Classifications and Characterisations of Snarks

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Abstract

This paper considers vertex-reductions of snarks. Each reduction naturally divides the class of snarks into three classes. The first class contains those snarks which are not reducible to a snark. The second contains those which are not reducible to a 3-colorable graph and the third one consists of the snarks which are reducible to a snark as well as to a 3-colorable graph. We characterise all these classes in terms of 2-factors.

Furthermore we characterise snarks which are divided into 3-colorable components by the removal of any edge cutset in terms of 3-critical subgraphs and show that there are infinitely many such snarks.

1 Introduction

We follow the terminology and notation of Bondy and Murty [2]. We consider proper edge colorings of 3-regular graphs. The set of 3-regular graphs is divided into two classes by a famous theorem of Vizing which states that each simple loopless graph is edge-colorable with Δ or $\Delta+1$ colors, see e.g. [2]. Class I contains the 3-colorable graphs and class II the graphs of chromatic index 4. Isaacs [12] called the graphs of class I *colorable* graphs and the graphs of class II *uncolorable* graphs. Uncolorable graphs are of great interest, since if there are counter-examples to some hard conjectures, e.g. Tutte's 5-flow conjecture, the double cover conjecture for 3-regular graphs (see e.g. [13]), then they must be uncolorable. Gardner [7] proposed to call nontrivial uncolorable graphs snarks. However it is not clear what is "nontrivial". The general opinion of a "nontrivial" snark seems to be that it is not reducible to a smaller snark by a well defined reduction. This topic has attracted considerable discussion in the literature and different types of reductions were investigated, see e. g. [4], [7], [12] and [15]. In none of these papers structural characterisations of the irreducible snarks are given.
As proposed in [16] and [17] another approach is to focus on the the structure of snarks, leaving the definition of a snark as broad as possible, and to find a recursive description of snarks.

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We pursue this approach and consider vertex-reductions of snarks. We define a *snark* to be a connected, bridgeless, simple, uncolorable graph.
A graph G is *vertex-reducible* if a simple graph G' can be obtained from G by removing two vertices together with all incident edges from G and adding an appropriate number of edges to restore 3-regularity. We also say G is vertex-reducible to G'. This reduction divides the class of snarks, denoted with $\mathcal{G}$, into three classes.

$\mathcal{B} := \{G|G \in \mathcal{G} \wedge G$ is not vertex-reducible to a snark$\}$

$\mathcal{S} := \{G|G \in \mathcal{G} \wedge G$ is not vertex-reducible to a colorable graph$\}$

$\mathcal{C} := \mathcal{G} \setminus (\mathcal{B} \cup \mathcal{S})$

The class $\mathcal{B}$ is the base, and the class $\mathcal{S}$ consists of strong snarks, that is snarks which are not vertex-reducible to a colorable graph. The class $\mathcal{C}$ consists of snarks which are vertex-reducible to a snark as well as to a colorable graph. One aim of this paper is to give a structural characterisation of these classes in terms of 2-factors.
We distinguish three kinds of vertex-reductions for the analysis of vertex-reductions of snarks. For $i = 1, 2$ we say G is (distance) *i-reducible* if it is vertex-reducible by removing two vertices v and w with $d_G(v, w) = i$. A graph G is (distance) *3-reducible* if it is vertex-reducible by removing two vertices v, w with $d_G(v, w) \geq 3$. 1-reductions are also called edge-reductions. The classes $\mathcal{B}_i$, $\mathcal{C}_i$ and $\mathcal{S}_i$ are defined as $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{S}$ with respect to i-reductions, for $i = 1, 2, 3$, and therefore

$$\mathcal{B} = \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3$$
$$\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3$$

Uncolorable graphs with bridge are of minor interest, since all conjectures mentioned above as well as the famous theorem of Tait (see e.g. [2]), which was the starting point for the study of snarks, are for bridgeless graphs. They are also not interesting from the view of reductions. It is easy to see that the only irreducible uncolorable graph with bridge can be obtained from two K_4 by inserting a vertex in an edge of each K_4 and joining these two vertices by an edge. All the other graphs with bridges are edge-reducible to a smaller graph with bridge, see [1].
It is natural to consider edge-reductions of snarks since each bridgeless (3-regular) graph $G \neq K_4$ can be reduced to a smaller bridgeless 3-regular graph by removing two adjacent vertices or by removing a square with a diagonal edge and restoring 3-regularity, see [1]. Edge-reductions are also directly related to Tutte's conjecture that every snark contains a subgraph contractable to the Petersen graph, see e.g. [17].
In section 2 we introduce the *color number* $c(G)$ of a graph G, the minimal cardinality of a color class of G. We show, for each even $n \geq 10$ there is a snark G with $c(G) = 2$. This invariant of a graph is a useful tool for the proofs of the characterisation theorems which are given in the sections 3, 4 and 5. In section 6 we investigate snarks of the bases. We show that the elements of $\mathcal{B}$ are at least cyclically 5-edge connected, and that $\mathcal{B}_1$ is an infinite set. In particular we show that each hypohamiltonian snark is an element of $\mathcal{B}_1$.

Furthermore we characterise snarks which are divided into 3-colorable components by the removal of any edge cutset in terms of 3-critical subgraphs.
Our results imply that Tutte's 5-flow conjecture as well as the double cover conjecture are true for snarks which are vertex-reducible to a colorable graph, c.f. [10], [11] and [13].

2 4-Colorings

Let G be an uncolorable graph, then each 4-coloring $c : EG \mapsto \{1, \cdots, 4\}$ of G partitions EG into color classes $X_i^c := \{e | e \in EG \wedge c(e) = i\}$, $i = 1, \cdots, 4$. We define the *color number* $c(G)$ of G to be

$$c(G) := \min\{|X_i^c| | c \text{ is 4-coloring of } G \wedge i \in \{1, \cdots, 4\}\}.$$

Clearly $c(G) \geq 2$ for each uncolorable graph G.
A 4-coloring c of G with $|X_i^c| = c(G)$ for an $i \in \{1, \cdots, 4\}$ is called a *minimal* 4-coloring of G. With no loss of generality we will always let X_4 to be a minimal color class of a 4-coloring under consideration, and we omit the superscript when there is no possibility of confusion.
Let c be a minimal 4-coloring of G. We define for $i = 1, 2, 3$ the subsets H_i of X_4 as follows: Suppose $e \in X_4$ and that $e_1, \cdots, e_4$ are the edges incident with e in G. Then

$$e \in H_i :\iff |\{e_j | c(e_j) = i \wedge j \in \{1, \cdots, 4\}\}| = 2$$

It is easy to see that $H_i \cap H_j = \emptyset$ if $i \neq j$, and $X_4 = H_1 \cup H_2 \cup H_3$.
Let $\{a, b, c\} = \{1, 2, 3\}$. An (a, b)-*path* is a path whose edges are alternately colored with colors a and b.

Lemma 2.1 *Let $X_4 = H_1 \cup H_2 \cup H_3$ be a minimal color class of an uncolorable graph and $e = vw \in H_a$. Then there is a (b, c)-path from v to w.*

Proof. Consider the (b, c)-path starting from v, consisting of the edges $e_1, e_2, \cdots, e_n$. Clearly, e_n is adjacent to an edge $e' \in X_4$. With no loss of generality we assume that e_1 is colored with color b. Suppose $e' \neq e$. Then we color e with b, e_1 with c, e_2 with b and so on up to e_n. But this is a 4-coloring c' of G with $|X_4^{c'}| < c(G)$, a contradiction.

□

Let $e = vw$ be the edge of Lemma 2.1, then e together with the (b, c)-path from v to w forms a circle C_e of odd length in G.

Lemma 2.2 *Let c be a minimal 4-coloring of an uncolorable graph G with minimal color class X_4. Then holds $C_{e_1} \cap C_{e_2} = \emptyset$ for two different edges $e_1, e_2 \in X_4$.*

Proof. Suppose $e \in C_{e_1} \cap C_{e_2}$. Let $e_1 \in H_1$. We get another minimal 4-coloring c' of G by recoloring C_{e_1} such that $c'(e) = 4$ and all the other edges of C_{e_1} are alternately colored with colors 2 and 3. Let $e_2 = v_2w_2 \in H_a$. Then there is no (b, c)-path from v_2 to w_2. This is a contradiction to Lemma 2.1.

□

Let S and S' be two disjoint subgraphs of G. We define the *distance* $d_G(S, S')$ between S and S' in G to be the number of edges of a shortest path between S and S' in G.

Lemma 2.3 *Let c be a minimal 4-coloring of an uncolorable graph G and $e_1, e_2 \in X_4$. Then holds $d_G(C_{e_1}, C_{e_2}) \neq 2$ for $e_1, e_2 \in H_a$.*

Proof. By Lemma 2.2, C_{e_1} and C_{e_2} are disjoint and therefore $d_G(C_{e_1}, C_{e_2}) \geq 1$. All the edges incident to C_{e_1} or C_{e_2} are colored with the same color. Therefore, if $d_G(C_{e_1}, C_{e_2}) = 2$ there is a vertex in G incident to two edges colored with the same color. This is a contradiction, since c is a proper coloring.

□

The Parity Lemma [12] is a very usefull tool for proofs in the field of uncolorable graphs.

Lemma 2.4 (Parity Lemma) *Let G be a graph that has been 3-colored with colors 1, 2 and 3. If a cutset consisting of n edges contains n_i edges of color i, for $i = 1, 2, 3$, then $n_1 \equiv n_2 \equiv n_3 \equiv n \pmod{2}$.*

We use this lemma to prove the following.

Lemma 2.5 *Let c be a minimal 4-coloring of an uncolorable graph G with minimal color class $X_4 = H_1 \cup H_2 \cup H_3$. Then holds: $|H_1| \equiv |H_2| \equiv |H_3| \equiv c(G) \pmod{2}$.*

Proof. Let $X_4 = \{e_1, e_2, \cdots, e_n\}$, with $e_i = v_iw_i$ for $i = 1, \cdots, n$. For each i, we put in a new vertex u_i in e_i and a "dangeling" edge f_i on u_i. If $e_i \in H_a$ we color u_iv_i and u_iw_i in an appropriate way with colors b and c and f_i with color a. Then the set of dangeling edges f_i forms a cut set of a colorable graph, e.g. if we join this graph to a copy of itself, colored in the same way. Thus the proposition follows from the Parity Lemma directly.

□

Let G be a graph and let $\mathcal{F}$ be a 2-factor of G. We denote the complementary 1-factor of $\mathcal{F}$ with $\overline{\mathcal{F}}$. Let $o_{\mathcal{F}}(G)$ be the number of odd circles in $\mathcal{F}$. If $o_{\mathcal{F}}(G) = 0$ we say $\mathcal{F}$ is an *even* 2-factor of G. If $o_{\mathcal{F}}(G) = m$ we denote the odd circles of $\mathcal{F}$ with $U_1^{\mathcal{F}}, U_2^{\mathcal{F}}, \cdots, U_m^{\mathcal{F}}$. We omit the superscript if there is no possibility of confusion. We define the *oddness* $\omega(G)$ of G to be

$$\omega(G) := \min\{o_{\mathcal{F}}(G) | \mathcal{F} \text{ is a 2-factor of G }\}.$$

Clearly, $\omega(G)$ is even and it holds $c(G) \leq \omega(G)$.

Lemma 2.6 *Let G be an uncolorable graph, then holds $\omega(G) = 2$ if and only if $c(G) = 2$.*

Proof. ($\Rightarrow$) Take a 2-factor of G with exactly two odd circles and color the edges of the even circles with colors 2 and 3 alternately. Color one edge in each odd circle with color 4 and the remaining edges of these two circles with colors 2 and 3 alternately. Color the edges of the complementary 1-factor with color 1. This is 4-coloring c of G with $|X_4| = 2$. Since G is uncolorable it holds $c(G) \geq 2$ and therefore $c(G) = 2$.

($\Leftarrow$) Let c be a minimal 4-coloring with $\{e_1, e_2\} = X_4$. By Lemma 2.5 holds $X_4 = H_a$ and C_{e_1}, C_{e_2} are odd circles. Then the (b,c)-circles and C_{e_1}, C_{e_2} are the elements of a 2-factor with exactly two odd circles. Thus $\omega(G) \leq c(G)$ and therefore $\omega(G) = 2$ since G is uncolorable.

□

Let c be a minimal 4-coloring of a snark G with $|X_4| = 2$. Let $\mathcal{F}$ be a 2-factor of G obtained from c as described in the second part of the proof of Lemma 2.6. We will say $\mathcal{F}$ is *induced by* c and denote it by $\mathcal{F}_c$. As a first application of the notion of the color number we show:

Theorem 2.7 *For all even $n \geq 10$, there is a snark G with $c(G) = \omega(G) = 2$.*

Proof. For each even $n \geq 10$ there are snarks of order n, cf. [3], [12]. We define for each even $n \geq 10$: $c(n) := min\{c(G) | G$ is an uncolorable graph of order $n\}$. Suppose there is an $n \geq 10$ such that $c(n) > 2$. Let G be a snark of order n with $c(G) = c(n)$. Let $e_1 = v_1u_1$ and $e_2 = v_2u_2$ be two edges of the minimal color class X_4 of a minimal 4-coloring of G.
By the pigeon hole principle there is one vertex of e_1, say v_1, and a vertex of e_2, say v_2, such that the edges incident to v_1 and v_2 and different from e_1 and e_2, respectively, are colored with the same two colors. Further we may assume $v_1v_2, u_1u_2 \notin EG$ with no loss of generality.
We consider the simple graph $G' = (G - \{e_1, e_2\}) + \{v_1v_2, u_1u_2\}$ which is of order n. The given 4-coloring of G can be modified to a 4-coloring c' on G' such that $|X_4^{c'}| \leq c(G) - 1$; since v_1v_2 can be colored with a color different to 4. Therefore G' is colorable or it contains a bridge or it is not connected.
It follows from Lemma 2.2 that C_{e_1} and C_{e_2} are disjoint circles. Therefore e_1 and e_2 are not both contained in an edge cutset of G with cardinality ≤ 3.
If none of e_1 and e_2 is contained in an 2-edge cut, then obviously G' is bridgeless. If one of e_1 and e_2 is an element of a 2-edge cutset then G' is bridgeless by its construction.
Thus G' is colorable. But when we reconstruct G from G' we can modify the coloring on G' to a 4-coloring c^* on G with $|X_4^{c^*}| = 2$, a contradiction.

□

3 1-Reductions

Lemma 3.1 *Let G be a snark. If an edge e is contained in a triangle, square or a 2-edge cut of G then there is no 2-factor of G with exactly two odd circles which are connected by e.*

Proof. Suppose there is a 2-factor $\mathcal{F}$ of G with exactly two odd circles U_1, U_2 which are connected by $e = v_1v_2$, and $v_i \in VU_i$ for $i = 1, 2$.
Suppose e is contained in a triangle. Then $VU_1 \cap VU_2 \neq \emptyset$, a contradiction.
Suppose e is contained in a square. Let $e' = v_1'v_2'$ be the edge of the square not adjacent to e and $v_i' \in VU_i$ for $i = 1, 2$. Then e' also connects U_1 and U_2 and $v_iv_i' \in EG$ for $i = 1, 2$. But $U' := (U_1 - v_1v_1') \cup (U_2 - v_2v_2') \cup \{e, e'\}$ is an even circle in G which is disjoint to each circle of $\mathcal{F}$ different to U_1 and U_2. Therefore $\mathcal{F}' := (\mathcal{F} \setminus \{U_1, U_2\}) \cup \{U'\}$ is an even 2-factor of G. Thus G is colorable, a contradiction.
Suppose e is contained in a 2-edge cut $\{e, e'\}$ of G whose removal divides G in two parts G_1, G_2. Then $e' \in \overline{\mathcal{F}}$, too. Let $e' = v_1'v_2'$ and $v_i, v_i' \in VG_i$ for $i = 1, 2$. But then $G_1 + v_1v_1'$ is a 3-regular graph of odd order, a contradiction.

□

Lemma 3.2 *For each snark G and each edge $e = vw \in EG$ the following statements are equivalent.*

i) G is 1-reducible to a colorable graph by removing v and w.

ii) There is a 2-factor of G with exactly two odd circles connected by e.

iii) There are two different 2-factors of G with exactly two odd circles connected by e.

iv) Each graph to which G is 1-reducible by removing v and w is colorable.

Proof. ($i) \Rightarrow ii)$): Let G be 1-reducible to a colorable graph G' by removing v and w. Let c' be a coloring of G'. We reconstruct G from G' and extend c' to a coloring -not necessarily proper- $\bar{c}$ on G as follows. Let $e_1^v, e_2^v \neq e$ be the edges incident to v, and $e_1^w, e_2^w \neq e$ be the edges incident to w. We color these four edges such that v and w are the only vertices which are incident to edges colored in the same color. Then we have $\bar{c}(e_1^v) = \bar{c}(e_2^v)$. Otherwise by the Parity Lemma e_1^w and e_2^w are colored in the same two different colors as e_1^v and e_2^v and therefore e can be colored with the third color. Hence G is colorable, a contradiction.
Similary we have $\bar{c}(e_1^w) = \bar{c}(e_2^w)$. Thus we can color e with a color different from $\bar{c}(e_1^v), \bar{c}(e_1^w)$, say c, and e_1^v, e_1^w can be colored with color 4 to obtain a 4-coloring c^* of G with $|X_4^{c^*}| = 2$. Thus e connects the two odd circles of $\mathcal{F}_{c^*}$.
($ii) \Rightarrow iii)$): Let c^* be the 4-coloring of G given above. Let U_1, U_2 be the odd circles of $\mathcal{F}_{c^*}$ and $e_2^w \in EU_2$. If $c^*(e_2^v) = a$ and $c^*(e_2^w) = b$ we interchange the colors a and b in U_2 to

obtain a 4-coloring c_1 of G with $c_1(e_2^w) = c_1(e_2^v) = a$ and $H_c = X_4^{c_1} = X_4^{c^*}$. The coloring c_1 colors all edges adjacent to e with color 4 or a. Thus e can also be colored with color b to obtain a proper 4-coloring c_2 of G. Then $\mathcal{F}_{c_1}$ and $\mathcal{F}_{c_2}$ are the two 2-factors with the desired properties. It is obvious that they are different.

(*iii*) $\Rightarrow$ *iv*)): Let U_1, U_2 be the two odd circles of $\mathcal{F}_{c_1}$, and let $v \in U_1$, $w \in U_2$. It does not matter how we join the neighbors of v and w to restore 3-regularity. In any case we obtain a graph G' and an even 2-factor of it. Thus G' is colorable.

(*iv*) $\Rightarrow$ *i*)): trivial.

□

Let G be a snark which is i-reducible to a colorable graph $\tilde{G}$. When we write that a coloring $\tilde{c}$ of $\tilde{G}$ can be *extended* to a (4-) coloring c on G we always mean that c can be obtained from $\tilde{c}$ in that way as we have obtained c^* from c' in the first part of the proof of Lemma 3.2. As an immediate consequence of Lemma 3.2 and Lemma 3.1 we obtain:

Lemma 3.3 *For each snark G the following statements are equivalent.*

i) $G \in \mathcal{G} \setminus \mathcal{S}_1$

ii) $\exists e \in EG : e$ *is not contained in a triangle, square or 2-edge cut of G, and there are two different 2-factors of G with exactly two odd circles connected by e.*

Theorem 3.4 *For each snark G the following statements are equivalent.*

i) $G \in \mathcal{B}_1$

ii) *G is at least cyclically 4-edge connected, girth$(G) \geq 5$ and for each $e \in EG$ there are two different 2-factors of G with exactly two odd circles connected by e.*

Proof. (*i*) $\Rightarrow$ *ii*)): It follows from Lemmas 3.2 and 3.1 that G is 3-edge connected, girth$(G) \geq 5$ and that for each edge e there are two different 2-factors of G with exactly two odd circles connected by e.

We have to show that G is at least cyclically 4-edge connected. Suppose there is a 3-edge cut E_3 of G whose removal divides G into two components. Both components contain a circle. Let G_1 and G_2 be the graph obtained from G by contracting the first and the second component to a single vertex, respectively. Isaacs [12] showed one of G_1 and G_2 is a snark. Since each component contains a circle of length ≥ 5 G is 1-reducible to graphs G_1' and G_2' by removing two adjacent vertices from the first and the second component, respectively. Note that we can choose the adjacent vertices in the components such that they are not adjacent to an edge of E_3. In both graphs E_3 is still a 3-edge cut and both graphs are colorable, since $G \in \mathcal{B}_1$. But this implies G_1 and G_2 are colorable, a contradiction.

(*ii*) $\Rightarrow$ *i*)): This follows from Lemma 3.2 directly.

□

Theorem 3.5 *For each snark G the following statements are equivalent.*

i) $G \in \mathcal{S}_1$

ii) For all 2-factors $\mathcal{F}$ of G holds: If $o_{\mathcal{F}}(G) = 2$ then $d_G(U_1^{\mathcal{F}}, U_2^{\mathcal{F}}) \geq 3$.

Proof. ($i) \Rightarrow ii)$): It follows from Lemma 3.3 that $G \in \mathcal{S}_1$ if and only if each edge $e \in EG$ is contained in a 2-edge cut or it is contained in a circle of length ≤ 4 or for all 2-factors $\mathcal{F}$ of G holds: If $o_{\mathcal{F}}(G) = 2$ then e does not connect $U_1^{\mathcal{F}}$ and $U_2^{\mathcal{F}}$. Now the statement follows from Lemmas 2.3, 2.6 and 3.1.
($ii) \Rightarrow i)$): This follows from Lemma 3.2 directly.

□

The class $\mathcal{C}_1$ is characterised by Lemma 3.3 and Theorem 3.4.

4 2-Reductions

In this section we show the classifications induced by 1- and 2-reductions are identical.

Lemma 4.1 *If a snark G contains a 2-edge cut then it is 2-reducible to a snark.*

Proof. Let $e = vw$ and $e' = v'w'$ be the edges of a 2-edge cut of G, and let $G - \{e, e'\}$ consisting of the components G_v and G_w with $v, v' \in VG_v$ and $w, w' \in VG_w$.
Clearly, in each component there are two vertices with distance 2. Assume G is 2-reducible by removing these vertices. Then holds for $i = 1, 2$: If all the graphs obtained from G by such an i-reduction are colorable, then the graphs $G_v + \{vv'\}$ and $G_w + \{ww'\}$ are colorable. This implies that G is colorable, see [12]. Contrary to our assumption that G is a snark.
Now we have to consider the case when G is not 2-reducible to a simple graph by removing two vertices of one of the components, say G_w. This implies that G_w is a square with a diagonal edge. Then we remove a vertex incident to the diagonal edge and v. To restore 3-regularity we join the two vertices of the square which are not adjacent in G. The two vertices of G_w which do not have degree 3 yet are joined with the neighbors of v. Each coloring of this graph can be extended to a coloring of G, by using the Parity Lemma. Thus it is not colorable.

□

Lemma 4.2 *If a snark G contains a triangle or a square then it is 2-reducible to a snark.*

Proof. Let $e = vu \in EG$ and let v_1, v_2 and u_1, u_2 be the other neighbors of v and u, respectively. Suppose e is contained in a triangle, that is $v_1 = u_1$. We may assume that G is 3-edge connected, since for otherwise we can apply Lemma 4.1. Therefore $v_2 \neq u_2$,

$v_1v_2, v_1u_2 \notin EG$ and $d_G(u, v_2) = 2$. Let v_2' and v_2'' be the other neighbors of v_2. Then G is 2-reducible to $G' = (G - \{u, v_2\}) + \{v_2'v, v_2''v, u_2v_1\}$ which is a simple, bridgeless graph. If G' is a snark, we are done. Otherwise G' is colorable. However, any coloring of G' is extendable to a coloring of G. Contrary to our assumption that G is a snark. Thus G contains no triangle.
Suppose e is contained in a square. With no loss of generality we may assume $v_1u_1 \in EG$. Then $G' = (G - \{u, v_1\}) + \{v_1'v, u_1v, u_1u_2\}$ is simple and bridgeless, where v_1' is the other neighbor of v_1. It is not hard to see each coloring of G' yields a coloring of G, by using the Parity Lemma.

□

Theorem 4.3 $\mathcal{B}_1 = \mathcal{B}_2$

Proof. ($\subseteq$) Let $G \in \mathcal{B}_1$ and let $v, w \in VG$ with $d_G(v, w) = 2$. It follows from Theorem 3.4 that $|N(v) \cap N(w)| = 1$. Let $\{u\} = N(v) \cap N(w)$, and let v_1, v_2 and w_1, w_2 be the other neighbors of v and w, respectively. For the edges uv and uw holds condition *ii)* of Theorem 3.4. Thus each simple graph G', obtained from G by removing v and w has an even 2-factor and therefore $G \in \mathcal{B}_2$.

($\supseteq$) Let $e = uv$ be an edge of $G \in \mathcal{B}_2$. Let u_1, u_2 and v_1, v_2 be the other neighbors of u and v, respectively. Let v_1', v_1'' be the neighbors of v_1. It follows from Lemmas 4.1 and 4.2 that G is 3-edge connected and girth$(G) \geq 5$. The proof that G is cyclically 4-edge connected is similar to the proof of this property in Theorem 3.4. Therefore G is 2-reducible to $G' = (G - \{u, v_1\}) + \{u_1u_2, vv_1', vv_1''\}$ which is a simple, bridgeless, colorable graph.
Let c' be a coloring of G'. It holds $c'(vv_1') \neq c'(vv_1'')$. After reconstructing G from G' we extend this coloring to a 4-coloring c on G such that $c(uu_1) = c(vv_1) = 4$, $c(uu_2) = c'(u_1u_2)$, $c(v_1'v_1) = c'(v_1'v)$, $c(v_1''v_1) = c'(v_1''v)$, and uv should be colored in a color different from $c(vv_2), c(uu_2)$ and 4. Thus e connects the two odd circles of $\mathcal{F}_c$. Since e is choosen arbitrarily it follows from Lemma 3.2 and Theorem 3.4 that $G \in \mathcal{B}_1$.

□

Theorem 4.4 $\mathcal{S}_1 = \mathcal{S}_2$

Proof. ($\subseteq$) Let $G \notin \mathcal{S}_2$. Then G is 2-reducible to a colorable graph G'. As in the proof above, a coloring on G' implies that G has a 2-factor with exactly two odd circles which are connected by an edge. Thus $G \notin \mathcal{S}_1$, by Theorem 3.5.

($\supseteq$) Let $G \notin \mathcal{S}_1$. By Lemma 3.3 there is an edge $e = vw$ of G and two different 2-factors with exactly two odd circles which are connected by e. Thus we may assume that at least one odd circle has lenght ≥ 5. So we consider a 2-factor $\mathcal{F}$ of G with exactly two odd circles U_1, U_2 such that $v \in VU_1$ and $w \in VU_2$ and $|VU_2| \geq 5$. Let v_1, v_2 and w_1, w_2 be the other neighbors of v and w, respectively, where $v_1, v_2 \in U_1$ and $w_1, w_2 \in U_2$. Since e is not contained in a circle of length ≤ 4 it holds $d_G(v, w_1) = 2$. Let w_1', w_1'' be the other neighbors of w_1 and $w_1w_1'' \in \overline{\mathcal{F}}$. The vertex w_1' is incident to at most one of v_1, v_2, say $v_1w_1' \notin EG$. Then G is 2-reducible to $G' = (G - \{v, w\}) + \{v_1w_1', v_2w, w_1''w\}$ which obviously has an even 2-factor. Thus $G \notin \mathcal{S}_2$.

□

By the definition of $\mathcal{C}_1$ and $\mathcal{C}_2$ and Theorems 4.3, 4.4 it follows that $\mathcal{C}_1 = \mathcal{C}_2$.

5 3-Reductions

This kind of vertex-reductions is interesting for snarks which are not very dense. The Petersen graph is the only snark which has no two vertices with distance ≥ 3. Therefore it is not 3-reducible. The characterisation of $\mathcal{S}_1$ shows that snarks of oddness 2 could be 1-reducible to a colorable graph. We will see that this is not true for 3-reductions.

Lemma 5.1 *For each snark $G \neq P$ the following statements are equivalent:*

i) $G \in \mathcal{G} \setminus \mathcal{S}_3$

ii) $c(G) = \omega(G) = 2$

Proof. ($i) \Rightarrow ii)$): Let G be vertex-reducible to a graph G' with coloring c', by removing the vertices v and w. We reconstruct G from G' and modify c' to a coloring -not necessarily proper- c_1 on G. Let e_1^v, e_2^v, e_3^v be the edges incident to v and let e_1^w, e_2^w, e_3^w be the edges incident to w.
If $c_1(e_1^v) = c_1(e_2^v) = a$, $c_1(e_3^v) = b$ then follows by the Parity Lemma $c_1(e_1^w) = c_1(e_1^v) = a$ or $= c$ and $c_1(e_3^w) = b$. We extend this coloring to a 4-coloring c_2 on G with $X_4^{c_2} = \{e_1^v, e_1^w\}$. Therefore $\mathcal{F}_{c_2}$ contains exactly two odd circles U_1 and U_2, and hence $\omega(G) = c(G) = 2$.
If $c(e_i^v) = a$ for $i = 1, 2, 3$ it follows $c(e_i^w) = a$ for $i = 1, 2, 3$ or $c(e_{i1}^w) = c(e_{i2}^w) = b$ and $c(e_{i3}^w) = a$, by the Parity Lemma. In both cases there is an (a, c)-path from v to w in G. We exchange the colors a and c along this path to obtain a situation discussed above.
($ii) \Rightarrow i)$): First we claim that there is a 2-factor of G with exactly two odd circles U_1, U_2 such that there are two vertices $v \in VU_1$, $w \in VU_2$ with $d_G(v, w) \geq 3$. Suppose this claim is not true. Then let $\mathcal{F}$ be a 2-factor of G with exactly two odd circles U_1 and U_2. Since each vertex of U_1 is adjacent to at most one vertex of U_2, it is easy to see that $|VU_1 \cup VU_2| \leq 10$, and every vertex of U_1 is adjacent to a vertex of U_2. Thus $\mathcal{F}$ has no even circle and $G = P$, a contradiction.
Thus there is a 2-factor of G with exactly two odd circles U_1 and U_2 and there are two vertices $v \in VU_1$ and $w \in VU_2$ with $d_G(v, w) \geq 3$. Let v_1, v_2, v_3 and w_1, w_2, w_3 be the neighbors of v and w, respectively. Let $v_1, v_2 \in VU_1$ and $w_1, w_2 \in VU_2$ be the neighbors of v in U_1 and w in U_2, respectively. If $v_3w_3 \notin EG$ then one of $(G - \{v, w\}) + \{v_1v_2, w_1w_2, v_3w_3\}$, $(G - \{v, w\}) + \{v_1w_1, v_2w_2, v_3w_3\}$ and $(G - \{v, w\}) + \{v_1w_2, v_2w_1, v_3w_3\}$

is simple and bridgeless, and has an even 2-factor. Therefore $G \in \mathcal{G} \setminus \mathcal{S}_3$. Thus, if there are two vertices $v \in VU_1$ and $u \in VU_2$ with $d_G(v,u) > 3$ we are done.

Thus we may assume that $d_G(x,y) \leq 3$ for all vertices $x \in VU_1$ and $y \in VU_2$, and there are vertices $v \in VU_1$ and $w \in VU_2$ with $d_G(v,w) = 3$ and $v_3w_3 \in EG$.
The vertex v_2 has at most one neighbor in U_2, so we may assume $v_2w_2 \notin EG$. Since $\overline{\mathcal{F}}$ is an 1-factor of G it holds $v_1v_3, w_1w_3 \notin EG$. Thus, if v_3w_3 is an edge of an (even) circle of $\mathcal{F}$ different to U_1 and U_2 then G is 3-reducible to $(G - \{v,w\}) + \{v_1v_3, w_1w_3, v_2w_2\}$ which is a simple bridgeless graph with even 2-factor and we are done.

Thus we assume now that v_3w_3 is an edge of U_1 or U_2, say of U_1. We claim that $VG = VU_1 \cup VU_2$. Otherwise there is vertex a of one of the odd circles, say $a \in VU_1$, which has a neighbor in an even circle of $\mathcal{F}$. It follows that $d_G(a,w) \leq 2$ for all vertices w of U_2 and therefore $|VU_2| \leq 2$, a contradiction.
Since G is 3-regular, w_3 has only one neighbor x_3 in U_1 different from v_3. Thus there is at most one vertex x in U_1 with $d_G(w,x) = 3$ and $w_3x_3 \in EG$. This implies $|VU_1| \leq 7$. Since all neighbors of any vertex of U_1 are elements of $VU_1 \cup \{w_1, w_2, w_3\}$ it follows $|VU_2| = 3$. Therefore $|VG| \leq 10$ and hence G is no snark, since G contains two vertices v, with $d_G(v,w) = 3$ and therefore $G \neq P$.

□

The following characterisation of $\mathcal{S}_3$ is an immediate consequence of Lemma 5.1 and 2.6.

Theorem 5.2 *For each snark G the following statements are equivalent*

i) $G \in \mathcal{S}_3$

ii) $c(G) \geq 3$

iii) $\omega(G) \geq 4$

It follows from Theorem 3.5 and 5.2 that $\mathcal{S}_3 \subseteq \mathcal{S}_1$. Therefore the following corollary holds by the definition of $\mathcal{S}$ and Theorem 4.4.

Corollary 5.3 $\mathcal{S} = \mathcal{S}_3$

Corollary 5.4 *For each even $n \geq 10$ there is a snark $G \in \mathcal{G} \setminus \mathcal{S}_3$ of order n.*

Proof. It follows from Theorem 2.7 that for each even $n \geq 10$ there is a snark G of order n with $c(G) = \omega(G) = 2$. Thus $G \in \mathcal{G} \setminus \mathcal{S}_3$ by Theorem 5.2.

□

Concerning the bases we are more interested in the structure of $\mathcal{B}$ than in the structure of $\mathcal{B}_3$. By Theorem 3.4 we know that the graphs of $\mathcal{B}$ are cyclically 4-edge connected. It is easy to see that cyclically 4-edge connected graph is 3-reducible by removing two vertices with distance ≥ 3. So we restrict to such graphs in the following characterisation of $\mathcal{B}_3$. In the theorem we denote the neighbors of vertices v and w with v_1, v_2, v_3 and w_1, w_2, w_3, respectively.

Theorem 5.5 *For each cyclically 4-edge connected snark $G \neq P$ the following statements are equivalent:*

i) $G \in \mathcal{B}_3$

ii) For any two vertices $v, w \in VG$ with $d_G(v,w) \geq 3$ and any $v_1 \in N(v)$ and $w_1 \in N(w)$ with $v_1w_1 \notin EG$ one of the following holds:

1. *There is a 2-factor $\mathcal{F}$ of G with exactly two odd circles U_1, U_2, such that $v \in U_1$, $w \in U_2$ and $vv_1, ww_1 \in \overline{\mathcal{F}}$.*
2. *$\exists e_1 \in \{v_2v_3, w_2w_3\}$ $\exists e_2 \in \{v_2w_2, v_3w_3\}$ $\exists e_3 \in \{v_2w_3, v_3w_2\}$: If $(G - \{v,w\}) + e_i$ is simple, then it has an even 2-factor $\mathcal{F}$, with $e_i \in \bigcup_{C \in \mathcal{F}} EC$, for $i = 1, 2, 3$.*
3. *$G - \{v, w\}$ has an even 2-factor.*

Proof. Let v and w be two vertices of G with $d_G(v,w) \geq 3$ and $v_1w_1 \notin EG$. To restore 3-regularity we have to put in two new edges in $(G - \{v,w\}) + \{v_1w_1\}$. Now there are three possibilities to obtain a 3-regular graph. We can add $E_1 = \{v_1w_1, v_2v_3, w_2w_3\}$ or $E_2 = \{v_1w_1, v_2w_2, v_3w_3\}$ or $E_3 = \{v_1w_1, v_2w_3, v_3w_2\}$ to $(G - \{v,w\}) + \{v_1w_1\}$. For $i = 1,2,3$ let $G_i = (G - \{v,w\}) + E_i$. As mentioned above at least one of the G_i is simple. We consider the simple G_i's, and in the following we denote the edges of E_i with e_1, e_2, e_3 for each $i = 1, 2, 3$, where $e_1 = v_1w_1$.

($i) \Rightarrow ii)$): Each of the G_i's is colorable. Since G is a snark there is no coloring which colors e_1, e_2 and e_3 with three different colors for G_2 and G_3. If there is such a coloring for G_1, we reconstruct G from G_1 and extend the coloring to a 4-coloring c' such that $c'(v_1v) = c'(w_1w) = a$, $c'(v_2v) = b$, $c'(w_2w) = c$, and $c'(v_3v) = c'(w_3w) = 4$. The 2-factor $\mathcal{F}_{c'}$ contains exactly two odd circles C_{v_3v} and C_{w_3w}. Furthermore $v_1v, w_1w \in \overline{\mathcal{F}_{c'}}$. Therefore condition 1 of *ii)* is satisfied.
If there is a coloring for one of the G_i which colors e_1 with color a and e_2, e_3 with color b we argue analogously.
If there is a coloring for one of the G_i which colors e_1, e_2 and e_3 with the same color, say a, then the set of (b,c)-circles is an even 2-factor of $G - \{v,w\}$. Therefore condition 3 of *ii)* is satisfied.
Thus for each of G_1, G_2 and G_3 there are only colorings c_i of that kind such that $c_i(e_1) = a$ and one of e_2 and e_3, say e_2, is also colored with a and e_3 is colored with b. But then the set of (b,c)-circles is an even 2-factor in $(G - \{v,w\}) + e_3$. Therefore condition 2 of *ii)* is satisfied.

($ii) \Rightarrow i)$): Let G be 3-reducible to G' by removing vertices v and w. Clearly one neighbor, say v_1, of v must be incident to a neighbor, say w_1, of w in G'. Thus $G' = G_i$ for some $i \in \{1,2,3\}$.
If condition 1 is satisfied, then G' has an even 2-factor and therefore it is colorable.
If condition 2 is satisfied then there is $e \in E_i$ such that $e \neq v_1w_1$ and $(G - \{v,w\}) + e$ has an even 2-factor $\mathcal{F}$, with $e \in \bigcup_{C \in \mathcal{F}} EC$. We color the edges of the circles of $\mathcal{F}$ with colors

b and c alternately, and the edges between these circles with color a. Then the two edges added to $(G - \{v, w\}) + e$ to obtain G' can be colored with color a and we get a coloring of G'.
If condition 3 is satisfied, we color the edges of the 2-factor $\mathcal{F}$ of $G - \{v, w\}$ with two colors alternately, and the edges of $\overline{\mathcal{F}}$ with the third color. Then the edges, added to $G - \{v, w\}$ to obtain G', can be colored with the third color. Thus G' is colorable.

□

Now $\mathcal{C}_3$ is characterised by Lemma 5.1 and Theorem 5.2.

6 Further Investigations

In this section we use the results obtained in the preceding sections for further investigations on the structure of snarks. We are mainly interested in the structure of snarks of the base.
First we consider snarks which have cycle seperating 4-edge cuts. Let G be a snark which is divided into two components G_1 and G_2 by removing a 4-edge cutset. We show that G is vertex-reducible to a snark which contains one of the components as a spanning subgraph. This implies that each graph of $\mathcal{B}$ is at least cyclically 5-edge connected.
To prove this we need the following construction due to Isaacs [12]. The *dot product* $G_1 \cdot G_2$ of two snarks G_1 and G_2 is constructed as follows:
1. take $G'_1 = G_1 - \{ab, cd\}$, where ab, cd are two nonadjacent edges of G_1;
2. take $G'_2 = G_2 - \{x, y\}$, where x, y are adjacent vertices in G_2;
3. let w, z and u, v be the other neighbors of x and y in G_2, respectively. Then $G_1 \cdot G_2$ is the graph $G = (V, E)$ with $V = VG'_1 \cup VG'_2$ and $E = EG'_1 \cup EG'_2 \cup \{aw, bz, cu, dv\}$.
The following theorem is proved in [4] and in [8].

Theorem 6.1 ([4],[8]) *Let G be a snark with a cutset of four edges whose removal leaves components G_1 and G_2. Then either*

(a) one of G_1 or G_2 is not 3-edge-colorable; or

(b) both G_1 and G_2 can be "extended" to snarks, by adding two vertices to one of them and edges to both in such a way that G is their dot product.

Theorem 6.2 *Let G be a snark with a cutset of four edges whose removal leaves two components G'_1 and G'_2. Then G is vertex-reducible to a snark which contains one of the components as a spanning subgraph.*

Proof. Case (a): One of the components, say G'_1, is uncolorable. Then we carry out vertex-reductions in G'_2 as long as possible. It is easy to see that we can do it in a way such that the resulting graph is G'_1 with two additional edges.

Case (b): Both components are colorable. By Theorem 6.1 G_1' and G_2' can be extended to snarks G_1 and G_2 such that $G = G_1 \cdot G_2$. Let G, G_i and G_i' $(i = 1, 2)$ be as in the description of the dot product above and let u_1, u_2 and v_1, v_2 be the other neighbors of u and v in G_2, respectively. We shall show G is vertex-reducible to G_1.
If u_1u_2 or $v_1v_2 \in EG$, say $u_1u_2 \in EG$, then by Lemmas 3.2 and 3.1, G is 1-reducible to a smaller snark by removing u_1 and u_2. We can carry out this reduction until we obtain a graph G' where the neighbors of u and of v in $G' - VG_1'$ are not adjacent, respectively. Note that G_1' is completely unchanged, and let still u_1, u_2 and v_1, v_2 denote the neighbors of u and v in $G' - VG_1'$, respectively. We reduce G' to the graph

$$\tilde{G}_1 = (G' - \{u, v\}) + \{u_1u_2, v_1v_2, cd\}.$$

We now show $\tilde{G}_1$ is a snark. Suppose to the contrary $\tilde{G}_1$ is colorable. Since $\{aw, bz\}$ is a 2-edge cut of $\tilde{G}_1$ it follows by the Parity Lemma that this two edges are of the same color. This implies that G_1 is colorable, which contradicts the fact that G_1 is a snark.
By the same argument $\tilde{G}_1$ is vertex-reducible to a snark $\tilde{G}_2$ by removing two vertices of $V\tilde{G}_1 \setminus VG_1'$. Continuing this process we obtain a sequence $\tilde{G}_1, \tilde{G}_2, \cdots, \tilde{G}_k$ of snarks, and each of them contains G_1' as a subgraph. This process stops, when $\tilde{G}_k$ is not vertex-reducible to a smaller graph by removing two vertices of $V\tilde{G}_k \setminus VG_1'$. This is only the case when $\tilde{G}_k - VG_1'$ is a square with a diagonal edge. Let a_1, a_2 be the neighbors of a in G_1' and s_1, s_2, s_3, s_3 be the vertices of the square and $s_1s_2, s_2s_3, s_3s_4, s_4s_1, s_1a, s_3b, s_2s_4 \in E\tilde{G}_k$. Then $\tilde{G}_k$ is vertex-reducible to

$$\tilde{G}_{k+1} = (\tilde{G}_k - \{a, s_2\}) + \{s_1s_3, a_1s_1, a_2s_4\}$$

It is easy to see that $\tilde{G}_{k+1}$ is a snark, by using the Parity Lemma. Analogously

$$\tilde{G}_{k+2} = (\tilde{G}_{k+1} - \{s_1, s_3\}) + \{bs_4, a_1s_4\}$$

is a snark. Finally we build

$$\tilde{G}_{k+3} = (\tilde{G}_{k+2} - \{b, s_4\}) + \{a_1a_2, b_1b_2\}.$$

It holds $\tilde{G}_{k+3} = G_1$ and the statement is proved.

□

Corollary 6.3 *Each $G \in \mathcal{B}$ is at least cyclically 5-edge connected.*

Proof. Let $G \in \mathcal{B}$. By Theorem 3.4 G is at least cyclically 4-edge connected. If G is cyclically 4-edge connected then it is vertex-reducible to a smaller snark by Theorem 6.2. Therefore G must be at least cyclically 5-edge connected.

□

If the removal of an edge cutset divides a snark G into components and if one of these components is not 3-colorable then clearly G is vertex-reducible to smaller snarks by removing vertices from the other components. Therefore snarks which are divided only into 3-colorable components by the removal any edge cutset are of interest.
A snark is called *k-edgecut-irreducible* if all components obtained from G by removing an edge cutset of cardinality $< k$ are 3-colorable; $k \in \mathbb{N}$, and, following [15], a snark G is called *critical* if $G - \{v, w\}$ is 3-colorable for any two adjacent vertices v, w of G, and it is called *bicritical* if $G - \{v, w\}$ is 3-colorable for any two vertices v, w of G.
Clearly, if $G \in \mathcal{B}_1$ then G is critical. On the other hand if G is critical then follows from the Parity Lemma that it is 1-reducible to a colorable graph by removing any two adjacent vertices. Thus $G \in \mathcal{B}_1$ by Lemma 3.2 and Theorem 3.4, and hence the following characterisation of $\mathcal{B}_1$ is true.

Theorem 6.4 *For each snark G the following statements are equivalent.*

i) $G \in \mathcal{B}_1$

ii) G is critical.

Nedela and Škoviera [15] showed that a snark is critical if and only if it is G is k-edgecut-irreducible for $k \in \{5, 6\}$. More general they proved the following:

Theorem 6.5 ([15]) *For each snark G the following statements are equivalent.*

i) G is bicritical.

ii) G is k-edgecut-irreducible for every $k \geq 1$.

Since all snarks of $\mathcal{B}$ are bicritical they have the following structural property which characterises bicritical snarks. A graph G with $\Delta(G) = k$ is called *k-critical* if $\chi'(G) = k+1$ and $\chi'(G - e) < \chi'(G)$ for every edge e of G. It follows from a theorem of Vizing, see e.g. [6] (Theorem 10.3), that each snark contains a 3-critical subgraph.

Theorem 6.6 *For each snark $G = (V, E)$ the following statements are equivalent.*

i) G is bicritical.

ii) For all $v \in V$ the graph $G - v$ contains a (connected) spanning 3-critical subgraph and it contains no 3-critical subgraph of order $\leq |V| - 2$.

Proof. ($i) \Rightarrow ii)$): Let $v \in V$. Then $G - v$ is not 3-colorable and therefore it has a 3-critical subgraph of order $\leq |V| - 1$. Suppose it has a 3-critical subgraph H of smaller order than $|V| - 1$. Then there is a vertex $x \in G - v$ which is not a vertex of H and therefore $G - \{v, x\}$ is not 3-colorable. This contradicts the fact that G is bicritical.

$(ii) \Rightarrow i)$): Suppose there are $v, w \in V$ such that $G - \{v, w\}$ is not 3-colorable. Then $G - v$ contains a 3-critical subgraph which is of order $\leq |V| - 2$.

□

We show that there are infinitely many bicritical snarks and therefore $\mathcal{B}_1$ is an infinite set. A graph G is called *hypohamiltonian* if G is not hamiltonian and $G - v$ is hamiltonian for every vertex v of G.

Lemma 6.7 *Each hypohamiltonian snark is bicritical.*

Proof. Let G be a hypohamiltonian snark and let v, w be two vertices of G. Since $G - v$ is hamiltonian we can color one edge of the Hamilton circle incident to w with color 4, and the other edges of the Hamilton circle with colors 2 and 3 alternately. We color the chordal edges with color 1 to obtain a proper 4-coloring of $G - v$. This implies a 3-coloring on $G - \{v, w\}$ and the statement is proved.

□

The converse is not true. The Goldberg snarks on 22 vertices are bicritical and they are not hypohamiltonian, c.f. [5], [15].
Fiorini [5] proved that the flower snarks J_k, k odd and ≥ 5 (see [12]), are hypohamiltonian and mentioned that the double-star snark of Isaacs [12] on 30 vertices has this property, too. He also constructed an infinite family of cyclically 4-edge connected hypohamiltonion snarks by applying the dot product. Therefore all snarks of this family are vertex-reducible to smaller snarks by Lemma 6.2. Thus there are bicritical snarks which are not in $\mathcal{B}$. In particular the following statement is true.

Corollary 6.8 $\mathcal{B} \subset \mathcal{B}_1$

Remark: The Petersen graph is the smallest snark and therefore not reducible to smaller one. At present we do not know any other elements of $\mathcal{B}$. Cyclically 5-edge connected snarks are potential candidates, We checked the double star snark and some more classes e.g.: Goldberg snarks, Kochol snarks (see [14]). All these snarks are vertex-reducible to smaller ones.

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