DECOMPOSITIONS AND REDUCTIONS OF SNARKS

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ABSTRACT. According to M. Gardner [9], a snark is a non-trivial cubic graph whose edges cannot be properly coloured by three colours. The problem of what 'non-trivial' means is implicitly or explicitly present in most papers on snarks, and is the main motivation of the present paper. Our approach to the discussion is based on the following observation. If G is a snark with a k-edge-cut producing components G_1 and G_2, then either one of G_1 and G_2 is not 3-edge-colourable, or by adding a 'small' number of vertices to either component one can obtain snarks $\tilde{G}_1$ and $\tilde{G}_2$ whose order does not exceed that of G. The two situations lead to a definition of a k-reduction and k-decomposition of G. Snarks that for $m < k$ do not admit m-reductions, m-decompositions or both are k-irreducible, k-indecomposable and k-simple, respectively. The irreducibility, indecomposability and simplicity provide natural measures of non-triviality of snarks closely related to cyclic connectivity. The present paper is devoted to a detailed investigation of these invariants. The results give a complete characterization of irreducible snarks and characterizations of k-simple snarks for $k \leq 6$. A number of problems that have arisen in this research conclude the paper.

1. INTRODUCTION

Non-trivial cubic graphs whose edges cannot be properly coloured with three colours have been investigated for more than 100 years. The primary incentive for their study was the observation made by P. G. Tait [23] that the four-colour theorem is equivalent to the statement that every bridgeless planar cubic graph is 3-edge-colourable.

Since the four-colour theorem is true, we know that no non-trivial planar uncolourable cubic graphs exist. However, even non-planar graphs of this kind, such as the Petersen graph (shown in Fig.1), have been extremely difficult to find. In fact, until 1975 when Isaacs constructed two infinite families of them only four such graphs had been known. This led Martin Gardner [9] to "*propose calling nontrivial uncolorable trivalent graphs Snarks*", borrowing their name from the famous nonsense ballad "The Hunting of the Snark" by Lewis Carroll.

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The ambiguous word 'non-trivial' in this definition has attracted considerable discussion in the literature (see, e.g., [2, 4, 13, 21, 24, 25]). At any rate, 'non-trivial' means primarily a graph without a bridge. Besides having a bridge, however, a snark may possess some other features that make it 'trivial'; for instance if it is but a 'trivial' variation of another snark. It has been customary to consider trivial any snark that has a digon or a triangle because these are trivial features that can be added to or removed from a snark without altering its uncolourable property (see Fig.2 and Fig.3). Similarly, a quadrilateral in a snark can be replaced by two edges as shown in Fig.4 to produce a smaller snark. Therefore a non-trivial snark should have girth 5 or more.

It has also become standard to exclude snarks that contain an edge-cut consisting of two or three independent edges. It is easy to see that using such a cut a snark can be split into two smaller cubic graphs at least one of which is a snark (see Fig.5 and Fig.6). In other words, a snark with an independent (or equivalently, cycle-separating) 2-cut or 3-cut can be either 'reduced' to a smaller snark or 'decomposed' to two smaller snarks. In this manner, snarks with small cycle-separating edge-cuts also become 'trivial' modifications of those with fewer vertices. Therefore, a 'non-trivial' snark should be cyclically 4-edge-connected.

However, it has recently been shown that similar results hold for cut-sets of size 4 and 5 (see [2, 10, 21]). The following two theorems are the main results of Cameron, Chetwynd and Watkins [2, Theorem 1 and Theorem 2].

Theorem 1. *Let G be a snark with a cut-set of four edges whose removal leaves components G_1 and G_2. Then either*

(a) *one of G_1 and G_2 is not 3-edge-colourable; or*

(b) *both G_1 and G_2 can be "extended" to snarks by adding two vertices to one of them and edges to both in such a way that G is their dot-product.*

Recall that the dot-product of two snarks G and H is defined as follows. Take two non-adjacent edges $e = a_1a_2$ and $f = b_1b_2$ in G, and remove e and f from G. Further, take an edge xy in H whose neighbours are vertices a'_1 and a'_2 adjacent to x and vertices b'_1 and b'_2 adjacent to y, and remove x and y from H. To obtain the dot-product of G and H, join a_i to a'_i and b_i to b'_i, $i = 1, 2$. The resulting graph $G.H$ is a snark ([13, 2]).

Theorem 2. *Let G be a snark with a cut-set of five edges whose removal leaves components G_1 and G_2. Let G not be the Petersen graph. Then either*

(a) *one of G_1 or G_2 is not 3-edge-colourable; or*

(b) *both G_1 and G_2 can be "extended" to cubic graphs that are not 3-edge-colourable, by adding at most five vertices; at least one of these extended graphs is smaller than G.*

At the first glance one could deduce from the above decomposition results that apart from the Petersen graph snarks with cycle-separating 4-cuts and 5-cuts could also be considered trivial (cf. [2, 24, 25]). However, if we apply these theorems to the Isaacs flower snark I_5 (see Fig.10) we see that for each decomposition in the sense indicated above I_5 itself appears as one of the factors. Does not this mean that along with the Petersen graph, I_5 should also be included among non-trivial snarks?

If we insisted that the Petersen graph be the only non-trivial snark with cyclic connectivity not exceeding 5 then *cum grano salis* we would adopt the interpretation that every such snark is more or less a 'trivial' variation of the Petersen graph. But since the cyclic connectivity of every planar cubic graph is 5 or less, the proof of the four-colour theorem would be more or less a 'trivial' consequence of the non-planarity of the Petersen graph.

These observations show that a more careful approach to the idea of a snark decomposition is needed. For this purpose it is convenient to keep the basic notion of a snark as broad as possible. Therefore we allow a cubic graph to have both multiple edges and loops, and understand the term 'snark' merely as a synonym for a cubic graph that has no 3-edge-colouring. Theorem 1 and Theorem 2 and the previous investigations suggest that in our theory we have to consider two types of operations which simplify a given snark by producing one or two smaller snarks that keep the main features of the original.

The first type starts with a snark G, identifies an uncolourable part in it, removes the complement and replaces it by a small subgraph which is stuck to the remainder to get a snark G' with fewer vertices than G. The resulting snark G' is a *reduction* of G. The second type of operation splits G into two subgraphs, adds a small number of vertices and edges to each to get two snarks G_1 and G_2, each having a smaller order than G. The pair $\{G_1, G_2\}$ is a *decomposition* of G. (For the sake of convenience, we also allow *improper* reductions and decompositions which yield snarks having the same order as G.)

The primary aim of this paper is to give the concepts of snark reduction and snark decomposition precise meaning. This aim assumes, in particular, to take into account the number of edges employed in these operations. Therefore, if a reduction or a decomposition of a snark G involves disconnection of k edges, we speak of a k-reduction or a k-decomposition, respectively. Our attention is naturally directed to k-irreducible, k-indecomposable and k-simple snarks, those which for $m < k$ have no proper m-reduction, no proper m-decomposition, or have neither, respectively. We thus obtain three types of connectivity-like invariants reflecting different aspects of non-triviality of snarks, each in a potentially infinite scale. Their behaviour, as far as the results of this paper reveal it, can be summarized as follows.

For $1 \leq k \leq 4$, a snark is k-irreducible or k-simple if and only if it is either cyclically 4-connected or the dumbbell graph (see Fig.1). Thus for small values of k, the k-irreducibility and k-simplicity (but not k-indecomposability!) basically coincide with the cyclic k-connectivity. For larger values of k, the situation is more interesting. Since the dumbbell graph is k-simple for every $k \geq 1$, henceforth we ignore this graph.

We prove that a snark is k-irreducible for $k \in \{5, 6\}$ precisely when it is *critical* in the following sense: the suppression of every edge, indicated in Fig.7, results in a 3-edge-colourable graph. Equivalently, a snark is critical if the deletion of any two adjacent vertices produces a 3-edge-colourable graph. For $k \geq 7$, a snark is k-irreducible if and only if it is critical and the deletion of any two non-adjacent vertices yields a 3-edge-colouring. These results have two surprising consequences. Firstly, a 7-irreducible snark is k-irreducible for every $k \geq 1$. Secondly, the existence of an extensive proper reduction of a snark can be recognized locally, without identifying a 'large' edge-cut which separates an uncolourable subgraph. It is sufficient to find a pair of vertices whose deletion produces an uncolourable graph.

As regards the other two non-triviality invariants, our results are not complete. It turns out that indecomposability, if not combined with irreducibility, is quite difficult to handle. Therefore we concentrate on the simplicity of snarks. As we have mentioned above, for $k \le 4$ a snark is k-simple precisely when it is cyclically k-connected. For $k \ge 5$ we establish the following two characterizations. A snark is 5-simple if and only if it is critical and cyclically 5-connected. A snark is 6-simple if and only if it is critical, cyclically 5-connected and every cycle-separating 5-edge-cut it contains will separate a 5-cycle.

Characterizations of k-simple snarks for larger values of k are not known, and will probably be difficult to obtain. Any attempt in this direction must be preceded by proving m-decomposition theorems for $m < k$ similar to Theorem 1 and Theorem 2. We show in this paper that such theorems always exist, but we leave the number of vertices needed for an extension of a subgraph to a snark undetermined. As the number of possibilities to consider grows fast with the size of cuts, this may be a very difficult problem.

Our paper is organized as follows. In the next section we introduce the necessary terminology and results used throughout the paper. Section 3 contains the definition of a k-reduction and a k-decomposition, and is concluded with the general k-decomposition theorem. In Section 4 we characterize k-irreducible snarks for arbitrary $k \ge 1$. The next two sections deal with characterizations of k-simple snarks for $k \le 6$. In the last section we discuss several problems related to the concepts introduced in this paper.

2. Multipoles and Tait colourings

The graphs we consider in this paper are cubic (3-valent), and may have loops and multiple edges. For the purpose of this paper, however, it is convenient to allow also dangling edges, edges with one end-vertex and one free end, which arise when an edge-cut is deleted from a graph but the resulting 'dangling edges' are retained. Following Fiol [6], we call such graphs multipoles. We often reverse this process: we compose the usual graphs from multipoles. From this point of view, it is very useful to allow also edges which do not have any end-vertex at all: isolated edges.

In order to keep a reasonable level of formalism, we make these ideas more precise. A *multipole* is a pair $M = (V(M), E(M))$ of disjoint finite sets, the *vertex-set* $V(M)$ and the *edge-set* $E(M)$ of M. Every edge $e \in E(M)$ has two *ends* and every end of e may or may not be incident with a vertex. If the ends of e are incident with two distinct vertices, then e is a *link*. If both ends of e are incident with the same vertex then e is a *loop*. If one end of e is incident with some vertex but the other not, then e is a *dangling edge*. The remaining possibility is that neither end of e is incident with a vertex of M, and in this case e is called an *isolated edge*. An end of an edge e which is incident with no vertex will be called a *semiedge*. Thus a dangling edge gives rise to one semiedge while an isolated edge gives rise to two semiedges. The set of all semiedges of M will be denoted by $S(M)$.

In this paper, every multipole is assumed to be 3-valent, i.e., every vertex is incident with precisely three edge ends. We use $|M|$ to denote the *order* of a multipole M, the number of its vertices. If a multipole M has k semiedges then it is also called a *k-pole*; 0-poles are in fact the usual cubic graphs. Note that we allow the vertex-set of a multipole to be empty, in which case the multipole is a disjoint union of a number of isolated edges. For the consistency reasons, we allow also the edge-set of a multipole to be empty.

A multipole M is *ordered* if the set $S(M)$ of its semiedges is endowed with a linear order. If a k-tuple $(e_1, e_2, \ldots, e_k)$ represents the linear order of the semiedges of an ordered k-pole M, then we usually write $M = M(e_1, e_2, \ldots, e_k)$.

Let M be a multipole and let e and f be two edges of M, not necessarily distinct. Assume that e has a semiedge e' and f has a semiedge f' and that $e' \neq f'$. Then we can identify e' with f' and construct a new multipole M' as follows. If $e \neq f$, we discard e and f from $E(M)$ and replace them by a new edge g whose ends are the other end of e and the other end of f. Thus $E(M') = (E(M) - \{e, f\}) \cup \{g\}$ and g in fact arises from e and f by the identification of e' with f'. If $e = f$, then e is an isolated edge, and we set $E(M') = E(M) - \{e\}$. In this case we actually create an 'isolated loop' which does not have any ends at all and therefore is deleted. To put it differently, the identification of the semiedges of an isolated edge cancels that edge.

We say that M' arises from M by the *junction* of e' and f'.

Let $M = M(e_1, e_2, \ldots, e_k)$ and $N = N(f_1, f_2, \ldots, f_k)$ be two ordered k-poles. Then the *junction*, or more precisely the *k-junction* $M * N$ of M and N is the cubic graph which arises from the disjoint union $M \cup N$ by performing the junctions e_i with f_i, $i = 1, 2, \ldots, k$. If M and N are 0-poles, then their 0-junction is just the disjoint union. Note that the junction of two 2-poles, each consisting of an isolated edge, is the empty multipole.

The subgraph inclusion extends to multipoles in the obvious way. Note that if a multipole N arises from a multipole M by the junction of two semiedges then we consider M to be contained in N, i.e., $M \subseteq N$. The precise definition of the inclusion for multipoles is left to the reader.

If g_i is the edge of $G = M * N$ obtained by the junction of e_i with f_i, then the set $S = \{g_i;\, i = 1, 2, \ldots, k\}$ is a k-edge-cut in G. Reversing the process, every k-edge-cut in G gives rise to two k-poles M and N such that $G = M * N$. In this case we say that M and N are the k-poles *separated* by S.

Since in the rest of the paper we will be mostly using ordered multipoles, henceforth the term 'multipole' will mean 'ordered multipole' unless the opposite follows from the immediate context.

Let M be a multipole and let $\varphi\colon E(M) \to \{1, 2, 3\}$ be a mapping assigning to each edge of M one of the elements 1, 2, and 3 called *colours*. Then φ induces an assignment of colours to the ends of edges in M. The mapping φ is called a *3-edge colouring*, or simply *a colouring* of M, if for every vertex v of M the three ends incident with v are assigned pairwise distinct colours. A multipole will be called *colourable* if it admits a colouring. If it has no colouring, then we say that M is *uncolourable*. An uncolourable cubic graph will be called a *snark*.

Sometimes it is convenient to consider the colours 1, 2 and 3 as non-zero elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, whenever needed, we shall identify 1, 2, 3 and 0 with the elements $(0, 1)$, $(1, 0)$, $(1, 1)$ and $(0, 0)$ in $\mathbb{Z}_2 \times \mathbb{Z}_2$, respectively. The advantage of this approach consists in the fact that we can use the well-defined sum of colours to reflect colouring properties of the mapping $\varphi\colon E(M) \to \{1, 2, 3\}$.

For instance, if $\partial(v)$ denotes the set of edge ends incident with a vertex v of M, then φ is a colouring of M if and only if

$$\sum_{x\in\partial(v)} \varphi(x) = 0, \tag{2.1}$$

for every vertex v of M. However, the latter is nothing but to say that φ is a nowhere-zero $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow of M. If we want to emphasize that a colouring φ of M takes colours in $\mathbb{Z}_2 \times \mathbb{Z}_2$ or is to be understood as a $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow, we shall say that φ is a *Tait colouring* of M.

The following two statements are consequences of (2.1); Lemma 2.2 is actually a restatement of the well-known Parity Lemma. Their proofs can be found in [18]. For the flow-free proofs of the Parity Lemma the reader is referred to [5, 13].

Lemma 2.1. *Let φ be a Tait colouring of a multipole M. Then*

$$\sum_{s\in S(M)} \varphi(s) = 0. \qquad \square$$

Lemma 2.2. (Parity Lemma) *Let M be a k-pole that has been coloured with colours* 1, 2 *and* 3. *If k_i is the number of semiedges coloured i, then*

$$k_1 \equiv k_2 \equiv k_3 \equiv k \pmod 2. \qquad \square$$

For our further needs in this paper, we mention several standard examples of snarks and fix their notation. The smallest snark is the *dumbbell graph Db*, and the smallest 2-connected snark is the *Petersen graph Ps* of order 10. These two snarks are depicted in Fig.1. The next two interesting snarks B_1 and B_2 have order 18 and were constructed by Blanuša [1]; they are shown in Fig.8.

¿From among a number of known infinite families of snarks we recall the *flower snarks* I_n discovered by Isaacs [13] and independently by Grinberg (unpublished, see [10]). They can be constructed as follows. Let Y be the 6-pole depicted in Fig.9. Then I_n is the cubic graph that arises from the disjoint union of n copies Y_i of Y by identifying the semiedge y_j in Y_i with the semiedge x_j in Y_{i+1}, where $j = 1, 2, 3$ and the indices $i \in \{1, 2, \ldots, n\}$ are taken modulo n. If $n \geq 3$ is odd then I_n is a snark; I_5 is shown in Fig.10.

In addition to this family, Isaacs discovered an interesting sporadic snark which he christened the *double-star*. We denote it by *Ds*. It is shown in Fig.11.

3. Snark extensions, reductions and decompositions

In this section we give precise definitions of a snark decomposition and reduction based on the concept of an extension of a multipole to a snark.

Let $M = M(e_1, e_2, \ldots, e_k)$ be a k-pole. Define the *colouring set* of M to be the set

$$\mathrm{Col}(M) = \{(\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_k));\ \varphi \text{ a Tait colouring of } M\}.$$

Clearly, $|\,\mathrm{Col}(M)| \leq 3^k$ for every k-pole M. If $|\,\mathrm{Col}(M)| = 0$, then M is uncolourable.

The other extreme situation occurs when every assignment $(\varphi(e_1), \varphi(e_2), \dots, \varphi(e_k))$ of three colours to the semiedges of a multipole M such that $\sum \varphi(e_i) = 0$ extends to a Tait colouring of M. In this case M is said to be *colour-complete*.

Two k-poles $M = M(e_1, e_2, \dots, e_k)$ and $N = N(f_1, f_2, \dots, f_k)$ will be called *colour-disjoint* if $\mathrm{Col}\,(M) \cap \mathrm{Col}\,(N) = \varnothing$. Thus in order to construct a snark it is sufficient to find a pair of colour-disjoint multipoles.

In order to get a deeper knowledge of the structure of snarks, it is important to investigate for a given k-pole M the class of all k-poles colour-disjoint with M. Therefore we define

$$\mathcal{C}(M) = \{N;\ N \text{ a colourable } k\text{-pole colour-disjoint with } M\}.$$

Now we can introduce the main concepts of this paper. We shall say that a multipole M *can be extended* (or *extends*) to a snark if $\mathcal{C}(M) \neq \varnothing$. Among snark extensions of M, those with minimum order will be of a special interest. More precisely, if N is a member of $\mathcal{C}(M)$ having minimum order, then the junction $M^\circ = M * N$ will be called a (*snark*) *completion* of M. Observe that the multipole N need not be unique (for instance, when M is uncolourable), and hence a multipole may have several non-isomorphic snark completions, all of the same order.

A multipole that does not extend to a snark will be called *colour-closed*. For instance, every colour-complete multipole is colour-closed. It is plausible, however, that there also exist incomplete colour-closed multipoles although no such have been found. We shall discuss this point in the final section of this paper.

Let G be a snark and let $G = M * N$ be a junction of two k-poles M and N ($k \geq 0$). If one of M and N, say M, is uncolourable then M can be extended to a snark $\tilde{M} \supseteq M$ of order $|\tilde{M}| \leq |G|$ (in the worst case we can take $\tilde{M} = G$). We call $\tilde{M}$ a *k-reduction* of G.

Now let M and N be either both colourable or both uncolourable. Then clearly each of M and N can be extended to a snark $\tilde{M}$ and $\tilde{N}$, respectively. Since G is a snark, we can take $\tilde{M}$ and $\tilde{N}$ so that $|\tilde{M}| \leq |G|$ and $|\tilde{N}| \leq |G|$. A pair $\{\tilde{M}, \tilde{N}\}$ arising in this way and satisfying the condition $|\tilde{M}| \leq |G|$ and $|\tilde{N}| \leq |G|$ will be called a *k-decomposition* of G.

Note that if both M and N are uncolourable and $\tilde{M}$ and $\tilde{N}$ are as above, then at the same time $\{\tilde{M}, \tilde{N}\}$ is a k-decomposition and each of $\tilde{M}$ and $\tilde{N}$ is a k-reduction of G.

The definitions we have just given imply that whenever S is a k-edge-cut in a snark G then either a decomposition or a reduction along S can be applied to produce snarks whose order does not exceed that of G. This suggests that there should exist a general statement which contains the essence of the known decomposition results, in particular Theorem 1 and Theorem 2 by Cameron, Chetwynd and Watkins [2]. To this end, we can prove the following.

Theorem 3.1. *Let G be a snark and let $k \geq 1$ be an integer. Then there exists an integer function $\kappa(k)$ such that if $G = M * N$ is a k-junction of two k-poles, then one of the following statements holds.*

(a) *One of M and N is not colourable.*

(b) *Both M and N can be extended to snarks $\tilde{M}$ and $\tilde{N}$ by applying the junction with k-poles M' and N', each having at most $\kappa(k)$ vertices. Moreover, $|\tilde{M}| \le |G|$ and $|\tilde{N}| \le |G|$.*

Remark. Note that in Theorem 1, Theorem 2, and Theorem 3.1 part (a) refers to reduction while part (b) refers to decomposition in the sense introduced above.

Proof. We start with the definition of the function $\kappa(k)$. Consider the equivalence relation defined on the set of all k-poles by declaring two k-poles to be equivalent if $\mathrm{Col}(A) = \mathrm{Col}(B)$. This equivalence relation decomposes the set of all k-poles into *finitely* many classes. Moreover, if A and B are in the same class then $\mathcal{C}(A) = \mathcal{C}(B)$ by the definition. For every equivalence class $[A]$ where $\mathcal{C}(A)$ is non-empty choose a representative $A' \in \mathcal{C}(A)$ with the minimum order and call it a *minimum complement* for A. Clearly, if A and B are equivalent k-poles then $A' = B'$. Let $\mathcal{B}$ be the set of all minimum complements. Since $\mathcal{B}$ is finite, there is an element in $\mathcal{B}$ having maximum order. Let m be this maximum. Then set $\kappa(k) = m$.

Now we can finish the proof. If one of M and N is not colourable, then there is nothing to be proved. So assume that each of M and N is colourable. Since G is a snark, M and N are colour-disjoint and $M \in \mathcal{C}(N)$ and $N \in \mathcal{C}(M)$. Set $\tilde{M} = M * M'$ and $\tilde{N} = N * N'$. Thus $\tilde{M}$ is a completion of M and $\tilde{N}$ is a completion of N. By the definition of the function $\kappa(k)$ we have $|M'| \le \kappa(k)$ and $|N'| \le \kappa(k)$. Moreover, since $M \in \mathcal{C}(N)$ and $N \in \mathcal{C}(M)$ it is clear that $|\tilde{M}| = |M| + |M'| \le |M| + |N| = |G|$, and similarly $|\tilde{N}| \le |G|$. $\square$

The idea that there exist a function $\kappa(k)$ bounding the order of a minimum complement of a k-pole extendable to a snark first appears in Fiol [6]. The same paper contains definitions equivalent to our colour-complete and colour-disjoint multipoles, but the latter concept is implicit also in Preissmann's work [21].

In the next sections we shall discuss reductions and decompositions of snarks in a greater detail, reductions considering first. Note that our definitions allow us to reduce a snark G to itself or to decompose it into two snarks, one of which is G. Of course, in a typical reduction or decomposition these trivial cases do not happen. However, for the consistency reasons we are unable to avoid them completely.

4. Irreducible snarks

Let G be a snark. A k-reduction H of G is *proper* if $|H| < |G|$. A snark is *k-irreducible* ($k \ge 1$) if it has no proper m-reduction for each $m < k$. Observe that a k-irreducible snark is also r-irreducible for every $r \le k$; in particular it is 1-irreducible and hence connected.

One of the possible ways how a proper reduction of a snark G can be performed is the following. We find in G a k-pole $M \subseteq G$, $|M| \ge 2$, which is colour-complete. Let N be the k-pole such that $G = M * N$. Then N must be uncolourable, for otherwise a colouring of N would induce an assignment of colours to the semiedges of M and hence a colouring of M. But then the union of the two colourings yields a colouring of G, a contradiction. Since N is uncolourable, we can extend it to a snark $\tilde{N}$ by employing the junction with a k-pole having one or none vertex (depending on the parity of k), and forget the irrelevant arbitrarily colourable part M of G. Thus $\tilde{N}$ is a proper reduction of G.

Note that a similar reduction can be performed by using a colour-closed multipole $M \subseteq G$ instead of a colour-complete one. However, to check whether a multipole is colour-closed may be extremely difficult even in relatively small cases.

It is easy to see that the k-pole consisting of a cycle of length k and the dangling edges attached to it is colour-complete for $k = 2$, 3 and 4, but not for $k \geq 5$. Thus the familiar reductions of a snark by suppressing a digon, a triangle (both unique in result), and a quadrilateral (3 different possibilities) are special cases of the reduction procedure described above. Using this fact we arrive at the well-known observation that a 'non-trivial' snark should have girth at least 5. More precisely we have:

Proposition 4.1. *For $2 \leq k \leq 5$, the girth of a k-irreducible snark other than the dumbbell graph is at least k.* $\square$

Proof. As we have already mentioned, a snark containing a cycle of length m, $2 \leq m \leq 4$, admits a proper m-reduction. Therefore a k-irreducible snark with $k \leq 5$ cannot contain cycles of length smaller than k other than loops.

Let G be a snark containing a loop, and let b be a bridge incident with it. Consider the 1-poles M and N obtained from $G - b$, where M consists of a loop and the incident dangling edge. If N is replaced by a 1-pole N' isomorphic with M, then $M * N'$ is a 1-reduction of G. This reduction is proper whenever G is not the dumbbell graph. Thus Db is the only 2-irreducible and hence the only k-irreducible snark, $k \geq 1$, which contains a loop. $\square$

With the obvious exceptions of an isolated edge and a *claw*, the unique acyclic 3-pole, every colour-complete multipole contains a cycle. This follows from the fact that adjacent semiedges in the remaining acyclic multipoles restrict the possible colourings and thus prevent them from being colour-complete. Therefore, in the reduction process of a snark cycle-separating edge-cuts and the concept of cyclic connectivity play a key role.

Recall that an edge-cut S in a connected cubic graph G is said to be *cycle-separating* if at least two components of $G - S$ contain cycles. A cubic graph is called *cyclically k-connected* if it has no cycle-separating m-edge-cut for every $m < k$. The largest integer k such that G is cyclically k-connected, provided it exists, is called the *cyclic connectivity* of G and is denoted by $\zeta(G)$; otherwise we set $\zeta(G) = |E(G)| - |V(G)| + 1$. Note that there are only three small connected cubic graphs where the latter case applies (see [19]), namely, K_4, $K_{3,3}$, and the graph consisting of two vertices and three parallel edges. For these the cyclic connectivity equals the cycle rank (Betti number).

Nonetheless, there are two other types of edge-cuts that are also very important for snark reductions, namely a cut separating an edge and a cut separating two non-adjacent vertices. This is reflected in the following definitions.

A pair $\{u, v\}$ of distinct vertices of a snark G will be called *non-removable*) if $G - \{u, v\}$ is colourable. If the end-vertices of an edge e form an pair, then e itself will be said to be *non-removable*. Otherwise, a pair of vertices or an edge will be called *removable*.

We define a snark G to be (edge-) *critical* if every pair of distinct adjacent vertices is non-removable. Similarly, we shall say that G is *cocritical* if every pair of distinct non-adjacent vertices is non-removable. If G is both critical and cocritical then we say that G is *bicritical*.

When dealing with the edge-criticality of snarks it is not necessary to switch from cubic graphs to 4-poles. We shall see that the deletion of two adjacent vertices is in fact equivalent with the *suppression* of the edge they form. By this we mean the deletion of an edge $e = uv$ (without creating semiedges in $G - e$) and the subsequent contraction of one edge at each of the two resulting 2-valent vertices (see Fig.7). We denote the resulting graph by $G \sim e$.

Proposition 4.2. *Let G be a snark and let $e \in E(G)$. Then e is removable if and only if $G \sim e$ is a snark.*

Proof. Suppose $G \sim e$ is a snark but, on the contrary, e is non-removable. The semiedges of $G - \{u, v\}$ are in two pairs, those in one pair being formerly incident with the same vertex u or v in G. Since G is a snark, every colouring of $G - \{u, v\}$ assigns identical colours to the semiedges in at least one of the pairs. By the Parity Lemma (Lemma 2.2), the same must be true for the other pair, too. Thus any colouring of $G - \{u, v\}$ induces a colouring of $G \sim e$, a contradiction. The converse is trivial. $\square$

Corollary 4.3. *A snark G is critical if and only if $G \sim e$ is colourable for every edge e of G.* $\square$

The following important theorem gives a characterization of k-irreducible snarks for every $k \geq 1$ in terms of criticality of snarks.

Theorem 4.4. *Let G be a snark. Then the following statements hold true.*

(a) *If $1 \leq k \leq 4$, then G is k-irreducible if and only if it is either cyclically k-connected or the dumbbell graph.*

(b) *If $k \in \{5, 6\}$, then G is k-irreducible if and only if it is critical.*

(c) *If $k \geq 7$, then G is k-irreducible if and only if it is bicritical.*

Proof. (a) The statement is obvious if $k = 1$.

Let $k \in \{2, 3, 4\}$ and let G be a k-irreducible snark. We show that if G is not cyclically k-connected then it is the dumbbell graph. Assume that G contains a cycle-separating m-cut that separates two m-poles M and N with $m < k \leq 4$. From the Parity Lemma (Lemma 2.2) we can easily deduce that at least one of M or N is uncolourable. Without loss of generality let M be uncolourable. Then a completion M° of M has $|M| + 1 \leq |G|$ vertices. Since G is k-irreducible, $|M| + 1 = |G|$ for otherwise M° would be a proper reduction of G. Thus $|N| = 1$. It follows that either N is a 1-pole H obtained by cutting the bridge in the dumbbell graph or N is a 3-pole obtained by a disjoint union of H with an isolated edge. In both cases N is uncolourable. We may therefore repeat the above consideration with the roles of M and N interchanged and conclude that $|G| = |N| + 1 = |N^\circ| = 2$. Consequently, G is the dumbbell graph.

Now let G be a cyclically k-connected snark for some $k \in \{2, 3, 4\}$. Then an m-cut in G with $m < k$ does not separate two cycles. Consequently, it separates either an isolated edge or a claw. However, neither an isolated edge nor a claw can be replaced by a smaller multipole in order to form a proper reduction of G. Hence G is a k-irreducible snark.

(b) Let $k = 5$ or $k = 6$. If G is k-irreducible then it is clearly critical. For the converse, let G be a critical snark and let $G = M * N$ be a junction where M is an uncolourable m-pole with $m < k$. We show that G is a completion of M, i.e., N cannot be replaced by a smaller m-pole.

First assume that N contains a loop. Then either G is the dumbbell graph or G contains a link $e = uv$ not adjacent to a specified loop f. In the latter case, $G - \{u, v\}$ contains f and hence is not colourable, contrary to assumption. It follows that G is the dumbbell graph, and the statement (b) holds in this case.

We can therefore assume that N has no loops. If N contains two adjacent vertices u and v, then $G - \{u, v\}$ is colourable by the edge-criticality. But then so is $M \subseteq G - \{u, v\}$, a contradiction. Therefore N has no pair of adjacent vertices and, unless G is the dumbbell graph, it contains no loops. It follows that N must be either an isolated edge or a claw or two isolated edges or the disjoint union of a claw and an isolated edge. In each case, N cannot be replaced with a smaller m-pole completing M to a cubic graph. Thus G has no proper m-reduction for $1 \leq m \leq 5$ proving that it is 6-irreducible.

(c) If G is k-irreducible for some $k \geq 7$ then G is obviously bicritical. Conversely, let G be a bicritical snark and let $G = M * N$ be a junction where M is an uncolourable m-pole with $1 \leq m < k$ and $k \geq 7$. We may clearly assume that G is not the dumbbell graph. Since G is critical, part (b) combined with (a) implies that G is cyclically 4-connected and hence loopless. We show that N is a union of $m/2$ isolated edges if m is even, or a union of a claw and $(m-3)/2$ isolated edges if m is odd. Observe that $M \subseteq G - \{u, v\}$ for every two vertices u and v of N. This implies that, firstly, N has no pair of adjacent vertices since G is critical and, secondly, N has at most one vertex since G is cocritical. Thus N is as claimed, and hence G is a completion of M. It follows that G has no proper m-reduction for $m < k$; in other words, it is k-irreducible. □

Corollary 4.5. *The following three statements hold true:*

(a) *A snark is 5-irreducible if and only if it is 6-irreducible.*

(b) *A snark is k-irreducible for some $k \geq 7$ if and only if it is 7-irreducible.*

(c) *A snark is k-irreducible for every $k \geq 1$ if and only if it is 7-irreducible.* □

When comparing Theorem 4.4 and Corollary 4.5 with the definition of a k-irreducible snark we see that surprisingly there exist only six different ranks of snark irreducibility. For $k = 1, 2, 3$ and 4, k-irreducibility basically coincides with the cyclic k-connectivity. For $k = 5$ or $k = 6$ it is the same as the criticality. For $k \geq 7$ k-irreducibility means bicriticality and implies k-irreducibility for every $k \geq 1$.

These facts suggest the following terminology.

Definition. A snark will be called *irreducible* if it has no proper reduction; else, it will be called *reducible*.

With this definition, parts (b) and (c) of Corollary 4.5 allow a less involved restatement.

Corollary 4.6. *For a snark G, the following statements are equivalent.*

(a) *G is irreducible.*

(b) *G is 7-irreducible.*

(c) *Every non-trivial edge-cut in G, i.e., one different from three edges incident with a vertex, separates G into colourable components.*

We already know that the set of irreducible snarks is non-empty. In fact, this set is infinite.

Proposition 4.7. *The flower snark I_n is irreducible for each odd $n \geq 5$.*

Proof. By Theorem 4.4 it is sufficient to prove that the snark I_n is bicritical for every $n \geq 5$. For $n = 5$ and $n = 7$ the validity of the statement can be checked directly (we have done it using computer). If $n \geq 9$, we employ induction with respect to n. Consider the flower snark I_n, $n \geq 9$. Let M_k denote the 6-pole arising from the union of k copies Y_i of the multipole Y (see Fig.9) by identifying the semiedge y_j in Y_i with the semiedge x_j in Y_{i+1} for $i = 1, 2, \ldots, k-1$ and $j = 1, 2, 3$. Examining the colouring properties of Y it can be checked that $\mathrm{Col}\,(M_{2m}) = \mathrm{Col}\,(M_2)$ for every $m \geq 1$. On the other hand, it is easy to see that regardless of the choice of two vertices u and v in I_n the multipole $I_n - \{u, v\}$ contains a copy of M_4. Replacing this copy of M_4 by a copy of M_2 we obtain $I_{n-2} - \{u, v\}$ which is colourable by the induction hypothesis. Since $\mathrm{Col}\,(M_4) = \mathrm{Col}\,(M_2)$ it follows that $I_n - \{u, v\}$ is colourable, too. Hence I_n is bicritical, and we are done. $\square$

To finish this section, we mention some interesting properties of irreducible snarks.

The next result improves Proposition 4.1 and shows that the restriction to cyclically 4-connected snarks of girth at least 5 adopted previously by several authors was natural.

Proposition 4.8. *Let G be an irreducible snark other than the dumbbell graph. Then G is cyclically 4-connected and has girth at least 5. Moreover, this is best possible.*

Proof. If G is an irreducible snark different from Db then it is 4-irreducible and hence cyclically 4-connected by Theorem 4.4. Proposition 4.1 yields that G has girth at least 5.

To prove that these bounds are best possible, consider the Blanuša snark B_1 of order 18 (see Fig.8). It is easy to see that B_1 has girth 5 (as has the Petersen graph) and cyclic connectivity 4. Moreover, it can be verified that B_1 is bicritical and hence irreducible. (The same holds for B_2.) $\square$

Snark reductions have been recently studied by Steffen [22] from another point of view. Because of a different context, we mention just a simple consequence of much stronger structural results established in [22].

Let G be a bridgeless cubic graph. The *oddness* of G, denoted by $\omega(G)$, is the minimum number of odd cycles in a 2-factor of G. Clearly, $\omega(G)$ is always even and vanishes precisely when G is colourable.

Theorem 4.9. [22] *Let G be a bridgeless snark containing an edge e such that $G \sim e$ is colourable. Then $\omega(G) = 2$.*

Corollary 4.10. *If G is a 5-irreducible snark then $\omega(G) = 2$.*

Celmins and Swart in [3] used the dot-product to construct snarks with cyclic connectivity 4 and arbitrarily large oddness. By Theorem 4.4, these snarks are 4-irreducible. Thus, in a certain sense, Corollary 4.10 cannot be improved.

It may also be interesting to note that the Cycle Double Cover Conjecture (see [14] for the formulation) has been verified by Huck and Kochol [12] for every snark of oddness 2 and by Huck [11] for every snark with oddness at most 4.

5. Characterization of k-simple snarks: $k \leq 5$

A k-decomposition $\{K, L\}$ of a snark G will be called *proper*, if $|K| < |G|$ and $|L| < |G|$. A snark which has no proper m-decomposition for each $m < k$ will be called *k-indecomposable*. A k-indecomposable snark is always connected.

It is natural to ask how do k-indecomposable snarks look like, especially when a k-decomposition theorem is known. However, the alternative of two statements (a) and (b) in Theorem 3.1 as well as in Theorem 1 and Theorem 2 suggests that instead of doing that it may be more convenient to characterize snarks to which neither statement (a) (reduction) nor statement (b) (decomposition) can be properly applied. Therefore we define a snark to be *k-simple* if it is both k-irreducible and k-indecomposable, and henceforth concentrate on them.

The dumbbell graph and the Petersen graph are easily seen to be k-simple for every $k \geq 1$. They are the only such graphs known so far. If the integer k is small we can say much more: we are able to characterize k-simple snarks. The major part of the rest of this paper deals with these characterizations. In this section we give a characterization of k-simple snarks for $k \leq 5$, and the whole next section will be devoted to characterizing 6-simple snarks.

As with reductions, the decomposition process relies mainly upon cycle-separating cuts. To see this, it is necessary to show that acyclic cuts do not lead to proper decompositions. Cuts of size not exceeding 5 are handled in the next lemma. A similar statement holds for 6-cuts (to be proved in the next section) and possibly for 7-cuts.

Lemma 5.1. *Let G be a k-irreducible snark, $k \leq 5$, and let $G = M * N$ be an m-junction where $m < k$ and N is acyclic. Then G is a completion of M.*

Proof. The lemma is trivial for $m \leq 3$ and consequently for $k \leq 4$. So we may assume that $k = 5$ and $m = 4$. Let B be a 4-pole consisting of two isolated edges, and let D be a 4-pole D consisting of an edge together with four dangling edges incident with its end-vertices (see Fig.12). Now, if the statement is false, then it is easy to see that $N = D$ and $M * B$ is a snark for some ordering of semiedges in B; we shall see that this cannot happen. Without loss of generality we assume that the semiedges of D are ordered as in Fig.12. Since G is 5-irreducible, M is colourable for otherwise an extension of M by two isolated edges would be a proper 4-reduction of G. Obviously, N is colourable, too. Moreover, M and N must be colour disjoint because G is a snark. From Lemma 2.1 or Lemma 2.2 (the Parity Lemma) it follows that up to permutation of colours every element in the colouring set of a 4-pole belongs to one of the following four types: 1111, 1122, 1212, and 1221. The colouring set of every colourable 4-pole has at least two different types, and they are related to one another by an alteration along a Kempe chain beginning and ending with a semiedge. Hence, out of the four possible types, M has two and N has the other two. Obviously, D only admits colourings of types 1212 and 1221. It follows that M has the remaining two types 1111 and 1122, in particular it has type 1111. But then the junction of M with any ordering of B produces a colourable graph. Thus to extend G to a snark one needs two vertices. In other words, G is a completion of M. $\square$

Now we are ready to prove our characterization theorems for k-simple snarks where $k \leq 5$.

Theorem 5.2. *Let G be a snark other than the dumbbell graph and let $1 \leq k \leq 4$. Then the following statements are equivalent.*

(1) *G is k-simple.*

(2) *G is k-irreducible.*

(3) *G is cyclically k-connected.*

Proof. The implication (1) $\Rightarrow$ (2) follows directly from the definition, and (2) $\Rightarrow$ (3) is a consequence of Theorem 4.4. It remains to establish the implication (3) $\Rightarrow$ (1). Let G be a k-connected snark where $k \leq 4$. Then G is k-irreducible by Theorem 4.4. We have to show that G is k-indecomposable. Suppose that G has a proper m-decomposition $\{G_1, G_2\}$ with $m < k$. Then there is an m-junction $M * N = G$ where M extends to the snark G_1 and N extends to the snark G_2. Since G is cyclically k-connected one of M and N, say N, must be acyclic. Thus N is either an isolated edge or a claw (cf. Lemma 5.1). In both cases $|G_1| = |G|$, a contradiction. $\square$

The next theorem is basically the same as Theorem 1 of Cameron, Chetwynd and Watkins [2].

Theorem 5.3. *Let G be a snark with a 4-edge-cut which separates 4-poles M and N, both colourable. Then M and N can be extended to snarks $\tilde{M}$ and $\tilde{N}$, respectively, by adding at most two vertices to each. Moreover, G is the dot-product of $\tilde{M}$ and $\tilde{N}$.*

Proof. Since $G = M * N$ is a snark, the 4-poles M and N must be colour-disjoint. We show that M and N can be extended to snarks $\tilde{M}$ and $\tilde{N}$ by employing a junction of one of them with a 4-pole B consisting of two isolated edges and of the other with a 4-pole D consisting of an edge together with four dangling incident with its end-vertices.

One of M and N has a colouring of type 1111. Assume it is M, and let σ be a colouring of the other type present in $\mathrm{Col}(M)$. It is then clear that $M * B$ is colourable for any ordering of B. Therefore we join M to D in such a way that each pair of semiedges which receives the same colour under σ will be attached to the same vertex of D. Then $M * D$ is obviously a snark.

We further show that there is an ordering of B such that $N * B$ is a snark, too. By our assumption, N does not admit colourings of type 1111. Let σ and τ be colourings of M belonging to the two different types present in $\mathrm{Col}(M)$. Then there always exists a pair $\{x, y\}$ of semiedges in N such that $\sigma(x) \neq \sigma(y)$ and $\tau(x) \neq \tau(y)$. The junction of x with y together with the junction of the remaining two semiedges of N has the same effect as $N * B$ for some ordering of B. The way how these junctions were performed implies that $N * B$ is a snark.

The fact that G is the dot-product of $\tilde{M} = M * D$ and $\tilde{N} = N * B$ is obvious. $\square$

The preceding theorem shows that the dot-product is the reverse to a 4-decomposition (cf. [2, 10, 21]). It is interesting to note that the dot-product is well-defined for an arbitrary pair of bridgeless snarks. As we shall see later, this situation has no direct analogue in 5-decompositions.

Theorem 5.4. *Let G be a snark other than the dumbbell graph. Then the following statements are equivalent.*

(1) *G is 5-simple.*

(2) *G is 5-indecomposable.*

(3) *G is cyclically 5-connected and critical.*

Proof. (1) $\Rightarrow$ (3) Let G be 5-simple. Then G is critical by Theorem 4.4. Thus it remains to prove that it is cyclically 5-connected. Since G is 4-irreducible, Theorem 4.4 yields that it is cyclically 4-connected. By way of contradiction suppose that G is not cyclically 5-connected. Then $\zeta(G) = 4$, in particular G is bridgeless. Let M and N be the 4-poles separated by a cycle-separating 4-edge-cut in G. It follows from Proposition 4.1 that $|M| \geq 5$ and $|N| \geq 5$. If one of M and N is uncolourable, say M, then an extension of M by two isolated edges is a proper reduction of G, contrary to our assumption. Therefore both M and N are colourable. Now, Theorem 5.3 implies that M and N extend to snarks $\tilde{M}$ and $\tilde{N}$, respectively, by adding at most two vertices to each. Since $|M| \geq 5$ and $|N| \geq 5$, the pair $\{\tilde{M}, \tilde{N}\}$ forms a proper 4-decomposition of G. This contradicts the 5-simplicity of G.

(3) $\Rightarrow$ (2) In order to prove that G is 5-indecomposable assume that G has a proper m-decomposition $\{G_1, G_2\}$ for some $m \leq 4$. Let $G = M * N$ be an m-junction where M extends to the snark G_1 and N extends to the snark G_2. Since G is cyclically 4-connected, Theorem 5.2 implies that G is 4-simple. Consequently $m = 4$. However, G is also cyclically 5-connected. It follows that one of M and N, say N, is acyclic. As G is critical, it is 5-irreducible, and we can employ Lemma 5.1 to conclude that $|G_1| = |M * N| = |G|$. Consequently, the decomposition $\{G_1, G_2\}$ is not proper, a contradiction.

(2) $\Rightarrow$ (1) Assume that G is 5-indecomposable. To prove that G is 5-simple we have only to show that it is 5-irreducible. By Theorem 4.4 it is sufficient to verify that G is critical. Suppose not. It follows that there exists an edge $e = uv$ such that $G - \{u, v\}$ is uncolourable. By Proposition 4.2, $G \sim e$ is a snark. Hence $\{G \sim e, Db\}$ is a proper decomposition of G, contradicting our assumption. $\square$

The comparison of Theorem 5.2 and Theorem 5.4 leads to a curious conclusion. For $k \leq 4$ the k-simplicity of a snark is equivalent with the k-irreducibility but not with the k-indecomposability. In contrast to this, the 5-simplicity is equivalent with the 5-indecomposability but not with 5-irreducibility. The flower snark I_3 is 4-indecomposable but it has a proper 3-reduction since it contains a triangle. Thus 4-indecomposability does not imply 4-simplicity. On the other hand, the Blanuša snark B_1 is 5-irreducible but since it is a dot-product of two copies of the Petersen graph it admits a 4-decomposition. Therefore, 5-irreducibility does not imply 5-simplicity.

One may ask whether the condition of criticality in statement (3) of Theorem 5.4 cannot replaced by a requirement on cyclic connectivity stronger than 5-connectivity. The answer is no. In [20] we construct examples of cyclically 6-connected snarks that are not critical. This shows that high cyclic connectivity alone is not sufficient to force a snark to be critical or and hence 5-irreducible. Thus, cyclic k-connectivity and k-irreducibility provide two essentially different measures of non-triviality of snarks if $k \geq 5$.

6. Characterization of 6-simple snarks

The aim of this section is to characterize 6-simple snarks. Since a 6-simple snark is 5-simple, by Theorem 5.4 it must be cyclically 5-connected and critical. These two conditions, however, are not enough for a snark to be 6-simple. Indeed, both are met by the double-star snark which has a 5-decomposition into two copies of the flower snark I_5. On the other hand, cyclic 6-connectivity is too much to require because I_5 is 6-simple and still cyclically 5-connected. Therefore a detailed analysis of decompositions along 5-edge-cuts is necessary in order to achieve our goal. This analysis will show that the characterization requires only one additional property; namely, that every cycle-separating 5-edge-cut be trivial. By a *trivial* cycle-separating 5-edge-cut in a cubic graph we mean one for which one of the resulting components is a 5-cycle (with dangling edges).

Theorem 6.1. *Let G be a snark other then the dumbbell graph. Then G is 6-simple if and only if it is critical, cyclically 5-connected, and every cycle-separating 5-edge-cut is trivial.*

In order to prove Theorem 6.1 we shall investigate Tait colourings of 5-poles. By the Parity Lemma, in each colouring of a 5-pole M three of the semiedges receive the same colour and the other two, the *lonely edges*, receive the remaining two colours. If $M = M(e_1, e_2, \dots, e_5)$, we can define the *colouring graph* $\mathrm{Cg}(M)$ of M as follows: the vertex-set of $\mathrm{Cg}(M)$ will be the set $\{v_1, v_2, \dots, v_5\}$ with v_i adjacent to v_j if there is a colouring of M for which e_i and e_j are the lonely edges. Of course, $\mathrm{Cg}(M)$ depends on the labelling, but any two labellings yield isomorphic graphs. The colouring graphs $\mathrm{Cg}(M)$ have the following properties.

Lemma 6.2. ([2, 21]) *Let M be a 5-pole. Then $\mathrm{Cg}(M)$ has no vertex of valency 1.*

Proof. The statement is obviously true if $\mathrm{Cg}(M)$ is edgeless. So we assume that $\mathrm{Cg}(M)$ has some edge. Let v_i be any non-isolated vertex of $\mathrm{Cg}(M)$ and let v_j be a vertex adjacent to v_i. Then there exists a colouring φ of M for which e_i and e_j are the lonely edges. Take a Kempe chain in M starting with e_j and ending with a semiedge $e_k \neq e_i$ so that e_i, e_j and e_k have different colours under φ. By interchanging the colours on the Kempe chain we obtain a colouring for which e_i and e_k are lonely, but not e_j. Thus v_i is adjacent in $\mathrm{Cg}(M)$ to a vertex $v_k \neq v_i$ which implies that v_i has valency at least 2. □

Lemma 6.3. ([2, 21]) *Let $M = M(e_1, e_2, \dots, e_5)$ and $N = N(f_1, f_2, \dots, f_5)$ be two 5-poles. Then the junction $M * N$ is a snark if and only if $\mathrm{Cg}(M)$ and $\mathrm{Cg}(N)$ are edge-disjoint.*

Proof. If $M * N$ is colourable then a colouring of $M * N$ induces colourings of M and N under which the lonely edges have the same indices. So $\mathrm{Cg}(M)$ and $\mathrm{Cg}(N)$ have an edge in common. If, conversely, $v_i v_j$ is an edge in both $\mathrm{Cg}(M)$ and $\mathrm{Cg}(N)$, then there is a colouring φ of M under which e_i and e_j are lonely and a colouring ψ of N under which f_i and f_j are lonely. If necessary, we permute the colours of N in order to obtain a new colouring ψ' such that $\varphi(e_i) = \psi'(f_i)$ and $\varphi(e_j) = \psi'(f_j)$. Since the colourings φ and ψ' agree on the lonely edges, they must agree on every pair of the corresponding semiedges. Thus φ and ψ' combine to a colouring of $M * N$. □

The following lemma is straightforward.

Lemma 6.4. (i) *Let $P = P(e_1, e_2, \ldots, e_5)$ be the 5-pole consisting of a single vertex and three dangling edges incident with it and of one additional isolated edge. Let the semiedges be labelled as in Fig.13(a). Then $\mathrm{Cg}(P)$ consists of a 3-cycle and two isolated vertices labelled as in Fig.13(b).*

(ii) *Let $Q = Q(e_1, e_2, \ldots, e_5)$ be the 5-pole consisting of a path of length 2 with five dangling edges incident to its vertices. Let the semiedges of Q be labelled as in Fig.14(a). Then $\mathrm{Cg}(Q)$ consists of a 4-cycle and one isolated vertex labelled in accordance with Fig.14(b).*

(iii) *Let $R = R(e_1, e_2, \ldots, e_5)$ be the 5-pole consisting of a 5-cycle and of five dangling edges incident with its vertices. Let R have the semiedges labelled in accordance with Fig.15(a). Then $\mathrm{Cg}(R)$ is a 5-cycle labelled as in Fig.15(b).*

(iv) *The 5-poles P, Q and R defined above are the only 5-poles of order at most 5 that do not contain cycles of length less than 5. Moreover, R is the only 5-pole M of order at most 5 for which $\mathrm{Cg}(M)$ is a 5-cycle.* □

Lemma 6.5. (cf. [21, Lemma]) *Let $M = M(e_1, e_2, \ldots, e_5)$ be a 5-pole for which $\mathrm{Cg}(M)$ is a 5-cycle $v_1 v_2 \ldots v_5$. Then, up to isomorphism, M has the unique completion to a snark, namely the junction with the 5-pole R from Lemma 6.4 (iii) whose semiedges have been reordered to yield the order $(e_1, e_4, e_2, e_5, e_3)$.*

Proof. It is clear from Lemma 6.2 and Lemma 6.3 that M can be extended to a snark only using the junction with a 5-pole N for which $\mathrm{Cg}(N)$, too, is a 5-cycle. By Lemma 6.4, the only possibility for a completion is $N = R$ up to a permutation of the semiedges. Suppose that R has the ordering of the semiedges as in Fig.15(a). By symmetry we may assume that e_1 in M is identified with e_1 in R. Since we want the junction of M and R to be a snark, after permuting the semiedges of R and the corresponding relabelling of $\mathrm{Cg}(R)$ the colouring graphs $\mathrm{Cg}(M)$ and $\mathrm{Cg}(R)$ must become edge-disjoint. More precisely, they must form the two 5-cycles whose union is the complete graph K_5. Hence, the only possibility is to identify e_i in M with $e_{\pi(i)}$ in R, $i = 1, 2, \ldots, 5$, where π is the permutation $(1)(2, 4, 5, 3)$. Thus there is only one way of completing M to a snark. □

The next lemma extends the validity of Lemma 5.1 to $k \leq 6$.

Lemma 6.6. *Let G be a k-irreducible snark, $k \leq 6$, and let $G = M * N$ be an m-junction where $m < k$ and N is acyclic. Then G is a completion of M.*

Proof. If $m < k \leq 5$, the result follows from Lemma 5.1. So it remains to verify the statement for $k = 6$ and $m = 5$. Let G be a 6-irreducible (and by Theorem 4.4 also

5-irreducible) snark satisfying the assumptions of the lemma. Let P denote a 5-pole consisting of the disjoint union of a claw with an isolated edge (Fig.13(a)), and let Q be a 5-pole consisting of a path uvw of length 2 together with five dangling edges incident with its vertices (Fig.14(a)). Now, if the statement is false, then $N = Q$ and $M * P$ is a snark for some ordering of semiedges of P; we shall see that this cannot happen. Since $N = Q$, Lemma 6.4 (ii) says that $\mathrm{Cg}(N)$ is a 4-cycle F in the complete graph K_5. The semiedges $e_1, e_2, \dots, e_5$ of M fall into three groups $\{e_1, e_2\}$, $\{e_3\}$, $\{e_4, e_5\}$, those in one group being formerly incident with the same vertex u, v, and w in Q, respectively. In this notation, $F = v_1v_4v_2v_5$. The complement $\overline{F}$ of F in K_5 consists of two triangles $T = v_3v_1v_2$ and $T' = v_3v_4v_5$ with the vertex v_3 in common. By Lemma 6.3, $\mathrm{Cg}(M) \subseteq \overline{F}$. Since G is 6-irreducible $\mathrm{Cg}(M)$ cannot be edgeless. Hence, Lemma 6.2 implies that $\mathrm{Cg}(M)$ contains one of T and T'. In view of the symmetry, we can assume that $T \subseteq \mathrm{Cg}(M)$. Consider the 4-pole $G - \{u, v\}$. The criticality of G yields that $G - \{u, v\}$ is colourable. A colouring of $G - \{u, v\}$ induces a colouring of $M \subseteq G - \{u, v\}$ where the semiedges e_4 and e_5 are coloured by different colours. Therefore at least one of these semiedges is lonely. In other words, either v_4 or v_5 is of non-zero valency in $\mathrm{Cg}(M)$. Taking into account Lemma 6.2, this can only happen when $T' \subseteq \mathrm{Cg}(M)$. Consequently, $\mathrm{Cg}(M) = T \cup T' = \overline{F}$. However, we have assumed that $M * P$ is a snark. Hence, by Lemma 6.3 $\mathrm{Cg}(P) \subseteq F$. But since Lemma 6.4 (i) shows that $\mathrm{Cg}(P)$ is a single 3-cycle and F is a 4-cycle, this is impossible. We have arrived at a contradiction, which proves the result. $\square$

The next result basically coincides with Theorem 2 of Cameron, Chetwynd and Watkins [2].

Theorem 6.7. *Let G be a snark with a 5-edge-cut which separates 5-poles M and N, both colourable. Then M and N can be extended to snarks by adding at most 5 vertices to each.*

Proof. It suffices to establish the lemma only for M. We accomplish that by specifying an appropriate 5-pole $K \in \{P, Q, R\}$ and an appropriate permutation of its semiedges for which $M * K$ is a snark. Let $M = M(e_1, e_2, \dots, e_5)$. Lemma 6.2 implies that $\mathrm{Cg}(N)$ contains a cycle C, necessarily of length 3, 4, or 5. By Lemma 6.3, C and $\mathrm{Cg}(M)$ have no edges in common. If C is a 3-cycle, say $v_1v_2v_3$, then we take $K = P$ and identify the semiedge e_i in M with e_i in P, $i = 1, 2, \dots, 5$. If $C = v_1v_2v_3v_4$, we take $K = Q$ and identify e_i in M with $e_{\pi(i)}$ in Q, $i = 1, 2, \dots, 5$, where π is the permutation $(1)(2, 4, 5, 3)$. If $C = v_1v_2v_3v_4v_5$, we take $K = R$ and identify e_i in M with e_i in R, $i = 1, 2, \dots, 5$. By Lemma 6.4, in each case $\mathrm{Cg}(K) = C$. Lemma 6.3 now implies that the junction $M * K$ with the given permutation of semiedges is a snark. $\square$

The following theorem constitutes the final step towards the proof of the characterization of 6-simple snarks presented in Theorem 6.1. With the help of the preceding lemmas it characterizes 6-simple snarks with cyclic connectivity 5. Part (2) of the statement reflects the fact that for snarks G and G' with specified 5-cycles $C \subseteq G$ and $C' \subseteq G'$ there exist a unique way of 5-composition that results in a snark. If this is applied to two copies of the flower snark I_5, then the result is the double-star Ds. Thus, a more detailed investigation of 5-poles whose colouring graph is a 5-cycle would be very useful.

On the other hand, it is not always the case that a new snark can be obtained from snarks G_1 and G_2 by removing from each of them one of the 5-poles P and Q instead of R (see Lemma 6.4) and performing junction. An example where the resulting graph is always colourable can be constructed if $G_1 = G_2 = Ps$, P is removed from G_1 and Q is removed from G_2. The result is a cubic graph H of order 16 with girth 5. Since every snark of order 16 contains a cycle of length less than 5 ([7, 8]), H is colourable. Thus a 5-composition of snarks reverse to the 5-decomposition cannot be applied without restrictions.

In order to state the next theorem we need one more definition. A k-decomposition $\{\tilde{M}, \tilde{N}\}$ of a snark $G = M * N$ is called *cyclic* if each of the k-poles M and N contains a cycle.

Theorem 6.8. *Let G be a 5-irreducible snark with $\zeta(G) = 5$. Then the following statements are equivalent.*

(1) *G is 6-simple.*

(2) *Each cyclic 5-decomposition of G consists of the Petersen graph and G itself.*

(3) *Every cycle-separating 5-edge-cut is trivial.*

Proof. (1) $\Rightarrow$ (3) Assume to the contrary that G does contain a cycle-separating 5-edge-cut S which is non-trivial. Thus if M and N are the 5-poles separated by S, then $|M| \geq 6$ and $|N| \geq 6$. If any of them, say M, were uncolourable then a completion M° of M would have order $|M^\circ| \leq |M| + 1 < |G|$ and therefore would provide a proper 5-reduction of G. Hence both M and N are colourable. Now Theorem 6.7 implies that M and N extend to snarks $\tilde{M}$ and $\tilde{N}$, respectively, by adding at most 5 vertices to each. But then $\{\tilde{M}, \tilde{N}\}$ is a proper decomposition of G, a contradiction.

(3) $\Rightarrow$ (2) Let S be a cycle-separating 5-edge-cut in G which yields 5-poles M and N, where N is a 5-cycle (with dangling edges). Clearly, M is colourable for otherwise a completion of M would be a proper 5-reduction of G. By Lemma 6.4 (iii), $\mathrm{Cg}(N)$ is a 5-cycle and Lemma 6.3 shows that $\mathrm{Cg}(M)$, too, must be a 5-cycle. Lemma 6.5 implies that each of M and N has a unique completion to a snark. Let M° and N° be these completions. As M can be completed to G and N to the Petersen graph Ps, we obtain that $M^\circ = G$ and $N^\circ = Ps$.

(2) $\Rightarrow$ (1) Condition (2) says that G has no proper cyclic 5-decomposition. Since G is 5-irreducible, Lemma 6.6 implies that G has no proper 5-decomposition at all. Hence G is a 6-simple snark. □

Proof of Theorem 6.1. ($\Rightarrow$) If G is a 6-simple snark other than the dumbbell graph, then Theorem 5.4 shows that G is cyclically 5-connected. Assume that there is a cycle-separating 5-cut S in G. Then $\zeta(G) = 5$, and Theorem 6.8 states that S is trivial.

($\Leftarrow$) Let G be a critical cyclically 5-connected snark different from Db and let each cycle-separating 5-cut in G be trivial. If $\zeta(G) = 5$ then G is 6-simple by Theorem 6.8. Otherwise G is cyclically 6-connected. Then, let $\{\tilde{M}, \tilde{N}\}$ be a k-decomposition of G with $k \leq 5$. Since G has no cycle-separating k-cuts, one of M and N, say N, is an acyclic k-pole. Lemma 6.6 now yields that $|\tilde{M}| \geq |M^\circ| = |G|$, implying that $\{\tilde{M}, \tilde{N}\}$ is not a proper decomposition. Hence G is a 6-simple snark. $\Box$

7. Concluding remarks

We conclude this paper with a discussion of several problems which have emerged from our present study.

In Section 4 we have indicated that a deeper study of colour-complete and colour-closed multipoles may be important for understanding the structural properties of irreducible snarks. Cycles of length 2, 3 and 4 together with one dangling edge attached to every vertex are obvious examples of colour-complete multipoles. Fig.16 shows two 'less-trivial' examples, M_1 of girth 5, and M_2 of girth 6. Taking two copies of M_1 and performing a partial junction by identifying a pair of semiedges in one copy with a pair of semiedges in the other copy we can form an infinite family of colour-complete 4-poles.

On the other hand, we do not know any example of a colour-closed multipole that is not colour-complete.

Problem 1. *Find a colour-closed k-pole which is not colour-complete ($k \geq 4$).*

The smallest non-trivial case is $k = 4$. It is easy to see (cf. the proof of Lemma 5.1) that every colourable 4-pole M admits colourings of at least two of four possible types. If M is colour-closed then it admits at least three types, say 1122, 1212 and 1221. If M is not colour-complete then it does not have a colouring of type 1111. The problem is to construct such a 4-pole.

If an n-pole is not colour-complete then by our definitions in Section 3 it can be extended to a snark. As shown in the proof of Theorem 3.1, the smallest number of vertices needed to extend an n-pole to a snark is a well-defined integer function $\kappa(n)$. Our next problem concerns this function.

Problem 2. *Is $\kappa(n)$ a recursive function?*

The results of this paper as well as previous investigations (see [2, 6, 10, 21]) imply that $\kappa(1) = 1$, $\kappa(2) = 0$, $\kappa(3) = 1$, $\kappa(4) = 2$, and $\kappa(5) = 5$. However, we have no idea how to continue with the computation or bounding of this function. The next step is therefore to bound or determine $\kappa(6)$.

Problem 3. *Find a bound on $\kappa(6)$.*

Finding such a bound (which does exist by Theorem 3.1) would immediately imply a 6-decomposition theorem. This, in turn, would possibly lead to a characterization of 7-simple snarks.

We expect that on the way from the 6-decomposition theorem towards to a characterization of 7-simple snarks it will be necessary to prove that acyclic 7-cuts do not give rise to proper 7-decompositions. This amounts in extending the validity of Lemma 5.1 and Lemma 6.6 to 7-cuts, which is our next problem.

Problem 4. *Prove the following statement. Let G be a k-irreducible snark, $k \leq 7$, and let $G = M * N$ be an m-junction where $m < n$ and N is acyclic. Then G is a completion of M.*

One of the cases that have to be dealt with in order to solve this problem in the positive consists in showing that if u and v are two non-adjacent vertices of an irreducible snark G, then $G - \{u, v\}$ cannot be extended to a snark using three isolated edges. We think, however, that sufficiently large acyclic cuts may be used to obtain proper decompositions, i.e., the statement in Problem 4 is false if the restriction on the size of cuts is relaxed.

The trouble with 7-simple snarks is that the dumbbell graph and the Petersen graph are the only such snarks known so far. Note that the flower snarks I_n, $n \geq 5$ odd, are 6-simple by Theorem 6.1 but not 7-simple since they have a proper 6-decomposition into two flower snarks of smaller order. A recent construction of new cyclically 6-connected 6-simple snarks [20] does not provide 7-simple snarks either. These examples thus suggest the following question.

Problem 5. *Do there exist 7-simple snarks other than the dumbbell graph and the Petersen graph?*

In view of our characterization of 6-simple snarks given in Theorem 6.1 it would of interest to have an infinite family of 6-simple snarks with cyclic connectivity 5. Among the known snarks only Ps and I_5 have this property. Using the *superposition* of snarks, a new powerful technique for constructing snarks introduced in [18] (see also [16, 17]), we are able to construct infinitely many of them [20]. Such snarks are interesting also due to the 5-composition operation which was discussed in the preceding section.

The capability of snark superposition is nicely illustrated with the recent disproof of the famous Girth Conjecture for snarks. This conjecture is due to Jaeger and Swart [15] and claims that there are no cyclically 4-connected snarks of girth greater than 6. Using the superposition Kochol [17] showed that for any integer $g > 6$ there exists a cyclically 4-connected and a cyclically 5-connected snark of girth g. The smallest such snarks for $g = 7$ have 298 and 506 vertices, respectively.

It turns out, however, that all Kochol's snarks of large girth are reducible. In fact, they contain removable edges and hence admit 4-reductions – although these may eventually decrease girth.

We believe that this phenomenon is not limited to Kochol's examples only, but occurs in general. In other words, we think that irreducible snarks of large girth do not exist. Therefore we propose the following improved conjecture.

Conjecture. *There exists no irreducible snark of girth greater than* 6.

Among the notions related to the snark irreducibility that of a critical snark plays probably the very central role. While critical snarks appear to be rare, we believe that most irreducible snarks should be cocritical. No snark is known that would be critical but not bicritical – in other words, 6-irreducible but not 7-irreducible.

Problem 6. *Does there exist a* 6*-irreducible snark that is not irreducible? Equivalently, does there exist a snark that is critical but not bicritical?*

Irreducibility imposes strong restrictions on snarks. For instance, there are no irreducible snarks of orders 12, 14 and 16 (see [7, 8]) but there do exist such snarks of order 18 (Blanuša snarks B_1 and B_2), 20 (the flower snark I_5) and 22 (Goldberg-Loupekine snarks GL_1 and GL_2, see Fig.17). No irreducible snark of order 24 has been known. Brinkmann (personal communication) generated all snarks of order not exceeding 26. On 24 vertices there are two cyclically 5-connected snarks and 36 additional cyclically 4-connected snarks of girth greater than or equal to 5. The two cylically 5-connected ones reduce by edge suppression to GL_1 and GL_2, respectively. Each of the remaining snarks has a 4-decomposition into two snarks, one of which is necessarily of order 12, 14 or 16. From the results of [7, 8] it follows that such a snark contains a digon, a triangle or a quadrilateral and hence is reducible. Since the original snark is the dot-product of those constituting its 4-decomposition, it is easy to see that it is reducible, too. Thus there exists no irreducible snark of order 24. This suggests the following problem.

Problem 7. *For which even numbers $n \geq 10$ does there exist an irreducible snark of order n. In particular, does there exist an irreducible snark of each sufficiently large order?*

We expect the second part of Problem 7 to have positive answer. In order to be able to attack this problem, a catalogue of small irreducible snarks would be helpful. Employing the computations perfomed by G. Brinkmann and proceeding as with snarks of order 24 we deduce that the only irreducible snarks of order not exceeding 24 are the following ones: Db, Ps, B_1, B_2, I_5, GL_1, and GL_2. In contrast to this, the number of snarks which are cyclically 4-connected, have girth at least 5 and their order does not exceed 24 is 67.

These numbers suggest that the reduction process results in a very restricted set of snarks that are more primitive than the others. This set becomes even more restricted if snark decompositions are involved, too. The purpose of snark decompositions and reductions is to better understand the structure of snarks and eventually to characterize them. Cameron et al. [2] proposed that "*a characterization of snarks could take the form of identifying 'prime' snarks and the ways that they can be composed. Thus any snark could be constructed in a finite number of steps from 'prime' snarks*". Although they did not offer a definition of a 'prime' snark we think that what they had in mind might be similar to our simple snarks.

The ways of construction of snarks from simple snarks include k-compositions and k-extensions, operations reverse to k-decompositions and k-reductions, respectively. Whereas k-compositions with $k \leq 4$ as for instance the dot-product are easy to handle, our discussion preceding Theorem 6.8 shows that already 5-compositions may be difficult, not to speak about k-compositions with $k \geq 6$. Theorem 4.4 characterizing irreducible snarks gives somewhat better chances to reductions rather than decompositions because of their local character. It implies that every reducible snark G contains a pair $\{u, v\}$ of vertices of G such that $G - \{u, v\}$ extends to a snark H by adding two or three isolated edges according as u is or is not adjacent to v, respectively. Conversely, H contains two or three edges which can be disconnected and two new vertices, possibly joined by an edge, can be added to construct a larger snark. Call G an *elementary extension* of H. Clearly, every snark can be obtained from an irreducible snark by a series of elementary extensions. The problem is to localize those pairs and triples of edges in a snark where this operation can be applied. Thus we have

Problem 8. *For a given snark G, characterize all elementary extensions of G.*

This problem, especially for irreducible snarks, seems to be difficult but even partial solutions to it may be interesting.

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