ON THE TOPOLOGICAL HOCHSCHILD HOMOLOGY OF $\mathbf{Z}/p^k\mathbf{Z}$

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The topological Hochschild homology (THH) was introduced by M. Bökstedt [1]. It was proved in [9] that THH and MacLene homology [7] are isomorphic. Recently B.I. Dundas and R. McCarthy [4] proved that THH and stable K-theory in the sense of F.Waldhausen are isomorphic. Using topological methods Bökstedt has calculated $THH(\mathbf{Z})$ and $THH(\mathbf{Z}/p\mathbf{Z})$ [2]. The last result is closely related with L. Breen's calculation [3]. Recently V. Franjou, J. Lannes and L. Schwartz give algebraical calculations of $THH(\mathbf{Z}/p\mathbf{Z})$ and $THH(\mathbf{Z},\mathbf{Z}/p\mathbf{Z})$ [5]. In this paper we give a similar calculation for $THH(\mathbf{Z}/p^k\mathbf{Z},\mathbf{Z}/p\mathbf{Z})$. For another calculation of THH see [8].

The main result of this paper is the following

THEOREM. Let $k > 1$. Then

$$\begin{aligned} dim\; THH_{2n}(\mathbf{Z}/p^k\mathbf{Z},\mathbf{Z}/p\mathbf{Z}) &= 1+[\frac{n}{p}] \\ dim\; THH_{2n-1}(\mathbf{Z}/p^k\mathbf{Z},\mathbf{Z}/p\mathbf{Z}) &= [\frac{n}{p}] \end{aligned}$$

Moreover the natural map

$$\alpha_k : THH_*(\mathbf{Z},\mathbf{Z}/p\mathbf{Z}) \longrightarrow THH_*(\mathbf{Z}/p^k\mathbf{Z},\mathbf{Z}/p\mathbf{Z})$$

is a monomorphism in each dimension and for any $m > k$ the restriction of the natural map

$$\beta_{mk} : THH_*(\mathbf{Z}/p^m\mathbf{Z},\mathbf{Z}/p\mathbf{Z}) \longrightarrow THH_*(\mathbf{Z}/p^k\mathbf{Z},\mathbf{Z}/p\mathbf{Z})$$

on the $Im(\alpha_m)$ is an isomorphism and is zero on the complement of $Im(\alpha_m)$.

Remark. Thus $THH(\mathbf{Z},\mathbf{Z}/p\mathbf{Z}) \longrightarrow\; lim\; THH(\mathbf{Z}/p^k\mathbf{Z},\mathbf{Z}/p\mathbf{Z})$ is an isomorphism.

Proof. We prefer to work with MacLane cohomology. Then theorem will follow from duality and [9]. For any small category $\mathbf{C}$ and ring R we denote the category of functors from $\mathbf{C}$ to R-mod by $\mathbf{C}$-R-mod. By Theorem A of [6] one has an isomorphism

$$H^*_{ML}(S,S) = Ext^*_{\mathbf{C}-S}(I,I)$$

where $\mathbf{C}$ is a full subcategory of the category S-mod containing all $S^n, n \geq 0$ and functor I is the inclusion $\mathbf{C} \longrightarrow S$-mod. For any functor

$$U : \mathbf{B} \longrightarrow \mathbf{A}$$

we write $U^* : \mathbf{A}$-S-mod$\longrightarrow \mathbf{B}$-$S$-mod for the induced functor. If U has a left adjoint F, then for each $X : \mathbf{A} \longrightarrow S$-mod and $Y : \mathbf{B} \longrightarrow S$-mod one has an obvious isomorphism:

$$Ext^*_{\mathbf{B}-S}(U^*X, Y) \cong Ext^*_{\mathbf{A}-S}(X, F^*Y).$$

As a particular case we obtain for any homomorphism of rings $f : R \longrightarrow S$ a natural isomorphism (compare also with 9.2 of [5])

$$H^*_{ML}(R, S) \cong Ext^*_{\mathbf{C}-R}(f^*I, f^*I)$$

where $f^* : \mathbf{C}$-S-$mod \longrightarrow \mathbf{C}$-$R$-$mod$ denotes the functor induced by f. It follows from the Grothendieck spectral sequence for composite functors, that for any functors

$$F : \mathbf{C} \longrightarrow S - mod, \; E : \mathbf{C} \longrightarrow R - mod$$

there exist a spectral sequence

$$E_2^{pq} = Ext^p_{\mathbf{C}-S}(F, Ext^q_R(S, E(-))) \Longrightarrow Ext^{p+q}_{\mathbf{C}-R}(f^*F, E).$$

Moreover the spectral sequence is compatible with natural left $Ext^*_{\mathbf{C}-R}(E, E)$-module and right $Ext^*_{\mathbf{C}-S}(F, F)$-modele structures.

Now we take $E = f^*F$ and $F = I$. It follows from the above isomorphism that $H^*_{ML}(R, S)$ has natural ring structure and there is a spectral sequence

$$E_2^{pq} = H^p_{ML}(S, Ext^q_R(S, S)) \Longrightarrow H^{p+q}_{ML}(R, S)$$

which is compatible with the left $H^*_{ML}(R, S)$-module and right $H^*_{ML}(S, S)$-module structures. We'll consider the case when $S = \mathbf{Z}/p\mathbf{Z}$ and $R = \mathbf{Z}$ or $\mathbf{Z}/p^k\mathbf{Z}$. The corresponding spectral sequences will be writen as $E^{**}_*(\mathbf{Z})$ and $E^{**}_*(\mathbf{Z}/p^k\mathbf{Z})$. In both cases one has an isomorphism of right $H^*_{ML}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$-modules:

$$(1) \qquad E_2^{*q}(R) = Ext^q_R(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z}) \otimes H^*_{ML}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$$

The left $H^*_{ML}(R, \mathbf{Z}/p\mathbf{Z})$-module structure on $E_2^{p*}(R)$ is induced from the ring homomorphism

$$\gamma_R^* : H^*_{ML}(R, \mathbf{Z}/p\mathbf{Z}) = Ext^*_{\mathbf{C}-R}(f^*I, f^*I) \longrightarrow Ext^*_R(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z}),$$

which is nothing but "take value on $\mathbf{Z}/p\mathbf{Z} \in \mathbf{C}$".

Let us denote by Λ the following graded algebra:

$$\Lambda = \mathbf{Z}/p\mathbf{Z}[e_0, e_1, ..., e_h, ...]/(e_h^p = 0; h \in \mathbf{N})$$

where $deg(e_h) = 2p^h, h \in \mathbf{N}$. Then the spectral sequence $E_*^{**}(\mathbf{Z})$ has the following properties [5]:

i) There is an isomorphism of graded algebras

$$H^*_{ML}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z}) \cong \Lambda.$$

ii) $E_2^{*q}(\mathbf{Z}) = 0$, if $q > 1$.

iii) If $q = 0$ (resp. $q = 1$), then $E_2^{*q}(\mathbf{Z})$ is a free graded right Λ-module, with rank 1 and $1 \otimes 1$ (resp. $u \otimes 1$) as a free generator. Here u is a canonical generator of $Ext^1_{\mathbf{Z}}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$. The bidegrees of such generators are (0,0) and (0,1) respectively.

iv) $d_2(u \otimes 1) = 1 \otimes e_0, d_r = 0, r > 0$.

v) If $q = 0, 1$, then $E_\infty^{*q} = E_3^{*q}$ is a free right $\Lambda/(e_0)$-module with rank 1, whose generators have bidegrees $(0, 0)$ in the case $q = 0$ and $(2p - 2, 1)$ in the case $q = 1$.

Now we consider the spectral sequence $E_*^{**}(\mathbf{Z}/p^k\mathbf{Z})$. We denote by $u_q, q \geq 0$ the canonical generator of

$$E_2^{0q}(\mathbf{Z}/p^k\mathbf{Z}) = Ext^q_{\mathbf{Z}/p^k\mathbf{Z}}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z}) \cong \mathbf{Z}/p\mathbf{Z}.$$

Then it is well known that in this Yoneda algebra one has:

$$u_0 = 1, u_{2n} = u_2^n, u_{2n+1} = u_2^n u_1.$$

According to (1) we know that $E_2^{*q}(\mathbf{Z}/p^k\mathbf{Z})$ is a free right Λ-module with rank 1 and a base u_q. The ring homomorphism $\mathbf{Z} \longrightarrow \mathbf{Z}/p^k\mathbf{Z}$ yields the homomorphism of spectral sequences

$$E_*^{**}(\mathbf{Z}/p^k\mathbf{Z}) \longrightarrow E_*^{**}(\mathbf{Z}),$$

which is an isomorphism on E_2^{*q} for $q = 0, 1$ and is 0 for $q > 1$. Hence all differentials which land on the lines $q = 0$, and $q = 1$ are zero, except those which are starting from the line

$q = 1$. In the last case these has same form as in $E_*^{**}(\mathbf{Z})$. Thus $d_2(u_1) = u_0 e_0, d_2(u_0) = 0$ and $E_2^{02}(\mathbf{Z}/p^k\mathbf{Z}) = H^2_{ML}(\mathbf{Z}/p^k\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$. Hence $\gamma^2_{\mathbf{Z}/p^k\mathbf{Z}}$ is an isomorphism. Let $v = (\gamma^2)^{-1}(u_2)$. Then $u_2 = vu_0$ and we obtain $u_{2n} = v^n u_0, u_{2n+1} = v^n u_1$. Observe that

$$d_2(u_{2n}) = 0, d_2(u_{2n+1}) = v^n u_0 e_0 \ and \ d_r = 0, \ for \ r > 2,$$

because $d_r(u_1) = d_r(u_0) = 0$ for $r > 2$, by a dimension argument. Thus $E_\infty^{*q}(\mathbf{Z}/p^k\mathbf{Z}) = E_3^{*q}(\mathbf{Z}/p^k\mathbf{Z})$ is a free module over the algebra $\Lambda/(e_0)$, with rank 1, whose generators has bidegree $(0, q)$, if q is even and $(2p-2, q)$, if q is odd. ¿From this and the fact that the homomorphism

$$E_2^{pq}(\mathbf{Z}/p^k\mathbf{Z}) \longrightarrow E_2^{pq}(\mathbf{Z}/p^{k+1}\mathbf{Z})$$

is zero, for $q > 1$, the theorem follows.

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