First Sincere Module

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1 Introduction

Throughout this note all algebras are assumed to be connect, basic finite-dimensional (associative) algebras with 1 over an algebraically closed field k and all modules are finite generated left modules. A preprojective component $\mathcal{P}$ of the Auslander-Reiten quiver of an algebra A is called a sincere preprojective component if there is a sincere indecomposable module M in $\mathcal{P}$. The main purpose of this note is to show the following

Theorem *Let $\mathcal{P}$ be a sincere preprojective component of the Auslander-Reiten quiver of an algebra A. Assume that the orbit graph of $\mathcal{P}$ is a tree. Then there exists unique sincere indecomposable module M in $\mathcal{P}$, such that $Hom_A(M, X) \neq 0$ for all sincere indecomposable A-module X.*

Note that in case A is a tree algebra the result was proved already by Unger in [**U**]. Thus the above result is a generalization of the main result in [**U**]. The module M determined by the theorem above is preprojective. Moreover all preprojective, sincere modules are successors of M. For this reason we call M the first sincere module. We will use the terminology of

[**R**]. We say that an indecomposable module V is a predecessor of a slice $\mathcal{S}$ if there exist some $[X] \in \mathcal{S}$ such that V is a predecessor of X. Let $\mathcal{P}$ be a sincere preprojective component of the Auslander-Reiten quiver of an algebra A. We denote by Ψ the module class consisting of all sincere indecomposable modules in $\mathcal{P}$. For an indecomposable module V in $\mathcal{P}$ and a slice $\mathcal{S}$ in $\mathcal{P}$, we denote by $\mathcal{S} \preceq V$ if V is a successor of some indecomposable module which lays on $\mathcal{S}$, and denote by $\mathcal{S} \preceq \Psi$ if $\mathcal{S} \preceq X$ for each $X \in \Phi$. For the slice $\mathcal{S}$, we put $\tau\mathcal{S}$=$\{\ \tau[X] \mid [X] \in \mathcal{S}\}$ where τ is the Auslander-Reiten translate.

The following result is a direct conseqence of the theorem since the orbit graph of a directed sincere algebra is a tree [**BLS**].

Corollary *Let A be a sincere directed algebra. Then there exists a unique sincere indecomposable module M such that $Hom_A(M, X) \neq 0$ for all sincere indecomposable modules X, and there exists a unique sincere indecomposable module N such that $Hom_A(X, N) \neq 0$ for all sincere indecomposable modules X.*

2 Proof of Theorem

Lemma 1 *Let $\mathcal{P}$ be a preprojective component of the Auslander-Reiten quiver of an algebra A, P a indecomposable projective module, and $\mathcal{S}$ a slice on $\mathcal{P}$. Then*

$$\{[X] \in \mathcal{S} | Hom_A(P, X) \neq 0\}$$

is connected.

Proof. Because $\mathcal{S}$ is a slice, by tilting theory, there is a hereditary algebra C and a tilting module ${}_CT$ with $\mathrm{End}_C(T) = A$ and

$$\{\mathrm{Hom}_C({}_CT, I) \mid I \textit{ is an indecomposable injective module}\} = \{[X] \in \mathcal{S}\}.$$

The functor $F = \mathrm{Hom}_C({}_cT, -)$ gives an equivalence between the full subcategory $\mathcal{G}$ of $C-$mod with objects generated by ${}_CT$ and the full subcategory

$\mathcal{Y}$ of $A-$mod with objects cogenerated by $D(T_A)$, and the inverse functor of F is given by $G =$
${}_CT_A \otimes -$ from $\mathcal{Y}$ to $\mathcal{G}$. As we know, $G(P)$ is an indecomposable direct summand of T. Thus,

$$\begin{aligned} & \{G(X) | [X] \in \mathcal{S} \text{ and } \mathrm{Hom}_A(P, X) \neq 0\} \\ = & \{Y \text{ indecomposable injective module} \mid \mathrm{Hom}_A(G(P), Y) \neq 0\} \\ = & \mathrm{supp}({}_CG(P)) \end{aligned}$$

is connected. This implies that

$$\{[X] \in \mathcal{S} | \mathrm{Hom}_A(P, X) \neq 0\}$$

is connected. □

Lemma 2 *Let $\mathcal{P}$ be a preprojective component of the Auslander-Reiten quiver of an algebra A and $\mathcal{S}$ a slice on $\mathcal{P}$. Assume that Y is a sincere indecomposable module which is not a predecessor of $\mathcal{S}$. Then*

$$\{[X] \in \mathcal{S} | Hom_A(X, Y) \neq 0\}$$

is connected.

Proof. In case Y is an injective module, the result holds by Lemma 1 since $\{[X] \in \mathcal{S}|\ \mathrm{Hom}_A(X,Y) \neq 0\}$=$\{[X] \in \mathcal{S}|\mathrm{Hom}_A(P,X) \neq 0\}$ where P is the projective cover of the socle of Y. So we can assume that Y is not injective. It is easy to see that $\tau^-\mathcal{S}$ is a slice on $\mathcal{P}$ because that $\mathcal{S}$ is a slice and the sincere module Y is not a predecessor of $\mathcal{S}$. By tilting theory, there is a hereditary algebra C and a tilting module ${}_CT$ with $\mathrm{End}_C(T) = A$ and $\{\ \mathrm{Ext}^1_C(T,P)|P$ is a indecomposable projective C-module $\} = \{[X] \in \tau^-\mathcal{S}\}$. The functor F'=$\mathrm{Ext}^1_C({}_CT, -)$ gives an equivalence between the full subcategory $\mathcal{F}$ of C-mod with objects cogenerated by $\tau_C T$ and the full subcategory $\mathcal{X}$ of A-mod with objects generated by $\tau^- D(T)$, and the inverse functor of F' is given by G'=$\mathrm{Tor}^A_1(T, -)$ from $\mathcal{X}$ to $\mathcal{F}$. Thus,

$$
\begin{aligned}
& \{G'(X) | [X] \in \tau^- \mathcal{S} \text{ and } \mathrm{Hom}_A(X, \tau^- Y) \neq 0\} \\
= & \ \{P \text{ indecomposable injective module} \mid \mathrm{Hom}_A(P, G'(\tau^- Y)) \neq 0\} \\
= & \ \mathrm{supp}({}_C G'(\tau^- Y))
\end{aligned}
$$

is connected. This implies that

$$\{[X] \in \tau^- \mathcal{S} | \mathrm{Hom}_A(X, \tau^- Y) \neq 0\}$$

is connected and

$$\{[X] \in \mathcal{S} | \mathrm{Hom}_A(X, Y) \neq 0\}$$

is also conected. $\square$

Lemma 3 *Let $\mathcal{P}$ be a sincere preprojective component of the Auslander-Reiten quiver of an algebra A. Assum that the orbit graph of $\mathcal{P}$ is a tree. Then there exists an unique sincere indecomposable module M such that all sincere indecomposable modules in $\mathcal{P}$ are successors of M.*

Proof. First, we take an indecomposable projective module U with the property that all indecomposable projective modules except U are not successors of $\sum([U] \to)$. Then the following assertions are clear.

(i). $\sum([U] \to) \preceq \Psi$;

(ii). $\sum([U] \to)$ is sincere.

So, the set

$$
\begin{aligned}
\Omega = & \ \{X \text{ indecomposable module in } \mathcal{P} \mid \\
& \quad \textstyle\sum([X] \to) \preceq \Psi \text{ and } \sum([X] \to) \text{ is sincere } \}
\end{aligned}
$$

is not empty and Ω is finite. Then, we take M in Ω with the most number of predecessors. We claim that M is sincere. For, if M is not sincere, then there is some primitive idempotent $e(i)$ of A, such that $\mathrm{Hom}_A(P(i), M) = 0$ where $P(i) := Ae(i)$. Since $\sum([M] \to)$ is sincere, by Lemma 1 we can take an indecomposable module Y lying on $\sum([M] \to)$ with conditions (a). $\mathrm{Hom}_A(P(i), Y) \neq 0$ and (b). the indecomposable module X lying $\sum([M] \to)$

with $\mathrm{Hom}_A(P(i), X) \neq 0$ is a successor of Y. Now we want to prove $Y \in \Omega$. In order to prove that $\sum([Y] \rightarrow) \preceq \Psi$, it is enough to show the assertion that $\mathrm{Hom}_A(P(i), X) = 0$ for any $[X]$ which is a successor of $\sum([M] \rightarrow)$ and is not a successor of $\sum([Y] \rightarrow)$. If this assertion is not true, then there is some $[X]$ which is a successor of $\sum([M] \rightarrow)$ and is not a successor of $\sum([Y] \rightarrow)$ with $\mathrm{Hom}_A(P(i), X) \neq 0$. So there is a section including $[Y]$, $[X]$ and the direct predecessor $[Z]$ of $[Y]$ lying on $\sum([M] \rightarrow)$. By Lemma 1, we see $\mathrm{Hom}_A(P(i), Z) \neq 0$. A contradication with (b). Thus we have proved that $\sum([Y] \rightarrow) \preceq \Psi$. And this implies that $\sum([Y] \rightarrow)$ is sincere. So $Y \in \Omega$. It is a contradication since the number of predecessors of Y is bigger than the number of predecessors of M. The uniqueness follows from $\mathcal{P}$ directing. $\square$

Proof of Theorem. By Lemma 3, there is a unique sincere indecomposable module M in $\mathcal{P}$ such that all sincere modules in $\mathcal{P}$ are successor of M. For given a sincerer indecomposable module Y, we suppose $\mathrm{Hom}_A(M, Y) = 0$. So M is not a projective module. Let

$$0 \rightarrow \tau M \rightarrow \oplus_{j=1}^{l} M_j^- \rightarrow M \rightarrow 0$$

is the Auslander-Reiten sequence. Note that M is sincere. This implies that $\oplus_{j=1}^{l} M_j^-$ is sincere, so $l > 1$. For convenience we will identity a slice $\mathcal{S}$ with the subquiver of the Auslander-Reiten quiver consisting of all $[X]$ which belong to $\mathcal{S}$. Since $\sum(\rightarrow [M])$ is a tree, so $\sum(\rightarrow [M])\backslash\{[M]\}$ is a disjoint union of some sections $\sum_j, 1 \leq j \leq l$, and $\sum_j$ has the unique sink $[M_j^-]$. We claim that there exist two different $\sum_s$ and $\sum_t$ such that $\mathrm{Hom}_A(\oplus_{Z \in \sum_s} Z, Y) \neq 0$ and $\mathrm{Hom}_A(\oplus_{Z \in \sum_t} Z, Y) \neq 0$. For, given an indecomposable projective module $P(i)$, we have $\mathrm{Hom}_A(P(i), Y) \neq 0$. And given non-zero morphism from $P(i)$ to X factors through the slice $\sum(\rightarrow [M])$. Further, factors through some $\sum_s$. This means $\mathrm{Hom}_A(\oplus_{Z \in \sum_s} Z, Y) \neq 0$. Now we consider the non-sincere module M_s^-. There is an indecomposable projective module $P(j)$ with $\mathrm{Hom}_A(P(j), M_s^-) = 0$. Note $i \neq j$ from Lemma 1. Thus, given an non-zero morphism from $P(j)$ to Y can not factor through $\sum_s$ from Lemma 1 again. So it factors through another section $\sum_t$. This means $\mathrm{Hom}_A(\oplus_{Z \in \sum_t} Z, Y) \neq 0$.

Now, because that the orbit graph of $\mathcal{P}$ is a tree, we see $\mathrm{Hom}_A(M, Y) \neq 0$ by Lemma 2. A contradication. $\Box$

Remark The following example shows that the condition that the orbit graph of $\mathcal{P}$ is a tree in Theorem is necessary.

Example Let A be an algebra given by the quiver

$$\begin{tikzcd} & 3 & \\ 1 \arrow[ur] \arrow[dr] & & 2 \arrow[ul] \arrow[dl] \\ & 4 & \end{tikzcd}$$

We denote by $P(i)$ the indecomposable projective module corresponding to the vertex i. Then $\tau^{-}P(1), \tau^{-}P(2)$ are sincere and $\mathrm{Hom}_A(\tau^{-}P(1), \tau^{-}P(2)) = 0$, $\mathrm{Hom}_A(\tau^{-}P(2), \tau^{-}P(1)) = 0$.

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