Lie Algebras determined
by Finite Auslander-Reiten Quivers

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Let Λ be a finite-dimensional associative k- algebra with 1 over some field k and Γ the Auslander-Reiten quiver of Λ. Denote by Γ_0 the vertex set of Γ. Set $\mathcal{L}(\Gamma) = \oplus_{x\in\Gamma_0}\mathbf{Z}x$, the free abelian group generated by Γ_0. It is interesting to make $\mathcal{L}(\Gamma)$ into a Lie algebra using the representation theory of algebras. C.M.Ringel in [Rin1,2] showed that if there exist the Hall polynomials of Λ (or Γ), such as directed algebras, then $\mathcal{L}(\Gamma)$ has a Lie structure via Hall algebras over finite fields. On the other hand, Ch.Riedtmann in [**3**], for any representation-finite algebra over $\mathbf{C}$, also obtained a Lie structure of $\mathcal{L}(\Gamma)$ using the Euler-Poincaré characteristic of some locally closed algebraic varieties. In case Λ is a representation-finite hereditary with some Dynkin class, two kinds of the above Lie structures are coincident and $\mathbf{C}\otimes\mathcal{L}(\Gamma)$ is isomorphic to the positive part n_+ of a $\mathbf{Z}$-form of the complex semisimple Lie algebra with the same Dynkin class (see [**7**] and [**3**]). But in general, we don't know if any representation-finite algebra has Hall polynomials. C.M.Ringel conjectured in [**6**] this is true. Even if we have the existance of Hall polynomials, we don't know if such two Lie structures are coincident.

In this investigation, for any finite connected Auslander-Reiten quiver Γ with valuation 1, using the methods of C.M.Ringel in [**Rin1,3**] and Ch.Riedtmann in [**4**], we obtain a Lie structure of $\mathcal{L}(\Gamma)$ as a Lie subalgebra of $\mathcal{H}(\Gamma)_1$, where $\mathcal{H}(\Gamma)_1$ is the degenerate Hall algebra, see Section 1 below, and similarly with [**4**] we have $\mathcal{L}(\Gamma)\simeq\mathcal{L}(\tilde{\Gamma})/G$, the "orbit" Lie algebra of the universal cover $\tilde{\Gamma}$ of Γ (see [**4**] or Section 1 below), where $\mathcal{L}(\tilde{\Gamma})$ is the Lie algebra determined by $\tilde{\Gamma}$ via Hall algebras of $\tilde{\Gamma}$. If Γ has Hall polynomials, then $\mathcal{L}(\Gamma)$ has the Lie structure in the sense of Ringel [**6**]. In this case, as a consequence, we obtain $\mathcal{L}(\Gamma)=\mathcal{L}(\tilde{\Gamma})/G$

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1 Prelemenaries and Statement of Theorem

Throughout this paper, by an algebra Λ we meant a finite-dimensional algebra over some field k, which is associative and unitary, and by a Λ-module M we meant a finitely generated unitary Λ-module M.

Given a finite field k and a k-algebra Λ. We recall the integral Hall algebra $\mathcal{H}(\Lambda)$ of Λ from [**5**]. Let $N_1, N_2, \cdots, N_t$ and M be Λ-modules. We denote by $F^M_{N_1N_2\cdots N_t}$ the munber of the filtrations

$$M = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_t = 0$$

of M such that $U_{i-1}/U_i \simeq N_i$, for $1 \leq i \leq t$. Let $\mathcal{H}(\Lambda)$ be the free abelian group with basis $(u_{[M]})_{[M]}$, indexed by the set of isomorphism classes of Λ-modules. Instead of $u_{[M]}$ we will also write u_M. We define on $\mathcal{H}(\Lambda)$ a multiplication by the following rule:

$$u_{[N_1]}u_{[N_2]} = \sum_{[M]} F^M_{N_1 N_2} u_{[M]};$$

note that on the right, we deal with a finite sum. In this way, $\mathcal{H}(\Lambda)$ become an associative ring with $1 = u_{[0]}$, and is called the integral Hall algebra of Λ.

Given Λ-modules N_1, N_2 and M. We denote by $\mathrm{Ext}^1_\Lambda(N_1, N_2)_M$ the subset of $\mathrm{Ext}^1_\Lambda(N_1, N_2)$ consisting of all exact sequences of form $0 \to N_2 \to M \to N_1 \to 0$. Then we have the following relation given in [**2**].

Proposition 1.1

$$F^M_{N_1 N_2} = \frac{|Ext^1_\Lambda(N_1, N_2)_M||Aut_\Lambda M|}{|Aut_\Lambda N_1||Aut_\Lambda N_2||Hom_\Lambda(N_1, N_2)|}$$

Here, for any set N, we denote its cardinality by $|N|$

Throughout this paper, we shall assume that Γ is a translation quiver with trivial valuation (that is, valuation 1), which is finite and connected with vertex set Γ_0. Denote by $\mathcal{B}$ the set of all functions $\Gamma_0 \to \mathbf{N}_0$. By abuse of language, we denote the characteristic function for $x \in \Gamma_0$ just by x. For any $a, b \in \mathcal{B}$, we can define $a + b \in \mathcal{B}$ by $(a+b)(x) = a(x) + b(x)$ for any $x \in \Gamma_0$. Thus for any $a \in \mathcal{B}$ we have a standard decomposition $a = \sum_{x\in\Gamma_0} a(x)x$. Denote $\mu(a) = \sum_{x\in\Gamma_0} a(x)$.

Γ will be said to be an Auslander-Reiten quiver provided there exist a field k and a k-algebra Λ auch that $\Gamma = \Gamma_\Lambda$, where Γ_Λ is the Auslander-Reiten quiver of Λ. In this case, for any $x \in \Gamma_0$, we choose a representative $M(x) = M(\Lambda, x)$ of indecomposable Λ-modules corresponding to x, and for any $a \in \mathcal{B}$, we denote by $M(a) = M(\Lambda, a)$ the Λ-module $\oplus_{x\in\Gamma_0} a(x)M(x)$. Note that Γ is of trivial valuation. So in case Γ is an Auslander-Reiten quiver, then for any field k, there exists a unique standard k-algebra via the mesh category $k(\Gamma)$ of Γ, denoted by $\Lambda(k)$, such that $\Gamma = \Gamma_{\Lambda(k)}$. In this case, for any $x \in \Gamma_0$ and any $a \in \mathcal{B}$, we also denote $M(k, x) = M(\Lambda(k), x)$ and $M(k, a) = M(\Lambda(k), a)$.

In [**2**], we obtained the following result.

Proposition 1.2 *Let Γ be an Auslander-Reiten quiver. Then*

i). For any $a, b \in \mathcal{B}$, there exists an integer $h(a, b) \in \mathbf{N}_0$ such that

$$|Hom_{\Lambda(k)}(M(k, a), M(k, b))| = |k|^{h(a,b)}$$

for any finite field k.

ii). For any $a \in \mathcal{B}$, we have

$$|Aut_{\Lambda(k)}M(k, a)| = \prod_{x,y\in\Gamma_0,\ x\neq y} |k|^{h(x,y)a(x)a(y)} \prod_{x\in\Gamma_0} \prod_{i=1}^{a(x)} (|k|^{h(x,x)a(x)} - |k|^{(h(x,x)-1)a(x)+(i-1)})$$

for any finite field k.

Let Γ be an Auslander-Reiten quiver. Let $\mathcal{K}$ be an infinite set of some finite fields up to isomorphism. As in [**7**], we consider the product $\prod_{k\in\mathcal{K}}\mathcal{H}(\Lambda(k))$ of rings. Consider the elments $u_a=(u_a^{(k)})_{k\in\mathcal{K}}$ for $a\in\mathcal{B}$ and $q=(q^{(k)})_{k\in\mathcal{K}}$, where $q^{(k)}=|k|=|k|u_0^{(k)}$. Denote by $\mathcal{H}(\Gamma,q)$ the subring of $\prod_{k\in\mathcal{K}}(\Lambda(k))$ generated by q and $u_a, a\in\mathcal{B}$, We shall call $\mathcal{H}(\Gamma,q)$ the generic Hall algebra of Γ on $\mathcal{K}$. Note that q is a center element of $\mathcal{H}(\Gamma,q)$. We shall call the quotient ring $\mathcal{H}(\Gamma,q)/(q-1)\mathcal{H}(\Gamma,q)$, denoted by $\mathcal{H}(\Gamma)_1$, the degenerate Hall algebra on $\mathcal{K}$. Put $p:\mathcal{H}(\Gamma,q)\to\mathcal{H}(\Gamma)_1$ to be the canonical map. By abuse of notations we denote $p(u_a)$ by u_a for any $a\in\mathcal{B}$. Note that an infinite set $\mathcal{K}$ ensure that subgroups $\sum_{x\in\Gamma_0}\mathbf{Z}u_x$ and $\sum_{a\in\mathcal{B}}\mathbf{Z}u_a$ of $\mathcal{H}(\Gamma)_1$ both are free. We denote $\mathcal{L}(\Gamma)=\sum_{x\in\Gamma_0}\mathbf{Z}u_x\simeq\oplus_{x\in\Gamma_0}\mathbf{Z}u_x$.

Remark. If Γ has Hall polynomials, then we can directly define the generic Hall algebra of Γ and the degenerate Hall algebra of Γ by specializing the indetermine T to 1 from the generic Hall algebra(see [Rin1,2]); however an infinite set $\mathcal{K}$ also ensure that these different definitions in fact are coincident. In this case, for any infinite set $\mathcal{K}$ of some finite fields up to isomorphism, $\mathcal{L}(\Gamma)$ is a Lie subalgebra of $\mathcal{H}(\Gamma)_1$, where $\mathcal{H}(\Gamma)_1$ is the degenerate Hall algebra either on $\mathcal{K}$ or in the sense of Ringel [**6**].

Let Γ be an Auslander-Reiten quiver. We shall consider the universal cover of Γ([**1**]). Let $F:\tilde{\Gamma}\to\Gamma$ be the universal covering with the fundamental group G. For any field k we denote by $\tilde{\Lambda}(k)$ the standard k-algebra of $\tilde{\Gamma}$; for any $\tilde{x}\in\tilde{\Gamma}_0$ we take $\tilde{M}(\tilde{x})=\tilde{M}(k,\tilde{x})$, a representative of the corresponding indecomposable $\tilde{\Lambda}(k)$-modules to $\tilde{x}$, such that $\{\tilde{M}(k,\tilde{x})|\tilde{x}\in\tilde{\Gamma_0}\}$ is stable unter the action of the fundamental group G on mod$\tilde{\Lambda}(k)$. We denote by ind$\tilde{\Lambda}(k)$ the full subcategory of mod$\tilde{\Lambda}(k)$ whose objects are the set $\{\tilde{M}(k,\tilde{x})|\tilde{x}\in\tilde{\Gamma}_0\}$. In addition, we denote by ind$\Lambda(k)$ the full subcategory of mod$\Lambda(k)$ whose objects are the set

$$\{M(k,x)\mid x\in\Gamma_0\}.$$

Note that $\Lambda(k)$ is standard. Thus we have an equivalence ind$\tilde{\Lambda}(k)/G$ $\simeq$ind$\Lambda(k)$ and therefore we can identify ind$\Lambda(k)$ with ind$\tilde{\Lambda}(k)/G$. Now F induces a covering functor, denoted still by F:

$$\mathrm{ind}\tilde{\Lambda}(k)\to\mathrm{ind}\Lambda(k)$$

which sends each $\tilde{M}(\tilde{x})$ to $M(x)$ provided $F\tilde{x}=x$. Following [**4**], for any $X,Y\in\{\tilde{M}(\tilde{x})\mid\tilde{x}\in\tilde{\mathcal{B}}\}$, F induces k-isomorphisms

$$\oplus_{g\in G}\mathrm{Ext}^1_{\tilde{\Lambda}(k)}(X,gY)\tilde{\to}\mathrm{Ext}^1_{\Lambda(k)}(FX,FY)$$

and

$$\oplus_{g\in G}\mathrm{Ext}^1_{\tilde{\Lambda}(k)}(gX,Y)\tilde{\to}\mathrm{Ext}^1_{\Lambda(k)}(FX,FY).$$

As in [**4**], we shall consider the " orbit" Lie algebra. In [**2**], we have shown that $\tilde{\Gamma}$ has Hall polynomials and thus $\mathcal{L}(\tilde{\Gamma})$ is a Lie subalgebra of $\mathcal{H}(\tilde{\Gamma})_1$ (here the degenerate algebra is on all finite fields up to isomorphism). Now we define a Lie multeplication on $\mathcal{L}(\Gamma)$ as follows:

$$[u_x,u_y]=\sum_{g\in G}F([u_{g\tilde{x}},u_{\tilde{y}}]),\ \text{for any } x,y\in\Gamma,$$

where F is $\mathbf{Z}$-linear map from $\mathcal{L}(\tilde{\Gamma})$ to $\mathcal{L}(\Gamma)$ defined by $Fu_{\tilde{z}}=u_z$ provided $F\tilde{z}=z$. In this way, we easily prove that $\mathcal{L}(\Gamma)$ become a Lie algebra, denoted by $\mathcal{L}(\tilde{\Gamma})/G$.

Now we can state our main resulte as follows.

Theorem. *For any finite Auslander-Reiten quiver Γ, there exists an infinite set $\mathcal{K}$ of some finite fields up to isomorphism such that $\mathcal{L}(\Gamma)$ is a Lie subalgebra of $\mathcal{H}(\Gamma)_1$ and just is $\mathcal{L}(\tilde{\Gamma})/G$, where $\mathcal{H}(\Gamma)_1$ is the degenerate Hall algebra on $\mathcal{K}$.*

From the above Remark, we directly have

Corollary. *If Γ has Hall polynomials, then the Lie algebra $\mathcal{L}(\Gamma)$ in the sense of Ringel* [**6**] *is canonically isomorphic to $\mathcal{L}(\tilde{\Gamma})/G$*

2 Conditions for $\mathcal{L}(\Gamma)$ being a Lie subalgebra of $\mathcal{H}(\Gamma)_1$

Given an algebra Λ over some field k with the Auslander-Reiten quiver Γ. Let M be a Λ-module and $M \simeq M(\Lambda, a)$ for some $a \in \mathcal{B}$. We denote $\mu(M) = \mu(a)$. In [**2**], we obtained the following result which is similar with one of Riedtmann's in [**3**].

Proposition 2.1 *Let $A_1, A_2, \cdots, A_t$ and B be Λ-modules. Assum that A_1, $A_2, \cdots, A_t$ and B correspond respectively to $a_1, a_2, \cdots, a_t$ amd b in $\mathcal{B}$. We have*

i). If $\mu(B) \geq \sum_{i=1}^{t} \mu(A_i)$ but $B \not\simeq \oplus_{i=1}^{t} A_i$, then $|k| - 1$ divides $F^{B}_{A_1 A_2 \cdots A_t}$;

ii). If $B \simeq \oplus_{i=1}^{t} A_i$, then $|k| - 1$ divides

$$F^{B}_{A_1 A_2 \cdots A_t} - \prod_{x \in \Gamma_0} \frac{b(x)!}{a_1(x)! a_2(x)! \cdots a_t(x)!}.$$

From now and on, we will assume that Γ is an Auslander-Reiten quiver.

Proposition 2.2 *Let $\mathcal{K}$ be an infinite set of some finite fields up to isomorphism. Assume that $\mathcal{H}(\Gamma)_1$ is the degenerate Hall algebra of Γ on $\mathcal{K}$. If for any $x, y, z \in \Gamma_0$ there exists an integer γ^z_{xy} such that $|k| - 1$ divides*

$$F^{M(k,z)}_{M(k,x) M(k,y)} - \gamma^z_{xy}$$

for any $k \in \mathcal{K}$, then $\mathcal{L}(\Gamma)$ is a Lie subalgebra of $\mathcal{H}(\Gamma)_1$.

Proof. For any $x \neq y \in \Gamma_0$, from Proposition 2.1 we have in $\mathcal{H}(\Gamma)_1$

$$\begin{aligned} u_x u_y &= \textstyle\sum_{z \in \Gamma_0} (F^{M(k,z)}_{M(k,x) M(k,y)})_{k \in \mathcal{K}} u_z + u_{(x+y)} \\ &= \textstyle\sum_{z \in \Gamma_0} \gamma^z_{xy} u_z + u_{(x+y)}. \end{aligned}$$

Thus $[u_x, u_y] = \sum_{z \in \Gamma_0} (\gamma^z_{xy} - \gamma^z_{yx}) u_z \in \mathcal{L}(\Gamma)$. That is, $\mathcal{L}(\Gamma)$ is a Lie subalgebra of $\mathcal{H}(\Gamma)_1$.

Instead of the universal cover we also consider other covers of Γ. Let H is a normal subgroup of G. Denote by $\bar{G}$ the quotient group G/H. Then we have a covering functor $\bar{F} : \bar{\Gamma} = \Gamma/H \to \Gamma$ with the fundamental group $\bar{G}$. Similarly with Section 1 for any field k we induces the covering $\bar{F} : \text{ind}\bar{\Lambda}(k) \to \text{ind}\Lambda(k)$, where $\bar{\Lambda}(k)$ is the standard k-algebra determined by $\bar{\Gamma}$. Now let $\mathcal{K}$ be an infinite set of some finite fields up to isomorphism and let $\mathcal{H}(\bar{\Gamma})_1$ be the degenerate Hall algebra of $\bar{\Gamma}$ on $\mathcal{K}$. If for any $\bar{x}, \bar{y}, \bar{z} \in \bar{\Gamma}_0$ there exists an integer $\bar{\gamma}_{\bar{x}\bar{y}}^{\bar{z}}$ such that $|k|-1$ divides

$$F_{\bar{M}(k,\bar{x})\bar{M}(k,\bar{y})}^{\bar{M}(k,\bar{z})} - \bar{\gamma}_{\bar{x}\bar{y}}^{\bar{z}}$$

for any $k \in \mathcal{K}$, then from Proposition 2.2 we know that $\mathcal{L}(\bar{\Gamma}) = \oplus_{\bar{x}\in\bar{\Gamma}_0}\mathbf{Z}u_{\bar{x}}$ is a Lie subalgebra of $\mathcal{H}(\bar{\Gamma})_1$. So we similarly have the "orbit" lie algebra $\mathcal{L}(\bar{\Gamma})/\bar{G}$. With these notions, we have

Proposition 2.3 *Let $\mathcal{K}_1$ be an infinite set of some finite fields up to isomorphism and let $\mathcal{H}(\bar{\Gamma})_1$ be the degenerate Hall algebra of $\bar{\Gamma}$ on $\mathcal{K}_1$. Assume that for any $\bar{x}, \bar{y}, \bar{z} \in \bar{\Gamma}_0$ there exists an integer $\bar{\gamma}_{\bar{x}\bar{y}}^{\bar{z}}$ such that $|k|-1$ divides*

$$F_{\bar{M}(k,\bar{x})\bar{M}(k,\bar{y})}^{\bar{M}(k,\bar{z})} - \bar{\gamma}_{\bar{x}\bar{y}}^{\bar{z}}$$

for any $k \in \mathcal{K}_1$. If there exists an infinite subset $\mathcal{K}$ of $\mathcal{K}_1$ such that for any $\bar{x}, \bar{y}, \bar{z} \in \bar{\Gamma}_0$ and for any $k \in \mathcal{K}$, $|k|-1$ divides

$$F_{M(k,x)M(k,y)}^{M(k,z)} - \sum_{\bar{g},\bar{h}\in\bar{G}} \bar{\gamma}_{\bar{g}(\bar{x})\bar{y}}^{\bar{h}(\bar{z})},$$

where $\bar{F}\bar{x} = x, \bar{F}\bar{y} = y$ and $\bar{F}\bar{z} = z$, then $\mathcal{L}(\Gamma)$ is a Lie subalgebra of $\mathcal{H}(\Gamma)_1$ and just is $\mathcal{L}(\bar{\Gamma})/\bar{G}$, where $\mathcal{H}(\Gamma)_1$ is the degenerate Hall algebra of Γ on $\mathcal{K}$

Note that the above $\sum_{\bar{g},\bar{h}\in\bar{G}} \bar{\gamma}_{\bar{g}(\bar{x})\bar{y}}^{\bar{h}(\bar{z})}$ is a finite sum.

Proof. For any $x, y, z \in \Gamma_0$ we arbitrarily take $\bar{x}, \bar{y}, \bar{z} \in \bar{\Gamma}_0$ with $\bar{F}\bar{x} = x, \bar{F}\bar{y} = y$ and $\bar{F}\bar{z} = z$, and denote $\gamma_{xy}^z = \sum_{\bar{g},\bar{h}\in\bar{G}} \bar{\gamma}_{\bar{g}(\bar{x})\bar{y}}^{\bar{h}(\bar{z})}$. We have from Proposition 2.2 that $\mathcal{L}(\Gamma)$ is a Lie subalgebra of $\mathcal{H}(\Gamma)_1$. And in $\mathcal{L}(\Gamma)$ as a Lie subalgebra of $\mathcal{H}(\Gamma)_1$, we have $[u_x, u_y] = \sum_{z\in\Gamma_0}(\gamma_{xy}^z - \gamma_{yx}^z)u_z$. On the other hand, in $\mathcal{L}(\bar{\Gamma})/\bar{G}$,

$$\begin{array}{rcl} [u_x, u_y] & = & \sum_{\bar{g}\in\bar{G}} F[u_{\bar{g}(\bar{x})}, u_{\bar{y}}] \\ & = & \sum_{\bar{g}\in\bar{G}} F(\sum_{\bar{z}\in\bar{\Gamma}_0}(\bar{\gamma}_{\bar{g}(\bar{x})\bar{y}}^{\bar{z}} - \bar{\gamma}_{\bar{y}\bar{g}(\bar{x})}^{\bar{z}})u_{\bar{z}}) \\ & = & \sum_{z\in\Gamma_0}\sum_{\bar{h},\bar{g}\in\bar{G}}(\bar{\gamma}_{\bar{g}(\bar{x})\bar{y}}^{\bar{h}(\bar{z})} - \bar{\gamma}_{\bar{y}\bar{g}(\bar{x})}^{\bar{h}(\bar{z})})u_z. \end{array}$$

Note that we have $\sum_{\bar{h},\bar{g}\in\bar{G}} \bar{\gamma}_{g(\bar{x})\bar{y}}^{\bar{h}(\bar{z})} = \gamma_{xy}^z$ and also $\sum_{\bar{h},\bar{g}\in\bar{G}} \bar{\gamma}_{\bar{y}\bar{g}(\bar{x})}^{\bar{h}(\bar{z})} = \gamma_{yx}^z$. So $\mathcal{L}(\Gamma) = \mathcal{L}(\bar{\Gamma})/\bar{G}$.

We still assume that H is a normal subgroup of G and denote $\bar{G} = G/H$. Assume that L is a normal subgroup of G with $H \subseteq L$ and denote $\bar{\bar{G}} = G/L$. We denote the residue class of an element g in G modulo H by $\bar{g}$ and the residue class of g modulo L by $\bar{\bar{g}}$. So we have the canonical group isomorphisms $\bar{L} \simeq L/H$ and $\bar{\bar{G}} = G/L \simeq \bar{G}/\bar{L}$. In addition, let $\bar{F} : \bar{\Gamma} = \Gamma/H \to \Gamma$, $\bar{F}' : \bar{\Gamma} \to \bar{\bar{\Gamma}} = \Gamma/L = \bar{\Gamma}/\bar{L}$, and $\bar{F}'' : \bar{\bar{\Gamma}} \to \Gamma$ be the corresponding coverings. Naturally we have $\bar{F} = \bar{F}'\bar{F}''$.

With the above notations, we have

Proposition 2.4 *Let k be a finite field. And let $\bar{x}, \bar{y}, \bar{y} \in \bar{\Gamma}_0$. Denote $\bar{F}\bar{x} = x$, $\bar{F}\bar{y} = y$, $\bar{F}\bar{z} = z$, $\bar{F}'\bar{x} = \bar{\bar{x}}$, $\bar{F}'\bar{y} = \bar{\bar{y}}$, and $\bar{F}'\bar{z} = \bar{\bar{z}}$. If $|k| - 1$ divides*

$$F^{M(k,z)}_{M(k,x)M(k,y)} - \sum_{\bar{\bar{g}},\bar{\bar{h}} \in \bar{\bar{G}}} F^{\bar{\bar{h}}(\bar{M}(k,\bar{\bar{z}}))}_{\bar{\bar{g}}(\bar{M}(k,\bar{\bar{x}}))\bar{M}(k,\bar{\bar{y}})},$$

and divides

$$F^{\bar{g}_2(\bar{M}(k,\bar{z}))}_{\bar{g}_1(\bar{M}(k,\bar{x}))\bar{M}(k,\bar{y})} - \sum_{\bar{l}_1,\bar{l}_2 \in \bar{L}} F^{\bar{l}_2\bar{g}_2(\bar{M}(k,\bar{z}))}_{\bar{l}_1\bar{g}_1(\bar{M}(k,\bar{x}))\bar{M}(k,\bar{y})}$$

for any $g_1, g_2 \in G$. Then $|k| - 1$ divides

$$F^{M(k,z)}_{M(k,x)M(k,y)} - \sum_{\bar{g},\bar{h} \in \bar{G}} F^{\bar{h}(\bar{M}(k,\bar{z}))}_{\bar{g}(\bar{M}(k,\bar{x}))\bar{M}(k,\bar{y})}$$

.

Proof. Let $\bar{\mathcal{S}} = \{$ a representative of $\bar{L}\bar{g} \mid \bar{g} \in \bar{G}$ $\}$ and thus $\bar{\mathcal{S}} \subseteq \bar{G}$. Then modulo $|k| - 1$,

$$\begin{aligned}
\textstyle\sum_{\bar{g},\bar{h} \in \bar{G}} F^{\bar{h}(\bar{M}(k,\bar{z}))}_{\bar{g}(\bar{M}(k,\bar{x}))\bar{M}(k,\bar{y})} &\equiv \textstyle\sum_{\bar{g}_1,\bar{g}_2 \in \bar{\mathcal{S}}} \sum_{\bar{l}_1,\bar{l}_2 \in \bar{L}} F^{\bar{l}_2\bar{g}_2(\bar{M}(k,\bar{z}))}_{\bar{l}_1\bar{g}_1(\bar{M}(k,\bar{x}))\bar{M}(k,\bar{y})} \\
&\equiv \textstyle\sum_{\bar{\bar{g}}_1,\bar{\bar{g}}_2 \in \bar{\mathcal{S}}} F^{\bar{\bar{g}}_2(\bar{M}(k,\bar{\bar{z}}))}_{\bar{\bar{g}}_1(\bar{M}(k,\bar{\bar{x}}))\bar{M}(k,\bar{\bar{y}})} \\
&\equiv \textstyle\sum_{\bar{\bar{g}},\bar{\bar{h}} \in \bar{\bar{G}}} F^{\bar{\bar{h}}(\bar{M}(k,\bar{\bar{z}}))}_{\bar{\bar{g}}(\bar{M}(k,\bar{\bar{x}}))\bar{M}(k,\bar{\bar{y}})} \\
&\equiv F^{M(k,z)}_{M(k,x)M(k,y)}
\end{aligned}$$

The proof is completed.

3 Proof of Theorem in some cases

For any positive integer $n \in \mathbf{N}$, we denote

$$\mathcal{K}(n) = \{ \text{ a representative } k \text{ of a finite field isoclass } \mid (|k| - 1, n!) = 1 \ \}.$$

Obviously, if $m, n \in \mathbf{N}$ with $n \leq m$, we can suitably choose $\mathcal{K}(n)$ and $\mathcal{K}(m)$ such that $\mathcal{K}(m) \subseteq \mathcal{K}(n)$. So we can think that for $m, n \in \mathbf{N}$, $n \leq m$ if and only if $\mathcal{K}(m) \subseteq \mathcal{K}(n)$. Note that $\mathcal{K}(1)$ is the set of all finite fields under isomorphism.

Lemma 3.1 *For any $n \in \mathbf{N}$, $\mathcal{K}(n)$ is an infinite set.*

Proof. Let $n! = 2^{t_0} p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}$ be the primary decomposition. Set

$$m_i = i(p_1 - 1)(p_2 - 1) \cdots (p_s - 1) + 1, \quad i = 1, 2, \cdots.$$

We write $2^{m_i} - 1 = 2^{m_i - 1} + 2^{m_i - 1} - 1$. Note that $p_j | (2^{m_i - 1} - 1)$ but $p_j \not| 2^{m_i - 1}$, $j = 1, 2, \cdots, s$, and that $2 \not| 2^{m_i} - 1$. Thus $(n!, 2^{m_i} - 1) = 1$, $i = 1, 2, \cdots$. However we know that for each m_i, there exists a finite field k_i with $|k_i| = 2^{m_i}$. This induces our Lemma.

To prove Theorem, we need the following discussion.

Let k be a finite field. Let M, N, X be indecomposable $\Lambda(k)$-modules. From Proposition 1.1 and 1.2 we have $F^X_{N,M} \equiv |\mathrm{Ext}^1_{\Lambda(k)}(N,M)_X|/(|k|-1) \bmod (|k|-1)$ with the right integer. On the other hand, $\mathrm{Ext}^1_{\Lambda(k)}(N,M)$ is a k-space and thus k^* acts naturally on $\mathrm{Ext}^1_{\Lambda(k)}(N,M)$ as the following way: for any $t \in k^*$ and for any $\varepsilon : 0 \to M \xrightarrow{f} Y \xrightarrow{g} N \to 0 \in \mathrm{Ext}^1_{\Lambda(k)}(N,M)$, $t\varepsilon : 0 \to M \xrightarrow{t^{-1}f} Y \xrightarrow{g} N \to 0$. We see that $\mathrm{Ext}^1_{\Lambda(k)}(N,M)_X$ is stable under this k^*-action and the induced k^*-action on $\mathrm{Ext}^1_{\Lambda(k)}(N,M)_X$ is free. Therefore $|Ext^1_{\Lambda(k)}(N,M)_X|/(|k|-1) = |Ext^1_{\Lambda(k)}(N,M)_X/k^*|$. Hence we have

Lemma 3.2 . $|k|-1$ *divides*

$$F^X_{NM} - |Ext^1_{\Lambda(k)}(N,M)_X/k^*|$$

.

In the following proofs, we will adopt the scheme of Riedtmann's proofs in [**4**].

At first, for some $\tilde{x}, \tilde{y} \in \tilde{\Gamma}_0$ we consider the condition $(\star)$ about some field k' (may be an infinite field):

$$\{g \in G \mid \mathrm{Ext}^1_{\tilde{\Lambda}(k')}(g\tilde{M}(k',\tilde{x}).\tilde{M}(k',\tilde{y})) \neq 0\ \}$$

has at most one element. Note that the condition $(\star)$ is independent on the field k', that is, if $\tilde{x}, \tilde{y}$ satisfy the condition $(\star)$ about some field, then they also satisfy the condition $(\star)$ about any field.

Proposition 3.1 *Let $\tilde{x}, \tilde{y} \in \tilde{\Gamma}_0$. Assume that $\tilde{x}, \tilde{y}$ satisfy the condition $(\star)$ about some field. Then for any finite field k and for any $\tilde{z} \in \tilde{\Gamma}_0$, $|k|-1$ divides*

$$F^{M(k,z)}_{M(k,x)M(k,y)} - \sum_{g,h \in G} F^{h(\tilde{M}(k,\tilde{z}))}_{g(\tilde{M}(k,\tilde{x}))\tilde{M}(k,\tilde{y})},$$

where $F\tilde{x} = x$, $F\tilde{y} = y$ and $F\tilde{z} = z$.

Proof. By the assumption F induces the isomorphism

$$\mathrm{Ext}^1_{\tilde{\Lambda}(k)}(g'\tilde{M}(k,\tilde{x}).\tilde{M}(k,\tilde{y})) \xrightarrow{\sim} \mathrm{Ext}^1_{\Lambda(k)}(M(k,x).M(k,y))$$

for some $g' \in G$. This meanwhile implies $\mathrm{Ext}^1_{\tilde{\Lambda}(k)}(h'\tilde{M}(k,\tilde{x}).\tilde{M}(k,\tilde{y})) = 0$ for any $h' \in G$ with $h' \neq g'$. Thus $|k|-1$ divides

$$|\ \mathrm{Ext}^1_{\Lambda(k)}(M(k,x).M(k,y))_{M(k,z)}/k^*| - \sum_{h \in G} |\ \mathrm{Ext}^1_{\tilde{\Lambda}(k)}(g'\tilde{M}(k,\tilde{x}).\tilde{M}(k,\tilde{y}))_{h\tilde{M}(k,\tilde{z})}/k^*|$$

and

$$\sum_{g,h \in G} |\ \mathrm{Ext}^1_{\tilde{\Lambda}(k)}(g\tilde{M}(k,\tilde{x}).\tilde{M}(k,\tilde{y}))_{h\tilde{M}(k,\tilde{z})}/k^*| = \sum_{h \in G} |\ \mathrm{Ext}^1_{\tilde{\Lambda}(k)}(g'\tilde{M}(k,\tilde{x}).\tilde{M}(k,\tilde{y}))_{h\tilde{M}(k,\tilde{z})}/k^*|.$$

The proof follows from Lemma 3.2.

Corollary. *Assume that for any* $\tilde{x}, \tilde{y} \in \tilde{\Gamma}_0$*,*$\tilde{x}, \tilde{y}$ *satisfy the condition* $(\star)$ *about some field, Take* $\mathcal{K}(1)$ *and let* $\mathcal{H}(\Gamma)_1$ *be the degenerate Hall algebra on* $\mathcal{K}(1)$*. Then* $\mathcal{L}(\Gamma)$ *is a Lie subalgebra of* $\mathcal{H}(\Gamma)_1$ *and* $\mathcal{L}(\Gamma)$ *just is* $\mathcal{L}(\tilde{\Gamma})/G$*.*

Proof. For any $x, y, z \in \Gamma_0$, take $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{\Gamma}_0$ with $F\tilde{x} = x$, $F\tilde{y} = y$ and $F\tilde{z} = z$. Note that for any $g, h \in G$ there is a integer $\gamma^{h(\tilde{z})}_{g(\tilde{x})\tilde{y}} = \varphi^{h(\tilde{z})}_{g(\tilde{x})\tilde{y}}(1)$, where $\varphi^{h(\tilde{z})}_{g(\tilde{x})\tilde{y}}(T) \in \mathbf{Z}[T]$ is the Hall polynomial, such that $|k| - 1$ divides

$$F^{h(\tilde{M}(k,\tilde{z}))}_{g(\tilde{M}(k,\tilde{x}))\tilde{M}(k,\tilde{y})} - \gamma^{h(\tilde{z})}_{g(\tilde{x})\tilde{y}}$$

for any finite field k. From Proposition 3.1 $|k| - 1$ divides

$$F^{M(k,z)}_{M(k,x)M(k,y)} - \sum_{g,h \in G} \gamma^{h(\tilde{z})}_{g(\tilde{x})\tilde{y}}$$

for any finite field k. The proof follows from Proposition 2.3.

Second, we consider the case $G \simeq \mathbf{Z}$.

Proposition 3.2 *Assume that* $G =< g >$ *be an infinite cyclic group. Then there exists a* $\mathcal{K}(n)$ *such that for any* $k \in \mathcal{K}(n)$ *and for any* $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{\Gamma}_0$*,* $|k| - 1$ *divides*

$$F^{M(k,z)}_{M(k,x)M(k,y)} - \sum_{g,h \in G} F^{h(\tilde{M}(k,\tilde{z}))}_{g(\tilde{M}(k,\tilde{x}))\tilde{M}(k,\tilde{y})},$$

where $F\tilde{x} = x$*,* $F\tilde{y} = y$ *and* $F\tilde{z} = z$*. In this case,* $\mathcal{L}(\Gamma)$ *is a Lie subalgebra of* $\mathcal{H}(\Gamma)_1$ *and* $\mathcal{L}(\Gamma)$ *just is* $\mathcal{L}(\tilde{\Gamma})/G$*, where* $\mathcal{H}(\Gamma)_1$ *is the degenerate Hall algebra on* $\mathcal{K}(n)$*.*

Proof. Note that Γ is a finite quiver. So there exists an integer $n \in \mathbf{N}_0$ such that for any field k and for any $x, y \in \Gamma_0$, F induces the isomorphism

$$\oplus_{i=0}^{n-1} \operatorname{Ext}^1_{\Lambda(k)}(g^i \tilde{M}(k, \tilde{x}').\tilde{M}(k, \tilde{y}')) \xrightarrow{\sim} \operatorname{Ext}^1_{\Lambda(k)}(M(k, x).M(k, y))$$

for suitable $\tilde{x}', \tilde{y}' \in \tilde{\Gamma}_0$ with $F\tilde{x}' = x$ and $F\tilde{y}' = y$. Now fix such n and arbitarily take one $k \in \mathcal{K}(n)$. We define a k^*-action on $\oplus_{i=0}^{n-1} \operatorname{Ext}^1_{\Lambda(k)}(g^i \tilde{M}(k, \tilde{x}').\tilde{M}(k, \tilde{y}'))$ as follows: $t \circ (\varepsilon_j)_{j=0}^{n-1} = (t^j \varepsilon_j)_{j=0}^{n-1}$ for any $(\varepsilon_j)_{j=0}^{n-1} \in \oplus_{i=0}^{n-1} \operatorname{Ext}^1_{\Lambda(k)}(g^i \tilde{M}(k, \tilde{x}').\tilde{M}(k, \tilde{y}'))$ and for any $t \in k^*$. This induces a k^*-action on $\operatorname{Ext}^1_{\Lambda(k)}(M(k, x).M(k, y))$ by F. Following 5.2 Corollary in [**4**], $\operatorname{Ext}^1_{\Lambda(k)}(M(k, x).M(k, y))_{M(k,z)}$ is stable unter this k^*-action. So this induces a k^*-action on $\operatorname{Ext}^1_{\Lambda(k)}(M(k, x).M(k, y))_{M(k,z)}/k^*$. Take an arbitary $\tilde{z} \in \tilde{\Gamma}_0$ with $F\tilde{z} = z$. We claim that k^* freely acts on

$$(\operatorname{Ext}^1_{\Lambda(k)}(M(k, x).M(k, y))_{M(k,z)}/k^*) \backslash \bigcup_{j=0}^{n-1} \bigcup_{i=-\infty}^{\infty} F(\operatorname{Ext}^1_{\Lambda(k)}(g^j \tilde{M}(k, \tilde{x}').\tilde{M}(k, \tilde{y}'))_{g^i \tilde{M}(k,\tilde{z})}/k^*).$$

In fact, for any $t \in k^*$ and for any $(\varepsilon_j)_{j=0}^{n-1} \in \oplus_{i=0}^{n-1} \operatorname{Ext}^1_{\Lambda(k)}(g^i \tilde{M}(k, \tilde{x}').\tilde{M}(k, \tilde{y}'))$ with at least two entries non-zero, such as $\varepsilon_{j_1}, \varepsilon_{j_2}$, $0 \le j_1 < j_2 \le n - 1$, if there are $t, s \in k^*$ such that $t\circ(\varepsilon_j)_{j=0}^{n-1} = (s\varepsilon_j)_{j=0}^{n-1}$, then $t^{j_1} = t^{j_2}$. However $(|k^*|, n!) = 1$ and thus $(|k^*|, j_2 - j_1) = 1$. So $t = 1$ and therefore our claim follows. This implies that $|k| - 1$ divides

$$| \operatorname{Ext}^1_{\Lambda(k)}(M(k,x).M(k,y))_{M(k,z)}/k^*| - \sum_{i,j\in\mathbf{Z}} | \operatorname{Ext}^1_{\tilde\Lambda(k)}(g^j\tilde M(k,\tilde x').\tilde M(k,\tilde y'))_{g^i\tilde M(k,\tilde z)}/k^*|.$$

However,

$$\sum_{i,j\in\mathbf{Z}} | \operatorname{Ext}^1_{\tilde\Lambda(k)}(g^j\tilde M(k,\tilde x').\tilde M(k,\tilde y'))_{g^i\tilde M(k,\tilde z)}/k^*| = \sum_{i,j\in\mathbf{Z}} | \operatorname{Ext}^1_{\tilde\Lambda(k)}(g^j\tilde M(k,\tilde x).\tilde M(k,\tilde y))_{g^i\tilde M(k,\tilde z)}/k^*|.$$

By Lemma 3.2 $|k| - 1$ divides

$$F^{M(k,z)}_{M(k,x)M(k,y)} - \sum_{g,h\in G} F^{h(\tilde M(k,\tilde z))}_{g(\tilde M(k,\tilde x))\tilde M(k,\tilde y)}.$$

Finally, note that for any $g, h \in G$ there is a integer $\gamma^{h(\tilde z)}_{g(\tilde x)\tilde y} = \varphi^{h(\tilde z)}_{g(\tilde x)\tilde y}(1)$, where $\varphi^{h(\tilde z)}_{g(\tilde x)\tilde y}(T) \in \mathbf{Z}[T]$ is the Hall polynomial, such that $|k| - 1$ divides

$$F^{h(\tilde M(k,\tilde z)}_{g(\tilde M(k,\tilde x))\tilde M(k,\tilde y)} - \gamma^{h(\tilde z)}_{g(\tilde x)\tilde y}.$$

Therefore $|k| - 1$ divides

$$F^{M(k,z)}_{M(k,x)M(k,y)} - \sum_{g,h\in G} \gamma^{h(\tilde z)}_{g(\tilde x)\tilde y}.$$

From Proposition 2.3, $\mathcal{L}(\Gamma)$ is a Lie subalgebra of $\mathcal{H}(\Gamma)_1$ and $\mathcal{L}(\Gamma)$ just is $\mathcal{L}(\tilde\Gamma)/G$.

4 Proof of Theorem in general case

We denote by G' the commutator subgroup of G and by $G^{(i)}$ the higher commutator subgroup of G, $i \in \mathbf{N}$, defined inductively by

$$G^{(i+1)} = [G^{(i)}, G^{(i)}].$$

They are normal subgroups of G. Note that G is a free group [**1**] . So $G^{(i)}$ also is free provided $G^{(i)} \neq < 1 >$, the unit element subgroup of G, and $\cap_{i=1}^{\infty} G^{(i)} = \{1\}$.

Now Put $\bar\Gamma = \tilde\Gamma/G'$ and $\bar G = G/G'$ and denote by $\bar F$ the covering $\bar\Gamma \to \bar\Gamma/\bar G$. Then we have

Proposition 4.1 *If there is a $\mathcal{K}(m)$ such that for any $\bar x, \bar y, \bar z \in \bar\Gamma_0$ there exists an integer $\bar\gamma^{\bar z}_{\bar x\bar y}$ such that $|k| - 1$ divides*

$$F^{\bar M(k,\bar z)}_{\bar M(k,\bar x)\bar M(k,\bar y)} - \bar\gamma^{\bar z}_{\bar x\bar y}$$

for any $k \in \mathcal{K}(n)$ and thus $\mathcal{L}(\bar\Gamma)$ is a Lie algebra, then there exits a $\mathcal{K}(n) \subseteq \mathcal{K}(m)$ such that for any $\bar x, \bar y, \bar z \in \bar\Gamma_0$ and for any $k \in \mathcal{K}(n)$, $|k| - 1$ divides

$$F^{M(k,z)}_{M(k,x)M(k,y)} - \sum_{\bar g,\bar h\in\bar G} \bar\gamma^{\bar h(\bar z)}_{\bar g(\bar x)\bar y},$$

where $\bar F\bar x = x, \bar F\bar y = y$ and $\bar F\bar z = z$. In this case, $\mathcal{L}(\Gamma)$ is a Lie subalgebra of $\mathcal{H}(\Gamma)_1$ and just is $\mathcal{L}(\bar\Gamma)/\bar G$, where $\mathcal{H}(\Gamma)_1$ is the degenerate Hall algebra of Γ on $\mathcal{K}(n)$.

Proof. For any $k \in \mathcal{K}(m)$ and For any $x, y \in \Gamma_0$, choose $\bar{x}, \bar{y} \in \bar{\Gamma}_0$ with $\bar{F}\bar{x} = x$ and $\bar{F}\bar{y} = y$. let $\mathcal{S}$ be a finite set

$$\mathcal{S} = \{\bar{g} \in \bar{G} | \mathrm{Ext}^1_{\bar{\Lambda}(k)}(\bar{g}\bar{M}(k, \bar{x}).\bar{M}(k, \bar{y})) \neq 0\}.$$

As $\bar{G}$ is free abelian, there exists a group homomorphism $\lambda : \bar{G} \to \mathbf{Z}$ whose restriction to $\mathcal{S}$ is injective. Denote by $F' : \bar{\Gamma} \to \bar{\Gamma}/\ker\lambda = \bar{\bar{\Gamma}}$ and $F'' : \bar{\Gamma}/\ker\lambda \to (\bar{\Gamma}/\ker\lambda)/(\bar{G}/\ker\lambda) = \bar{\Gamma}/\bar{G} = \Gamma$ the coverings. Of course we have $\bar{F} = F'F''$. Set $F'\bar{x} = \bar{\bar{x}}$ and $F'\bar{y} = \bar{\bar{y}}$.

We denote the residue class of an element g in G modulo G' by $\bar{g}$ and the residue class of $\bar{g}$ modulo $\ker\lambda$ by $\bar{\bar{g}}$.

By Proposition 3.2 there exists a $\mathcal{K}(n)$ such that for any $k \in \mathcal{K}(n)$, $|k| - 1$ divides

$$F^{M(k,z)}_{M(k,x)M(k,y)} - \sum_{\bar{\bar{g}}, \bar{\bar{h}} \in \bar{G}/\ker\lambda} F^{\bar{\bar{h}}(\bar{\bar{M}}(k,\bar{\bar{z}}))}_{\bar{\bar{g}}(\bar{\bar{M}}(k,\bar{\bar{x}}))\bar{\bar{M}}(k,\bar{\bar{y}})}.$$

On the other hand, we claim that, by the choice of λ, for any $\bar{g} \in \bar{G}$ there is a unique $\bar{l} \in \ker\lambda$ such that $\mathrm{Ext}^1_{\bar{\Lambda}(k)}(\bar{l}(\bar{g}\bar{M}(k, \bar{x})).\bar{M}(k, \bar{y})) \neq 0$ provided $\mathrm{Ext}^1_{\bar{\bar{\Lambda}}(k)}(\bar{\bar{g}}\bar{\bar{M}}(k, \bar{\bar{x}}).\bar{\bar{M}}(k, \bar{\bar{y}})) \neq 0$. Indeed, if for $\bar{l}_1, \bar{l}_1 \in \ker\lambda$,

$$\mathrm{Ext}^1_{\bar{\Lambda}(k)}(\bar{l}_i(\bar{g}\bar{M}(k, \bar{x})).\bar{M}(k, \bar{y})) \neq 0, \ \ i = 1, 2,$$

then $\bar{l}_i\bar{g} \in \mathcal{S}$ and $\bar{\bar{l}}_1\bar{\bar{g}} = \bar{\bar{g}} = \bar{\bar{l}}_2\bar{\bar{g}}$. However $\lambda|_{\mathcal{S}}$ is injective. So $\bar{l}_1\bar{g} = \bar{l}_2\bar{g}$ and thus $\bar{l}_1 = \bar{l}_2$. Our claim follows.

Therefore by Proposition 3.1, 3.2 and 2.4, for any $k \in \mathcal{K}(n)$, $|k| - 1$ divides

$$F^{M(k,z)}_{M(k,x)M(k,y)} - \sum_{\bar{g}, \bar{h} \in \bar{G}} F^{\bar{h}(\bar{M}(k,\bar{z}))}_{\bar{g}(\bar{M}(k,\bar{x}))\bar{M}(k,\bar{y})}$$

and thus divides

$$F^{M(k,z)}_{M(k,x)M(k,y)} - \sum_{\bar{g}, \bar{h} \in \bar{G}} \gamma^{\bar{h}(\bar{z})}_{\bar{g}(\bar{x})\bar{y}}.$$

The last consequence can be obtained from Proposition 2.3.

Proposition 4.2 *There exists some $t \in \mathbf{N}$ such that for any finite field k and for any $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{\Gamma}_0$, $|k| - 1$ divides*

$$F^{\bar{M}(k,\bar{F}\tilde{z})}_{\bar{M}(k,\bar{F}\tilde{x})\bar{M}(k,\bar{F}\tilde{y})} - \sum_{g,h \in G^{(t)}} F^{h(\tilde{M}(k,\tilde{z}))}_{g(\tilde{M}(k,\tilde{x}))\tilde{M}(k,\tilde{y})},$$

where $\bar{F} : \tilde{\Gamma} \to \tilde{\Gamma}/G^{(t)} = \bar{\Gamma}$ is the covering with the fundamental group $G^{(t)}$. Consequently, $\mathcal{L}(\bar{\Gamma})$ is a Lie subalgebra of $\mathcal{H}(\bar{\Gamma})_1$ and just is $\mathcal{L}(\tilde{\Gamma})/G^{(t)}$, where $\mathcal{H}(\bar{\Gamma})_1$ is the degenerate Hall algebra of $\bar{\Gamma}$ on $\mathcal{K}(1)$.

Proof. we set

$$\mathcal{S}(\tilde{x}, \tilde{y}) = \{g \in G \mid \mathrm{Ext}^1_{\tilde{\Lambda}(k)}(g(\tilde{M}(k, \tilde{x})), \tilde{M}(k, \tilde{y})) \neq 0\}$$

and

$$\mathcal{T}(\tilde{x},\tilde{y}) = \{g_1 g_2^{-1} \mid g_1, g_2 \in \mathcal{S}(\tilde{x},\tilde{y})\}.$$

Clearly we have $\mathcal{S}(h\tilde{x}, l\tilde{y}) = l\mathcal{S}(\tilde{x},\tilde{y})h^{-1}$ and $\mathcal{T}(h\tilde{x}, l\tilde{y}) = l\mathcal{T}(\tilde{x},\tilde{y})l^{-1}$. Since Γ is a finite quiver, Γ_0 contains only finitely many G-orbits and thus the set

$$\mathcal{T} = \bigcup_{\tilde{x},\tilde{y}\in\tilde{\Gamma}_0} \mathcal{T}(\tilde{x},\tilde{y})$$

is a finite union of conjugacy classes in G. As $\cap_{i\in\mathbf{N}} G^{(i)} = \{1\}$, there exixts some $t \in \mathbf{N}$ such that $\mathcal{T} \cap G^{(t)} = \{1\}$. Therefore the set

$$\mathcal{S}(\tilde{x},\tilde{y}) \cap G^{(t)} = \{g \in G^{(t)} \mid \mathrm{Ext}^1_{\tilde{\Lambda}(k)}(g(\tilde{M}(k,\tilde{x})), \tilde{M}(k,\tilde{y})) \neq 0\ \}$$

contains at most one element for all $\tilde{x},\tilde{y} \in \tilde{\Gamma}_0$ and for all finite field k. From Proposition 3.1 and its Corollary the proof is completed.

Proof of Theorem. We fix the integer t in Proposition 4.2. Then from Proposition 4.2 and 2.4, and applying Proposition 4.1 in which we replace G by $G^{(t-1)}$, and Γ by $\tilde{\Gamma}/G^{(t-1)} = (\tilde{\Gamma}/G^{(y)})/(G^{(t-1)}/G^{(t)})$. there exist $\mathcal{K}(n_1) \subseteq \mathcal{K}(n_0)$ with $n_0 = 1$ and integers $\gamma^{z^{(t-1)}}_{x^{(t-1)}y^{(t-1)}}$ for all $x^{(t-1)}, y^{(t-1)}, z^{(t-1)} \in \Gamma/G^{(t-1)}$ such that $|k| - 1$ divides

$$F^{M^{(t-1)}(k,z^{(t-1)})}_{M^{(t-1)}(k,x^{(t-1)})M^{(t-1)}(k,y^{(t-1)})} - \gamma^{z^{(t-1)}}_{x^{(t-1)}y^{(t-1)}}$$

for any $k \in \mathcal{K}(n_1)$, and $\mathcal{L}(\tilde{\Gamma}/G^{(t-1)}) = \mathcal{L}(\tilde{\Gamma}/G^{(t)})/(G^{(t-1)}/G^{(t)}) = (\mathcal{L}(\tilde{\Gamma})/G^{(t)})/(G^{(t-1)}/G^{(t)})$ is a Lie subalgebra of $\mathcal{H}(\tilde{\Gamma}/G^{(t-1)})_1$, where $\mathcal{H}(\tilde{\Gamma}/G^{(t-1)})_1$ is the degenerated Hall algebra of $\tilde{\Gamma}/G^{(t-1)}$ on $\mathcal{K}(n_1)$. However we easily prove $(\mathcal{L}(\tilde{\Gamma})/G^{(t)})/(G^{(t-1)}/G^{(t)}) = \mathcal{L}(\tilde{\Gamma})/G^{(t-1)}$ and thus $(\mathcal{L}(\tilde{\Gamma})/G^{(t-1)}) = \mathcal{L}(\tilde{\Gamma})/G^{(t-1)}$.

Now we denote $G^{(0)} = G$. Similarly, by induction on i we can obtain $\mathcal{K}(n_1), \mathcal{K}(n_2), \cdots,$ $\mathcal{K}(n_t)$ with $n_1 \leq n_2 \leq \cdots \leq n_t$ and integers $\gamma^{z^{(t-i)}}_{x^{(t-i)}y^{(t-i)}}$ for all $x^{(t-i)}, y^{(t-i)}, z^{(t-i)} \in \Gamma/G^{(t-i)}$ such that $|k| - 1$ divides

$$F^{M^{(t-i)}(k,z^{(t-i)})}_{M^{(t-i)}(k,x^{(t-i)})M^{(t-i)}(k,y^{(t-i)})} - \gamma^{z^{(t-i)}}_{x^{(t-i)}y^{(t-i)}}$$

for any $k \in \mathcal{K}(n_i)$, and $\mathcal{L}(\tilde{\Gamma}/G^{(t-i)}) = \mathcal{L}(\tilde{\Gamma})/G^{(t-i)}$ is a Lie subalgebra of $\mathcal{H}(\tilde{\Gamma}/G^{(t-i)})_1$, where $\mathcal{H}(\tilde{\Gamma}/G^{(t-i)})_1$ is the degenerated Hall algebra of $\tilde{\Gamma}/G^{(t-i)}$ on $\mathcal{K}(n_i)$. In particular, take $\mathcal{K} = \mathcal{K}(n_t)$ and denote by $\mathcal{H}(\Gamma)_1$ the degenerated Hall algebra of Γ on $\mathcal{K}$. Then $\mathcal{L}(\Gamma) = \mathcal{L}(\tilde{\Gamma})/G$ is a Lie subalgebra of $\mathcal{H}(\Gamma)_1$.

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