) and a construction F commutes with the tensor product, lgd.ind$^+$ $F = 1$, lgd $F(L) = 1$, (for example,$F(L) = L[X, X^{-1}; \sigma]$ or $F(L) = L[X; \sigma, \partial]$), then $F(L)$ is a tensor homological minimal algebra with respect to $\mathcal{LFN}$ resp. to $\mathcal{SC}$.

Proof. 1. It is enough to show that lgd $K(H) \otimes C \leq$ lgd C for any algebra $C \in \mathcal{LFN}$ (respectively, $\mathcal{SC}$) such that lgd $C < \infty$. An algebra $B = K(H) \otimes C$ is the localization of $A = K[H] \otimes C$ at the multiplicatively subset $S = K[H]\backslash\{0\}$, i.e. $B = S^{-1}A$, therefore $\mathrm{lgd}\, C \leq \mathrm{lgd}\, B \leq \mathrm{lgd}\, A = \mathrm{lgd}\, K[H] \otimes C = 1 + \mathrm{lgd}\, C < \infty$.

Since $C \in \mathcal{LFN}$ (respectively, $\mathcal{SC}$), then $A \in \mathcal{LFN}$ (respectively, $\mathcal{SC}$). By Theorem 3.5 each simple A-module equals to $M_\lambda = K_\lambda \otimes N$ for some $N \in \hat{C}$ and $K_\lambda = K[H]/(H-\lambda)$, $\lambda \in K$. Under the localization all the simple A-modules are vanished, i.e. $S^{-1}M_\lambda = 0$ for any $M_\lambda \in \hat{A}$, hence, by Theorem 4.4 ([MR], p. 245) lgd $B <$ lgd $A = 1+$ lgd C or, equivalently, lgd $B \leq$ lgd C.

2. Since lgd $F(L) = 1 =$ lgd L+lgd.ind F, L is (lgd^+, F)-extremal and THM with respect to $\mathcal{LFN}$ or $\mathcal{SC}$, $F(L)$ is the partner of L. Now the proof follows from Theorem 5.2.(1). ∎

Proposition 5.9 *1. Each Artinian Schurian algebra A is tensor weak (respectively, homological) minimal with respect to Alg (respectively, $\mathcal{LN}$).*

2. Each finite dimensional algebra A over an algebraically closed field is tensor weak (respectively, homological) minimal with respect to Alg (respectively, $\mathcal{LN}$).

Proof. 1. Let d=wd (respectively, d=lgd). Since A is Artinian and Schurian, by Corollary 3.4 (where $\mathbf{a}$ is the set of finitely generated A-modules) each finitely generated A-module is finite-split with respect to Alg. As we pointed out at the beginning of this section each finite-split with respect to $\Omega = Alg$ module (for example, ${}_AA$) is tensor d-restricted with respect to Alg. Therefore for each algebra C any cyclic $A \otimes C$-module has the flat (respectively, projective) dimension less or equals to dA+dC, hence, d$A \otimes C \leq$dA+dC. On the other hand, for each (respectively, left Noetherian) algebra C: d$A \otimes C \geq$dA+dC, i.e. d$A \otimes C =$dA+dC.

2.Each finite dimensional algebra over algebraically closed field is Artinian and Schurian. Now the proof follows from 1. ∎

6 Tensor d-minimality of generalized Weyl algebras and the Weyl algebra A_n

Proof of Theorem 1.4.

1. If a ring R is left Noetherian and an R-module M is finitely generated then wd $R =$lgd R and $\mathrm{fd}_R M = \mathrm{pd}_R M$. So we can consider the both cases simultaneously. Let d be wd or lgd.

Since $\mathrm{d}(A \otimes C) \geq$d$A$+d$C$ for each $C \in \Omega$, we may assume that d$A < \infty$, d$C < \infty$. Denote by Λ the tensor product $A \otimes C$ of algebras A and C, then for d=lgd the algebra Λ is left Noetherian ($A \otimes \Omega \subseteq \mathcal{LN}$). The algebra $R = B \otimes C = S^{-1}\Lambda$ is the localization of Λ at S. Since B is the tensor d-minimal algebra with respect to Ω, then d$R =$dB+dC. Let $\mu = \sup\{\mathrm{fd}_A N \,|\, N \,\mathrm{is\, cyclic}\, S-\mathrm{torsion}\ A-\mathrm{module}\}$ and $\mu_1 = \sup\{\mathrm{fd}_\Lambda L \,|\, L$ is cyclic S-torsion Λ-module $\}$. By Corollary 4.8

$$\mathrm{d}\Lambda = \max\{\mu_1, \mathrm{d}R\}.$$

Prove that $\mu_1 = \mu + \mathrm{d}C$. Let $\delta = \mathrm{fd}$ (respectively, $\delta = \mathrm{pd}$), then

$$\delta_{A\otimes C}(X \otimes Y) = \delta_A X + \delta_C Y$$

for any (respectively, left Noetherian) algebras A, C and any (respectively, finitely generated) modules ${}_AX, {}_CY$, so $\mu_1 \geq \mu + \mathrm{d}C$.

On the other hand, let L be a non-zero cyclic S-torsion Λ-module, then $L = \Lambda l$ for some $l \in \Lambda$ and $M := Al$ is cyclic S-torsion A-module such that map

$$M \otimes C \to L,\ al \otimes c \to acl, \text{ for all } a \in A, c \in C,$$

is an epimorphism of Λ-modules.

Since A-module M is finite-split with respect to Ω, $L = \cup_{i\in I} L_i$ is I-ordered Λ-module for some well-ordered finite set I such that $L_i/L_{<i} \simeq M_i \otimes N_i$, the tensor product of modules ${}_AM_i$, ${}_CN_i$. It is clear that for each $i \in I$ the A-module M_i is S-torsion, therefore

$$\delta_\Lambda L \leq \sup\{\delta_\Lambda(L_i/L_{<i}) = \delta_A M_i + \delta_C N_i,\ i \in I\} \leq \sup\{\mathrm{fd}_A M \mid {}_AM \text{ is } S-\text{torsion}\} + \mathrm{wd}\, C =$$

$\mu + \mathrm{d}\, C$, i.e. $\mu_1 \leq \mu + \mathrm{d}C$.

Finally, $\mathrm{d}(A\otimes C) = \mathrm{d}\Lambda = \max\{\mu_1, \mathrm{d}R\} = \max\{\mu + \mathrm{d}C, \mathrm{d}B + \mathrm{d}C\} = \max\{\mu, \mathrm{d}B\} + \mathrm{d}C = \mathrm{d}A + \mathrm{d}C$, where by Corollary 4.8 $\mathrm{d}A = \max\{\mu, \mathrm{d}B\}$.

2. This is a corollary of 1 and Proposition 3.2. ∎

An algebra A has the *(left) restricted minimum condition* if each proper factormodule of ${}_AA$ has the finite length.

Corollary 6.1 *1. Suppose that a construction of algebras F commutes with tensor product, lgd.ind F =lgd $F(L) = 1$ where $L = K(H)$ is the field of rational functions in one variables over an algebraically closed uncountable (respectively, algebraically closed) field K and $F(L) = S^{-1}A$ is a localization of a left Noetherian (respectively, somewhat commutative) algebra A at multiplicatively closed subset S of AW such that each cyclic S-torsion A-module is finite-split. Then A is tensor homological minimal with respect to $\mathcal{LFN}$ (respectively, $\mathcal{SC}$).*

2. If a Schurian left Noetherian (respectively, Schurian somewhat commutative) algebra A with restricted minimum condition contains a multiplicatively closed subset S such that the localization $S^{-1}A$ of A at S equals to the skew Laurent polynomial ring $L[X, X^{-1}, \sigma]$ or to $L[X; \sigma, \partial]$ then A is tensor homological minimal with respect to $\mathcal{LFN}$ (respectively, $\mathcal{SC}$).

Proof follows immediately from Lemma 5.8 and Theorem 1.4. ∎

Let D be a commutative ring and σ its automorphism. We say that σ *satisfies the condition*

(C) if $\sigma^i(\mathbf{p}) \neq \mathbf{p}$ for any maximal ideal $\mathbf{p}$ of D and any non-zero integer i.

Proposition 6.2 *,[Bav 2]. If $A = D(\sigma, a)$, $a \neq 0$, is the generalized Weyl algebra of degree 1 with Dedekind ring D and an automorphism σ which satisfies (C). Then A is the ring with restricted minimum condition.* ∎

Corollary 6.3 *Let A be as in Proposition 6.2 where $D = T^{-1}K[H]$ is a localization of the polynomial ring $K[H]$ in one variable at the multiplicatively closed subset T of $K[H]$, K is algebraically closed. Then*

1. If K is uncountable, then A is tensor homological minimal algebra with respect to $\mathcal{LFN}$.

2. If A is somewhat commutative, then A is tensor homological minimal with respect to $\mathcal{SC}$.

Proof. The localization $B = S^{-1}A$ of A at $S = K[H]\backslash 0$ is the skew Laurent polynomial ring $B = F_{sl}(L) = L[X, X^{-1}; \sigma]$. Now we complete the proof by using Corollaries 6.1.(2), 6.2 and Theorem 3.5. ▪

Proof of Corollary 1.5. All these algebras are finitely generated Noetherian and satisfied the assumptions of Corollary 6.3. ▪

7 The global dimension of the generalized Weyl algebras of degree 1 with basic Dedekind ring (proof of Theorem 1.6)

Let $A = D(\sigma, a)$ be the generalized Weyl algebra of degree 1 with basic Dedekind ring D, with any defining automorphism σ of D and with any defining element a. If $a \neq 0$, then the ideal $(a) = Da$ of D decomposes into the product $(a) = \mathbf{p}_1^{n_1} \dots \mathbf{p}_s^{n_s}$ of the distinct maximal ideals $\mathbf{p}_i$ of D, i.e. $\mathbf{p}_i \neq \mathbf{p}_j$, if $i \neq j$.

The purpose of this section is to prove Theorem 1.6. The main idea of the proof is to reduce this problem (by Lemma 7.7) in such a way to find the projective dimensions of the simple D-torsion A-modules (see Theorem 7.2, below).

Embeding in an Euclidean ring. Let A be as above and the defining element $a \neq 0$. Denote by k the field of fractions of D, i.e. $k = S^{-1}D$ is the localization of D $S = D\backslash\{0\}$. Then the localization $B = S^{-1}A$ of A at S is isomorphic to the ring of *skew Laurent* polynomials

$$B \simeq k[X, X^{-1}; \sigma] := \{\sum \alpha_i X_i, \alpha_i \in k, \text{ almost all } \alpha_i = 0\}, \text{ [MR]}.$$

The ring of Laurent polynomials is Euclidean with respect to the "length" function l given by $l(\alpha X^m + \dots + \beta X^n) = n - m$, where $\alpha, \beta \in k$ are not zero and $m < \dots < n$. Since each left (right) non-zero ideal J is principal, hence ${}_BJ \simeq B$ (B is a domain) and gl.dim $B = 1$.

If M is an A-module, then the set $\mathrm{tor}(M) = \{m \in M : sm = 0 \text{ for some } s \neq 0 \in D\}$ of D-torsion elements is the A-submodule , the *D-torsion* submodule. If Q is an isomorphism-invariant property of simple A-modules, then $\hat{A}(Q) = \{[M] \in \hat{A} : M$ has property $Q\}$.

Specm(D). The cyclic group G, generated by σ, acts in an obvious way on the set Specm(D) of maximal ideals of D. So, each orbit $\mathcal{O}$ equals to $\mathcal{O}(\mathbf{p}) = \{\sigma^i(\mathbf{p}), i \in \mathbf{Z}\}$ for some maximal ideal $\mathbf{p}$ of D. An orbit is *cyclic* of length n (respectively, *linear*) if it contains a finite (respectively, infinite) number $n = |\mathcal{O}|$ of elements. The set of cyclic (linear) orbits denote by Cyc (Lin). An orbit $\mathcal{O}$ is *degenerated*, if it contains an ideal $\mathbf{p}$ such that $a \in \mathbf{p}$ (such ideals are called *marked*). Denote by $Cycd$ and $Lind$ ($Cycn$ and $Linn$) the set of all (non-)degenerated cyclic and linear orbits respectively. The degenerated orbit which contains more than one marked ideal are called *strong degenerated* and we denote by $Cycds$ and $Linds$ the set of all such cyclic and linear orbits respectively.

Each linear orbit $\mathcal{O}(\mathbf{p})$, $\mathbf{p} \in \mathcal{O}$, by the map $\mathcal{O}(\mathbf{p}) \to \mathbf{Z}, \sigma^i(\mathbf{p}) \to i$, can be identified with the set of integers $\mathbf{Z}$. Therefore, for $\mathcal{O}(\mathbf{p}) \in Lin$ we can use, without mentioning it explicitly, all the definitions and notations which are employed for $\mathbf{Z}$ (such that the order, the segment, the semiaxis, etc.). For example, $\sigma^i(\mathbf{p}) \leq \sigma^j(\mathbf{p})$ iff $i \leq j$; $(-\infty, \mathbf{p}] = \{\sigma^i(\mathbf{p}), i \leq 0\}$. For a cyclic orbit $\mathcal{O}$ of the length n we use only the definition of the segment $(\mathbf{q}, \mathbf{p}] = \{\sigma^i(\mathbf{q}), 0 < i \leq j \leq n\}$ where j is the minimal positive integer such that $\sigma^j(\mathbf{q}) = \mathbf{p}\}$. Marked ideals $\mathbf{p}_1 < \ldots < \mathbf{p}_s$ (respectively, $\mathbf{p}_1$, $\mathbf{p}_2 = \sigma^{i(2)}(\mathbf{p}_1), \ldots, \mathbf{p}_s = \sigma^{i(s)}(\mathbf{p}_1)$, $i(2) < \ldots < i(s)$) of the generated linear (respectively, cyclic) orbit $\mathcal{O}$ divide it into $s+1$ (respectively, s) parts,

$$\Gamma_1 = (-\infty, \mathbf{p}_1],\ \Gamma_2 = (\mathbf{p}_1, \mathbf{p}_2], \ldots, \Gamma_{s+1} = (\mathbf{p}_s, \infty),$$

$$(\text{respectively, } \Gamma_1 = (\mathbf{p}_s, \mathbf{p}_1], \Gamma_2 = (\mathbf{p}_1, \mathbf{p}_2], \ldots, \Gamma_s = (\mathbf{p}_{s-1}, \mathbf{p}_s]).$$

We say that maximal ideals $\mathbf{p}$ and $\mathbf{q}$ are *equivalent* ($\mathbf{p} \sim \mathbf{q}$) if they belong either to a non-degenerated orbit or to some Γ_i.

Subrings $\Lambda_n, n \geq 1$. For each integer $n \geq 1$ denote by Λ_n the subring of A generated by D, X^n and Y^n, then

$$\Lambda_n = D(\sigma^n, (-n, n))$$

is a generalized Weyl algebra of degree 1. Let $\mathcal{O}$ be a cyclic orbit of the length n, then any ideal $\mathbf{p}$ of $\mathcal{O}$ generates the two-sided ideal $(\mathbf{p}) = \mathbf{p}\Lambda_n = \Lambda_n\mathbf{p}$ of Λ_n (since $\sigma^n(\mathbf{p}) = \mathbf{p}$) and the factorrings $\Lambda_n/(\mathbf{p}), \Lambda_n/(\mathbf{q})$ (and $D/\mathbf{p}, D/\mathbf{q}$) are isomorphic for all $\mathbf{p}, \mathbf{q} \in \mathcal{O}$. Denote by $\Lambda_{\mathcal{O}}$ and by $k_{\mathcal{O}}$ the isoclasses of rings $\Lambda_n/(\mathbf{p})$ and of $D/\mathbf{p}$, $\mathbf{p} \in \mathcal{O}$, respectively. Then $\Lambda_{\mathcal{O}}$ is the skew Laurent polynomial ring $\Lambda_{\mathcal{O}} = k_{\mathcal{O}}[t, t^{-1}; \sigma^n]$, if $\mathcal{O}$ is non-degenerated ($t = X^n + (\mathbf{p})$); $\Lambda_{\mathcal{O}} = k_{\mathcal{O}}(\sigma^n; 0) = k_{\mathcal{O}} < x, y >$, the GWA of degree 1, if $\mathcal{O}$ is degenerated ($x = X^n + (\mathbf{p}), y = Y^n + (\mathbf{p})$). In the last case $\Lambda_{\mathcal{O}}$ has two subrings $\Lambda_{\mathcal{O},+} = k_{\mathcal{O}} < x >= k_{\mathcal{O}}[x; \sigma^n]$ and $\Lambda_{\mathcal{O},-} = k_{\mathcal{O}} < y >= k_{\mathcal{O}}[y; \sigma^{-n}]$, each of them is a skew polynomial ring, moreover, is an Euclidean ring. Since two-sided ideals (x) and (y) annihilate each other, $(x)(y) = (y)(x) = 0$, then

$$\hat{\Lambda_{\mathcal{O}}} = \hat{\Lambda_{\mathcal{O},+}} \sqcup \hat{\Lambda_{\mathcal{O},-}} \sqcup [\Lambda_{\mathcal{O}}/(x, y) = k_{\mathcal{O}}] \text{ (the disjoint union)}$$

(see [Bav 2] for details). Define

$$\tilde{\Lambda_{\mathcal{O}}} = \hat{\Lambda_{\mathcal{O}}} \backslash [k_{\mathcal{O}}], \text{ if } \mathcal{O} \in Cycd;\ \hat{\Lambda_{\mathcal{O}}}, \text{ if } \mathcal{O} \in Cycn.$$

Weight A**- modules.** A-module V is *weight* if ${}_DV$ is semisimple, i.e.

$$V = \oplus_{\mathbf{p} \in \mathrm{Specm}(D)} V_{\mathbf{p}}$$

where $V_{\mathbf{p}} = \{v \in V : \mathbf{p}v = 0\} = \{$ the sum of simple D-submodules which are isomorphic to ${}_D(D/\mathbf{p})\}$, the *component* of V of the *weight* $\mathbf{p}$. The *support* $\mathrm{Supp}(V)$ of the weight module V is the set of maximal ideals $\mathbf{p}$ such that $V_{\mathbf{p}} \neq 0$.

Since

$$XV_{\mathbf{p}} \subseteq V_{\sigma(\mathbf{p})} \text{ and } YV_{\mathbf{p}} \subseteq V_{\sigma^{-1}(\mathbf{p})}, \tag{7.1}$$

each weight A-module V decomposes into the direct sum of A-submodules

$$V = \oplus\{\, V_{\mathcal{O}} \,|\, \mathcal{O} \,\text{is an orbit}\,\},$$

where $V_{\mathcal{O}} = \oplus\{\, V_{\mathbf{p}} \,|\, \mathbf{p} \in \mathcal{O}\,\}$. Obviously, for each maximal ideal $\mathbf{p}$ of D the module

$$A(\mathbf{p}) := A/A\mathbf{p} \simeq A \otimes D/\mathbf{p} = \bigoplus\nolimits_{i\in\mathbf{Z}} v_i \otimes D/\mathbf{p} \tag{7.2}$$

is weight (${}_D(v_i \otimes D/\mathbf{p}) \simeq D/\sigma^i(\mathbf{p})$) with support $\mathcal{O}(\mathbf{p})$.

- Each simple D-torsion A-module M is a factormodule of $A(\mathbf{p})$ for some maximal ideal $\mathbf{p}$ of D, hence, M is weight.

Since M is D-torsion and the ring D with restricted minimum condition, the module ${}_DM$ contains a simple D-submodule $N \simeq D/\mathbf{p}$. Since ${}_AM$ is simple we have an epimorphism of A-modules $A(\mathbf{p}) \to AN = M$.

Lemma 7.1 *Each finitely generated D-torsion A-module M has a finite chain of submodules $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ such that every factor M_i/M_{i-1} is a factormodule of $A(\mathbf{p}_i)$ for some $\mathbf{p}_i \in$Specm D.*

Proof. Repeat the arguments above we see that ${}_DM$ contains a simple D-module $N_1 = D/\mathbf{p}_1$ for some maximal ideal $\mathbf{p}_1$ of D and A-module $M_1 := AN_1$ is a factormodule of $A(\mathbf{p}_1)$. If $M = M_1$, there is nothing to prove. Otherwise, using the same arguments for the factormodule M/M_1 we can find a submodule M_2 of M such that $M_1 \subset M_2$ and M_2/M_1 is a factormodule of $A(\mathbf{p}_2)$ for some maximal ideal $\mathbf{p}_2$ and so on. Since A is the Noetherian and M is the finitely generated A-module, so the procedure of finding M_kth is stopped after n steps, say. ∎

Theorem 7.2 *(Classification of simple D-torsion A-modules, [Bav 2], $a \neq 0$.) Each simple D-torsion A-module is isomorphic to one from 1 to 7 below and vice versa.*

A. (Modules which are determined by their support). *The map*

$$(Lin \cup Cycd)/ \sim \to \hat{A}(D - \text{torsion}),\, \Gamma \to [L(\Gamma)],$$

is injective, Supp $L(\Gamma) = \Gamma$, where

1. *if $\Gamma \in Linn$, then $L(\Gamma) = A/A\mathbf{p}, \mathbf{p} \in \Gamma$;*

2. *if $\Gamma = (-\infty, \mathbf{p}] \in Lind$, then $L(\Gamma) = A/A(\mathbf{p}, \Gamma)$;*

3. *if* $\Gamma = (\sigma^{-n}(\mathbf{p}), \mathbf{p}] \in$ *Lind* $\cup$ *Cycd*, $n \in \mathbf{N}$, *then* $L(\Gamma) = A/A(\mathbf{p}, X, Y^n)$, $l_D(L(\Gamma)) = n$;

4. *if* $\Gamma = (\mathbf{p}, \infty) \in$*Lind, then* $L(\Gamma) = A/A(\sigma(\mathbf{p}), Y)$.

   B. (Additional series). *The map*

$$\sqcup_{\mathcal{O}\in Cyc} \tilde{\Lambda_{\mathcal{O}}} \to \hat{A}(D-\text{torsion}),\ [N] \to [L(N) = A \otimes_{\Lambda_n} N],\ n = |\mathcal{O}|,$$

is injective. Moreover, let $\mathbf{p}$ *be any ideal from* $\mathcal{O}$, *then*

5. *if* $\mathcal{O} \in$ *Cycn*, $N = \Lambda_n/\Lambda_n(\mathbf{p}, f)$ *for some element* $f = \sum_{i\geq 0}^m \alpha_i X^{ni}$ *such that all* $\alpha_i \in D$,$\alpha_0, \alpha_m \notin \mathbf{p}$, $m > 0$, *and the image* $\bar{f}$ *of* f *under the epimorphism* $\Lambda_n \to \Lambda_{\mathcal{O}}$ *is an irreducible element, then* $L(N) \simeq A/A(\mathbf{p}, f)$;

6. *if* $\mathcal{O} \in$*Cycd*, $N = \Lambda_n/\Lambda_n(Y^n, \mathbf{p}, f)$ *for some element* $f = \sum_{i\geq 0}^m \alpha_i X^{ni}$ *such that all* $\alpha_i \in D$,$\alpha_0, \alpha_m \notin \mathbf{p}$, $m > 0$, *and the image* $\bar{f}$ *of* f *under the epimorphism* $\Lambda_n \to \Lambda_{\mathcal{O},+}$ *is an irreducible element, then* $L(N) \simeq A/A(Y^n, \mathbf{p}, f)$;

7. *if* $\mathcal{O} \in$*Cycd* , $N = \Lambda_n/\Lambda_n(X^n, \mathbf{p}, f)$ *for some element* $f = \sum_{i\geq 0}^m \alpha_i Y^{ni}$ *such that all* $\alpha_i \in D$, $\alpha_0, \alpha_m \notin \mathbf{p}$, $m > 0$ *and the image* $\bar{f}$ *of* f *under the epimorphism* $\Lambda_n \to \Lambda_{\mathcal{O},-}$ *is an irreducible element, then* $L(N) \simeq A/A(X^n, \mathbf{p}, f)$. ▪

Remark. If $[N] \in \tilde{\Lambda_{\mathcal{O}}}$, $n = |\mathcal{O}|$, we can choose any maximal ideal $\mathbf{p} \in \mathcal{O}$, since $\Lambda_{\mathcal{O}} \simeq \Lambda_n/(\mathbf{p})$, we can say that N is a Λ_n-module, then the tensor product $A \otimes_{\Lambda_n} N$ in Theorem 7.2 is well-defined.

Proof of Theorem 1.6.

Case, $a = 0$. We have the following exact sequences of A-modules;

$$0 \to AX \to^{\alpha} A \to^{\varphi} AY \to 0,\ \alpha(u) = u,\ \varphi(v) = vY, \tag{7.3}$$

$$0 \to AY \to^{\beta} A \to^{\psi} AX \to 0,\ \beta(u) = u,\ \psi(v) = vX. \tag{7.4}$$

The sequence (7.3) is non-split. Suppose the contrary, then $\chi\alpha = id$ for some A-homomorphism $\chi : A \to AX, u \to u\gamma$, $\gamma \in AX$; hence, $X = \chi\alpha(X) = X\gamma$, a contradiction (since in the set $AX = \sum_{i\geq 1} DX^i$ there is no such element γ). Then the A-modules AX and AY are not projective, so (7.4) is also non-split, hence gl.dim $A = \infty$.

Case, $a \neq 0$. *From now on* $a \neq 0$.

Lemma 7.3 *1. Let* D *be a commutative ring, the defining element* a *be regular and* $\mathbf{p}$ *be a two-sided ideal of* D *such that* $a \in \mathbf{p}$. *Then the following sequences of* A*-modules are exact*

$$0 \to A(\sigma(\mathbf{p}), Y) \to^{\varphi} A\mathbf{p} \oplus A \to^{\psi} A(\mathbf{p}, X) \to 0, \tag{7.5}$$

where $\varphi(\omega) = (\omega X, \omega), \psi(u, v) = u - vX$,

$$0 \to A(\mathbf{p}, X) \to^{\varphi} A\sigma(\mathbf{p}) \oplus A \to^{\psi} A(\sigma(\mathbf{p}), Y) \to 0, \tag{7.6}$$

where $\varphi(\omega) = (\omega Y, \omega), \psi(u, v) = u - vY$.

2. *The sequence (7.5) (or (7.6)) is split iff* $\mathbf{p}(1-p_0) \subseteq Da$ *for some* $p_0 \in \mathbf{p}$.

3. *If, additionally, D is Dedekind and* $\mathbf{p}$ *is maximal, then (7.5) (or (7.6)) is not split iff* $a \in \mathbf{p}^2$, *i.e. exists* $n_i \geq 2$.

Proof. 1. The exactness of (7.5) follows from the equality

$$A\mathbf{p} \cap AX = A(\sigma(\mathbf{p}), Y)X. \tag{7.7}$$

The inclusion "$\supseteq$" is clear, since $\sigma(\mathbf{p})X = X\mathbf{p}$, $YX = a \in \mathbf{p}$. Conversely, let u belongs to the left side of (7.7), then $u = vX$ for some $v \in A$. Let us prove that $v \in A(\sigma(\mathbf{p}), Y)$. Since

$$A(\sigma(\mathbf{p}), Y) = \sum\nolimits_{i<0} v_i D + \sum\nolimits_{i\geq 0} v_i \sigma(\mathbf{p}), \tag{7.8}$$

then, without less of generality, we can suppose that $v = \sum_{i\geq 0} v_i\alpha_i$ where all $\alpha_i \in \sigma(\mathbf{p})$. From the equality $vX = \sum_{i\geq 0} v_{i+1}\sigma^{-1}(\alpha_i) \in A\mathbf{p}$ we conclude that all $\alpha_i \in \sigma(\mathbf{p})$, i.e. $v \in A(\sigma(\mathbf{p}), Y)$.

The exactness of (7.6) follows from the equality

$$A\sigma(\mathbf{p}) \cap AY = A(\mathbf{p}, X)Y \tag{7.9}$$

which can easily be established in the same way.

2. The sequence (7.5) is split iff φ has a left inverse homomorphism, say, χ, i.e. $\chi\varphi = id$. Then $\chi(up, v) = u\chi_0(p) + v\gamma$ for all $u, v \in A$ where $\chi_0 = \chi|_{\mathbf{p}\oplus 0}$, $\gamma = \chi(0,1) = \sum \gamma_i v_i$, $\gamma_i \in D$, if $i < 0$; $\gamma_i \in \sigma^{i+1}(\mathbf{p})$, if $i \geq 0$. The condition $\chi\varphi = id$ is equivalent to two equalities

$$Y = \chi\varphi(Y) = \chi(YX = a, Y) = \chi_0(a) + Y\gamma,$$
$$q = \chi\varphi(q) = \chi(qX = X\sigma^{-1}(q), q) = X\chi_0(\sigma^{-1}(q)) + q\gamma, \text{for all } q \in \sigma(\mathbf{p}). \tag{7.10}$$

Since a is regular in D, hence X is regular in A (since A is a graded ring). Multiplying the first equality by X on the left we get the second for $q = \sigma(a)$, so the first is the consequence of (7.10).

Using (7.8), write χ_0 as a sum of D-homomorphisms, $\chi_0 = \sum_{i\in\mathbf{Z}} \chi_{0,i}v_i$ where $\chi_{0,i}$ maps $\mathbf{p}$ to D, if $i < 0$; and to $\sigma^{i+1}(\mathbf{p})$, if $i \geq 0$. Substituting χ_0 for sum of D-homomorphisms in (7.10) and writing the 0'th component we get

$$(\sigma\chi_{0,-1}\sigma^{-1})(q)\sigma(a) + q\gamma_0 = q, \text{ for all } q \in \sigma(\mathbf{p})$$

or equivalently,

$$\chi_{0,-1}(p)a = p(1-\sigma^{-1}(\gamma_0)), \text{ for all } p \in \mathbf{p},$$

i.e. $\mathbf{p}(1-p_0) \subseteq Da$ for $p_0 = \sigma^{-1}(\gamma_0) \in \mathbf{p}$.

On the other hand, if $\mathbf{p}(1-p_0) \subseteq Da$ for some $p_0 \in \mathbf{p}$, then put $\chi_0(p) = \chi_{0,-1}(p) := p(1-p_0)a^{-1}$, $p \in \mathbf{p}$, (a is regular) and $\gamma = p_0$ we can satisfy (7.10), i.e. (7.5) is split.

3. If D is Dedekind and $\mathbf{p}$ is maximal, then it is easy to see that $\mathbf{p}(1-p_0) \subseteq Da$, $p_0 \in \mathbf{p}$, is equivalent to $a \notin \mathbf{p}^2$. ∎

The generalized Weyl algebra $A = D(\sigma, a)$ is a $\mathbf{Z}$-graded ring, i.e. $A = \oplus_{i\in\mathbf{Z}}(Dv_i = v_iD)$, therefore, $D \to A$ is the flat chosen extension, then by Corollary 4.4.(3) and Lemma 4.3.(3)

$$\mathrm{pd}_D(A \otimes_D N) = \mathrm{pd}_A(A \otimes_D N) = \mathrm{pd}_D N$$

for any finitely generated D-module N and $1 \leq \mathrm{lgd}\, D \leq \mathrm{lgd}\, A$. Since D is Dedekind, then lgd $D = 1$, so each ideal $\mathbf{p}$ of D is a projective D-module, then, using equalities above, we get $0 = \mathrm{pd}_D\mathbf{p} = \mathrm{pd}_A(A \otimes_D \mathbf{p} \simeq A\mathbf{p})$. By Lemma 7.3, lgd $A = \infty$ if $n_i \geq 2$ for some i.

Case, $n_1 = \ldots = n_s = 1$, **i.e.** $Da = \mathbf{p}_1 \ldots \mathbf{p}_s$, **the product of distinct maximal ideals, or** a **is invertible in** D**.** Then the left ideals $A(\mathbf{p}, X)$ and $A(\sigma(\mathbf{p}), Y)$ from Lemma 7.3 are projective, hence the projective dimension of the modules from Theorem 7.2.(2,4) is 1.

Lemma 7.4 *If a is as above, then lgd $A = \mu$, where $\mu = \sup\{\mathrm{pd}_A M \,|\, M$ is a simple D-torsion A-submodule $\}$.*

Proof. By Corollary 4.8 lgd $A = \max\{1 = \mathrm{lgd}\, B, \mu_1\}$, where $\mu_1 = \sup\{\mathrm{pd}_A M \,|\, M$ is a cyclic D-torsion A-module $\}$. By Lemma 7.1 $\mu_1 = \sup\{\mathrm{pd}_A M \,|\, M$ is a factormodule of $A(\mathbf{p})$ for some maximal ideal $\mathbf{p}$ of D $\}$.

Since B is Euclidean, every two non-zero left ideals of B have non-zero intersection. The same holds for A (since B is the localization of A and A is the subring of B), hence, for each projective non-zero ideal P of A the factormodule A/P has the projective dimension $1 = \mathrm{pd}_A(A/P)$. Since $_AA$ is not simple, each simple A-module has the projective dimension more or equal to 1, therefore, $1 \leq \mu \leq \mu_1$ and lgd $A = \mu_1$. It is enough to prove that $\mu = \mu_1$. Let M be a factormodule of $A(\mathbf{p})$ for some maximal ideal $\mathbf{p}$ of D. Consider all the possible situations.

If $\mathcal{O}(\mathbf{p}) \in$ **Linn**, then by Theorem 7.2 A-module $A(\mathbf{p})$ is simple.

If $\mathcal{O}(\mathbf{p}) \in$ **Lind** and $\mathbf{p}_1 < \ldots < \mathbf{p}_s$ are all marked ideals (i.e. $a \in \mathbf{p}_i$ for all i) of $\mathcal{O}(\mathbf{p})$ which divide it into $s+1$ parts, $\Gamma_1 = (-\infty, \mathbf{p}_1]$, $\Gamma_2 = (\mathbf{p}_1, \mathbf{p}_2], \ldots, \Gamma_{s+1} = (\mathbf{p}_s, \infty)$.

The D-module $A(\mathbf{p})$ decomposes into the direct sum of pairwise non-isomorphic simple D-modules

$$A(\mathbf{p}) = \oplus_{i\in\mathbf{Z}} A(\mathbf{p})_i,\ {}_DA(\mathbf{p})_i = v_i \otimes D/\mathbf{p} \simeq D/\sigma^i(\mathbf{p}).$$

Using the relations $YX = a$ and $XY = \sigma(a)$ and (7.1) we see that the homomorphisms of abelian groups

$$X_i : A(\mathbf{p})_i \to A(\mathbf{p})_{i+1},\ u \to Xu \ (\text{respectively}, Y_i : A(\mathbf{p})_{i+1} \to A(\mathbf{p})_i,\ u \to Yu\,)$$

are either isomorphisms or zeros. Moreover, $X_i = 0$ (respectively, $Y_i = 0$) iff $\sigma^i(\mathbf{p})$ is the marked ideal such that $\sigma^i(\mathbf{p}) \leq \mathbf{p}$ (respectively, $\sigma^i(\mathbf{p}) \geq \mathbf{p}$).

Since $A(\mathbf{p})$ is semisimple D-module with non-isomorphic simple submodules $A(\mathbf{p})_i$, each proper A-submodule N of $A(\mathbf{p})$ has the support $\Gamma_1 \cup \ldots \cup \Gamma_i$, $\Gamma_j \cup \Gamma_{j+1} \cup \ldots \cup \Gamma_{s+1}$ or $\Gamma_1 \cup \ldots \cup \Gamma_i \cup \Gamma_j \cup \Gamma_{j+1} \cup \ldots \cup \Gamma_{s+1}$ where $i < k < j$ and vice versa.

Submodules with the supports as above are isomorphic, respectively, to the following A-modules

$$V(\mathbf{p}_i, -) = A/A(\mathbf{p}_i, X),\ V(\mathbf{p}_j, +) = A/A(\sigma(\mathbf{p}_j), Y) \text{ and } V(\mathbf{p}_i, -) \oplus V(\mathbf{p}_j, +),$$

hence, by Lemma 7.3 they have projective dimension 1.

The factormodule $A(\mathbf{p})/(V(\mathbf{p}_i,-)\oplus V(\mathbf{p}_j,+))$ has a finite length as D-module (since its support is $\Gamma_{i+1}\cup\ldots\cup\Gamma_{j-1}$), therefore it has the projective dimension no more than μ. The factormodule $A(\mathbf{p})/V(\mathbf{p}_i,-)$ (respectively, $A(\mathbf{p})/V(\mathbf{p}_j,+)$) contains the submodule $L_{i,-}=\sum\{A(\mathbf{p})_{\mathbf{q}}\,|\,\mathbf{q}\in\Gamma_{i+1}\cup\ldots\cup\Gamma_{k-1}\}$ (respectively, $L_{j,+}=\sum\{A(\mathbf{p})_{\mathbf{q}}\,|\,\mathbf{q}\in\Gamma_{k+1}\cup\ldots\cup\Gamma_{j-1}\}$) of finite D-length which may be 0, such that the sequence

$$0\to L_{i,-}\to A(\mathbf{p})/V(\mathbf{p}_i,-)\to V(\mathbf{p}_k,+)\to 0\ (0\to L_{j,+}\to A(\mathbf{p})/V(\mathbf{p}_j,+)\to V(\mathbf{p}_k,-)\to 0)$$

is exact.

Now the projective dimensions of the A-modules $A(\mathbf{p})/V(\mathbf{p}_i,-)$ and $A(\mathbf{p})/V(\mathbf{p}_j,+)$ are less or equal than $\max\{1,\mu\}=\mu$.

If $\mathcal{O}(\mathbf{p})\in\mathbf{Cycn}$, $n=|\mathcal{O}(\mathbf{p})|$, then $A(\mathbf{p})=\oplus\{A(\mathbf{p})_{\mathbf{q}}\,|\,\mathbf{q}\in\mathcal{O}(\mathbf{p})\}$, where for each $\mathbf{q}=\sigma^i(\mathbf{p})$, $i=0,\ldots,n-1$, the D-module

$$A(\mathbf{p})_i:=A(\mathbf{p})_{\sigma^i(\mathbf{p})}=\sum{}_{j\in\mathbf{Z}}\,v_{i+kj}\otimes D/\mathbf{p}$$

is the sum of simple D-modules each of them is isomorphic to $D/\sigma^i(\mathbf{p})$. So, every A-submodule N of $A(\mathbf{p})$ decomposes into the direct sum $N=\oplus_{i=0}^{n-1}N_i$ of D-submodules where $N_i=N\cap A(\mathbf{p})_i$. Obviously, the ring $A(\mathbf{p})_0$ equals to the skew Laurent polynomial ring $\Lambda_n/(\mathbf{p})\simeq D/\mathbf{p}[x,x^{-1};\sigma^n]$, hence, it has the restricted minimum condition. For each $i=0,\ldots,n-1$, $A(\mathbf{p})_0$-module $A(\mathbf{p})_i$ is free of rank 1 and the maps

$$X_i:A(\mathbf{p})_i\to A(\mathbf{p})_{i+1},\ u\to Xu,\ \ Y_i:A(\mathbf{p})_{i+1}\to A(\mathbf{p})_i,\ u\to Yu\,,$$

are the isomorphisms of the abelian groups. Hence, for each non-zero A-submodule N of $A(\mathbf{p})$, the factormodule $A(\mathbf{p})/N=\oplus_{i=0}^{n-1}A(\mathbf{p})_i/N_i$ has a finite D-length. Therefore each factormodule of $A(\mathbf{p})$ has the projective dimension less or equal to $\max\{\mathrm{pd}_A A(\mathbf{p})=1,\mu\}=\mu$.

If $\mathcal{O}(\mathbf{p})\in\mathbf{Cycd}$. It is easy to see that A-modules $A(\mathbf{p})$ and $A(\mathbf{q})$ are isomorphic iff $\mathbf{p}\sim\mathbf{q}$. So, we may assume that $\mathbf{p}$ is the marked ideal. Then the following sequence of A-modules is exact :

$$0\to L^+:=A/A(\sigma(\mathbf{p}),Y)\to{}^{\chi}A(\mathbf{p})\to{}^{\phi}L^-:=A/A(\mathbf{p},X)\to 0, \qquad (7.11)$$

where $\chi(u+A(\sigma(\mathbf{p}),Y))=uX$, $\phi(v+A\mathbf{p})=v+A(\mathbf{p},X)$. Namely, by (7.7)

$$\mathrm{Ker}\,\phi=A(\mathbf{p},X)/A\mathbf{p}\simeq AX/AX\cap A\mathbf{p}\simeq AX/A(\sigma(\mathbf{p}),Y)X\simeq A/A(\sigma(\mathbf{p}),Y)$$

(X is not a zero divisor in A). By Lemma 7.3 the ideals $A(\sigma(\mathbf{p}),Y)$ and $A(\mathbf{p},X)$ are projective, hence, $\mathrm{pd}_A L^{\pm}=1$. The ring A contains the subrings $A_{\pm}=D<X_1^{\pm}>=D[X_1^{\pm};\sigma^{\pm n}]$ which are skew polynomial rings. The A_+-submodule (respectively, the A_--submodule) $\sum_{i>0}v_i\otimes D/\mathbf{p}$ (respectively, $\sum_{i\le 0}v_i\otimes D/\mathbf{p}$) of $A(\mathbf{p})$ and the module L^+ (respectively, L^-) are isomorphic and each proper A_+-factormodule (respectively, A_--factormodule) of it has a finite D-length. Now for each A-submodule N of $A(\mathbf{p})$ by (7.11) we can build the exact sequence of A-modules:

$$0\to N^+\to A(\mathbf{p})/N\to N^-\to 0,$$

where $N^- = L^-/\phi(N)$ and $N^+ = L^+/\chi^{-1}(\chi(L^+)\cap N)$, such that each $N^\pm$ either equals to $L^\pm$ or has the finite D-length. So, $\mathrm{pd}_A A(\mathbf{p})/N \le \max\{\mathrm{pd}_A N^+, \mathrm{pd}_A N^-\} \le \max\{1,\mu\} = \mu$. ∎

Lemma 7.5 *If the maximal ideals* $\mathbf{p}$ *and* $\mathbf{q} = \sigma^{-n}(\mathbf{p})$, $n \in \mathbf{N}$, *of the ring* D *contain* a *and the ideal* Da *is the product* $\mathbf{p}_1 \dots \mathbf{p}_s$ *of distinct maximal ideals, then the projective dimension of the left ideal* $A(\mathbf{p}, X, Y^n)$ *is equal to 1 (hence, the projective dimension of the module* $L(\Gamma)$ *from Theorem 7.2.(3) equals to 2).*

Proof. Since all the ideals $\mathbf{p}_i$ are distinct, then by Lemma 7.3 the left ideals $A(\mathbf{p}, X)$ and $A(\mathbf{q}, X)$ are projective A-modules. It follows from the easily checked equality

$$A(\mathbf{p}, X) \cap AY^n = A(\mathbf{q}, X)Y^n$$

that the following sequence of A-modules is exact:

$$0 \to A(\mathbf{q}, X) \to {}^{\varphi} A(\mathbf{p}, X) \oplus A \to {}^{\psi} A(\mathbf{p}, X, Y^n) \to 0 \tag{7.12}$$

where $\varphi(\omega) = (\omega Y^n, \omega)$ and $\psi(u, v) = u - vY^n$. Thus, the projective dimension of the ideal $A(\mathbf{p}, X, Y^n)$ is equal to 1 iff (7.12) is non-split. Suppose that (7.12) is split, i.e. the monomorphism φ has a left inverse homomorphism χ: $\chi\varphi = id$, where

$$\chi(up + vX, \omega) = u\chi_0(p) + v\gamma + \omega\delta, \text{ for all } u, v, \omega \in A$$

and some $\gamma, \delta \in A(\mathbf{q}, X)$ and some D-homomorphism χ_0 from $\mathbf{p}$ to $A(\mathbf{q}, X)$. Since $Xp = \sigma(p)X \in A\mathbf{p} \cap AX$ for all $p \in \mathbf{p}$, therefore

$$X\chi_0(p) = \sigma(p)\gamma, \text{ for all } p \in \mathbf{p}. \tag{7.13}$$

As a D-module the left ideal $A(\mathbf{q}, X)$ is a direct sum of D-submodules

$${}_D A(\mathbf{q}, X) = \sum_{i \le 0} \sigma^i(\mathbf{q})v_i + \sum_{i>0} Dv_i, \tag{7.14}$$

so the D-homomorphism χ_0 decomposes into a finite direct sum of D-homomorphisms:

$$\chi_0(p) = \sum_{i \in \mathbf{Z}} \chi_{0,i}(p)v_i, \;\; p \in \mathbf{p},$$

where $\chi_{0,i}$ maps $\mathbf{p}$ to $\sigma^i(\mathbf{q})$, if $i \le 0$, and to D, otherwise.

By (7.14) $\gamma = \sum \gamma_i v_i$ and $\delta = \sum \delta_i v_i$ where γ_i, δ_i belong to $\sigma^i(\mathbf{q})$, if $i \le 0$; and to D, otherwise. Substitute χ_0 and γ for their decompositions in (7.13) and consider the $(n+1)$'th component we have $\chi_{0,n}(p) = p\sigma^{-1}(\gamma_{n+1})$ for all $p \in \mathbf{p}$, hence,

$$\chi_{0,n}(\mathbf{p}) \subseteq \mathbf{p}. \tag{7.15}$$

Since $\chi\varphi = id$, so

$$q = \chi\varphi(q) = Y^n\chi_0(\sigma^n(q)) + q\delta, \text{ for all } q \in \mathbf{q}.$$

Considering the 0'th component of the above equality we come to a contradictrion: $\mathbf{q} \subseteq Y^n\chi_{0,n}(\sigma^n(\mathbf{q}))X^n + \mathbf{q}\delta_0 \subseteq (\sigma^{-n}\chi_{0,n}\sigma^n)(\mathbf{q})a\sigma^{-1}(a)\dots\sigma^{-n+1}(a) + \mathbf{q}^2$ (since $\delta_0 \in \mathbf{q}) \subseteq \mathbf{q}^2$ (by (7.15) and $a \in \mathbf{q}$), a contradiction. ∎

Lemma 7.6 *Let* $\mathbf{p}$ *be a maximal ideal of* D *such that* $\sigma^n(\mathbf{p}) = \mathbf{p}$ *for some integer* $n \geq 1$, $f = \sum \alpha_i v_{ni}$ *where* i *runs through either non-negative or non-positive integers,* $\alpha_i \in D$ *for all* i, $\alpha_0 \notin \mathbf{p}$. *Then* $A\mathbf{p} \cap Af = A\mathbf{p}f$.

Proof. Since $\mathbf{p}v_{ni} = v_{ni}\sigma^{-ni}(\mathbf{p}) = v_{ni}\mathbf{p}$ for any i, then $A\mathbf{p}f \subseteq A\mathbf{p} \cap Af$.

Conversely, let $u \in A\mathbf{p} \cap Af$, then $u = bf$ for some $b \in A$. Prove that $b \in A\mathbf{p}$. Suppose the contrary, then $b = c + d$ for some $d \in A\mathbf{p}$ and $c = c_i + \ldots + c_j \in A$, $i < \ldots < j$, such that $c_k \in A_k$ for all $k = i, \ldots, j$ and $c_i = v_i\beta$, $\beta \in D\backslash\mathbf{p}$, in the first case; $c_j = v_j\beta$, $\beta \in D\backslash\mathbf{p}$, in the second case. Since $bf \in A\mathbf{p}$, then $cf \in A\mathbf{p}$ and, using the grading of A, we have $c_i\alpha_0 = v_i\beta\alpha_0 \in A\mathbf{p}$ and $c_j\alpha_0 = v_j\beta\alpha_0 \in A\mathbf{p}$ for the first and the second case respectively. Hense $\beta\alpha_0 \in \mathbf{p}$, a contradiction, since $\mathbf{p}$ is prime and β and α_0 do not belong to $\mathbf{p}$. ▪

Lemma 7.7 *Suppose that a maximal ideal* $\mathbf{p}$ *of* D *belongs to the degenerated cyclic orbit* $\mathcal{O}(\mathbf{p})$ *of the length* n. *Then the left ideal* $A(\mathbf{p}, f_\pm, v_{\mp n})$ *equals to* $A(\mathbf{p}, f_\pm)$ *for any element* $f_\pm = \sum_{i\geq 0} \alpha_i v_{\pm ni} \in A\backslash D$ *such that* $\alpha_0 \notin \mathbf{p}$, *where all* $\alpha_i \in D$; *and* $\mathrm{pd}_A A(\mathbf{p}, f_\pm, v_{\mp n}) \leq 1$ *(hence, the projective dimension of the module from Theorem 7.2.(6,7)* ≤ 2.*)*

Proof. It is enough to prove that $v_{\mp n} \in A(\mathbf{p}, f_\pm)$. Since $\mathcal{O}(\mathbf{p})$ is degenerated, then $< \mp n, \pm ni > \in \mathbf{p}$, for all $i \geq 1$, hence

$$v_{\mp n}\alpha_0 = v_{\mp n}(\alpha_0 - f_\pm) + v_{\mp n}f_\pm = v_{\mp n}f_\pm + \sum_{i\geq 1} \sigma^{\mp n}(\alpha_i)v_{\pm n(i-1)} < \mp n, \pm ni > \in A(\mathbf{p}, f_\pm).$$

Since $\mathbf{p}$ is maximal and $\alpha_0 \notin \mathbf{p}$, then $p_0 + d\alpha_0 = 1$ for some $p_0 \in \mathbf{p}$, $d \in D$. Therefore

$$v_{\mp n} = v_{\mp n}p_0 + \sigma^{\mp n}(d)v_{\mp n}\alpha_0 \in A(\mathbf{p}, f).$$

By Lemma 7.6 $A\mathbf{p} \cap Af = A\mathbf{p}f$, $f = f_\pm$, so we have the projective resolution (7.16) (see below), i.e. $\mathrm{pd}_A A(\mathbf{p}, f) \leq 1$. ▪

Denote by S_a the monoid generated by the set $\{\sigma^i(a), i \in \mathbf{Z}\}$, then S_a is the multiplicatively closed subset of A and the localization $B_a = S_a^{-1}A$ of A at S_a is the skew Laurent polynomial ring $B_a = D_a[X, X^{-1}; \sigma]$, $D_a = S_a^{-1}D$.

Lemma 7.8 *If a maximal ideal* $\mathbf{p}$ *of* D *belongs to the non-degenerated orbit* $\mathcal{O}(\mathbf{p})$ *of the length* n. *Then the left ideal* $A(\mathbf{p}, f)$ *has the projective dimension 1 for any element* $f = \sum_{i\geq 1}^m \alpha_i X^{ni}$ *such that* α_0 *and* $\alpha_m \notin \mathbf{p}$, $m > 0$, *all* $\alpha_i \in D$ *(hence, the projective dimension of the module from Theorem 7.2.(5) is 2).*

Proof. By Lemma 7.6 $A\mathbf{p} \cap Af = A\mathbf{p}f$, so we have the projective resolution:

$$0 \to A\mathbf{p} \to^{\varphi} A\mathbf{p} \oplus A \to^{\psi} A(\mathbf{p}, f) \to 0,\ \varphi(\omega) = (\omega f, \omega),\ \psi(u, v) = u - vf, \qquad (7.16)$$

hence, $\mathrm{pd}_A A(\mathbf{p}, f) \leq 1$. It is not difficult (but rather boring) to prove the non-splitness of this resolution doing in the same way as in Lemma 7.3. But we use another approach.

Since D_a is not a field, lgd $D_a = 1$. It is easy to see that A-module $M = A/A(\mathbf{p}, f)$ is the finitely generated semisimple D-module with Supp $M = \mathcal{O}(\mathbf{p})$. Therefore, we can consider M as B_a-module, $M = B_a \otimes {}_A M$, which is the finitely generated semisimple D_a-module, then pd ${}_{D_a}M = 1$ and by Corollary 9.7 ([MR], p. 272), pd ${}_{B_a}M = 2$. Since $A \to B_a$ is the flat chosen extension, then by Corollary 4.4.(3) and Lemma 4.3.(3) pd ${}_A M =$pd ${}_{B_a}M = 2 \Leftrightarrow \mathrm{pd}_A A(\mathbf{p}, f) = 1$. ∎

Now we are ready to complete the proof of Theorem 1.6. If $Cyc \cup Linds \neq \emptyset$, then by Lemmas 7.5, 7.8 gl.dim $A = 2$, and gl.dim $A = 1$, otherwise (by Lemma 7.3).

So we have

$$\text{gl.dim}\, A = \begin{cases} \infty & \text{, if } a = 0 \text{ or } n_i \geq 2 \text{ for some } i; \\ 2 & \text{, if } a \neq 0,\ n_1 = \ldots = n_s = 1 \text{ or } a \text{ is invertible and } Cyc \cup Linds \neq \emptyset; \\ 1 & \text{, otherwise.} \end{cases}$$

8 Back to the Examples

In sec. 8 let K be *an algebraically closed field* and, when it is stated otherwise, $A = K[H](\sigma, a)$, $\sigma(K) = \lambda H + \mu$, be the GWA of degree 1 from Example 1 (see sec. 2). By the bijection Specm $K[H] \leftrightarrow K$, $(H - \gamma) \leftrightarrow \gamma$, $\gamma \in K$, we identify the set of maximal ideals of $K[H]$ and K. Under this identification denote by τ the action of σ on $K =$ Specm $K[H]$, i.e. $\tau(\gamma) = (\gamma - \mu)/\lambda$ for $\gamma \in K$. Since K is algebraically closed, then

$$a = \delta(H - \lambda_1)^{n_1} \ldots (H - \lambda_s)^{n_s},\ \delta \in K, \tag{8.1}$$

where $\{\lambda_i\}$ are the distinct roots of the polynomial a. Denote by K^* the set $K \backslash \{0\}$. Then Theorem 8.1 is a corollary of Theorem 1.6.

Theorem 8.1 *Let A and K be as above. Then the global dimension of A is*

1. *char $K = 0$,*

$$\text{gl.dim}\, A = \begin{cases} \infty & \textit{, if } a = 0 \textit{ or } a \textit{ has a repeated root;} \\ 2 & \textit{, if the roots of } a \neq 0 \textit{ are distinct or } a \textit{ is invertible and} \\ & (\lambda, \mu) \notin (1, K^*), \textit{ otherwise, i.e. } (\lambda, \mu) \in (1, K^*), \textit{ there exist} \\ & \textit{some integers } i \neq j,\ k \textit{ such that } \lambda_i = \tau^k(\lambda_j) = \lambda_j - k\mu; \\ 1 & \textit{, otherwise.} \end{cases}$$

2. char $K \neq 0$,

$$\text{gl.dim}\, A = \begin{cases} \infty & \textit{, if a=0 or a has a repeated root;} \\ 2 & \textit{, otherwise.} \end{cases}$$ ∎

Corollary 8.2 *([Hod 1], [Bav 1,4]).Let K be an algebraically closed field of characteristic zero and A be as in Theorem 8.1 where $\sigma(H) = H - 1$. Then*

$$\text{gl.dim}\, A = \begin{cases} \infty & \textit{, if } a = 0 \textit{ or } a \textit{ has a repeated root;} \\ 2 & \textit{, if the roots of } a \neq 0 \textit{ are distinct and } \lambda_i - \lambda_j \in \mathbf{Z} \textit{ for some } i \textit{ and } j; \\ 1 & \textit{, otherwise.} \end{cases}$$ ∎

Corollary 8.2 gives us the well-known result

$$\text{gl.dim}\, U(\lambda) = \begin{cases} \infty & \text{, if } \lambda = -1/4; \\ 2 & \text{, if } \lambda = (n^2-1)/4,\ n \in \mathbf{N}; \\ 1 & \text{, otherwise,} \end{cases}$$

first proved by Stafford in [St], where $U(\lambda)$ is the infinite dimensional primitive factor of $Usl(2)$ (see, Example 3 in sec. 2).

Corollary 8.3 *1. The global dimension of the quantum Weyl algebra $A_1(q)\,(q \neq 0, 1)$ and of the quantum plane Λ is 2.*

2. The global dimension of the algebra of functions $A(S^2_\lambda)$ on the quantum 2-dimensional sphere is

$$\text{gl.dim}\ A(S^2_\lambda) = \begin{cases} \infty & \text{, } if\ c = -d; \\ 2 & \text{, } if\ c \neq -d. \end{cases}$$

3. Let $A = K[H, (H - \mu/(1-\lambda)^{-1}](\sigma, a)$, $\sigma(H) = \lambda H + \mu$, be the GWA from Example 2 (sec. 2), where $\lambda_1, \dots, \lambda_s$ are the distinct "roots" of a (see (8.1), $\lambda_i \neq \mu/(1-\lambda)$ for all i) and $\delta = \nu(H - \mu/(1-\lambda))^t$ for some $t \in \mathbf{Z}$, $\nu \in K$. Then

$$\text{gl.dim}\ A = \begin{cases} \infty & \text{, if } a = 0 \text{ or } a \text{ has a repeat root;} \\ 2 & \text{, if the roots of } a \neq 0 \text{ are distinct or } a \text{ is invertible and either } \lambda \neq 1 \\ & \quad \text{is a root of 1 or } (\lambda, \mu) = (1, 0); \text{ or } (\lambda, \mu) \in (1, K^*),\ \text{char } K > 0; \\ & \quad \text{or } \lambda_i = \tau^k \lambda_j \text{ for some } i \neq j \text{ and an integer } k; \\ 1 & \text{, otherwise .} \end{cases}$$

4. Suppose that $\lambda \neq 0 \in K$ and $q \neq 0 \in K$ such that $q^4 \neq 1$. Let $\mathcal{H}_q(\lambda)$ be the factoralgebra of the quantum Heisenberg algebra $\mathcal{H}_q$ (see the end of the sec. 2). Then

$$\text{gl.dim}\ H_q(\lambda) = \begin{cases} 2 & \text{, if } q \text{ is a root of 1;} \\ 1 & \text{, if } q \text{ is not a root of 1.} \end{cases} \blacksquare$$

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