Double of the Yangian and rational R-matrices

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Abstract

Studying the algebraic structure of the double $\mathcal{D}Y(g)$ of the yangian $Y(g)$ we present the triangular decomposition of $\mathcal{D}Y(g)$ and a factorization for the canonical pairing of the yangian with its dual inside $\mathcal{D}Y(g)$. As a consequence we obtain an explicit formula for the universal R-matrix of $\mathcal{D}Y(g)$ and demonstrate how it works in evaluation representations of $Y(sl_2)$. We interprete one-dimensional factor arising in concrete representations of R as bilinear form on highest weight polynomials of irreducible representations of $Y(g)$ and express this form in terms of Γ functions.

1 Introduction

Yangian $Y(g)$ of a simple Lie algebra g was introduced by V.Drinfeld [D1] as a deformation of universal enveloping algebra $U(g[t])$ of a current algebra $g[t]$. The yangians $Y(g)$ and quantum affine algebras $U_q(\hat{g})$ play the role of dynamical symmetries in quantum field theories [BL] [S]. Tensor products of finite-dimensional representations of yangians produce rational solutions of the Yang-Baxter equation; tensor products of finite-dimensional representations of quantum affine algebras produce trigonometric solutions of the Yang-Baxter equation. One can find out other deep parallels in representation theories of yangians and of quantum affine algebras. Nevertheless both of them have their own original features. Yangian $Y(g)$ is much more closer to classical Lie algebras, at least it contains universal enveloping algebra $U(g)$ as a subalgebra; moreover the yangians $Y(gl_n)$ could be defined entirely in terms of classical representation theory [O]. The structure of

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quantum affine $U_q(\hat{g})$ is more complicated. On the other hand, $U_q(\hat{g})$ inhabit the main properties of contragredient algebras; Chevalley generators and q-deformed Serre relations are permanent participants of the games with quantum affine algebras.

The theory of Cartan-Weyl basis [KT2] [KT3] allows to describe explicitely the universal R-matrix, one of the main object in physical applications. We have nothing of this for the yangians, though suitable modifications of classical methods of representation theory work for $Y(g)$ and $U_q(\hat{g})$ as well (see [Ch] [D1] [J] [NT] for instance). We want to make this gap smaller.

It is more reasonable to work with quantum double $\mathcal{D}Y(g)$ of the yangian, if we take in mind physical applications. The representations of $\mathcal{D}Y(g)$ do not differ much from the representations of yangian $Y(g)$: the extension of a representatin of $Y(g)$ to representation of $\mathcal{D}Y(g)$ can be achieved just by reexpansion of the currents from $Y(g)$ in other point of projective line. We present here a study of some algebraic properties of $\mathcal{D}Y(g)$, with the accent to the canonical pairing in the double.

We prove that $\mathcal{D}Y(g)$ itself and a Hopf pairing of $Y(g)$ with its dual inside $\mathcal{D}Y(g)$ admit triangular decomposition analogous to Gauss decomposition of ordinary matrices. This property gives possibility to describe the pairing quite explicitely. As a consequency we obtain explicite factorized expression for the universal R-matrix of $\mathcal{D}Y(g)$ (completely proved for $\mathcal{D}Y(sl_2)$ and partially in general case). To make the formulas more transparant, we present detailed calculations for $\mathcal{D}Y(sl_2)$, including the action of the universal R-matrix on evaluation representations.

The most interesting factor R_H of the universal R-matrix is concerned with (zero charge) Heisenberg subalgebra of $\mathcal{D}Y(g)$ which is a deformation of the currents to Cartan subalgebra h of g. Analogously to the case of $U_q(\hat{g})$ [KT2] [KST] the structure of R_H is governed by the q-analog of invariant scalar product in h; whenever R_H acts on representations of $\mathcal{D}Y(g)$ a variable q becomes a shift operator $T:\ Tf(x) = f(x+1)$ (for quantum affine algebras q goes to multiplicative shift $T_qf(x) = f(qx)$ in analogous situation).

After substitution of the universal R-matrix into tensor product of concrete representations of $\mathcal{D}Y(g)$ we obtain more then usual rational R-matrix; an additional information is concentrated in scalar phase factor (scalar S-matrix) which we interprete as bilinear multiplicative form on highest weight polynomials of finite-dimensional representations V of $Y(g)$ (or, equivalently, on $K_0(RepY(g))$. This form is a deformation of skewsymmetric form $\frac{<,>}{a-b}$ on irreducible evaluation representations of $g[t]$ where $<,>$ is invariant scalar product in h^*; a and b are the points where evaluation representations are living. We present an explicit expression of this form as some ratio of Γ functions defined by the structure of q-analog of invariant scalar product in h.

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2 Yangian $Y(g)$ and its quantum double

Let g be a simple Lie algebra with Cartan matrix $A_{i,j}$ and Chevalley generators $e_i = e_{\alpha_i}$, $h_i = h_{\alpha_i}$, $f_i = f_{\alpha_i} = e_{-\alpha_i}$, $i = 1, \ldots, r$. Yangian $Y(g)$ is a deformation of the universal enveloping algebra $U(g[t])$. It can be defined [D1], [D3] as a Hopf algebra with generators $e_{i,k} = e_{\alpha_i,k}$, $h_{i,k} = h_{\alpha_i,k}$, $f_{i,k} = f_{\alpha_i,k}$, $i = 1, \ldots, r$, $k = 0, 1, \ldots$ subjected to relations

$$[h_{i,k}, h_{j,l}] = 0, \qquad [h_{i,0}, e_{j,l}] = (\alpha_i, \alpha_j) e_{j,l},$$

$$[h_{i,0}, f_{j,l}] = -(\alpha_i, \alpha_j) f_{j,l}, \qquad [e_{i,k}, f_{j,l}] = \delta_{i,j} h_{i,k+l}$$

,

$$[h_{i,k+1}, e_{j,l}] - [h_{i,k}, e_{j,l+1}] = \frac{1}{2}(\alpha_i, \alpha_j)\{h_{i,k}, e_{j,l}\} \tag{2.1}$$

where $\{a, b\} = ab + ba$,

$$[h_{i,k+1}, f_{j,l}] - [h_{i,k}, f_{j,l+1}] = -\frac{1}{2}(\alpha_i, \alpha_j)\{h_{i,k}, f_{j,l}\}$$

$$[e_{i,k+1}, e_{j,l}] - [e_{i,k}, e_{j,l+1}] = \frac{1}{2}(\alpha_i, \alpha_j)\{e_{i,k}, e_{j,l}\}$$

$$[f_{i,k+1}, f_{j,l}] - [f_{i,k}, f_{j,l+1}] = -\frac{1}{2}(\alpha_i, \alpha_j)\{f_{i,k}, f_{j,l}\} \tag{2.2}$$

$$i \neq j,\ n_{i,j} = 1 - A_{i,j} \longrightarrow \left\{ \begin{array}{l} \mathrm{Sym}_{\{k\}}[e_{i,k_1}[e_{i,k_2} \ldots [e_{i,k_{n_{i,j}}}, e_{j,l}] \ldots]] = 0 \\ \mathrm{Sym}_{\{k\}}[f_{i,k_1}[f_{i,k_2} \ldots [f_{i,k_{n_{i,j}}}, f_{j,l}] \ldots]] = 0 \end{array} \right. \tag{2.3}$$

Subalgebra generated by $e_{i,0}$, $f_{i,0}$, $h_{i,0}$, $i = 1, 2, \ldots, r$ is naturally isomorphic to $U(g)$ by (2.1)-(2.2). Using this isomorphism we omit sometimes index zero for these generators and consider the elements e_γ, f_γ, h_γ, $\gamma \in \Delta_+(g)$ of Lie algebra g as elements of $Y(g)$. Here $\Delta_+(g)$ is a system of all positive roots of g and the root vectors e_γ, f_γ, h_γ are normalized by the condition $[e_\gamma, f_\gamma] = h_\gamma$.

One can deduce from original description of $Y(g)$ [D1] the following formulas of comultiplication for basic generators of $Y(g)$:

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad x \in g,$$

$$\Delta(e_{i,1}) = e_{i,1} \otimes 1 + 1 \otimes e_{i,1} + h_{i,0} \otimes e_{i,0} - \sum_{\gamma \in \Delta_+(g)} f_\gamma \otimes [e_{\alpha_i}, e_\gamma],$$

$$\Delta(f_{i,1}) = f_{i,1} \otimes 1 + 1 \otimes f_{i,1} + f_{i,0} \otimes h_{i,0} + \sum_{\gamma \in \Delta_+(g)} [f_{\alpha_i}, f_\gamma] \otimes e_\gamma,$$

$$\Delta(h_{i,1}) = h_{i,1} \otimes 1 + 1 \otimes h_{i,1} - \sum_{\gamma \in \Delta_+(g)} (\alpha_i, \gamma) f_\gamma \otimes e_\gamma \tag{2.4}$$

In this paper we are much more interested in quantum double $\mathcal{D}Y(g)$ of $Y(g)$ (see [D3] for definitions). The algebraic structure of $\mathcal{D}Y(g)$ can be described as follows [D1].

Let $C(G)$ be an algebra generated by the elements $e_{i,k}$, $f_{i,k}$, $h_{i,k}$, $i = 1, \ldots, r$, $k \in \mathbf{Z}$ subjected to relations (2.2)-(2.3). Algebra $C(g)$ admits $\mathbf{Z}$-filtration

$$\ldots \subset C_n \subset C_{n+1} \ldots \subset C(g)$$

defined by the condition $\deg e_{i,k} = \deg f_{i,k} = \deg h_{i,k} = k$. Let $\widehat{C}(g)$ be the corresponding formal completion of $C(g)$. Actually Drinfeld [D1] proved that the double $\mathcal{D}Y(g)$ is isomorphic to $\widehat{C}(g)$ as an algebra. Generators $e_{i,k}$, $f_{i,k}$, $h_{i,k}$, $i = 1,\ldots,r$, $k \geq 0$ define an inclusion $Y(g) \hookrightarrow \mathcal{D}Y(g)$. We denote sometimes its image by $Y_+(g)$ or shortly by Y when we need short notation. The dual Hopf algebra with opposite comultiplication $Y(g)^0$is isomorphic to $Y_-(g) = Y_-$ which is generated by formal series $\sum_{k<0} a_k$, $\deg a_k = k$, and in this sense is generated by the elements $e_{i,k}$, $f_{i,k}$, $h_{i,k}$, $i = 1,\ldots,r$, $k < 0$ (see also Remark to Theorem 4.2). To complete the description of $\mathcal{D}Y(g)$ one should describe the structure of comultiplication in $Y_-(g)$ and the pairing between $Y_+(g)$ and $Y_-(g)$. This will be done below for $g = sl_2$ and partially for general case.

3 Triangular decomposition of $\mathcal{D}Y(g)$

Let E', H' and F' be subalgebras without unit element of $Y(g)$ generated by the elements $e_{i,k}$, $i = 1,\ldots,r$, $k \geq 0$; $h_{i,k}$, $i = 1,\ldots,r$, $k \geq 0$; $f_{i,k}$, $i = 1,\ldots,r$, $k \geq 0$ correspondingly. We denote also by E, H and F the algebras E', H' and F' with added unit element. Following [CP1] one can deduce from (2.2)- (2.3) the following decomposition of $Y(g)$:

Proposition 3.1 *A multoplication in $Y(g)$ induces an isomorphism of vector spaces*

$$E \otimes H \otimes F \rightsquigarrow Y(g) \tag{3.1}$$

We are going to extend this decomposition to the double $\mathcal{D}Y(g)$ and factorize the natural pairing of $Y_+(g)$ and $Y^0(g) \simeq Y_-(g)$ with respect to this decomposition. First we summarize the properties of comultiplication in $Y = Y(g)$ which easily generalize by induction the formulas (2.4) (see also [CP1]).

Lemma 3.1

(i) For any $e \in E'$

$$\Delta(e) = e \otimes 1 \qquad mod Y \otimes E'; \tag{3.2}$$

(ii) For any $f \in F'$

$$\Delta(f) = 1 \otimes f \qquad mod F' \otimes Y. \tag{3.3}$$

In particular, we conclude that E is a right coideal in Y ($\Delta(E) \subset Y \otimes E$) and F is a right coideal ($\Delta(F) \subset F \otimes Y$) of $Y(g)$

Let also $\overline{E}$ be a subalgebra (without unit element) of $Y(g)$ generated by the elements $e_{i,k}$ and $h_{j,l}$, $i,j = 1,\ldots,r$, $k,l \geq 0$; $\overline{F}$ be a subalgebra (without unit element) of $Y(g)$ generated by the elements $f_{i,k}$ and $h_{j,l}$, $i,j = 1,\ldots,r$, $k,l \geq 0$. We have also

Lemma 3.2

(i) For any $e \in \overline{E}$

$$\Delta(e) = e \otimes 1 \qquad mod Y \otimes \overline{E}; \tag{3.4}$$

(ii) For any $f \in \overline{F}$

$$\Delta(f) = 1 \otimes f \qquad mod \overline{F} \otimes Y. \tag{3.5}$$

(iii) For any $h \in H'$

$$\Delta(h) = h \otimes 1 \quad mod Y \otimes \overline{E} = $$
$$= 1 \otimes h \quad mod \overline{F} \otimes Y. \tag{3.6}$$

Let $<,>$ denotes the canonical Hopf pairing of $Y_+(g)$=$Y(g)$ and its dual $Y^0(g)$. The Hopf property of $<,>$ in this case can be read as

$$< a, cd >=< \Delta(a), c \otimes d >, \qquad < ab, c >=< b \otimes a, \Delta(c) >$$

for any $a, b \in Y_+(g)$, and for any $c, d \in Y^0(g)$. Here $< a \otimes b, c \otimes d >=< a, c >< b, d >$.

Let now $E^* \subset Y^0(g)$ be defined as annulator $(Y_+\overline{F})^\perp$ of $Y_+\overline{F}$, that is

$$E^* = \{e^* \in Y^0 : < y\overline{f}, e^* >= 0 \ \ \forall y \in Y(g), \ \forall \overline{f} \in \overline{F}\}. \tag{3.7}$$

Analogously, we define

$$F^* = (\overline{E}Y_+)^\perp, \qquad \overline{E}^* = (Y_+F')^\perp, \qquad \overline{F}^* = (E'Y_+)^\perp \tag{3.8}$$

and

$$H^* = (E'Y_+)^\perp \cap (Y_+F')^\perp. \tag{3.9}$$

The following lemma follows directly from the definition of the Hopf pairing.

Lemma 3.3 *Let A and A^* be two Hopf algebras with a Hopf pairing $<,>: A \otimes A^* \to \mathbf{C}$. Let a subset $X \subset A$ satisfies the condition:*

$$\Delta(X) \subset X \otimes A + A \otimes X.$$

Then both $(AX)^\perp$ and $(XA)^\perp$ are subalgebras of A^*.

Corollary 3.1 *$E^*, \overline{E}^*, F, \overline{F}^*$ are subalgebras of $Y^0(g)$.*

The main result of this section is the following

Theorem 3.1 *For any $e \in E$, $h \in H$, $f \in F$; $e^* \in E^*$, $h^* \in H^*$, $f^* \in F^*$ there is a factorization of canonical pairing:*

$$< ehf, e^*h^*f^* >=< e, e^* >< h, h^* >< f, f^* > \tag{3.10}$$

As a consequence of Theorem 3.1 we have also

Corollary 3.2 *A multiplication in $Y^0(g)$ induces an isomorphism of vector spaces*

$$E^* \otimes H^* \otimes F^* \rightsquigarrow Y^0(g) \tag{3.11}$$

The proof of Theorem 3.1 is strongly based on the statements of Lemma 3.1 and of Lemma 3.2. For instance, we check first that

$$< e\overline{f}, e^*\overline{f}^* >=< e, e^* >< \overline{f}, \overline{f}^* >$$

for any $e \in E'$, $e^* \in E^*$, $\overline{f} \in \overline{F}$, $\overline{f}^* \in \overline{F}^*$. Indeed, we have by definition

$$< e\overline{f}, e^*\overline{f}^* >==< \Delta(e)\Delta(\overline{f}), e^* \otimes \overline{f}^* >=$$

$$=< (e \otimes 1 + \sum y_i \otimes e_i)(1 \otimes \overline{f} + \sum \overline{f}_j \otimes y_j), e^* \otimes \overline{f}^* > \tag{3.12}$$

where $e_i \in E'$, $\overline{f}_j \in \overline{F}$ by (3.2)-(3.5). The rhs of (3.12) is equal to $< e \otimes \overline{f}, e^* \otimes \overline{f}^* >$ by definition of E^* and $\overline{F}^*$. Then we prove analogously that $< eh, e^*h^* >=< e, e^* >< h, h^* >$ for any $e \in E'$, $h \in H'$, $e^* \in E^*$, $h^* \in H^*$ and then take off the primes.

Now we proceed with a more detailed study of the pairing in yangian double. In the next section we compute explicitely the pairing between the generators of $Y_+(g)$ and of $Y_-(g) \simeq Y^0(g)$.

4 Basic pairing for $\mathcal{D}Y(g)$

The aim of this section is to compute explicitely the pairing between generators $e_{i,k}$, $h_{i,k}$ and $f_{i,k}$ of $Y_+(g)$ and of $Y_-(g) \simeq Y^0(g)$. The answer will be written in terms of generating functions ("fields") $e_{i,\pm}(u)$, $h_{i,\pm}(u)$ and $f_{i,\pm}(u)$ of $Y_\pm(g)$:

$$e_{i,+}(u) = \sum_{k\geq 0} e_{i,k}u^{-k-1}, \qquad f_{i,+}(u) = \sum_{k\geq 0} f_{i,k}u^{-k-1},$$

$$h_{i,+}(u) = 1 + \sum_{k\geq 0} h_{i,k}u^{-k-1}$$

generate $Y_+(g)$ and

$$e_{i,-}(u) = -\sum_{k<0} e_{i,k}u^{-k-1}, \qquad f_{i,-}(u) = -\sum_{k<0} f_{i,k}u^{-k-1},$$

$$h_{i,-}(u) = 1 - \sum_{k<0} h_{i,k}u^{-k-1}$$

generate $Y_-(g)$.

Explicit calculations will be done for the case of $Y(sl_2)$ where we omit for simplicity everywhere an index 1 of a simple root (for instance, for generating functions we use the notations $e_\pm(u)$, $h_\pm(u)$ and $f_\pm(u)$).

The main result of this section for $\mathcal{D}Y(sl_2)$ may be formulated in the following theorem:

Theorem 4.1 *(i) Subalgebras E^*, H^* and F^* (see (3.7)-(3.9)) of $Y_-(sl_2)$ are generated by the fields $e_-(u)$, $h_-(u)$ and $f_-(u)$ correspondingly; (ii) The pairing of the generators of $Y_\pm(sl_2)$ is given by the relations*

$$< e_+(u), f_-(x) >= \frac{1}{u-x}, \qquad < f_+(u), e_-(x) >= \frac{1}{u-x}, \tag{4.1}$$

$$< h_+(u), h_-(x) >= \frac{u-x+1}{u-x-1} \qquad | x | \ll 1 \ll | u | \tag{4.2}$$

or, in terms of generators,

$$< e_k, f_{-l} >=< f_k, e_{-l} >= -\delta_{k,l-1}, \qquad k \geq 0,\ l > 0,$$

$$< h_k, h_{-l} >= -2\frac{k!}{(l-1)!(k-l+1)!} \qquad k \geq 0,\ l > 0$$

In general case we have analogous

Theorem 4.2 *(i) Subalgebras E^*, H^* and F^* (see (3.7)-(3.9)) of $Y_-(g)$ are generated by the fields $e_{i,-}(u)$, $h_{i,-}(u)$ and $f_{i,-}(u)$ correspondingly ($i = 1, \ldots, r$);*

(ii) The pairing of the generators of $Y_\pm(g)$ is given by the relations

$$< e_{i,+}(u), f_{j,-}(x) >= \frac{\delta_{i,j}}{u-x}, \qquad < f_{i,+}(u), e_{j,-}(x) >= \frac{\delta_{i,j}}{u-x}, \tag{4.3}$$

$$< h_{i,+}(u), h_{j,-}(x) >= \frac{u-x+\frac{1}{2}(\alpha_i,\alpha_j)}{u-x-\frac{1}{2}(\alpha_i,\alpha_j)} \qquad | x | \ll 1 \ll | u | \tag{4.4}$$

Remark. As a corollary of Theorem 4.2 we obtain a rigorous proof of Drinfeld's Theorem, mentioned in Section 2: the double $\mathcal{D}Y(g)$ is isomorphic to an algebra $\widehat{C}(g)$, in particular, dual to $Y(g)$ Hopf algebra with opposite comultiplication $Y^0(g)$ is isomorphic to $Y_-(g)$.

We can describe explicitely the yangian $Y(sl_2)$ and its quantum double in terms of generating functions. One can check that the defining relation (2.1)-(2.3) for $Y(sl_2)$ are equivalent to the following conditions on $e_+(u)$, $h_+(u)$, $f_+(u)$:

$$[h_+(u), h_+(v)] = 0$$

$$[e_+(u), e_+(v)] = \frac{(e_+(v)-e_+(u))^2}{v-u}, \qquad [f_+(u), f_+(v)] = \frac{(f_+(u)-f_+(v))^2}{u-v},$$

$$[h_+(u), e_+(v)] = \frac{\{h_+(u), e_+(v)-e_+(u)\}}{u-v},$$

$$[h_+(u), f_+(v)] = \frac{\{h_+(u), f_+(u)-f_+(v)\}}{u-v},$$

$$[e_+(u), f_+(v)] = \frac{h_+(u)-h_+(v)}{u-v}. \tag{4.5}$$

Moreover, A.I.Molev [M] showed that the comultiplication in $Y(sl_2)$ can be described as follows:

$$\Delta(e_+(u)) = e_+(u) \otimes 1 + \sum_{k=0}^{\infty} (-1)^k f_+^k(u+1) h_+(u) \otimes e_+^{k+1}(u) =$$

$$= e_+(u) \otimes 1 + \sum_{k=0}^{\infty} (-1)^k h_+(u) f_+^k(u-1) \otimes e_+^{k+1}(u), \tag{4.6}$$

$$\Delta(f_+(u)) = 1\otimes f_+(u) + \sum_{k=0}^{\infty}(-1)^k e_+^{k+1}(u)\otimes h_+(u)e_+^k(u+1) =$$

$$= 1\otimes f_+(u) + \sum_{k=0}^{\infty}(-1)^k e_+^{k+1}(u)\otimes e_+^k(u-1)h_+(u) \tag{4.7}$$

$$\Delta(h_+(u)) = \sum_{k=0}^{\infty}(-1)^k(k+1)f_+^k(u+1)h_+(u)\otimes h_+(u)e_+^{k+1}(u+1) =$$

$$= \sum_{k=0}^{\infty}(-1)^k(k+1)h_+(u)f_+^k(u-1)\otimes e_+^{k+1}(u-1)h_+(u), \tag{4.8}$$

The generators of $\mathcal{D}Y(sl_2)$ satisfy the relations

$$[h_\pm(u),h_\pm(v)]=0, \qquad [h_+(u),h_-(v)]=0$$

$$[e_\pm(u),e_\pm(v)] = \frac{(e_\pm(v)-e_\pm(u))^2}{v-u}, \qquad [f_\pm(u),f_\pm(v)] = \frac{(f_+(u)-f_+(v))^2}{u-v},$$

$$[h_\pm(u),e_\pm(v)] = \frac{\{h_\pm(u),e_\pm(v)-e_\pm(u)\}}{u-v},$$

$$[h_\pm(u),f_\pm(v)] = \frac{\{h_\pm(u),f_\pm(u)-f_\pm(v)\}}{u-v},$$

$$[e_\pm(u),f_\pm(v)] = \frac{h_\pm(u)-h_\pm(v)}{u-v},$$

$$[e_+(u),e_-(v)] = \frac{(e_+(v)-e_-(u))^2}{v-u}, \qquad [f_+(u),f_-(v)] = \frac{(f_+(u)-f_-(v))^2}{u-v},$$

$$[h_+(u),e_-(v)] = \frac{\{h_+(u),e_-(v)-e_+(u)\}}{u-v},$$

$$[h_-(u),e_+(v)] = \frac{\{h_-(u),e_+(v)-e_-(u)\}}{u-v},$$

$$[h_+(u),f_-(v)] = \frac{\{h_+(u),f_+(u)-f_-(v)\}}{u-v},$$

$$[h_-(u),f_+(v)] = \frac{\{h_+(u),f_-(u)-f_+(v)\}}{u-v},$$

$$[e_+(u),f_-(v)] = \frac{h_-(v)-h_+(u)}{u-v}, \qquad [e_-(u),f_+(v)] = \frac{h_+(v)-h_-(u)}{u-v}. \tag{4.9}$$

For the comultiplication in $Y_-(g)$ one can use formulas (4.6), (4.7) with subindex $+$ replaced everywhere by $-$.

The rest of the section is devoted to the proof of the Theorem 4.1. (the proof of Theorem 4.2 is quite analogous).

For the proof of Theorem 4.1 we need a stronger version of Lemmas 3.1 and 3.2 for $Y=Y(sl_2)$. One can find them in [CP1] or deduce directly from (4.6)-(4.8). In the notations of the previous section we have

Proposition 4.1 *[CP1] The following properties of comultiplication are valid for $Y(sl_2)$:*

$$\Delta(e_+(u)) = e_+(u)\otimes 1 + h_+(u)\otimes e_+(u) \mod YF\otimes E,$$
$$\Delta(f_+(u)) = 1\otimes f_+(u) + f_+(u)\otimes h_+(u) \mod F\otimes EY, \tag{4.10}$$

$$\Delta(h_+(u)) = h_+(u)\otimes h_+(u) \mod YF\otimes EY.$$

We can use rhs of (4.10) in order to compute for instance the terms of $\Delta(e_+(u)e_+(v))$ which give nonzero contribution to the pairing with elements of $E^*\otimes E^*$. More precisely, using commutation relations (4.3) we have:

$$\Delta(e_+(u)e_+(v)) = e_+(u)e_+(v)\otimes 1 + 1\otimes e_+(u)e_+(v)+$$
$$+\frac{u-v+1}{u-v-1}e_+(v)\otimes e_+(u) - \frac{2}{u-v-1}e_+(u-1)\otimes e_+(u) \qquad \mod YF\otimes E, \tag{4.11}$$

$$\Delta(f_+(u)f_+(v)) = f_+(u)f_+(v)\otimes 1 + 1\otimes f_+(u)f_+(v))+$$

$$+\frac{u-v-1}{u-v+1}f_+(v)\otimes f_+(u) - \frac{2}{u-v+1}f_+(v)\otimes f_+(v-1) \qquad \mod F\otimes EY, \tag{4.12}$$

$$\Delta(h_+(u)f_+(v)) = h_+(u)\otimes h_+(u)f_+(v) + h_+(u)f_+(v)\otimes h_+(u)h_+(v),$$

$$\Delta(e_+(u)h_+(v)) = e_+(u)h_+(v)\otimes h_+(v) + h_+(u)h_+(v)\otimes e_+(u)h_+(v) \mod YF\otimes EY \tag{4.13}$$

Let now $e^*(x)$ be a generating function of some elements $e_i^*\in E^*$, $e^*(x)=\sum_{i<0}e_i^*x^{-i-1}$ such that the pairing $<e_+(u), e^*(x)>= E(x-u)$, $\mid x\mid \ll 1 \ll\mid u\mid$ depends on $u-x$ only. Analogously, let $f^*(x)$ be a generating function of some elements $f_i^*\in F^*$, $f^*(x)=\sum_{i<0}f_i^*x^{-i-1}$ such that the pairing $<f_+(u), f^*(x)>= F(x-u)$, $\mid x\mid \ll 1 \ll\mid u\mid$ depends on $u-x$ only. Using (4.10), (4.11) one can prove the following

Proposition 4.2 *(i) The conditions*

$$<e_+(u), e^*(x)>= E(u-x) = \frac{\alpha}{u-x} \qquad \textit{for some } \alpha\in \mathbf{C} \tag{4.14}$$

and

$$[e^*(x), e^*(y)] = \frac{(e^*(x)-e^*(y)^2}{x-y} \tag{4.15}$$

are equivalent;

(ii) the conditions

$$<f_+(u), f^*(x)>= F(u-x) = \frac{\beta}{u-x} \qquad \textit{for some } \beta\in \mathbf{C} \tag{4.16}$$

and

$$[f^*(x), f^*(y)] = \frac{(f^*(y)-f^*(x)^2}{x-y} \tag{4.17}$$

are equivalent;

Let now $h^*(x) = 1 + \sum_{i<0} h_i^* x^{-i-1}$ be a generating function of some elements $h_i^* \in H^*$ such that
(i) a pairing $< h_+(u), h^*(x) >= H(u-x)$ depends on $u-x$ only;
(ii) the pairing $H(u-x)$ is multiplicative on $h_+(u)$: $< h_+(u)h_+(v), h^*(x) >= H(v-x)H(u-x)$ (the last condition is natural due to multiplicative structure of $\Delta(h_+(u))$ mod $YF \otimes EY$).

The calculations analogous to ones from Proposition 4.2 demonstrate the following kharacterization of the pairing $H(u-x)$:

Proposition 4.3 *(i) Let $e^*(x)$ satisfies the conditions of Proposition 4.2. Then the equalities*

$$< h_+(u), h^*(x) >= \frac{u-x+1}{u-x-1} \tag{4.18}$$

and

$$[h^*(x), e^*(y)] = \frac{\{h^*(x), e^*(y) - e^*(x)\}}{x-y}; \tag{4.19}$$

(ii) Let $f^*(x)$ satisfies the conditions of Proposition 4.2. Then the equalities

$$< h_+(u), h^*(x) >= \frac{u-x+1}{u-x-1} \tag{4.20}$$

and

$$[h^*(x), f^*(y)] = \frac{\{h^*(x), f^*(x) - f^*(y)\}}{x-y}; \tag{4.21}$$

are equivalent.

Analogously, one can see that under the conditions of Proposition 4.2 and of Proposition 4.3 the following relation is valid:

$$[f^*(x), e^*(y)] = \alpha\beta \frac{h^*(x) - h^*(y)}{x-y}. \tag{4.22}$$

Comparing (4.14)-(4.22) with (4.9) we conclude that $\alpha\beta = 1$ and we can identify $h^*(x)$ with $h_-(x)$, $e^*(x)$ with $\gamma f_-(x)$ and $f^*(x)$ with $\gamma^{-1} e_-(x)$ for some $\gamma \in \mathbf{C}^*$. We should find the constant γ to make the proof of Theorem 4.1 complete. Such an information can be extracted from Yang R-matrix $R = 1 + \frac{P}{a-b}$ acting in tensor product $V(a) \otimes V(b)$ of two-dimensional representations of $\mathcal{D}Y(sl_2)$. The action of generators of $\mathcal{D}Y(sl_2)$ in $V(c)$ wih a basis v_1, v_2 can be described by the formulas:

$$e_i(v_1) = 0, \quad e_i(v_2) = c^i v_1, \quad f_i(v_2) = 0, \quad f_i(v_1) = c^i v_2, \quad ,$$

$$h_i(v_1) = c^i v_1, \quad h_i(v_2) = -c^i v_2, \quad ,$$

According to Theorem 3.1 (see also further Proposition 5.1, the reformulation in terms of the universal R-matrix) we take Gauss decomposition of Yang R-matrix: $R = R_E R_H R_F$ and find that

$$R_E = 1 + \frac{1}{a-b} e_0 \otimes f_0 = 1 - \sum_{i \geq 0} e_i \otimes f_{-i-1}$$

which gives $< e_i, f_{-i-1} >= -1$, $i \geq 0$ or, equivalently $< e_+(u), f_-(x) >= \frac{1}{u-x}$. Thus $\gamma = 1$ which complete the proof of Theorem 4.1.

Remarks. 1. Actually a variant of pairing (4.3)-(4.4) was computed by Drinfeld [D4]. It appeared as one of the basic points of his quantization of $g[t]$.

2. The pairing (4.3)-(4.4) may be considered as a deformation of the classical pairing in $g[[t^{-1}, t]$ given by rational r-matrix $r = \frac{\Omega}{u}$ where Ω is divided Casimir operator. Formulas (4.3)-(4.4) show that this pairing remains unchanged for the currents to nilpotent subalgebras and changes by shifts $\pm\frac{1}{2}(\alpha_i, \alpha_j)$ in (de)nominators of the pairing functions of the current to Cartan subalgebras.

5 The universal R-matrix for $\mathcal{D}Y(g)$

Let us remind that the universal R-matrix [D3] for a quasitriangular Hopf algebra $\mathcal{A}$ is an invertible element R of some extension of $\mathcal{A} \otimes \mathcal{A}$ satisfying the conditions

$$\Delta'(x) = R\Delta(x)R^{-1} \qquad \forall x \in \mathcal{A}, \tag{5.1}$$

$$(\Delta \otimes id)R = R^{13}R^{23}, \qquad (id \otimes \Delta)R = R^{13}R^{12} \tag{5.2}$$

where $\Delta' = \sigma\Delta$, $\sigma(u \otimes v) = v \otimes u$ is an opposite comultiplication in $\mathcal{A}$. If $\mathcal{A}$ is a quantum double of a Hopf algebra $\mathcal{A}_+$, $\mathcal{A} \simeq \mathcal{A}_+ \otimes \mathcal{A}_-$, $\mathcal{A}_-$ being dual to $\mathcal{A}_+$ with an opposite comultiplication, then $\mathcal{A}$ admits a canonical presentation of the universal R-matrix $R = \sum e_i \otimes e^i$, where e_i and e^i are dual bases of $\mathcal{A}_+$ and of $\mathcal{A}_-$.

In the case of yangian $\mathcal{A}_+$ is $Y(g)$ and R is a canonical universal R-matrix in $\mathcal{D}Y(g)$. In the notations of section 3 we have, due to part (i) of Theorem 4.2 the following reformulation of Theorem 3.1.

Proposition 5.1 *Let $E_\pm$, $H_\pm$, $F_\pm$ be subalgebras of $\mathcal{D}Y(g)$ generated by $e_{i,\pm}$, $h_{i,\pm}$, $f_{i,\pm}$, $i = 1, \ldots, r$. Then the universal R-matrix R of $\mathcal{D}Y(g)$ can be factorized as*

$$R = R_E R_H R_F \tag{5.3}$$

where $R_E \in E_+ \otimes E_-$, $R_H \in H_+ \otimes H_-$, $R_F \in F_+ \otimes F_-$.

Explicit expressions of R_E and R_F for $\mathcal{D}Y(sl_2)$ can be found quite easily by inductive computation of the pairing for E_+ and E_-, F_+ and F_- (compare [Ro]):

$$< e_0^{n_0} e_1^{n_1} \cdots e_k^{n_k}, f_{-1}^{m_0} f_{-2}^{m_1} \cdots f_{-k-1}^{m_k} >=$$

$$= (-1)^{n_0 + \ldots n_k} \delta_{n_0,m_0} \delta_{n_1,m_1} \cdots \delta_{n_k,m_k} \cdot n_0! n_1! \cdots n_k!, \tag{5.4}$$

$$< f_k^{n_k} \cdots f_1^{n_1} f_0^{n_0}, e_{-k-1}^{m_k} \cdots e_{-2}^{m_1} \cdots e_{-1}^{m_0} >=$$

$$= (-1)^{n_0 + \ldots n_k} \delta_{n_0,m_0} \delta_{n_1,m_1} \cdots \delta_{n_k,m_k} \cdot n_0! n_1! \cdots n_k!, \tag{5.5}$$

As an immediate corollary of (5.4)-(5.5) we have

Theorem 5.1 *The factors R_E and R_F of the universal R-matrix for $\mathcal{D}Y(sl_2)$ can be written as*

$$R_E = \prod_{i\geq 0}^{\rightarrow} \exp(-e_i\otimes f_{-i-1}) = \exp(-e_0\otimes f_{-1})\cdot\exp(-e_1\otimes f_{-2})\cdot\ldots, \qquad (5.6)$$

$$R_F = \prod_{i\geq 0}^{\leftarrow} \exp(-f_i\otimes e_{-i-1}) = \ldots\cdot\exp(-f_1\otimes e_{-2})\cdot\exp(-f_0\otimes e_{-1}). \qquad (5.7)$$

For the general case of Theorem 5.1 see the end of this section (formulas (5.43), (5.44)).

The middle term R_H of the universal R-matrix R of $\mathcal{D}Y(g)$ has more delicate structure. One can find it directly after huge calculations but we prefer to use more elegant arguments of the connection between two realizations of $\mathcal{D}Y(g)$. The general scheme is as follows. Let $\widetilde{\mathcal{D}Y}(g)$ be a Hopf algebra isomorphic to $\mathcal{D}Y(g)$ as an algebra with a following comultiplication [D2], which naturally appears in a quantization of current algebra $g[t]$:

$$\begin{aligned}\tilde{\Delta}(e_i(u)) &= e_i(u)\otimes 1 + h_{i,-}(u)\otimes e_i(u),\\ \tilde{\Delta}(f_i(u)) &= 1\otimes f_i(u) + f_i(u)\otimes h_{i,+},\\ \tilde{\Delta}(h_{i,\pm}(u)) &= h_{i,\pm}(u)\otimes h_{i,\pm}(u)\end{aligned} \qquad (5.8)$$

where $e_i(u) = e_{i,+}(u) - e_{i,-}(u)$, $f_i(u) = f_{i,+}(u) - f_{i,-}(u)$ or $e_i(u) = \sum_{n\in Z} e_{i,n}u^{-n-1}$, $f_i(u) = \sum_{n\in Z} f_{i,n}u^{-n-1}$.

The arguments of [KT4] show that just as for $U_q(\hat{g})$ the Hopf algebra structure of $\widetilde{\mathcal{D}Y}(g)$ is connected with a Hopf algebra structure of $\mathcal{D}Y(g)$ via twisting of comultiplication by action of the longest (virtual) element w_0 of affine Weyl group. The elements of $H_\pm$ are stable under this action thus the pairing $< h_{i,+}(u), h_{j,-}(x) >$ is the same in $\mathcal{D}Y(g)$ and in $\widetilde{\mathcal{D}Y}(g)$. Formulas (5.8) show that the components of $K_{i,\pm}(u) = \log h_{i,\pm}(u)$ are primitive elements of $\widetilde{\mathcal{D}Y}(g)$. This allows us to write down immediately the pairing for the whole subalgebras $H_\pm$; the computation of the factor R_H reduces to diagonalization of the form $< K_{i,+}(u), K_{j,-}(x) >$. Explicit diagonalization will be done later, now we formulate a general statement about the structure of R_H.

Theorem 5.2 *Let $K_{i,\pm}(u) = \log h_{i,\pm}(u) = \sum k_{i,n,\pm}u^{-n-1}$. Let Ψ be a linear space generated by the elements $k_{i,n,+}$; Φ be a linear space generated by the elements $k_{i,n,-}$. Then the factor R_H of the universal R-matrix for $\mathcal{D}Y(g)$ has a form*

$$R_H = \exp\sum_a \psi_a\otimes\phi^a \qquad (5.9)$$

where $\sum_a \psi_a\otimes\phi^a$ is a canonical tensor in $\Psi\otimes\Phi$ with respect to the pairing

$$< K_{i,+}(u), K_{j,-}(x) >= \log\frac{u-x+\frac{1}{2}(\alpha_i,\alpha_j)}{u-x-\frac{1}{2}(\alpha_i,\alpha_j)}, \qquad |\, x \,|\ll 1 \ll|\, u \,| \qquad (5.10)$$

Proof There is no action of affine Weyl group on $\mathcal{D}Y(g)$: natural analogs of simple reflections map $\mathcal{D}Y(g)$ into a different algebra. Nevertheless the affine shifts in $\mathcal{D}Y(g)$ are well defined. Let, for instance, $\overline{w}_0$ be the following automorphism of $\mathcal{D}Y(g)$:

$$\overline{w}_0(e_{i,n}) = e_{i,n+1}, \qquad \overline{w}_0(f_{i,n}) = f_{i,n-1}, \qquad \overline{w}_0(h_{i,n}) = h_{i,n}. \qquad (5.11)$$

The arguments of [KT4] applied to $\mathcal{D}Y(g)$ give the following

Proposition 5.2 *A Hopf algebra $\widetilde{\mathcal{D}Y}(g)$ is isomorphic to $\mathcal{D}Y(g)$ with a comultiplication twisted by $w_0 = \lim_{n\to\infty} \overline{w}_0^n$. In other words, for any $x \in \mathcal{D}Y(g)$*

$$\tilde{\Delta}(x) = \Delta^{w_0}(x) := \lim_{n\to\infty} \overline{w}_0^n \otimes \overline{w}_0^n \Delta(\overline{w}_0^{-n} x) \tag{5.12}$$

for a suitable topology in $\mathcal{D}Y(g) \otimes \mathcal{D}Y(g)$ (see [KT4]).

The Hopf algebra $\widetilde{\mathcal{D}Y}(g)$ is by definition a double of $\widetilde{\mathcal{D}Y}_+(g)$ where $\widetilde{\mathcal{D}Y}_+(g)$ is generated by $e_{i,n}$, $n \in \mathbf{Z}$ and $h_{i,n}$, $n \geq 0$. Let $\widetilde{\mathcal{D}Y}_-(g)$ be generated by $f_{i,n}$, $n \in \mathbf{Z}$ and $h_{i,n}$, $n < 0$. Then $\widetilde{\mathcal{D}Y}_-(g)$ is isomorphic to a dual of $\widetilde{\mathcal{D}Y}_+(g)$ with an opposite comultiplication. Proposition 5.2 allows one to compute the pairing $\widetilde{\mathcal{D}Y}_+(g) \otimes \widetilde{\mathcal{D}Y}_+(g) \to \mathbf{C}$. Before computing this pairing let us say first some general words about Hopf pairings and automorphisms.

Let A and B be two Hopf algebras with a Hopf pairing $<,>: A \otimes B \to \mathbf{C}$. Let $w_A : A \to A'$ and $w_B : B \to B'$ be some isomorphisms of algebras. Then the algebras A' and B' can be canonically equipped with a structure of Hopf algebras if we define comultiplications $\Delta^{w_A} : A' \to A'$ and $\Delta^{w_B} : B' \to B'$ by the rules

$$\Delta^{w_A}(a') = w_A \otimes w_A \Delta(w_A^{-1} a'), \qquad \Delta^{w_B}(b') = w_B \otimes w_B \Delta(w_B^{-1} b'). \tag{5.13}$$

Moreover, the pairing

$$<,>: A' \otimes B' \to \mathbf{C}, \qquad < a', b' >=< w_A^{-1} a', w_B^{-1} b' > \tag{5.14}$$

is a Hopf pairing.

Taking $A = Y_+(g)$, $B = Y_-(g)$, $w_A = w_B = \overline{w}_0^n$ we obtain a Hopf pairing between $Y_+^n(g)$ and $Y_-^n(g)$ where $Y_+^n(g) = \overline{w}_0^n Y_+(g)$, $Y_-^n(g) = \overline{w}_0^n Y_-(g)$. One can easily see from (4.3), (4.4) that this pairing stabilizes when $n \to \infty$ and defines a Hopf pairing between $\widetilde{\mathcal{D}Y}_+(g)$ and $\widetilde{\mathcal{D}Y}_-(g)$ (which should actually coincide with that from a double structure of $\widetilde{\mathcal{D}Y}(g)$). This pairing looks like

$$< e_{i,n}, f_{j,-m-1} >= -\delta_{i,j}\delta_{m,n}, \tag{5.15}$$

$$< h_+(u), h_-(x) >= \frac{u - x + \frac{1}{2}(\alpha_i, \alpha_j)}{u - x - \frac{1}{2}(\alpha_i, \alpha_j)}, \qquad | x | \ll 1 \ll | u | \tag{5.16}$$

The pairing (5.16) coincides with (4.4) because the elements $h_{i,n}$ remain stable under the action of automorphism $\overline{w}_0$. Let $K_{i,\pm}(u) = \log h_{i,\pm}(u)$. Then the relation (5.5) show that

$$\tilde{\Delta}(K_{i,\pm}(u)) = K_{i,\pm}(u) \otimes 1 + 1 \otimes K_{i,\pm}(u), \tag{5.17}$$

$$< K_{i,+}(u), K_{i,-}(x) >= \log \frac{u - x + \frac{1}{2}(\alpha_i, \alpha_j)}{u - x - \frac{1}{2}(\alpha_i, \alpha_j)}, \qquad | x | \ll 1 \ll | u | \tag{5.18}$$

which means that the coefficients of $K_{i,\pm}(u)$ are primitive elements with respect to $\tilde{\Delta}$ (an element a is primitive with respect to $\tilde{\Delta}$ if $\tilde{D}(a) = a \otimes 1 + 1 \otimes a$). Now we are in the condition of the following simple general statement.

Let A and B be two dual Hopf algebras isomorphic as algebras to free commutative algebras $A \simeq \mathbf{C}[\Psi]$, $B \simeq \mathbf{C}[\Phi]$ where Ψ and Φ are vectorspaces of generators, such that all $\psi \in \Psi$ (or all $\phi \in \Phi$) are primitive elements. Let ψ_a and ϕ^a be dual bases of Ψ and Φ with respect to (nondegenerated) restriction of the pairing to $\Psi \otimes \Phi$. Then, once we choose some order of basic vectors, we have

$$< \psi_{a_1}^{n_1} \cdots \psi_{a_k}^{n_k}, (\phi^{a_1})^{m_1} \cdots (\phi^{a_l})^{m_l} >= \delta_{k,l}\delta_{n_1,m_1} \cdots \delta_{n_k,m_k} n_1! \cdots n_k!$$

or, in other words, the canonical tensor $R_{A\otimes B} = \sum a_i \otimes b^i$ of the pairing of A and B is an exponential of the canonical tensor $\Omega = \sum \psi_a \otimes \phi^a$ of the pairing of Ψ and Φ:

$$R_{A\otimes B} = \exp \Omega \tag{5.19}$$

which proves Theorem 5.2.

We describe further the canonical tensor $\Omega = \sum \psi_a \otimes \phi^a$ from Theorem 5.2 more explicitly, in other words we present concrete diagonalization of bilinear form (5.10). For illustration, we first do this for $\mathcal{D}Y(sl_2)$.

We have from (5.10)

$$< \frac{d}{du} K_+(u), K_-(x) >= \frac{1}{u-x+1} - \frac{1}{u-x-1} \tag{5.20}$$

If $\psi(u) = \sum_{i\geq 0} \psi_i u^{-i-1}$, $\phi(x) = \sum_{i<0} \phi_i x^{-i-1}$ are arbitrary fields from Ψ and Φ then the diagonal pairing $< \psi_i, \phi_{-j-1} >_{diag}= \delta_{i,j}$ in terms of generating functions looks like

$$< \psi(u), \phi(x) >_{diag}= \frac{1}{u-x}, \qquad \mid x \mid \ll 1 \ll \mid u \mid$$

Let A be linear operator in the space Φ. If we use the notation $< \psi(u), A\phi(x) >_{diag}$ as a pairing of $\psi(u)$ and $A\phi(x)$ under the condition that the pairing of $\psi(u)$ and $\phi(x)$ is known to be diagonal then (5.20) can be read as

$$< \frac{d}{du} K_+(u), K_-(x) >=< \frac{d}{du} K_+(u), (T - T^{-1}) K_-(x) >_{diag} \tag{5.21}$$

where $T : Tf(x) = f(x+1)$ is a shift operator. We have from (5.21):

$$< \frac{d}{du} K_+(u), (T - T^{-1})^{-1} K_-(x) >=< \frac{d}{du} K_+(u), K_-(x) >_{diag}= \frac{1}{u-x}$$

Now we can formally invert an operator $(T - T^{-1})$ as

$$(T - T^{-1})^{-1} = -T - T^3 - T^5 - \ldots \tag{5.22}$$

which gives the diagonalization of bilinear form (5.10):

$$\sum_{n\geq 0} < -\frac{d}{du} K_+(u), K_-(x+1+2n) >= \frac{1}{u-x} \tag{5.23}$$

Remark. The inverse (5.22) to a difference derivative $T - T^{-1}$ is only right-inverse operator (just as usual integral). We can define it in a different manner like

$$(T - T^{-1})^{-1} = T^{-1} + T^{-3} + T^{-5} - \ldots \tag{5.24}$$

for instance and obtain a diagonalization form

$$\sum_{n\geq 0} < \frac{d}{du}K_+(u), K_-(x-1-2n) >= \frac{1}{u-x} \tag{5.25}$$

Both formulas work for proper regions of finite-dimensional representations of $\mathcal{D}Y(g)$.

Using the notations $(\psi(u))_i = \psi_i$ for $\psi(u) = \sum \psi_i u^{-i-1}$ and

$$\mathrm{Res}_{u=x}\psi(u)\otimes\phi(x) = \sum_i \psi_i \otimes \phi_{-i-1}$$

we interprete (5.23) as the following expression of the factor R_H of the universal R-matrix for $\mathcal{D}Y(sl_2)$:

$$R_H = \prod_{n\geq 0} \exp \sum (-\frac{d}{du}K_+(u))_i \otimes (K_-(x+2n+1))_{-i-1}$$

$$= \prod_{n\geq 0} \exp \mathrm{Res}_{u=x}(-\frac{d}{du}K_+(u)i \otimes K_-(x+2n+1)) \tag{5.26}$$

or if we use (5.25),

$$R_H = \prod_{n\geq 0} \exp \sum_{i\geq 0} (\frac{d}{du}K_+(u))_i \otimes (K_-(x-2n-1))_{-i-1}$$

$$= \prod_{n\geq 0} \exp \mathrm{Res}_{u=x}(\frac{d}{du}K_+(u) \otimes K_-(x-2n-1)) \tag{5.27}$$

We can summarize the calculations of the universal R-matrix for $\mathcal{D}Y(sl_2)$ in the following theorem.

Theorem 5.3 *Let $K_\pm(u) = \log h_\pm(u)$. Then the universal R-matrix for $\mathcal{D}Y(sl_2)$ may be presented in factorized form*

$$R = R_E R_H R_F \tag{5.28}$$

where

$$R_E = \overrightarrow{\prod_{i\geq 0}} \exp(-e_i \otimes f_{-i-1}), \tag{5.29}$$

$$R_H = \prod_{n\geq 0} \exp Res_{u=x}(-\frac{d}{du}K_+(u) \otimes K_-(x+2n+1)), \tag{5.30}$$

$$R_F = \overleftarrow{\prod_{i\geq 0}} \exp(-f_i \otimes e_{-i-1}). \tag{5.31}$$

Let us return to the general case. Just as for sl_2-case, we have the following description of the pairing (5.10) in terms of the derivative of $K_{i,+}(u)$:

$$< \frac{d}{du}K_{i,+}(u), K_{j,-}(x) >= \frac{1}{u-x+\frac{1}{2}(\alpha_i,\alpha_j)} - \frac{1}{u-x-\frac{1}{2}(\alpha_i,\alpha_j)}. \tag{5.32}$$

It is more convenient to collect fields $K_{i,\pm}(u)$ to vector valued generating functions

$$\bar{K}_{\pm}(u) = \begin{pmatrix} K_{1,\pm}(u) \\ K_{2,\pm}(u) \\ \dots \\ K_{r,\pm}(u) \end{pmatrix} \qquad\qquad r = \operatorname{rank} g.$$

In terms of vector valued fields $\bar{\psi}(u)$ and $\bar{\phi}(x)$ the diagonal pairing $< \psi_{i,n}, \phi_{j,-m-1} >= \delta_{i,j}\delta_{n,m} >$ looks like

$$< \bar{\psi}(u), \bar{\phi}(x) >_{diag} = \frac{E}{u-x}$$

where E is $r \times r$ identity matrix.

Let now B be a symmetrized Cartan matrix of g with matrix elements being integers without common divisor, $B_{i,j} = (\alpha_i, \alpha_j)$ and $B(q)$ be a q-analog of B:

$$B_{i,j}(q) = [(\alpha_i, \alpha_j)]_q = \frac{q^{(\alpha_i,\alpha_j)} - q^{-(\alpha_i,\alpha_j)}}{q - q^{-1}} \tag{5.33}$$

Here we use standard notation $[a]_q = \frac{q^a - q^{-a}}{q-q^{-1}}$.

Let $D(q)$ be an inverse matrix to $B(q)$. One can see that $D(q)$ can be presented in a form

$$D(q) = \frac{1}{[l(g)]_q} C(q) \tag{5.34}$$

where $C(q)$ is a matrix with matrix coefficients $C_{i,j}(q)$ being polynomials of q and of q^{-1} with positive integer coefficients and $l(g)$ being positive integer. Actually the calculation of det $B(q)$ show that $l(g)$ is proportional to a dual Coxeter number $h\check{}$ of g (see also Table (7.8)below:

$$l(g) = h\check{} \quad \text{for } g = A_n, E_6, E_7, E_8, \qquad\qquad l(g) = 2h\check{} \quad \text{for } g = B_n, D_n, F_4,$$

$$l(g) = 3h\check{} \quad \text{for } g = G_2, \qquad\qquad l(g) = 4h\check{} \quad \text{for } g = C_n \tag{5.35}$$

In these notations the pairing (5.32) can be written as

$$< \frac{d}{du}\bar{K}_+(u), K_-(x) >=< \frac{d}{du}\bar{K}_+(u), \left((q-q^{-1})B(q)\right)|_{q=T^{\frac{1}{2}}} K_-(x) >_{diag} \tag{5.36}$$

where a shift operator $T : Tf(x) = f(x+1)$ is substituted inside rhs of (5.36) instead of q^2. Next we deduce that

$$< \frac{d}{du}\bar{K}_+(u), (T^{\frac{l(g)}{2}} - T^{-\frac{l(g)}{2}})^{-1} C(T^{\frac{1}{2}}) K_-(x) >= \frac{E}{u-x}. \tag{5.37}$$

Returning to original notations we get the following diagonalization of the pairing (5.10):

$$\sum_{n\geq 0} < -\frac{d}{du}K_{i,+}(u), \sum_l C_{l,j}(T^{\frac{1}{2}}) K_{l,-}(x + (n+\frac{1}{2})l(g)) >= \frac{\delta_{i,j}}{u-x}. \tag{5.38}$$

Let us note once more again that,just as in sl_2 case (5.27) it is possible to write down another diagonalization of the pairing (5.10) for any g. We can summarize the calculations in the following theorem.

Theorem 5.4 *Let $K_{i,\pm}(u) = \log h_{i,\pm}(u)$, B be a symmetrized Cartan matrix of g with matrix elements being integers without common divisor, $B_{i,j} = (\alpha_i, \alpha_j)$ and $B(q)$ be a q-analog of B. Let $D(q)$ be an inverse matrix to $B(q)$, and $C(q)$ be defined by the relation $C(q) = [l(g)]_q D(q)$, $l(g)$ from (5.35), $C_{i,j}(q) \in \mathbf{Z}[q, q^{-1}]$.*

Then the factor R_H of the universal R-matrix for $\mathcal{D}Y(g)$ can be presented as

$$R_H = \prod_{n\geq 0} \exp - \sum_{i,j=1,\ldots,r} \sum_{m\geq 0} \left(\frac{d}{du} K_{i,+}(u)\right)_m \otimes \left(C_{j,i}(T^{\frac{1}{2}}) K_{j,-}(x+(n+\frac{1}{2})l(g))\right)_{-m-1} =$$

$$= \prod_{n\geq 0} \exp - \sum_{i,j=1,\ldots,r} Res_{u=x} \left(\frac{d}{du} K_{i,+}(u) \otimes C_{j,i}(T^{\frac{1}{2}}) K_{j,-}(x+(n+\frac{1}{2})l(g))\right). \qquad (5.39)$$

In order to complete the description of the universal R-matrix for $\mathcal{D}Y(G)$ we should extend the description of their factors R_E and R_H from sl_2 case to the case of arbitrary simple Lie algebra g. Let us first change the notations for generators of $\mathcal{D}Y(g)$. Instead of $e_{i,n}$ we use $e_{\alpha_i+n\delta}$ and instead of $f_{i,n}$ we use $e_{-\alpha_i+n\delta}$. Denote also by $\hat{\Delta}_{Re}$ the set of all real roots of corresponding affine nontwissted Lie algebra. Let Ξ be a subset of $\hat{\Delta}_{Re}$. Recall that a total linear ordering $<$ of Ξ is called *normal* (or convex) [T] if for any three roots $\alpha, \beta, \gamma \in \Xi$, $\gamma = \alpha + \beta$ we have $\alpha < \gamma < \beta$ or $\beta < \gamma < \beta$.

Let Ξ_E and Ξ_F be the following subsets of $\hat{\Delta}_{Re}$:

$$\Xi_E = \{\gamma + n\delta \ \mid \gamma \in \Delta_+(g), n \geq 0\}, \qquad \Xi_F = \{-\gamma + n\delta \ \mid \gamma \in \Delta_+(g), n \geq 0\}, \qquad (5.40)$$

Here δ is a minimal imaginary root of $\hat{g}$. Let us equip Ξ_E and Ξ_F with two arbitrary normal orderings $<_E$ and $<_F$ satisfying the additional constraint

$$\gamma + n\delta <_E \gamma + m\delta, \quad \text{and} \quad -\gamma + n\delta >_F -\gamma + m\delta \qquad \text{if } n > m \qquad (5.41)$$

for any $\gamma \in \Delta_+(g)$. We can define the "root vectors" $e_{\pm\nu}$, $\nu \in \Xi_E \cup \Xi_F$ by induction following the instruction [KT3]

$$e_{\nu_3} = [e_{\nu_1}, e_{\nu_2}], \qquad e_{-\nu_3} = [e_{-\nu_2}, e_{-\nu_1}] \qquad (5.42)$$

if $\nu_1 < \nu_3 < \nu_2$ and (ν_1, ν_2) is a minimal segment in a sense of chosen orderings containing ν_3, and e_{ν_1}, e_{ν_2} are already being constructed. Analogously to the case $U_q(\hat{g})$ [KT3], [KT4] one can prove that the procedure (5.42) is correctly defined and the monomials

$$e_{\nu_1}^{n_1} e_{\nu_2}^{n_2} \cdots e_{\nu_k}^{n_k}, \quad \nu_1 <_E \nu_2 <_E \ldots <_E \nu_k, \quad \nu_i \in \Xi_E$$

form a basis of subalgebra $E \subset Y(g)$ and the monomials

$$e_{\nu_1}^{n_1} e_{\nu_2}^{n_2} \cdots e_{\nu_k}^{n_k}, \quad \nu_1 <_F \nu_2 <_F \ldots <_F \nu_k, \quad \nu_i \in \Xi_F$$

form a basis of subalgebra $F \subset Y(g)$.

Various arguments [KST], [KT1] show that the factors R_E and R_F of the universal R-matrix for $\mathcal{D}Y(g)$ should have the following form which we state here as a conjecture since we did not check the rigorous proof.

Conjecture. The factors R_E and R_F of the universal R-matrix for $\mathcal{D}Y(g)$ have the following form

$$R_E = \prod_{\eta\in\Xi_E}^{\rightarrow} \exp\left(-a(\eta)e_\eta \otimes e_{-\eta}\right), \tag{5.43}$$

$$R_F = \prod_{\eta\in\Xi_F}^{\rightarrow} \exp\left(-a(\eta)e_\eta \otimes e_{-\eta}\right) \tag{5.44}$$

where the products in (5.43) and in (5.44) are taken in given normal orderings $<_E$ and $<_F$ satisfying (5.43). Normalizing constants $a(\eta)$ are taken from the relations

$$[e_\eta, e_{-\eta}] = \frac{h_\gamma}{a(\eta)} \quad \text{if } \eta = \gamma + n\delta \in \Xi_E, \quad \gamma \in \Delta_+(g),$$

$$[e_{-\eta}, e_\eta] = \frac{h_\gamma}{a(\eta)} \quad \text{if } \eta = -\gamma + n\delta \in \Xi_F, \quad \gamma \in \Delta_+(g),$$

.

6 R-matrix for tensor products of evaluation representations of $Y(sl_2)$

In this section we demonstrate for the examples of evaluation representations of $Y(sl_2)$ how the general formulas (5.28), (5.39), (5.43), (5.44) forr the universal R-matrix work in concrete representations of the yangian $Y(sl_2)$. Analogous calculations for $U_q(\widehat{sl}_2)$ are presented in [KST].

One can easily check that an assignment

$$\phi : e_0 \to e, \;\; h_0 \to h, \;\; f_0 \to f, \qquad e_1 \to \frac{h-1}{2}e, \;\; f_1 \to f\frac{h-1}{2}, \tag{6.1}$$

extends to an epimorphism of algebras $Y(sl_2) \to U(sl_2)$. In terms of generating functions morphism ϕ can be written as

$$\phi e_+(u) = \left(u - \frac{h-1}{2}\right)^{-1} e, \qquad \phi f_+(u) = f\left(u - \frac{h-1}{2}\right)^{-1},$$

$$\phi h_+(u) = 1 + \left(u - \frac{h-1}{2}\right)^{-1} ef - \left(u - \frac{h+1}{2}\right)^{-1} fe, \qquad |\, u \,| \gg 1. \tag{6.2}$$

Morphism ϕ is also properly defined for $\mathcal{D}Y(sl_2)$ since one can interprete rhs of (6.2) as an expansion near zero:

$$\phi e_-(u) = \left(u - \frac{h-1}{2}\right)^{-1} e, \qquad \phi f_-(u) = f\left(u - \frac{h-1}{2}\right)^{-1},$$

$$\phi h_-(u) = 1 + \left(u - \frac{h-1}{2}\right)^{-1} ef - \left(u - \frac{h+1}{2}\right)^{-1} fe, \qquad |\, u \,| \ll 1. \tag{6.3}$$

Let V_λ and V_μ be two representations of Lie algebra sl_2 with highest weights λ and μ (Verma modules or their finite-dimensional quotions, for instance), let $V_\lambda(a)$ and $V_\mu(b)$ be corresponding evaluation representations. In concrete example when V_n is finite-dimensional representation ($\dim V_n = n+1$) with highest weight vector v_0 and a basis $v_0, v_1, \ldots, v_n$ we have for $V_n(a)$ by (6.2):

$$e_i v_k = k\left(a + \frac{n-2k+1}{2}\right)^i v_{k-1}$$

$$f_i v_k = (n-k)\left(a + \frac{n-2k-1}{2}\right)^i v_{k+1} \tag{6.4}$$

$$h_i v_k = \left((k+1)(n-k)\left(a + \frac{n-2k-1}{2}\right)^i - k(n-k+1)\left(a + \frac{n-2k+1}{2}\right)^i\right) v_k$$

For the calculation of R-matrix in $V_\lambda(a)\otimes V_\mu(b)$ it is sufficient to compute $(\phi\otimes\phi)(T_a\otimes T_b)R$ as a function of $a-b$ with values in $U(sl_2)\otimes U(sl_2)$. Here T_d is a shift operator in $\mathcal{D}Y(sl_2)$, $T_d e_\pm(u) = e_\pm(u-d)$, $T_d h_\pm(u) = h_\pm(u-d)$, $T_d f_\pm(u) = f_\pm(u-d)$. We do this first for the factors R_E and R_F. Let

$$y = a - b + \frac{h\otimes 1 - 1\otimes h}{2}.. \tag{6.5}$$

We think of y as of diagonal matrix acting in $V_\lambda(a)\otimes V_\mu(b)$. Substitution of (6.2) into (5.6) gives the following answer:

$$(\phi\otimes\phi)(T_a\otimes T_b)R_E =$$

$$= 1 + \frac{1}{y-1} e\otimes f + \frac{1}{2(y-1)(y-2)} e^2\otimes f^2 + \ldots + \frac{1}{n!(y-1)\cdots(y-n)} e^n\otimes f^n + \ldots \tag{6.6}$$

One can consider rhs of (6.6) as a difference analog of ordered exponential:

$$(\phi\otimes\phi)(T_a\otimes T_b)(R_E) =: \exp(y-1)^{-1}\cdot e\otimes f :_{T^{-1}}$$

where T is again a shift operator and

$$: \exp f(y)\cdot g(y) :_{T^{\pm 1}} = 1 + f(y)\cdot g(y) + \frac{1}{2} f(y) f(y\pm 1)\cdot g(y) g(y\pm 1) + \ldots$$

$$\ldots + \frac{1}{n!} f(y) f(y\pm 1)\ldots f(y\pm(n-1))\cdot g(y) g(y\pm 1)\ldots g(y\pm(n-1)) + \ldots$$

Analogously,

$$(\phi\otimes\phi)(T_a\otimes T_b)(R_F) =: \exp(y+1)^{-1}\cdot f\otimes e :_T =$$

$$= 1 + \frac{1}{y+1} e\otimes f + \frac{1}{2(y+1)(y+2)} f^2\otimes e^2 + \ldots + \frac{1}{n!(y+1)\cdots(y+n)} f^n\otimes e^n + \ldots \tag{6.7}$$

Note that for any weight vector of $V_\lambda\otimes V_\mu$ the series (6.6) and (6.7) is finite and have a form of operator $U(sl_2)\otimes U(sl_2)$ with rational coefficients.

The calculation of $(\phi\otimes\phi)(T_a\otimes T_b)(R_H)$ is more complicated. We perform this calculation directly in $V_\lambda\otimes V_\mu$ for simplicity. We can rewrite the action of $h_+(u)$ in $V_\lambda(a)$ (6.4) as

$$h_+(u)=\frac{(u-a-\frac{\lambda+1}{2})(u-a+\frac{\lambda+1}{2})}{(u-a-\frac{h+1}{2})(u-a-\frac{h-1}{2})},\qquad \mid u\mid\gg 1 \tag{6.8}$$

and analogously the action of $h_-(x)$ in $V_\mu(b)$ as

$$h_-(x)=\frac{(x-b-\frac{\mu+1}{2})(x-b+\frac{\mu+1}{2})}{(x-b-\frac{h+1}{2})(x-b-\frac{h-1}{2})},\qquad \mid x\mid\ll 1. \tag{6.9}$$

We get from (6.8) and (6.9):

$$\frac{d}{du}K_+(u)=d\log h_+(u)=$$

$$=\frac{1}{(u-a-\frac{\lambda+1}{2})}+\frac{1}{(u-a+\frac{\lambda+1}{2})}-\frac{1}{(u-a-\frac{h+1}{2})}-\frac{1}{(u-a-\frac{h-1}{2})},\quad \mid u\mid\gg 1, \tag{6.10}$$

and

$$K_-(x)=\log\frac{(x-b-\frac{\mu+1}{2})(x-b+\frac{\mu+1}{2})}{(x-b-\frac{h+1}{2})(x-b-\frac{h-1}{2})},\qquad \mid x\mid\ll 1. \tag{6.11}$$

The rest of computations can be easily performed using the followiing Lemma, where we assume, as usually, that $\mid x\mid\ll 1\ll\mid u\mid$.

Lemma 6.1

$$\prod_{n\geq 0}\exp Res_{u=x}\left(\frac{1}{u-\gamma}\cdot\log\frac{x-\alpha+2n+1}{x-\beta+2n+1}\right)=\frac{\Gamma\left(\frac{\gamma-\beta+1}{2}\right)}{\Gamma\left(\frac{\gamma-\alpha+1}{2}\right)}. \tag{6.12}$$

Denoting $h_1=h\otimes 1$, $h_2=1\otimes h$, $c=\frac{a-b}{2}$ we get finally the following horrible answer:

$$R_H\mid_{V_\lambda(a)\otimes V_\mu(b)}=\frac{\Gamma\left(c+\frac{\lambda-\mu}{4}+\frac{1}{2}\right)\Gamma\left(c-\frac{\lambda-\mu}{4}+\frac{1}{2}\right)\Gamma\left(c+\frac{\lambda+\mu}{4}+1\right)\Gamma\left(c-\frac{\lambda+\mu}{4}\right)}{\Gamma\left(c+\frac{\lambda-h_2}{4}+\frac{1}{2}\right)\Gamma\left(c-\frac{\lambda+h_2}{4}+\frac{1}{2}\right)\Gamma\left(c+\frac{\lambda+h_2}{4}+1\right)\Gamma\left(c-\frac{\lambda+h_2}{4}\right)}\cdot$$

$$\cdot\frac{\Gamma\left(c+\frac{h_1-h_2}{4}\right)\left(\Gamma\left(c+\frac{h_1-h_2}{4}+\frac{1}{2}\right)\right)^2\Gamma\left(c+\frac{h_1-h_2}{4}+1\right)}{\Gamma\left(c+\frac{h_1-\mu}{4}+\frac{1}{2}\right)\Gamma\left(c+\frac{h_1+\mu}{4}+\frac{1}{2}\right)\Gamma\left(c+\frac{h_1+\mu}{4}+1\right)\Gamma\left(c+\frac{h_1-\mu}{4}\right)} \tag{6.13}$$

If we suppose λ and μ to be integers and V_λ, V_μ be finite dimensional (dim $V_\lambda=\lambda+1$, dim $V_\mu=\mu+1$, then the whole R-matrix has rational coefficients up to a scalar factor. We can easily find this factor $\rho(\lambda,\mu)$ normalizing R-matrix in such a way that a matrix coefficient of tensor product of highest weight vector to itself is equal to one. This gives

$$\rho(\lambda,\mu)=\frac{\Gamma\left(\frac{a-b}{2}+\frac{\lambda+\mu}{4}+\frac{1}{2}\right)\Gamma\left(\frac{a-b}{2}-\frac{\lambda+\mu}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{a-b}{2}+\frac{\lambda-\mu}{4}+\frac{1}{2}\right)\Gamma\left(\frac{a-b}{2}+\frac{\mu-\lambda}{4}+\frac{1}{2}\right)} \tag{6.14}$$

In the next section we describe this scalar factor for arbitrary finite-dimensional representations of $Y(g)$.

7 The characters of the universal R-matrix and bilinear form on the weights of $h_+(u)$

Let V and W be two irreducible finite-dimensional representations of the yangian $Y(g)$. It can be proved by fusion procedure and by studying of the R-matrix for fundamental representations of the yangian that the R-matrix $R_{V,W}$ intertwining two coproducts Δ and Δ' in $V(a) \otimes V(b)$ can be presented as a matrix with rational coefficients of $a-b$ (here $V(a)$ and $W(b)$ are obtained from V and W by means of a natural shift automorphism of $Y(g)$). On the other hand, one can apply $\rho_{V(a)} \otimes \rho_{W(b)}$ to the universal R-matrix. The results should differ by a scalar phase factor $< V, W >$ which appears due to nonlinear conditions on the universal R-matrix:

$$(\Delta \otimes id)R = R^{13}R^{23}, \tag{7.1}$$

$$(id \otimes \Delta)R = R^{13}R^{12}. \tag{7.2}$$

The factor $< V, W >$ is by definition unique modulo rational functions of $(a-b)$ and plays an important role in scattering theory.

Unfortunately, the intriguing theory of finite-dimensional representations of yangians is too young and does not says much about representations: there is a classification of irreducible modules but their structure is almost unknown (including the dimensions and characters). Nevertheless the theory of highest weight developed by Drinfeld [D2] (see also [Ta1], [Ta2]) coupled with our description of the universal R-matrix allows to compute the factor $< V, W >$ for arbitrary irreducible finite-dimensional representations of yangian $Y(g)$.

Let us recall the basic definitions of highest weight polynomials of finite-dimensional representation of $Y(g)$.

Definition 7.1 *Let V be a $Y(g)$-module. A vector $v \in V$ is a highest weight vector if*

(i) $\quad e_{i,+}(u)v = 0, \qquad i = 1, \dots, r$

(ii) $\quad h_{i,+}(u)v = H_i(u)v \quad$ *(i.e.,v is an eigenvector for all $h_{i,n}, n \geq 0$).*

The functions $H_i(u)$, $i = 1, \dots r$ are called eigenfunctions of v.

Theorem 7.1 *[D1].*

a) Any irreducible finite-dimensional $Y(g)$-module is generated by a highest weight vector;

b) An irreducible finite-dimensional $Y(g)$-module V with a highest weight vector v is finite-dimensional iff the eigenfunctions $H_i(u)$ of v can be presented as ratio of polynomials

$$H_i(u) = \frac{P_i(u + \frac{1}{2}(\alpha_i, \alpha_i))}{P_i(u)} \tag{7.3}$$

Definition 7.2 *a) The polynomials $P_i(u)$, $i = 1, \dots, r$ defined by the condition (7.3) are called highest weight polynomials of a highest weight vector v (and of finite-dimensional representation V);*

b) Finite-dimensional representation of $Y(g)$ with highest weight polynomials $P_i(u) = u - a, P_j(u) = 1, j \neq i$ is called i-th fundamental representation of $Y(g)$.

We denote i-th fundamental representation of $Y(g)$ by $\omega_i(a)$.

Bycomponent multiplication of weight polynomials endows the set $E(g)$ of all irreducible finite-dimensional representations of $Y(g)$ with a structure of abelian (multiplicative) semigroup generated by fundamental representations. An element V of $E(g)$ can be presented as

$$V = \omega_1(a_{1,1})\cdots\omega_1(a_{1,i_1})\cdot\ldots\cdot\omega_r(a_{r,1})\cdots\omega_r(a_{r,i_r}) \tag{7.4}$$

which means that highest weight polynomials of V are $P_j(u) = (u - a_{j,1})\cdots(u - a_{j,i_j})$ and V can be realized as subfactor of tensor product of fundamental representations $\omega(a_{i,j})$ containing tensor product of highest weight vectors of fundamental representations. Analogously, the multiplication law $U = V\cdot W$ implies that an irreducible module U is a subfactor of $V\otimes W$ generated by image of tensor product of highest weight vectors of V and W.

Let now V and W be finite-dimensional irreducible representation of $Y(g)$ generated by highest weight vectors v and w. They can be endowed wih a structure of $Y_-(g)$ (and of $\mathcal{D}Y(g)$)-module just by reexpansion of matrix coefficients of $e_{i,+}(u)$, $h_{i,+}(u)$, $f_{i,+}(u)$, in $u = 0$. The general structure (5.3) of the universal R-matrix shows that $v\otimes u$ is an eigenvector of R. This gives a possibility to normalize scalar phase $< V, W >$ for all highest weight representations of $Y(g)$.

Definition 7.3 *A scalar function $< V, W >$ defined by the condition*

$$R(v\otimes w) = < V, W > v\otimes w \tag{7.5}$$

where R is universal R-matrix for $\mathcal{D}Y(g)$, v and w are highest weight vectors of V and W, is called the character of the universal R-matrix R corresponding to highest weight representations V and W.

The following lemma explaines that the character of the universal R-matrix may be considired as (multiplcative) bilinear form on $E(g)$.

Lemma 7.1 *Let $V, V_1, V_2;\ \ W, W_1, W_2\ \in E(g)$. Then*

$$< V_1\cdot V_2, W > = < V_1, W > \cdot < V_2, W >, \tag{7.6}$$

$$< V, W_1\cdot W_2 > = < V, W_1 > \cdot < V, W_2 > \tag{7.7}$$

The proof immediately follows from (7.1), (7.2).

The general expresion (5.39) for the factor R_H of the universal R-matrix allows to find out the character of R for arbitrary finite-dimensional representations of $Y(g)$. Indeed, the action of R on tensor product $v\otimes w$ of highest weight vectors reduces to the action of its factor R_H. If $P_i(u)$ are highest weight polynomials of v and $Q_j(u)$ are highest weight polynomials of w then the action of the fields $\frac{d}{du}K_{i,+}(u)$ on v is given by the expression

$$\frac{d}{du}K_i(u)v = \left(d\log P_i\left(u + \frac{1}{2}(\alpha_i,\alpha_i)\right) - d\log P_i(u)\right)v, \qquad |\, u\,| \gg 1;$$

$K_{-,j}(x)$ act on w as

$$K_{j,-}(x)w = \log\frac{Q_j(x + \frac{1}{2}(\alpha_i,\alpha_j))}{Q_j(x)}w, \qquad |\, x\,| \ll 1.$$

The rest is technical application of (5.39). Due to Lemma 7.1 it is sufficient to compute the characters, corresponding to tensor product of fundamental representations.

Let us recall the notations of previous section. Now again B is a symmetrized Cartan matrix of g with matrix elements being integers without common divisor, $B_{i,j}=(\alpha_i,\alpha_j)$ $i,j-1,\ldots,r$ and $B(q)$ is a q-analog of B; $D(q)$is an inverse matrix to $B(q)$ and $C(q)$ is a $r\times r$ matrix with coeffitients from $\mathbf{Z}[q,q^{-1}]$ defined by the condition $D(q)=\frac{1}{[l(g)]_q}C(q)$, where $l(g)$ is proportional to dual Coxeter number $h\check{}(g)$ (see (5.35)). A presentation (5.34), (5.35) of the inverse to q-analogue of symmetrized Cartan matrix follows from calculation of $\det B(q)$:

$$
\begin{array}{cccc}
g & \det B(q) & h\check{}(g) & l(g) \\
 & & & \\
A_l & [l+1]_q & l+1 & l+1 \\
B_l & \frac{[2(2l-1)]_q}{[2]_q[2l-1]_q} & 2l-1 & 2(2l-1) \\
C_l & \frac{[2]_q[2(l+1)]_q}{[l+1]_q} & l+1 & 4(l+1) \\
D_l & \frac{[2]_q[2l-2)]_q}{[l-1]_q} & 2l-2 & 2(2l-2) \\
E_6 & \frac{[2]_q[3]_q[12]_q}{[4]_q[6]_q} & 12 & 12 \\
E_7 & \frac{[2]_q[3]_q[18]_q}{[6]_q[9]_q} & 18 & 18 \\
E_8 & \frac{[2]_q[3]_q[5]_q[30]_q}{[6]_q[10]_q[15]_q} & 30 & 30 \\
F_4 & \frac{[2]_q[3]_q[18]_q}{[6]_q[9]_q} & 9 & 18 \\
G_2 & \frac{[2]_q[3]_q[12]_q}{[4]_q[6]_q} & 4 & 12
\end{array}
\tag{7.8}
$$

The calculations with R_H gives the following

Theorem 7.2 *Let $C_{i,j}(q)=\sum_k C^k_{i,j}q^k$, $C^k_{i,j}\in\mathbf{Z}_+$. The character $<\omega_i(a),\omega_j(b)>$ of universal R-matrix, corresponding to fundamental representations $\omega_i(a)$ and $\omega_j(b)$ is equal to*

$$
<\omega_i(a),\omega_j(b)>=
$$

$$
\prod_k\left(\frac{\Gamma\left(\frac{a-b}{l(g)}+\frac{l(g)-k}{2l(g)}\right)\Gamma\left(\frac{a-b}{l(g)}+\frac{l(g)-k+(\alpha_j,\alpha_j)-(\alpha_i,\alpha_i)}{2l(g)}\right)}{\Gamma\left(\frac{a-b}{l(g)}+\frac{l(g)-k-(\alpha_i,\alpha_i)}{2l(g)}\right)\Gamma\left(\frac{a-b}{l(g)}+\frac{l(g)-k+(\alpha_j,\alpha_j)}{2l(g)}\right)}\right)^{C^k_{i,j}}
\tag{7.9}
$$

For instance, for $Y(sl_2)$ the pairing (7.9) looks like

$$
<\omega(a),\omega(b)>=\frac{\Gamma\left(\frac{a-b}{2}+\frac{1}{2}\right)^2}{\Gamma\left(\frac{a-b}{2}\right)\Gamma\left(\frac{a-b}{2}+1\right)}
\tag{7.10}
$$

and, more generally,

$$< \prod_i \omega(a_i), \prod_j \omega(b_j) >= \prod_{i,j} \frac{\Gamma\left(\frac{a_i-b_j}{2}+\frac{1}{2}\right)^2}{\Gamma\left(\frac{a_i-b_j}{2}\right)\Gamma\left(\frac{a_i-b_j}{2}+1\right)} \tag{7.11}$$

which agrees with (6.14) since $V_n(a) = \omega(a - \frac{n-1}{2})\omega(a - \frac{n-3}{2}) \cdots \omega(a + \frac{n-1}{2})$ in $E(sl_2)$.

Quasiclassically $R(u) = 1 + \frac{\Omega}{u}$ and quasiclassical limit of the form (7.5) on highest weights of evaluation representation $V_\lambda(a)$ and $V_\mu(b)$ of Lie algebra $g[t]$ should be $\frac{<\lambda,\mu>}{a-b}$ where $<,>$ is invariant scalar product in h^* (h is Cartan subalgebra of g).

It will be interesting to obtain combinatorial and geometric interpretation of the form (7.5).

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