Some Hall Polynomials for Representation-finite Trivial Extension Algebras

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Let k be a finite field and assume that Λ is a finite dimensional associative k-algebra with 1. Denote by modΛ the category of all finitely generated (right) Λ-modules and by indΛ the full subcategory in which every object is a representative of the isoclass of an indecomposable (right) Λ-module. We are interested in the existance of the Hall polynomial φ^M_{NL} for any $L, M, N \in$modΛ (for the definition, see [**7,8**] or Section 1 below). In case Λ is directed, C. M. Ringel in [**7**] has shown that Λ has Hall polynomials, and in case Λ is cyclic serial, the same result has also been obtained by J. Guo [**4**]. It has been conjectured in [**8**] that any representation-finite k-algebra has Hall polynomials. In this investigation, we shall show that if Λ is a representation-finite trivial extension algebra, then, for any $L, M, N \in$modΛ with N indecomposable, Λ has the Hall polynomials φ^M_{LN} and φ^M_{NL}. Using these Hall polynomials, we can naturally structure the free abelian group with a basis indΛ, denoted by $K(\text{mod}\Lambda)$, into a Lie algebra and the universal enveloping algebra of $K(\text{mod}\Lambda) \otimes_{\mathbf{Z}} \mathbf{Q}$ is just $\mathcal{H}(\Lambda)_1 \otimes_{\mathbf{Z}} \mathbf{Q}$, where $\mathcal{H}(\Lambda)_1$ is the degenerated Hall algebra of Λ (see Section 5 below).

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1 Preliminaries

For M, $N_1, \ldots, N_t \in$modΛ, we denote by $F^M_{N_t \ldots N_1}$ the number of filtrations

$$0 = U_0 \subseteq U_1 \subseteq \ldots \subseteq U_t = M$$

of M such that $U_i/U_{i-1} \simeq N_i$ for all $1 \leq i \leq t$. In particular, F^M_{LN} is the number of the submodules U of M such that $U \simeq N$ and $M/U \simeq L$.

Let E be a field extension of k. For any k-space V, we denote by V^E the E-space $V \otimes_k E$. Of course, Λ^E naturally becomes a E-algebra. We recall from [**8**] that E is conservative for Λ if for any indecomposable $M \in$modΛ, $(\text{End}M/\text{radEnd}M)^E$ is a field. Under field isomorphisms we put

$$\Omega = \{ \text{ a finite field extension } E \text{ of } k \mid E \text{ is conservative for } \Lambda\}.$$

Note that if Λ is representation-finite, then Ω is an infinite set.

Proposition 1.1 *Let E be a field conservative extension of k. Denote by $(mod\Lambda)^E$ the full subcategory of $mod\Lambda$ which objects are M^E for all $M \in mod\Lambda$. Then*

i). $M \in mod\Lambda$ is indecomposable if and only if $M^E \in mod\Lambda^E$ is indecomposable. In particular, M is an indecomposable projective (respectively, injective) Λ-module if and only if M^E is an indecomposable projective (respectively, injective) Λ^E-module; and for any $M \in mod\Lambda$, $(radM)^E = radM^E$ and $(socM)^E = socM^E$.

ii). If $M \xrightarrow{f} N$ is a source (respectively, a sink) map in $mod\Lambda$, then $M^E \xrightarrow{f\otimes 1_E} N^E$ is also a source (respectively, a sink) map in $mod\Lambda^E$.

In particular, if Λ is representation-finite, then $(mod\Lambda)^E = mod\Lambda^E$.

Proof. Since E is conservative, we have $(\mathrm{radEnd}_\Lambda M)^E = \mathrm{radEnd}_{\Lambda^E} M^E$ for any $M \in \mathrm{ind}\Lambda$.

i). From the definition of E, it is clear that $M \in \mathrm{mod}\Lambda$ is indecomposable if and only if $M^E \in \mathrm{mod}\Lambda^E$ is indecomposable. In particular, M is an indecomposable projective Λ-module if and only if M^E is an indecomposable projective Λ^E-module. Therefore given a complete set of orthogenal primary idempotents $\{e_1, e_2, \ldots, e_n\}$ of Λ, we obtain a complete set of orthogenal primary idempotents $\{e_1 \otimes 1_E, e_2 \otimes 1_E, \ldots, e_n \otimes 1_E\}$ of Λ^E and

$$e_i \otimes 1_E(\mathrm{rad}\Lambda)^E e_i \otimes 1_E = \mathrm{rad}(e_i \otimes 1_E \Lambda^E e_i \otimes 1_E) = (\mathrm{rad}(e_i\Lambda e_i))^E = (e_i \mathrm{rad}\Lambda e_i)^E$$

for $i = 1, 2, \ldots, n$ and hence

$$\begin{aligned} \mathrm{rad}\Lambda^E &= \textstyle\sum_{i=1}^n e_i \otimes 1_E \mathrm{rad}\Lambda^E e_i \otimes 1_E + \sum_{i\neq j} e_i \otimes 1_E \Lambda^E e_j \otimes 1_E \\ &= (\textstyle\sum_{i=1}^n e_i \mathrm{rad}\Lambda e_i + \sum_{i\neq j} e_i \Lambda e_j)^E \\ &= (\mathrm{rad}\Lambda)^E. \end{aligned}$$

And for any $M \in \mathrm{mod}\Lambda$,

$$\mathrm{rad}M^E = \mathrm{rad}\Lambda^E M^E = (\mathrm{rad}\Lambda)^E M^E = (\mathrm{rad}M)^E.$$

By duality, we can consider the injective and the socle.

ii). By i), it is enough to show that if

$$\alpha: \ 0 \longrightarrow N \xrightarrow{f} L \xrightarrow{g} M \longrightarrow 0$$

is an Auslander-Reiten sequence in $\mathrm{mod}\Lambda$, then

$$\alpha^E: \ 0 \longrightarrow N^E \xrightarrow{f\otimes 1_E} L^E \xrightarrow{g\otimes 1_E} M^E \longrightarrow 0$$

is also an Auslander-Reiten sequence in $\mathrm{mod}\Lambda^E$. In fact, according to [**1**], we easily see that by the structure $N^E \simeq (DTrM)^E = DTrM^E$ and that α^E is not split; and, by the structure of Auslander-Reiten sequences, it is enough to see that α^E belongs to the (simple) socle of $\mathrm{End}M^E - \mathrm{End}N^E-$bimodule $\mathrm{Ext}^1_{\Lambda^E}(M^E, N^E)$, that is,

$$(\mathrm{radEnd}_{\Lambda^E} M^E)\alpha^E(\mathrm{radEnd}_{\Lambda^E} N^E) = 0.$$

But $(\mathrm{radEnd}M)\alpha(\mathrm{radEnd}N) = 0$. Therefore

$$(\mathrm{radEnd}_{\Lambda^E} M^E)\alpha^E(\mathrm{radEnd}_{\Lambda^E} N^E) = ((\mathrm{radEnd}_\Lambda M)\alpha(\mathrm{radEnd}_\Lambda N))^E = 0.$$

In particular, if Λ is representation-finite, then from i) and ii) we easily deduce $(\mathrm{mod}\Lambda)^E = \mathrm{mod}\Lambda^E$.

For any finite set M, we denote its cardinality by $|M|$.

Given Λ with Ω infinite. We recall from [**8**] that Λ has Hall polynomials provided for any $X, Y, Z \in \mathrm{mod}\Lambda$, there exists a polynomial $\varphi^Y_{ZX} \in \mathbf{Z}[T]$ such that for any $E \in \Omega$,

$$\varphi^Y_{ZX}(|E|) = F^{Y^E}_{Z^E X^E}.$$

Such φ^Y_{ZX} is called a Hall polynomial.

Remark. If Λ is representation-finite and if Λ has Hall polynomials, then for any $M, N_1, \ldots, N_t \in \mathrm{mod}\Lambda$, Λ has the polynomial $\varphi^M_{N_t \ldots N_1} \in \mathbf{Z}[T]$ such that for any $E \in \Omega$,

$$\varphi^M_{N_t \ldots N_1}(|E|) = F^{M^E}_{N_t^E \ldots N_1^E}.$$

In fact, it is enough to note that for any $1 \leq i \leq t$ we have

$$F^{M^E}_{N_t^E \ldots N_1^E} = \sum_{[L]} F^{M^E}_{N_t^E \ldots N_{i+1}^E L^E} F^{L^E}_{N_i^E \ldots N_1^E}$$

since Proposition 1.1, where $[L]$ goes through the isoclasses of all $L \in \mathrm{mod}\Lambda$. In this case, we also call $\varphi^M_{N_t \ldots N_1}$ a Hall polynomial.

Similarly, we have the following.

Proposition 1.2 *Let $\mathcal{C}$ be a full subcategory of modΛ closed under isomorphisms and assume that Λ is representation-finite. If for any $X, Y, Z \in$ modΛ with $X \in \mathcal{C}$, Λ has the Hall polynomials φ^Y_{XZ} and φ^Y_{ZX}, then for any $M \in$ modΛ and $N_1, \ldots, N_t \in$ modΛ with at most one $N_i \notin \mathcal{C}$, Λ has the Hall polynomial $\varphi^M_{N_t \ldots N_1}$.*

2 The Existance of Some Polynomials

In this section, we list some well-known results.

Lemma 2.1 *For any $M, N \in$ modΛ, taking $h(M,N) = dim_k Hom_\Lambda(M,N)$ and putting $\gamma_{MN} = T^{h(M,N)} \in \mathbf{Z}[T]$, we have*

$$\gamma_{MN}(|E|) = |Hom_{\Lambda^E}(M^E, N^E)|$$

for any $E \in \Omega$.

Proposition 2.1 *Let R be a finite local ring and $M_m(R)$ the matrix ring of all $m \times m$ matrices over R. Then the number of all invertible matrices in $M_m(R)$ is equal to*

$$\prod_{i=1}^{m}(|R|^m - |R|^{i-1}|radR|^{m-(i-1)}).$$

For the convenience of the reader, we give out the proof as follows.

Proof. Take $H \in M_m(R)$ and denote by $\alpha_1, \ldots, \alpha_m$ the row vectors of H. Since R is local, H is not invertible if and only if there exist α_j and $a_i \in R$, $1 \leq i \leq j-1$, and $\beta \in (\mathrm{rad}R)^{(m)}$ such that $\alpha_j = \sum_{i<j} a_i\alpha_i + \beta$.

Now assume that H is invertible. By induction on i, $1 \leq i \leq m$, we can obtain that there are $(|R|^m - |R|^{i-1}|\mathrm{rad}R|^{m-(i-1)})$ ways of the choice for α_i. this completes the proof.

Corollary 2.1.1 *For any $M \in mod\Lambda$, there exists a monic polynomial $\alpha_M \in \mathbf{Z}[T]$ such that for any $E \in \Omega$,*

$$\alpha_M(|E|) = |Aut_{\Lambda^E} M^E|.$$

Proof. Let $M = a_1M_1 \oplus \ldots \oplus a_lM_l$ with M_i indecomposale and pairwise non-isomorphic. Note that for any $E \in \Omega$, we have $\mathrm{radEnd}_{\Lambda^E} M_i^E = (\mathrm{radEnd}_\Lambda M_i)^E$. So taking $\beta_{M_i} = T^{h(M_i,M_i)-1} \in \mathbf{Z}[T]$, we have $\beta_{M_i}(|E|) = |\mathrm{radEnd}_{\Lambda^E} M_i^E|$. Set

$$\alpha_M = \prod_{i,j,\ i\neq j} \gamma_{a_iM_i,a_jM_j} \prod_{s=1}^{l}\prod_{t=1}^{a_s}(\gamma_{M_sM_s}^{a_s} - \gamma_{M_sM_s}^{t-1}\beta_{M_s}^{a_s-(t-1)}).$$

Then $\alpha_M \in \mathbf{Z}[T]$ is monic. And by Proposition 2.1 we can check $\alpha_M(|E|) = |\mathrm{Aut}_{\Lambda^E} M^E|$.

Lemma 2.2 *Assume that Λ is representation-finite. Then for any $M, N \in mod\Lambda$, there are the polynomials $\sigma_M^N, \eta_M^N \in \mathbf{Z}[T]$ such that for any $E \in \Omega$, $\sigma_M^N(|E|)$ is the number of the submodules of N^E isomorphic to M^E, and $\eta_M^N(|E|)$ is the number of the submodules U of N^E with N^E/U isomorphic to M^E.*

For the proof, we use Lemma 2.1 and Corollary 2.1.1 and refer to [**7**].

The following is direct.

Corollary 2.2.1 *Assume that Λ is representation-finite. Then for any $M, N, S \in mod\Lambda$ with S simple, Λ has the Hall polynomials φ_{SN}^M and φ_{NS}^M.*

We now consider some infinite dimension k-algebra without unit. Let R be an associative k-algebra and have a set of completely orthogenal primary idempotents $\{e_i |\ i \in I\}$ (I may be an infinite set) such that $R = \oplus_{i,j\in I} e_iRe_j$. Denote by $\mathrm{mod}R$ the category of all finitely generated (right) R-modules M with $MR = M$. We recall that R is locally bounded if $\dim_k e_iR < \infty$ for all $i \in I$ and that R is directed if in $\mathrm{mod}R$ there is no cycle $M = M_0 \xrightarrow{f_1} M_1 \to \ldots \to M_{t-1} \xrightarrow{f_t} M_t = M$ with $t > 0$, all M_i indecomposable and all $f_i \neq 0$ non-isomorphic.

Now let R be a locally bounded k-algebra and assume that for R there are infinitely many finite field extensions of k (under isomorphisms) which are conservative. Replacing Λ with R and representation-finite with locally representation-finite, then the all results above still hold. So from the proof of Theorem 1 in [**7**], we have the following.

Proposition 2.2 *Let R be a locally bounded k-algebra and assume that for R there are infinitely many finite field extensions of k (under isomorphisms) which are conservative. If R is directed, then R has Hall polynomials.*

3 Hall polynomials

For any $M \in \mathrm{mod}\Lambda$, we denote by $\mathbf{dim}M$ the dimension vector of M. Note that for any $E \in \Omega$, $\mathbf{dim}M$=$\mathbf{dim}M^E$.

Let $A, B, X \in \mathrm{mod}\Lambda$. As in [**5**], we denote by $\mathrm{Ext}^1_\Lambda(B, A)_X$ the set of all exact sequences in $\mathrm{Ext}^1_\Lambda(B, A)$ with middle term X, by $W(A, B; X)$ the set

$$\{(f, g) \in \mathrm{Hom}_\Lambda(A, X) \times \mathrm{Hom}_\Lambda(X, B) \mid f \text{ injective}, g \text{ surjective}, fg = 0\},$$

and by $V(A, B; X)$ the set of all filtrations $X \supseteq U \supseteq 0$ of X such that $U \simeq A$ and $X/U \simeq B$.

The group $\mathrm{Aut}_\Lambda A \times (\mathrm{Aut}_\Lambda B)^{op}$ acts freely on $W(A, B; X)$: for an automorphism ρ of A and σ of B and $(f, g) \in W(A, B; X)$, we set

$$(\rho, \sigma)(f, g) = (\rho f, g\sigma).$$

Assume $\mathbf{dim}X$=$\mathbf{dim}A$+$\mathbf{dim}B$. Then the polynomial map $\varphi : W(A, B; X) \to V(A, B; X)$ defined by $\varphi(f, g) = (0 \subseteq \mathrm{im}f \subseteq X)$ is surjective, and its fibers are all isomorphic to $\mathrm{Aut}_\Lambda A \times (\mathrm{Aut}_\Lambda B)^{op}$. Thus

$$|\mathrm{Aut}_\Lambda A||V(A, B; X)||\mathrm{Aut}_\Lambda B| = |W(A, B; X)|.$$

On the other hand, consider the map $\psi : W(A, B; X) \to \mathrm{Ext}^1_\Lambda(B, A)_X$ given by sending (f, g) to the class in $\mathrm{Ext}^1_\Lambda(B, A)_X$ of the exact sequence

$$0 \longrightarrow A \xrightarrow{f} X \xrightarrow{g} B \longrightarrow 0.$$

The fiber of $\psi(f, g)$ is the orbit in $W(A, B; X)$ under the action of $\mathrm{Aut}_\Lambda X$ defined by

$$\tau(f, g) = (f\tau, \tau^{-1}g)$$

for an automorphism τ of X. Note that the isotropy group of any point under this action is isomorphic to $\mathrm{Hom}_\Lambda(B, A)$. Thus

$$|\mathrm{Ext}^1_\Lambda(B, A)_X||\mathrm{Aut}_\Lambda X| = |W(A, B; X)||\mathrm{Hom}_\Lambda(B, A)|.$$

Therefore we obtain the following

Lemma 3.1. *For any $X, A, B \in mod\Lambda$, we have*

$$F^X_{BA} = \frac{|Ext^1_\Lambda(B, A)_X||Aut_\Lambda X|}{|Aut_\Lambda A||Aut_\Lambda B||Hom_\Lambda(B, A)|}.$$

By Lemma 3.1, Lemma 1.1, Corollary 2.1.1 and Lemma in [**7**], we conclude:

Proposition 3.1 *Assume that Λ is representation-finite. Then for any $L, M, N \in mod\Lambda$, Λ has the Hall polynomial φ^M_{NL} if and only if there exists a polynomial $\psi^M_{NL} \in \mathbf{Z}[T]$ such that for any $E \in \Omega$,*

$$\psi^M_{NL}(|E|) = |Ext^1_{\Lambda^E}(N^E, L^E)_{M^E}|.$$

We recall that if A is an algebra over some field k', denote by D the dual functor $\mathrm{Hom}_{k'}(-,k')$, then DA has a canonical structrue of $A-A-$bimodule, and the trivial extension algebra of A, denote by $A \propto DA$, is an k'-algebra whose basic k'-space is $A \oplus DA$, the direct product of k'-spaces A and DA, and whose multiplication is defined by the following:

$$(a,\varphi)(b,\psi) = (ab, a\psi + \varphi b)$$

for any $a, b \in A$ and $\varphi, \psi \in DA$. And the repetitive algebra of A is a matrix k'-algebra of form

$$\hat{A} = \begin{pmatrix} \ddots & \ddots & & & & 0 \\ & \ddots & \ddots & & & \\ & & A_{i+1} & DA_{i+1} & & \\ & & & A_i & DA_i & \\ & & & & \ddots & \ddots \\ 0 & & & & & \ddots \end{pmatrix},$$

with all $A_i = A$, in which every element is a matrix with at most finite entries non-zero. Note that $\hat{A}$ is an infinite dimensional k'-algebra but it is locally bounded.

Denote by ν the automorphism induced by the identity map $A_i \to A_{i+1}$. Then ν induces an automorphism of $\mathrm{mod}\hat{A}$, still denoted by ν. Following [**2**], we have a covering functor $F : \mathrm{mod}\hat{A} \longrightarrow \mathrm{mod}A \propto DA$ with the fundamental group $G =< \nu >$.

Now we can give out the main result as follows.

Theorem 3.1 *Assume that Λ is a representation-finite trivial extension k- algebra. Then for any $L, M, N \in \mathrm{mod}\Lambda$ with N indecomposable, Λ has the Hall polynomials φ^M_{LN} and φ^M_{NL}.*

Proof. Let $\Lambda = A \propto DA$. Following [**3**] or [**2**], A must be a tilted algebra of Dynkin type. Now, we consider the repetitive algebra $\hat{A}$ and the covering functor $F : \mathrm{mod}\hat{A} \longrightarrow \mathrm{mod}\Lambda$ with the fundamental group $G =< \nu >$. It is easy to see (or see [**6**]) that for any $X, Y \in \mathrm{mod}\hat{A}$, F induces k-isomorphisms

$$\bigoplus_{g\in G} \mathrm{Ext}^1_{\hat{A}}(X, gY) \xrightarrow{\sim} \mathrm{Ext}^1_{\Lambda}(FX, FY)$$

and

$$\bigoplus_{g\in G} \mathrm{Ext}^1_{\hat{A}}(gX, Y) \xrightarrow{\sim} \mathrm{Ext}^1_{\Lambda}(FX, FY).$$

Since Λ is the trivial extension of a tilted algebra, we know that for any indecomposable $\hat{A}$-modules X, Y, there is at most one $g \in G$ such that $\mathrm{Ext}^1_{\hat{A}}(X, gY) \neq 0$. From this and note that N is indecomposable, we deduce that there exist $X, Y \in \mathrm{mod}\hat{A}$ with $FX = L$ and $FY = N$ such that F induces a k-isomorphism

$$\mathrm{Ext}^1_{\hat{A}}(X, Y) \xrightarrow{\sim} \mathrm{Ext}^1_{\Lambda}(L, N).$$

Note that a finite field extention E of k for $\hat{A}$ is conservative if and only if $E \in \Omega$ and that $\hat{A}$ is directed and hence is locally representation-finite. So given $E \in \Omega$, F induces a covering functor

$$F^E = F \otimes_k 1_E : \mathrm{mod}\hat{A}^E = (\mathrm{mod}\hat{A})^E \longrightarrow (\mathrm{mod}\Lambda)^E = \mathrm{mod}\Lambda^E.$$

So F^E also induces the k-isomorphism

$$\mathrm{Ext}^1_{\hat{A}^E}(X^E, Y^E) \xrightarrow{\sim} \mathrm{Ext}^1_{\Lambda^E}(L^E, N^E).$$

Therefore, we have an isomorphism

$$\bigcup_{[H^E],\ F^E H^E = M^E} \mathrm{Ext}^1_{\hat{A}^E}(X^E, Y^E)_{H^E} \xrightarrow{\sim} \mathrm{Ext}^1_{\Lambda^E}(L^E, N^E)_{M^E}.$$

Note that the left above is a finite disjoint union. Hence

$$\begin{aligned} |\mathrm{Ext}^1_{\Lambda^E}(L^E, N^E)_{M^E}| &= \textstyle\sum_{[H^E],\ F^E H^E = M^E} |\mathrm{Ext}^1_{\hat{A}^E}(X^E, Y^E)_{H^E}| \\ &= \textstyle\sum_{[H],\ FH = M} |\mathrm{Ext}^1_{\hat{A}^E}(X^E, Y^E)_{H^E}|. \end{aligned}$$

However $\hat{A}$ is directed. By Proposition 2.2 and 3.1 there are polynomials $\psi^H_{XY} \in \mathbf{Z}[T]$ such that $\psi^H_{XY}(|E|) = |\mathrm{Ext}^1_{\hat{A}^E}(X^E, Y^E)_{H^E}|$. Put $\psi^M_{LN} = \sum_{[H],\ FH=M} \psi^H_{XY}$. Then $\psi^M_{LN} \in \mathbf{Z}[T]$ and $\psi^M_{LN}(|E|) = |\mathrm{Ext}^1_{\Lambda^E}(L^E, N^E)_{M^E}|$. By Proposition 3.1 Λ has the Hall polynomial φ^M_{LN}

Similarly we can prove that Λ has the Hall polynomial φ^M_{NL}.

Remark. From the proof we see that Theorem 3.1 still holds if we replace the representation-finite trivial extension by the following more general algebra Λ: Λ is representation-finite and there is a locally bounded k-algebra R which is directed, such that there exists a covering functor $F : \mathrm{mod}R \to \mathrm{mod}\Lambda$, and for any $M, N \in \mathrm{ind}\Lambda$ there exist $X, Y \in \mathrm{mod}R$ with $FX = M$ and $FY = N$ such that F induces the k-isomorphism

$$\mathrm{Ext}^1_R(X, Y) \xrightarrow{\sim} \mathrm{Ext}^1_\Lambda(M, N).$$

4 The characterization of some F^B_{CA}

Given $M, N \in \mathrm{mod}\Lambda$ with N indecomposable, denote $\mu(M : N)$ the number of direct summands of M isomorphic to N, that is, M is isomorphic to $\oplus_{X \in \mathrm{ind}\Lambda} \mu(M : X)X$. And denote $\mu(M) = \sum_{X \in \mathrm{ind}\Lambda} \mu(M : X)$. In addition, we denote by $D(M)$ the set of module isomorphism class pairs $([M_1], [M_2])$ such that $[M_1 \oplus M_2] = [M]$.

Now we assume that Λ is representation-finite.

The following result is similar with 2.2 Lemma in [**5**].

Proposition 4.1 *Given $E \in \Omega$ and let $A_1, A_2, \ldots, A_t$ and B be Λ-modules.*

i). If $\mu(B) \geq \sum_{i=1}^t \mu(A_i)$ but $B \not\simeq \oplus_{i=1}^t A_i$, then $|E| - 1$ divides $F^{B^E}_{A_1^E A_2^E \ldots A_t^E}$;

ii). If $B \simeq \oplus_{i=1}^t A_i$, then $|E| - 1$ divides

$$F^{B^E}_{A_1^E A_2^E \ldots A_t^E} - \prod_{X \in ind\Lambda} \frac{\mu(B : X)!}{\mu(A_1 : X)! \mu(A_2 : X)! \ldots \mu(A_t : X)!}.$$

Proof. At first, we assume $t=2$ and let $C=A_1$ and $A=A_2$. We use induction on $\mu(B)$.

i). If $\mu(B)=1$, *i*) holds trivially. So we assume $B=M_1\oplus M_2$ with both M_1,M_2 nonzero. By Proposition 3 in [**8**], $|E|-1$ divides

$$F^{B^E}_{C^EA^E}-\sum F^{M_1^E}_{(C')^E(A')^E}F^{M_2^E}_{(C'')^E(A'')^E},$$

where the sum on the right ranges over all pairs $([C'],[C''])\in D(C)$, and all pairs $([A'],[A''])\in D(A)$. However, in each term $F^{M_1^E}_{(C')^E(A')^E}F^{M_2^E}_{(C'')^E(A'')^E}$, $\mu(M_1)\geq\mu(C')+\mu(A')$ but $M_1\not\simeq C'\oplus A'$, or $\mu(M_2)\geq\mu(C'')+\mu(A'')$ but $M_2\not\simeq C''\oplus A''$. Thus $|E|-1$ divides $\sum F^{M_1^E}_{(C')^E(A')^E}F^{M_2^E}_{(C'')^E(A'')^E}$. Therefore $|E|-1$ divides $F^{B^E}_{C^EA^E}$.

ii). Assume that $B=B_1\oplus B_2$ with B_1 indecomposable, and furthermore B_1 is isomorphic to a direct summand of C. If B_1 also is isomorphic to a direct summand of A, then we have by Proposition 3 in [**8**] and *i*) that $|E|-1$ divides

$$F^{B^E}_{C^EA^E}-F^{B_1^E}_{B_1^E0}F^{B_2^E}_{C_2^EA^E}-F^{B_1^E}_{0B_1^E}F^{B_2^E}_{C^EA_2^E},$$

where $B_1\oplus C_2\simeq C$, $B_1\oplus A_2\simeq A$, and thus $B_2\simeq C_2\oplus A\simeq C\oplus A_2$. Therefore by induction

$$\begin{aligned}F^{B^E}_{C^EA^E} &\equiv \prod_{X\in\mathrm{ind}\Lambda}\tfrac{\mu(B_2:X)!}{\mu(C_2:X)!\mu(A:X)!}+\prod_{X\in\mathrm{ind}\Lambda}\tfrac{\mu(B_2:X)!}{\mu(C:X)!\mu(A_2:X)!}\\ &= \prod_{X\in\mathrm{ind}\Lambda,X\not\simeq B_1}\tfrac{\mu(B:X)!}{\mu(C:X)!\mu(A:X)!}\left(\tfrac{(\mu(B:B_1)-1)!}{(\mu(C:B_1)-1)!\mu(A:B_1)!}+\tfrac{(\mu(B:B_1)-1)!}{\mu(C:B_1)!(\mu(A:B_1)-1)!}\right)\\ &= \prod_{X\in\mathrm{ind}\Lambda}\tfrac{\mu(B:X)!}{\mu(C:X)!\mu(A:X)!}\end{aligned}$$

modulo $|E|-1$. Note that we just used $\mu(B:B_1)=\mu(C:B_1)+\mu(A:B_1)$. If B_1 is not isomorphic to any direct summand of A, then $\mu(A:B_1)=0$ and $\mu(B:B_1)=\mu(C:B_1)$. Similarly we deduce that $|E|-1$ divides

$$F^{B^E}_{C^EA^E}-\prod_{X\in\mathrm{ind}\Lambda}\frac{\mu(B:X)!}{\mu(C:X)!\mu(A:X)!}.$$

Now for a general t, note that we have

$$F^{B^E}_{A_1^EA_2^E\ldots A_t^E}=\sum_{[M],\ M\in\mathrm{mod}\Lambda}F^{M^E}_{A_1^EA_2^E\ldots A_{t-1}^E}F^{B^E}_{M^EA_t^E}.$$

By induction on t, we deduce *i*) and *ii*).

By Lemma in [**7**], we have directly the following

Corollary 4.1.1 *Given $A_1,A_2,\ldots,A_t,B\in mod\Lambda$. Assume that Λ has the Hall polynomial $\varphi^B_{A_1A_2\ldots A_t}$.*

i). If $\mu(B)\geq\sum_{i=1}^t\mu(A_i)$ but $B\not\simeq\oplus_{i=1}^tA_i$, then $\varphi^B_{A_1A_2\ldots A_t}(1)=0$,

ii). If $B\simeq\oplus_{i=1}^tA_i$, then

$$\varphi^B_{A_1A_2\ldots A_t}(1)=\prod_{X\in ind\Lambda}\frac{\mu(B:X)!}{\mu(A_1:X)!\mu(A_2:X)!\ldots\mu(A_t:X)!}.$$

5 The Lie algebras and universal enveloping algebras

In this section, we assume that Λ is representation-finite and for any $L, M, N \in \mathrm{mod}\Lambda$ with N indecomposable, Λ has the Hall polynomials φ^M_{LN} and φ^M_{NL}. Specially Λ is a representation-finite trivial extension algebra.

We recall the Hall algebra $\mathcal{H}(R)$ of a finitary ring R [**7**]. Denote by fin$R$ the category of all finite R-modules. Of course, if R is a finite dimensional k-algebra, then mod$R$=fin$R$. The Hall algebra $\mathcal{H}(R)$ is a free abelian group with a basis $\{u_{[M]} \mid M \in \mathrm{mod}R\}$, whose multiplication is

$$u_{[M]}u_{[N]} = \sum_{[L],\ L\in\mathrm{fin}R} F^L_{MN}u_{[L]}$$

for any $M, N \in \mathrm{fin}R$. Note that the right is a finite sum, and $\mathcal{H}(R)$ is an associative ring with the unit $u_{[0]}$.

As in [**9**], we define the product $\prod_{E\in\Omega}\mathcal{H}(\Lambda^E)$ of rings. Consider the elements $u_{[M]} = (u_{[M^E]})_{E\in\Omega}$ for $M \in \mathrm{mod}\Lambda$ and $q = (|E|)_{E\in\Omega}$. Denote by $\mathcal{H}(\Lambda, q)$ the subring of $\prod_{E\in\Omega}\mathcal{H}(\Lambda^E)$ generated by q and $u_{[M]}$, $M \in \mathrm{mod}\Lambda$. We shall call $\mathcal{H}(\Lambda, q)$ the generic Hall algebra of Λ. Denote by $\mathcal{I}(\Lambda, q)$ the subring of $\mathcal{H}(\Lambda, q)$ generated by q and $u_{[X]}$, $X \in \mathrm{ind}\Lambda$. Note that q is a center element of $\mathcal{H}(\Lambda, q)$. Consider the quotient ring $\mathcal{H}(\Lambda, q)/(q-1)\mathcal{H}(\Lambda, q)$, denoted by $\mathcal{H}(\Lambda)_1$, and put $p : \mathcal{H}(\Lambda, q) \to \mathcal{H}(\Lambda)_1$ to be the canonical map. Set $p(\mathcal{I}(\Lambda, q)) = \mathcal{I}(\Lambda)_1$. By abuse of notations we denote $p(u_{[M]})$ by $u_{[M]}$.

Now we arbitraryly fix an order on indΛ, say $\mathrm{ind}\Lambda = \{X_1, X_2, \ldots, X_m\}$.

Proposition 5.1 *In $\mathcal{H}(\Lambda)_1$ for any $X_{j_1}, X_{j_2}, \ldots, X_{j_t} \in ind\Lambda$, we have*

$$u_{[X_{j_1}]}u_{[X_{j_2}]}\ldots u_{[X_{j_t}]} = \sum_{\mu(L)<t} a(L)u_{[L]} + (\prod_{X\in ind\Lambda}\mu(M:X)!)u_{[M]},$$

with all $a(L) \in \mathbf{Z}$ and $M \simeq X_{j_1} \oplus X_{j_2} \oplus \ldots \oplus X_{j_m}$. In Particular, in $\mathcal{H}(\Lambda)_1$, for any $M \in mod\Lambda$ we have

$$u^{\mu(M:X_1)}_{[X_1]}u^{\mu(M:X_2)}_{[X_2]}\ldots u^{\mu(M:X_m)}_{[X_m]} = \sum_{\mu(L)<\mu(M)} a(L)u_{[L]} + (\prod_{i=1}^{m}\mu(M:X_i)!)u_{[M]},$$

with all $a(L) \in \mathbf{Z}$.

Proof. By Remark in Section 1 Λ has the Hall polynomials $\varphi^N_{X_{j_1}X_{j_2}\ldots X_{j_m}}$ for all $N \in \mathrm{mod}\Lambda$. The proof is completed by taking $a(L) = \varphi^L_{X_{j_1}X_{j_2}\ldots X_{j_m}}(1)$ since Corollary 4.1.1.

Corollary 5.1.1 $\mathbf{Q}\otimes_{\mathbf{Z}}\mathcal{I}(\Lambda)_1 = \mathbf{Q}\otimes_{\mathbf{Z}}\mathcal{H}(\Lambda)_1$

Proof. For any $M \in \mathrm{mod}\Lambda$, by Proposition 4.1 and by indution on $\mu(M)$ we deduce that $u_{[M]}$ is a $\mathbf{Q}$-linear combination of

$$\{u^{\mu(N:X_1)}_{[X_1]}u^{\mu(N:X_2)}_{[X_2]}\ldots u^{\mu(N:X_m)}_{[X_m]} \mid N \in \mathrm{mod}\Lambda\}.$$

Therefore $\mathbf{Q}\otimes_{\mathbf{Z}}\mathcal{I}(\Lambda)_1 = \mathbf{Q}\otimes_{\mathbf{Z}}\mathcal{H}(\Lambda)_1$.

Corollary 5.1.2 *$\mathcal{H}(\Lambda)_1$ is the free $\mathbf{Z}$-module $\oplus_{[M],\ M\in mod\Lambda}\mathbf{Z}u_{[M]}$.*

Proof. For any $L, N \in \mathrm{mod}\Lambda$, it is enough to show that in $\mathbf{Q}\otimes_{\mathbf{Z}}\mathcal{H}(\Lambda)_1$, $u_{[L]}u_{[N]}$ is a $\mathbf{Q}$-linear combination of $\{u_{[M]}|\ M\in \mathrm{mod}\Lambda\}$. By Corollary 5.1.1 $u_{[L]}u_{[N]}$ is a $\mathbf{Q}$-linear combination of some elements in $\mathcal{I}(\Lambda)_1$. By Proposition 5.1 $\mathcal{I}(\Lambda)_1 \subseteq \oplus_{[M],\ M\in\mathrm{mod}\Lambda}\mathbf{Z}u_{[M]}$. So $u_{[L]}u_{[N]}$ is a $\mathbf{Q}$-linear combination of $\{u_{[M]}|\ M\in \mathrm{mod}\Lambda\}$.

Theorem 5.1 *The free abelian subgroup $\oplus_{X\in ind\Lambda}\mathbf{Z}u_{[X]}$ of $\mathcal{H}(\Lambda)_1$, denoted by $K(mod\Lambda)$, is a Lie subalgebra of $\mathcal{H}(\Lambda)_1$. And $\mathcal{H}(\Lambda)_1\otimes_{\mathbf{Z}}\mathbf{Q}$ is the universal enveloping algebra of $K(mod\Lambda)\otimes_{\mathbf{Z}}\mathbf{Q}$.*

Proof. By Proposition 5.1 for any $X, Y \in \mathrm{ind}\Lambda$, $u_{[X]}u_{[Y]} - u_{[Y]}u_{[X]}$ belongs to $K(\mathrm{mod}\Lambda)$ and thus $K(\mathrm{mod}\Lambda)$ is a Lie subalgebra of $\mathcal{H}(\Lambda)_1$. Also by Proposition 5.1

$$\{u_{[X_1]}^{\mu(M:X_1)}u_{[X_2]}^{\mu(M:X_2)}\dots u_{[X_m]}^{\mu(M:X_m)}\mid M\in\mathrm{mod}\Lambda\}$$

is a $\mathbf{Z}$-independent subset of $\mathcal{I}(\Lambda)_1$ and for any $N\in\mathrm{mod}\Lambda$, $u_{[N]}$ is a $\mathbf{Q}$-linear combination of

$$\{u_{[X_1]}^{\mu(M:X_1)}u_{[X_2]}^{\mu(M:X_2)}\dots u_{[X_m]}^{\mu(M:X_m)}\mid M\in\mathrm{mod}\Lambda\}.$$

Therefore $\mathcal{H}(\Lambda)_1\otimes_{\mathbf{Z}}\mathbf{Q}$ is the universal enveloping algebra of $K(\mathrm{mod}\Lambda)\otimes_{\mathbf{Z}}\mathbf{Q}$ with the PBW-basis

$$\{u_{[X_1]}^{\mu(M:X_1)}u_{[X_2]}^{\mu(M:X_2)}\dots u_{[X_m]}^{\mu(M:X_m)}|\ M\in\mathrm{mod}\Lambda\}.$$

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