Iterated Monoidal Categories

Z. Fiedorowicz, R. Schwänzl, and R. Vogt

Introduction

For many years it has been known that there is a strong connection between coherence theory of categories and coherence problems in homotopy theory. Early work of Stasheff [15] and MacLane [8] showed that monoidal categories are analogous in a precise way to 1-fold loop spaces. Later a similar connection was noted between symmetric monoidal categories and infinite loop spaces. This connection was exploited with great success in algebraic K-theory. For instance the group completion of the nerve of a symmetric monoidal category is an infinite loop space, and the homotopy groups of this infinite loop space are the Quillen K-groups of that category, which provide algebraic information about the original category. Conversely this fact has also been used to construct new examples of infinite loop spaces and infinite loop maps of great interest to topologists.

It is striking to note that there appears to be a very similar connection between braided tensor categories and 2-fold loop spaces. It is shown in [4] that the group completion of the nerve of a braided tensor category is a 2-fold loop space. However the proof of this fact given there, while short and simple, is in some ways mysterious and unsatisfactory. For the proof notes that it is already known that the group completion of the nerve of a free braided tensor category is a free 2-fold loop space, and the general result follows from this. The result also raises new questions such as: what algebraic structure on a category corresponds to an n-fold loop structure for $2 \leq n < \infty$?

This paper gives another proof of the aforementioned result and sets up a framework for dealing with questions like this. The key idea is a generalization of the notion of braided tensor category, whose algebraic structure is more transparently analogous to the structure of a 2-fold loop space. This notion we call a *2-fold monoidal category.*

For a 2-fold loop space can be thought of as a loop space in the category of loop spaces. Noting the analogy between monoidal categories and 1-fold loop spaces, the analogous categorical concept is a category which is monoidal in the category of monoidal categories. This gives the notion of a 2-fold monoidal category. It is easy to see that a braided tensor category is a special case of this notion. Moreover it is easy to adapt the Segal-Thomason proof that the group completion of the nerve of a symmetric monoidal category is an infinite loop space, to prove the same connection between 2-fold monoidal categories and 2-fold loop spaces. These constructions iterate in an obvious manner to give notions of n-fold monoidal categories corresponding to n-fold loop spaces.

This paper is a preliminary draft which raises more questions than it answers and is meant to call attention to problems in this area. We will only discuss the 2-fold case in any detail. In the last section we will briefly sketch the generalization to n-fold monoidal categories for $n \geq 3$.

Basic Definitions

Definition 1: A *(strict) monoidal category* is a category $\mathcal{C}$ together with a functor $\Box : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object E such that

1. $\Box$ is strictly associative.
2. E is a strict 2-sided unit for $\Box$.

A monoidal functor $(F, \eta) : \mathcal{C} \to \mathcal{D}$ between monoidal categories consists of a functor F such that $F(E) = E$ together with a natural transformation

$$\eta_{A,B} : F(A)\Box F(B) \to F(A\Box B),$$

which satisfies the following conditions

1. Internal Associativity: The following diagram commutes

$$\begin{tikzcd} F(A)\Box F(B)\Box F(C) \arrow[r, "\eta_{A,B}\Box id_{F(C)}"] \arrow[d, "id_{F(A)}\Box\eta_{B,C}"] & F(A\Box B)\Box F(C) \arrow[d, "\eta_{A\Box B,C}"] \\ F(A)\Box F(B\Box C) \arrow[r, "\eta_{A,B\Box C}"] & F(A\Box B\Box C) \end{tikzcd}$$

2. Internal Unit Conditions: $\eta_{A,E} = \eta_{E,A} = id_{F(A)}$.

Given two monoidal functors $(F, \eta) : \mathcal{C} \to \mathcal{D}$ and $(G, \zeta) : \mathcal{D} \to \mathcal{E}$, we define their composite to be the monoidal functor $(GF, \xi) : \mathcal{C} \to \mathcal{E}$, where ξ denotes the composite

$$GF(A)\Box GF(B) \xrightarrow{\zeta_{F(A),F(B)}} G\big(F(A)\Box F(B)\big) \xrightarrow{G(\eta_{A,B})} GF(A\Box B).$$

(It is an exercise to check that ξ satisfies the associativity condition above.) We denote by $\mathbf{M}onCat$ the category of monoidal categories and monoidal functors. Note that this the usual product in **Cat** defines a product in **MonCat**.

Definition 2: A 2*-fold monoidal category* is a monoid in $\mathbf{M}onCat$. This means that we are given a monoidal category $(\mathcal{C}, \Box, E)$ and a monoidal functor $(\boxdot, \eta) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which satisfies

1. External Associativity: the following diagram commutes in $\mathbf{M}onCat$

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[r, "(\boxdot,\eta)\times id_{\mathcal{C}}"] \arrow[d, "id_{\mathcal{C}}\times(\boxdot,\eta)"] & \mathcal{C}\times\mathcal{C} \arrow[d, "(\boxdot,\eta)"] \\ \mathcal{C}\times\mathcal{C} \arrow[r, "(\boxdot,\eta)"] & \mathcal{C} \end{tikzcd}$$

2. External Unit Conditions: the following diagram commutes in $\mathbf{M}onCat$

$$\begin{tikzcd} \mathcal{C}\times E \arrow[r, "\subseteq"] \arrow[d, "\cong"] & \mathcal{C}\times\mathcal{C} \arrow[d, "(\boxdot,\eta)"] & E\times\mathcal{C} \arrow[l, "\supseteq"'] \arrow[d, "\cong"] \\ \mathcal{C} \arrow[r, "="] & \mathcal{C} & \mathcal{C} \arrow[l, "="'] \end{tikzcd}$$

Explicitly this means that we are given a second associative binary operation $\boxdot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, for which E is also a two-sided unit. Moreover we are given a natural transformation

$$\eta_{A,B,C,D} : (A \boxdot B)\Box(C \boxdot D) \to (A\Box C) \boxdot (B\Box D).$$

The internal unit conditions give $\eta_{A,B,E,E} = \eta_{E,E,A,B} = id_{A \boxdot B}$, while the external unit conditions give $\eta_{A,E,B,E} = \eta_{E,A,E,B} = id_{A \Box B}$. The internal associativity condition gives the commutative diagram

$$\begin{tikzcd}
(U \boxdot V)\Box(W \boxdot X)\Box(Y \boxdot Z) \arrow[r, "\eta_{U,V,W,X}\Box id_{Y\boxdot Z}"] \arrow[d, "id_{U\boxdot V}\Box\eta_{W,X,Y,Z}"] & ((U\Box W) \boxdot (V\Box X))\Box(Y \boxdot Z) \arrow[d, "\eta_{U\Box W,V\Box X,Y,Z}"] \\
(U \boxdot V)\Box((W\Box Y) \boxdot (X\Box Z)) \arrow[r, "\eta_{U,V,W\Box Y,X\Box Z}"] & (U\Box W\Box Y) \boxdot (V\Box X\Box Z)
\end{tikzcd}$$

The external associativity condition gives the commutative diagram

$$\begin{tikzcd}
(U \boxdot V \boxdot W)\Box(X \boxdot Y \boxdot Z) \arrow[r, "\eta_{U\boxdot V,W,X\boxdot Y,Z}"] \arrow[d, "\eta_{U,V\boxdot W,X,Y\boxdot Z}"] & ((U \boxdot V)\Box(X \boxdot Y)) \boxdot (W\Box Z) \arrow[d, "\eta_{U,V,X,Y} \boxdot id_{W\Box Z}"] \\
(U\Box X) \boxdot ((V \boxdot W)\Box(Y \boxdot Z)) \arrow[r, "id_{U\Box X} \boxdot \eta_{V,W,Y,Z}"] & (U\Box X) \boxdot (V\Box Y) \boxdot (W\Box Z)
\end{tikzcd}$$

Remark 1: Notice that we have natural transformations

$$\eta_{A,E,E,B} : A\Box B \to A \boxdot B \qquad \text{and} \qquad \eta_{E,A,B,E} : A\Box B \to B \boxdot A.$$

Remark 2: Recall that a braided monoidal category (also known as braided tensor category) is a category $\mathcal{C}$ together with a functor $\Box : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is strictly associative, has a strict 2-sided unit object E and with a natural commutativity isomorphism $c_{A,B} : A\Box B \xrightarrow{\cong} B\Box A$ satisfying the following properties:

1. Unit Condition: $c_{A,E} = c_{E,A} = id_A$.
2. Associativity Conditions: For any three objects A, B, C the following diagrams commute:

$$\begin{tikzcd}
A\Box B\Box C \arrow[rr, "c_{A\Box B,C}"] \arrow[dr, "id_A\Box c_{B,C}"'] & & C\Box A\Box B \\
& A\Box C\Box B \arrow[ur, "c_{A,C}\Box id_A"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A\Box B\Box C \arrow[rr, "c_{A,B\Box C}"] \arrow[dr, "c_{A,B}\Box id_C"'] & & B\Box C\Box A \\
& B\Box A\Box C \arrow[ur, "id_B\Box c_{A,C}"'] &
\end{tikzcd}$$

We claim that a braided monoidal category is exactly the same thing as a 2-fold monoidal category with $\Box = \boxdot$, η an isomorphism, and with

$$\eta_{A,B,E,C} = \eta_{A,E,B,C} = id_{A\Box B\Box C}.$$

Assuming that $\Box = \boxdot$ and that the natural isomorphism $\eta_{A,B,C,D}$ satisfies $\eta_{A,B,E,C} = \eta_{A,E,B,C} = id_{A\Box B\Box C}$, one proceeds as follows to show that we have a braided monoidal category. In the internal associativity diagram take $V = W = E$ and obtain that $\eta_{U,X,Y,Z} = id_U\Box\eta_{E,X,Y,Z}$. Then take $X = Y = E$ and obtain that $\eta_{U,V,W,Z} = \eta_{U,V,W,E}\Box id_Z$. Combining these two facts, one obtains that

$$\eta_{A,B,C,D} = id_A\Box c_{B,C}\Box id_D, \tag{$*$}$$

where $c_{B,C} = \eta_{E,B,C,E}$. Then take $U = Z = W = E$ in the internal associativity law to get the first associativity law for c, and take $U = Z = X = E$ to get the other one. With the additional conditions we have here the external associativity law is superfluous.

Conversely given a braided monoidal category, we can define a 2-fold monoidal structure by $(*)$.

Remark 3: Joyal and Street [6] considered a very similar concept to our notion of 2-fold monoidal category. They loosened our requirement that the two operations $\Box$ and $\boxdot$ be strictly associative with a strict unit by only requiring these conditions to hold up to coherent natural isomorphisms. More significantly they required the natural transformation $\eta_{A,B,C,D}$ to be an isomorphism. They then showed that such a category is naturally equivalent to a braided monoidal category. Briefly given such a category one obtains an equivalent braided monoidal category by discarding one of the two operations, say $\boxdot$, and defining the commutativity isomorphism for the remaining operation $\Box$ to be the composite

$$A \Box B \xrightarrow{\eta_{E,A,B,E}} B \boxdot A \xrightarrow{\eta^{-1}_{B,E,E,A}} B \Box A.$$

Our requirement that the operations be strictly associative and unital are not significant restrictions and were adopted for convenience and simplicity. One can always replace categories with operations which are associative and unital up to coherent natural isomorphisms by equivalent categories with strictly associative and unital operations. On the other hand not requiring $\eta_{A,B,C,D}$ to be an isomorphism imposes some attractive finiteness conditions on the free categories as will be discussed later. More importantly Joyal and Street showed that if one generalizes their notion of n-fold monoidal category to $n \geq 3$, the resulting categories are naturally equivalent to symmetric monoidal categories. Thus their nerves have group completions which are infinite loop spaces rather than n-fold loop spaces. By discarding the requirement that $\eta_{A,B,C,D}$ be an isomorphism, we avoid this problem.

We also note that 2-fold monoidal categories have features similar to braided tensor categories. For instance:

Yang-Baxter Hexagon: In a 2-fold monoidal category the following diagram commutes:

$$\begin{tikzcd}
& (B \boxdot A) \Box C \arrow[r, "\eta_{B,A,C,E}"] & (B \Box C) \boxdot A \arrow[dr, "\eta_{E,B,C,E} \boxdot id_A"] & \\
A \Box B \Box C \arrow[ur, "\eta_{E,A,B,E} \Box id_C"] \arrow[dr, "id_A \Box \eta_{E,B,C,E}"'] & & & C \boxdot B \boxdot A \\
& A \Box (C \boxdot B) \arrow[r, "\eta_{E,A,C,B}"] & C \boxdot (A \Box B) \arrow[ur, "id_C \boxdot \eta_{E,A,B,E}"'] &
\end{tikzcd}$$

Proof Sketch: By connecting the left hand vertex of the hexagon with the upper right hand vertex and the lower left hand vertex with the right hand vertex we can subdivide the hexagon into a rectangle

$$\begin{tikzcd}
A \Box B \Box C \arrow[r, "\eta_{E,A,B\Box C,E}"] \arrow[d, "id_A \Box \eta_{E,B,C,E}"] & (B \Box C) \boxdot A \arrow[d, "\eta_{E,B,C,E} \boxdot id_A"] \\
A \Box (C \boxdot B) \arrow[r, "\eta_{E,A,C\boxdot B,E}"] & C \boxdot B \boxdot A
\end{tikzcd}$$

and two triangles

$$\begin{tikzcd}
& (B \boxdot A) \Box C \arrow[dr, "\eta_{B,A,C,E}"] & \\
A \Box B \Box C \arrow[ur, "\eta_{E,A,B,E} \Box id_C"] \arrow[rr, "\eta_{E,A,B\Box C,E}"] & & (B \Box C) \boxdot A
\end{tikzcd}$$

$$\begin{tikzcd}
& C \boxdot (A \Box B) \arrow[dr, "id_C \boxdot \eta_{E,A,B,E}"] & \\
A \Box (C \boxdot B) \arrow[ur, "\eta_{E,A,C,B}"] \arrow[rr, "\eta_{E,A,C\boxdot B,E}"] & & C \boxdot B \boxdot A
\end{tikzcd}$$

The rectangle commutes by naturality of $\eta_{E,U,V,E}$. The first triangle commutes by taking $U = X = Z = E$, $V = A$, $W = B$, and $Y = C$ in the internal associativity diagram and applying the external unit condition. The second triangle commutes by taking $U = V = Z = E$, $W = A$, $Y = B$, and $X = C$ in the external associativity diagram and applying the internal unit condition.

Connection with 2-fold Loop Spaces

In this section we show that the group completion of the nerve of a 2-fold monoidal category is a 2-fold loop space. The proof closely mimics Thomason's [18] proof for the analagous connection between symmetric monoidal categories and infinite loop spaces. That proof in turn is based on Segal's ideas [14].

Segal showed that a space Y is homotopy equivalent to a 1-fold loop space if and only if one can construct a "bar construction on Y up to homotopy.This means a simplicial space $X_* : \Delta^{op} \to \mathbf{T}op$ with $X_1 = Y$ and satisfying

1. There is a homotopy equivalence $X_n \xrightarrow{\simeq} (X_1)^n$ induced by certain iterated face maps and x_0 is contractible.
2. $\pi_0(X_1)$ is a group under the multiplication induced by

$$(X_1)^2 \xleftarrow{\simeq} X_2 \xrightarrow{d_1} X_1$$

($\pi_0(X_1)$ is automatically a monoid by condition (1).)

Moreover he showed that the geometric realization $|X_*|$ is an up-to-homotopy delooping of $X_1 = Y$, ie. $\Omega|X_*| \simeq X_1 = Y$. It was subsequently shown [10] that if condition (2) is omitted, then under some mild additional homotopy commutativity assumption $H_*(\Omega|X_*|)$ is obtained from $H_*(X_1)$ by inverting the elements of $\pi_0(X_1) \subset H_0(X_1)$. This relation is usually referred to as saying that $\Omega|X_*|$ is the *group completion* of X_1. Simplicial spaces satisfying condition (1) are referred to as *special* Δ-*spaces*.

Segal also noted that one could formulate categorical versions of these concepts. For instance a *special* Δ-*category* is a simplicial category $\mathcal{C}_* : \Delta^{op} \to \mathbf{C}at$ satisfying

1. There is an equivalence of categories $\mathcal{C}_n \xrightarrow{\simeq} (\mathcal{C}_1)^n$ induced by certain iterated face maps and C_0 has a initial/terminal object.

Since the nerve construction preserves products and sends categorical equivalences to homotopy equivalences, the nerve of a special Δ-category is a special Δ-space.

Segal noted that a strictly monoidal category $\mathcal{C}$ naturally gives rise to a special Δ-category $\mathcal{C}_*$ with $\mathcal{C}_n = (\mathcal{C})^n$. If the monoidal structure is not strictly associative, then one can still construct a special Δ-category $\mathcal{C}_*$ but with $\mathcal{C}_n \simeq (\mathcal{C})^n$. Here one has to use the extra flexibility of allowing categorical equivalences rather than isomorphisms. (This is not critical in this case, since monoidal categories are equivalent to strictly associative ones. When one attempts to put symmetric monoidal categories in this framework one encounters the problem that commutativity can not be made strict.)

Segal's construction of special Δ-categories in the absence of strict algebraic relations (like associativity) was incomplete and ad hoc. This was remedied by Thomason [18], who noted that this construction could be done in two steps. First one can construct a *lax functor* $\mathcal{C}_* : \Delta^{op} \to \mathbf{C}at$ such that $\mathcal{C}_n = (C_1)^n$. Next one could use the result of Street [16], which states that for any category $\mathcal{I}$ and any lax functor $F : \mathcal{I} \to \mathbf{C}at$, one can construct an equivalent strict functor $\widehat{F} : \mathcal{I} \to \mathbf{C}at$. This functor $\widehat{F}$ is called the *Street rectification* of the original lax functor F. Applying this to the lax functor $\mathcal{C}_* : \Delta^{op} \to \mathbf{C}at$, one obtains a strict functor $\widehat{\mathcal{C}_*} : \Delta^{op} \to \mathbf{C}at$, which is the desired special Δ-category.

While Segal never explicitly considered 2-fold loop spaces, as noted by Dunn [3], his ideas can easily be adapted to this case. One needs to consider special Δ-spaces in the category of special Δ-spaces. These are the same thing as bisimplicial spaces $X_{**} : \Delta^{op} \times \Delta^{op} \to \mathbf{T}op$ satisfying the condition

1. There is a homotopy equivalence $X_{m,n} \xrightarrow{\simeq} (X_1)^{mn}$ induced by certain iterated face maps.

We call such functors *special* $\Delta \times \Delta$-*spaces*. From Segal's results in the 1-fold loop case, we easily see that for a special $\Delta \times \Delta$-space X_{**} that $\Omega^2|X_{**}|$ is a group completion of X_{**}. The notion of special $\Delta \times \Delta$-category can be formulated similarly.

Theorem 1: A functor $\lambda : \mathcal{D} \to \mathbf{M}onCat$ determines a lax functor

$$\tilde{\lambda} : \Delta^{op} \times \mathcal{D} \to \mathbf{C}at$$

such that the restriction of $\tilde{\lambda}$ to $[n] \times \mathcal{D}$ is the n-fold product of λ.

Proof Sketch: Let D-Cat denote the category whose objects are functors $\alpha : \mathcal{D} \to \mathbf{C}at$ and whose morphisms are *lax* natural transformations. A functor $\lambda : \mathcal{D} \to \mathbf{M}onCat$ can be reinterpreted as a monoid in D-Cat, and thus determines a bar construction in D-Cat. This is a functor $\hat{\lambda} : \Delta^{op} \to D\text{-}Cat$ whose adjoint is the functor $\tilde{\lambda}$ we want.

Corollary: A 2-fold monoidal category $\mathcal{C}$ determines a lax functor $\mathcal{C}_{**} : \Delta^{op} \times \Delta^{op} \to \mathbf{C}at$ such that $\mathcal{C}_{m,n} = \mathcal{C}^{mn}$.

Proof Sketch: By definition a 2-fold monoidal category $\mathcal{C}$ is a monoid in $\mathbf{M}onCat$ and hence determines a bar construction $\mathcal{C}_* : \Delta^{op} \to \mathbf{M}onCat$. By Theorem 1 this in turn determines a lax functor $\mathcal{C}_{**} : \Delta^{op} \times \Delta^{op} \to \mathbf{C}at$ such that $\mathcal{C}_{m,n} = \mathcal{C}^{mn}$.

Theorem 2: The group completion of the nerve of a 2-fold monoidal category is a double loop space.

Proof Sketch: By the Corollary to Theorem 1, we have a lax functor $\mathcal{C}_{**} : \Delta^{op} \times \Delta^{op} \to \mathbf{C}at$ such that $\mathcal{C}_{m,n} = \mathcal{C}^{mn}$. Now apply Street rectification to obtain a genuine functor $\widehat{\mathcal{C}}_{**} : \Delta^{op} \times \Delta^{op} \to \mathbf{C}at$, with $\widehat{\mathcal{C}}_{m,n} \simeq \mathcal{C}^{mn}$. Taking nerves we obtain a functor $B\widehat{\mathcal{C}}_{**} : \Delta^{op} \times \Delta^{op} \to \mathbf{T}op$, with $B\widehat{\mathcal{C}}_{m,n} \simeq (B\mathcal{C})^{mn}$. Thus $B\widehat{\mathcal{C}}_{**}$ is a special $\Delta \times \Delta$-space, and the result follows.

Free 2-fold Monoidal Categories and Coherence.

We propose to investigate the difference in generality between 2-fold monoidal categories and braided tensor categories. Our feeling is that in some sense the two notions are equivalent. We would conjecture that for any 2-fold monoidal category $\mathcal{C}$, there is a braided tensor category $\mathcal{C}'$ such that the nerves $B\mathcal{C}$ and $B\mathcal{C}'$ are homotopy equivalent, and that this equivalence could be realized in some natural way on the categorical level.

The natural place to start the investigation of such questions is to compare the free categories in each instance.

The free braided tensor category on one object is of course well known. It is simply the disjoint union of braid groups $\amalg_{n \geq 0} B_n$ and its nerve is the disjoint union $\amalg_{n \geq 0} K(B_n, 1)$ of Eilenberg-MacLane spaces.

The free 2-fold monoidal category $\mathcal{F}$ on one object (denoted 1) also breaks up into a disjoint union of subcategories $\amalg_{n \geq 0} \mathcal{F}_n$. $\mathcal{F}_n$ is the full subcategory of $\mathcal{F}$ whose objects are n-fold combinations (with various bracketings) of the generating object 1 using the two operations $\square$ and $\boxdot$.

Conjecture 1: The nerve $B\mathcal{F}_n$ is an Eilenberg-MacLane space $K(B_n, 1)$.

Since a braided tensor category is a special case of 2-fold monoidal category, there is a natural functor $\mathcal{F} \to \amalg_{n \geq 0} B_n$ which would induce the conjectured equivalence. The cases $n \leq 2$ of the conjecture are trivial. We have verified the first nontrivial case $n = 3$. We now propose a line of investigation which we hope should lead to a proof.

Definition 3: Let $\mathcal{P}_n$ denote the following category. The objects of $\mathcal{P}_n$ are partial orders α on the set $\{1, 2, \ldots, n\}$ satisfying the following two conditions:

1. α is a **suborder** of the standard order $1 < 2 < \ldots < n$.

2. α satisfies the interval condition: if $i <_\alpha j$, then the interval $\{k \mid i \leq k \leq j\}$ (where $<$ denotes the standard order) is a union of two intervals I and J such that $i \in I$, $k <_\alpha j$ for all $k \in I$; $j \in J$ and $i <_\alpha k$ for all $k \in J$.

A morphism $\sigma : \alpha \to \beta$ in $\mathcal{P}_n$ is a permutation $\sigma \in \Sigma_n$ satisfying the following three conditions:

(a) If $i <_\alpha j$ then $\sigma(i) <_\beta \sigma(j)$.

(b) For any i, $\sigma(i) \approx_\beta i$, where $\approx_\beta$ is the equivalence relation generated by β.

(c) If $i < j$ and $\sigma(i) > \sigma(j)$, then $\sigma(i) >_\beta \sigma(j)$.

Composition in $\mathcal{P}_n$ is given by composition of permutations.

It is easy to see that $\mathcal{P}_n$ is a category. It is not immediate that $\mathcal{P}_n$ actually is a finite category. To check this we first introduce a few definitions. If $\sigma : \alpha \to \beta$ is a morphism in $\mathcal{P}_n$, we call a pair (i, j) a *derangement* of σ if $i < j$ while $\sigma(i) > \sigma(j)$. We denote by $Der(\sigma)$ the set of all derangements of σ, and by $chaos(\sigma)$ the cardinality of $Der(\sigma)$.

Lemma: If $\sigma : \alpha \to \beta$ and $\tau : \beta \to \gamma$ are morphisms in $\mathcal{P}_n$, then

1. $Der(\tau\sigma) = Der(\sigma) \amalg (\sigma \times \sigma)^{-1} Der(\tau)$
2. $chaos(\tau\sigma) = chaos(\sigma) + chaos(\tau)$.

Proof. First note that if $(i, j) \in Der(\sigma)$, then by (c) we have $\sigma(i) >_\beta \sigma(j)$ and by (a) it follows that $\tau\sigma(i) >_\gamma \tau\sigma(j)$. Since γ is a suborder of the natural order, $\tau\sigma(i) > \tau\sigma(j)$. Thus we have $Der(\sigma) \subseteq Der(\tau\sigma)$.

Now suppose $(i, j) \in Der(\tau)$. Then we can't have $\sigma^{-1}(i) > \sigma^{-1}(j)$ for that would imply by (c) that $i <_\beta j$, and by (a) that would force $\tau(i) <_\gamma \tau(j)$ and thus $\tau(i) < \tau(j)$. Hence we must have $\sigma^{-1}(i) < \sigma^{-1}(j)$, so $(\sigma^{-1}(i), \sigma^{-1}(j))$ is a derangement of $\tau\sigma$. Hence $(\sigma \times \sigma)^{-1} Der(\tau) \subseteq Der(\tau\sigma)$.

Clearly $Der(\tau\sigma) \subseteq Der(\sigma) \cup (\sigma \times \sigma)^{-1} Der(\tau)$ and $Der(\tau\sigma) \cap Der(\sigma) \cap (\sigma \times \sigma)^{-1} Der(\tau) = \emptyset$. From this (1) follows and (2) is an immediate consequence of (1).

Proposition: The category $\mathcal{P}_n$ enjoys the following properties:

1. If $\sigma : \alpha \to \beta$ is a morphism other than an identity morphism, then the partial order β has strictly larger cardinality than α.
2. The nerve $\mathcal{P}_n$ is a finite complex.

Proof: By condition (a) of Definition 3, β has cardinality at least that of α. If σ is not the identity, then there is a derangement (i, j) of σ. By condition (c), $(\sigma(j), \sigma(i))$ is in β but not in the image under σ of α. This proves (1), and (2) is an easy consequence.

Definition 4: Define a functor $\Lambda_n : \mathcal{F}_n \to \mathcal{P}_n$ as follows. The cases $n = 0, 1$ are trivial. Assume by induction we have defined Λ_i for $i < n$. If $n > 1$ any object C in $\mathcal{F}_n$ can be decomposed as $C = A \Box B$ or $C = A \boxdot B$ where A is an object of $\mathcal{F}_p$, B is an object of $\mathcal{F}_q$, and (p, q) is a pair such that $0 < p < n$, $0 < q < n$, $p + q = n$. By induction $\Lambda_p(A)$ is an already defined partial order on $\{1, 2, \ldots, p\}$ and $\Lambda_q(B)$ is an already defined partial order on $\{1, 2, \ldots, q\}$. We can translate $\Lambda_q(B)$ to a partial order on $\{p + 1, p + 2, \ldots, n\}$. Now define $\Lambda_n(C)$ to be the following partial order on $\{1, 2, \ldots, n\}$. If $C = A \Box B$, just take $\Lambda_n(C)$ to be the disjoint union of the partial order $\Lambda_p(A)$ on $\{1, 2, \ldots, p\}$ and the partial order $\Lambda_q(B)$ on $\{p+1, p+2, \ldots, n\}$. If $C = A \boxdot B$, take $\Lambda_n(C)$ to be disjoint union of $\Lambda_p(A)$ and $\Lambda_q(B)$, together with the additional order relations $i <_{\Lambda_n(C)} j$ for every pair (i, j) such that $i \in \{1, 2, \ldots, p\}$ and $j \in \{p + 1, p + 2, \ldots, n\}$. It is easily checked that any choice of decomposition $C = A \Box B$ or $C = A \boxdot B$ does not affect the definition of the partial order $\Lambda_n(C)$.

Define Λ_n on morphisms as follows. If $f : A \to B$ is a morphism in $\mathcal{F}_n$, then f is some algebraic combination of the identity morphisms and natural transformation η. Relabel every occurence of the generating object 1 in the object A by the labels $1, 2, \ldots, n$ in order from left to right. After applying the natural transformation corresponding to f, the target object will have the labels permuted. Define $\Lambda_n(f)$ to be this permutation.

One can easily check inductively that the partial orders $\Lambda_n(A)$ satisfy conditions (1) and (2) of Definition 3. Noting that Λ_n takes morphisms of the form $f\Box g$ and $f \boxdot g$ to the block sum of the permutations corresponding to f and g, we see that it suffices to check conditions (a), (b), (c) of Definition 3 for morphisms given by the natural transformation η, which is obvious. Finally to check that Λ_n preserves compositions, it suffices to check that Λ_n preserves the naturality diagrams for η as well as the associativity diagrams. This is easily seen by inspection.

Coherence Conjecture for 2-fold Monoidal Categories: The functor $\Lambda_n : \mathcal{F}_n \to \mathcal{P}_n$ is faithful (ie. injective) on both objects and morphisms.

It can be easily seen that Λ_n is actually bijective on objects. A stronger version of this conjecture would be that Λ_n is an isomorphism of categories. However it is possible that the morphisms in the image of Λ_n satisfy additional conditions besides (a), (b), (c) which we may have overlooked. So at the moment we are unwilling to commit ourselves to the stronger version of the conjecture.

We now cite some evidence for this conjecture and its bearing on our previous conjecture.

First let us note that the mere existence of the functor Λ_n has some important consequences.

Proposition: The category $\mathcal{F}_n$ enjoys the following properties:

1. $\mathcal{F}_n$ is a finite category.
2. The nerve $B\mathcal{F}_n$ is a finite complex.

Proof: For an object A of $\mathcal{F}_n$, let $deg(A)$ denote the cardinality of the partial order $\Lambda_n(A)$, ie. the number of ordered pairs satisfying the relation. Let us say a morphism $f : A \to B$ in $\mathcal{F}_n$ is *degree monotonic* if $deg(A) < deg(B)$. It is easy to see that any f given by the natural transformation η is degree monotonic. It is also easy to see that if f and g are degree monotonic, so are $f\Box g$ and $f \boxdot g$. Of course it is obvious that composites of degree monotonic morphisms are degree monotonic. It follows that all morphisms of $\mathcal{F}_n$ are degree monotonic. From this (1) and (2) easily follow.

It follows that the coherence conjecture can be automatically checked for a fixed n in a finite number of steps. We have done this by hand for the case $n = 3$ and obtained:

Proposition: $\Lambda_3 : \mathcal{F}_3 \to \mathcal{P}_3$ is an isomorphism of categories.

We now indicate how this implies that $B\mathcal{F}_3 \simeq K(B_3, 1)$.

Definition 5: Let K_n denote the full subcategory of $\mathcal{P}_n$ whose objects are partial orders of the form generated by pairs $(i, i+1)$. There are 2^{n-1} objects in K_n obtained by filling in some (or all or none) of the spaces in the sequence

$$1 \;\; 2 \;\; 3 \;\; \ldots \;\; n$$

by $<$'s. (K_n is the image under Λ_n of the full subcategory of $\mathcal{F}_n$ whose objects have the form $A_1\Box A_2\Box\ldots\Box A_n$ with each A_i having the form $1 \boxdot 1 \boxdot \ldots \boxdot 1$.)

Proposition: $BK_n \simeq K(B_n, 1)$.

A proof of this fact is given in [5].

In fact remarkably the nerves BK_n arise in a different context. In [11] Milgram introduces a construction $J_2(X)$ on a based space X of the form

$$J_2(X) = (\coprod_{n\geq 0} P_n \times X^n)/\approx$$

Here P_n is the permutohedron, the convex hull of $\{\sigma(1, 2, \ldots, n) | \sigma \in \Sigma_n\}$ in $\mathbb{R}^n$, where Σ_n acts on $\mathbb{R}^n$ by permuting coordinates. Milgram showed that for X connected $J_2(X) \simeq \Omega^2 S^2(X)$. If we take $X = S^0$ (which of course is not connected), then we obtain

$$J_2(S^0) = \coprod_{n\geq 0} P_n/\approx$$

The equivalence relation on P_n identifies $x \in \partial P_n$ with $\sigma(x)$ in some, but not all circumstances. Remarkably $BK_n = P_n/\approx$. Indeed BK_n is the barycentric subdivision of the natural polyhedral decomposition of $P_n/\approx$.

There are several unpublished proofs of the equivalence of $P_n/\approx$ and $K(B_n, 1)$ due to Milgram and Tatsuoka [17]. M. Kapranov informs us that this result is also an easy consequence of results in Salvetti [13].

Proposition: The inclusion $K_3 \subset \mathcal{P}_3$ induces an equivalence of nerves $BK_3 \simeq B\mathcal{P}_3$.

Proof Sketch: This follows by an easy application of Quillen's Theorem A (cf. [12]).

It now follows from the results mentioned above that $B\mathcal{F}_3 \simeq K(B_3, 1)$. It is our hope that a similar method of proof will work for $n > 3$.

We conclude by showing that the $n = 3$ case implies

Proposition: For all n, $\pi_1(B\mathcal{F}_n) = B_n$.

Proof: We already have a functor $\mathcal{F}_n \to B_n$, inducing a homorphism $\pi_1(B\mathcal{F}_n) \to B_n$. Thus it suffices to produce elements $b_i \in \pi_1(B\mathcal{F}_n)$, for $i = 1, 2, \ldots, n-1$, which satisfy the braid relations and which map into the generators of B_n under this homomorphism, and then to show that these elements generate $\pi_1(B\mathcal{F}_n)$.

To define such elements, we first introduce some notation. Let $k \cdot 1$ denote the k-fold combination $1\Box 1\Box \cdots \Box 1$. We take this object as the basepoint of $B\mathcal{F}_k$. For $i = 1, 2, \ldots, n-1$ let A_i denote the object $(i-1)\cdot 1\Box(1 \boxdot 1)\Box(n-i-1)\cdot 1$ in $\mathcal{F}_n$. Let $f_i : n\cdot 1 \to A_i$ denote the following morphism in $\mathcal{F}_n$:

$$n\cdot 1 \xrightarrow{id\Box\eta_{E,1,1,E}\Box id} (i-1)\cdot 1\Box(1\boxdot 1)\Box(n-i-1)\cdot 1$$

Similarly let $g_i : n\cdot 1 \to A_i$ denote

$$n\cdot 1 \xrightarrow{id\Box\eta_{1,E,E,1}\Box id} (i-1)\cdot 1\Box(1\boxdot 1)\Box(n-i-1)\cdot 1$$

Then the diagram

$$n\cdot 1 \xrightarrow{f_i} A_i \xleftarrow{g_i} n\cdot 1$$

represents an element of $\pi_1(B\mathcal{F}_n)$ which we denote b_i. It is clear that b_i maps onto the i-th standard braid generator under the homomorphism $\pi_1(B\mathcal{F}_n) \to B_n$. Thus it is left to show that the b_i satisfy the braid relations.

First suppose $i < j-1$. We want to show $b_ib_j = b_jb_i$. To do this note that in this case we can find a pair of positive integers (p, q) with $p + q = n$ and with $i + 1 \le p$. Then we can write

$$\begin{aligned}
f_i &= f_i'\Box id_{q\cdot 1} : n\cdot 1 \to A_i'\Box q\cdot 1\\
g_i &= g_i'\Box id_{q\cdot 1} : n\cdot 1 \to A_i'\Box q\cdot 1\\
f_j &= id_{p\cdot 1}\Box f_k'' : n\cdot 1 \to p\cdot 1\Box A_k''\\
g_j &= id_{p\cdot 1}\Box g_k'' : n\cdot 1 \to p\cdot 1\Box A_k''
\end{aligned}$$

where $k = j - p$, and $A_i', f_i', g_i', A_k'', f_k'', g_k''$ denote the corresponding objects and morphisms in $\mathcal{F}_p$, resp. $\mathcal{F}_q$. From the commutative diagram

$$\begin{tikzcd}
n\cdot 1 \arrow[r, "f_i'\Box id"] \arrow[d, equal] & A_i'\Box q\cdot 1 \arrow[d, "id\Box f_k''"] & n\cdot 1 \arrow[l, "g_i'\Box id"'] \arrow[d, equal] \\
n\cdot 1 \arrow[r, "f_i'\Box f_k''"] & A_i'\Box A_k'' & n\cdot 1 \arrow[l, "g_i'\Box f_k''"'] \arrow[d, "id\Box f_k''"] \\
 & n\cdot 1 \arrow[u, "g_i'\Box g_k''"] \arrow[r, "id\Box g_k''"] & p\cdot 1\Box A_k'' \arrow[ul, "g_i'\Box id"']
\end{tikzcd}$$

we see that the diagram

$$n\cdot 1 \xrightarrow{f_i'\Box f_k''} A_i'\Box A_k'' \xleftarrow{g_i'\Box g_k''} n\cdot 1$$

represents the product b_ib_j. A similar argument shows that this diagram also represents b_jb_i.

Finally to obtain the braid relation between b_i and b_{i+1}, note that there is an obvious functor $\mathcal{F}_3 \to \mathcal{F}_n$ given on objects by $C \mapsto (i-1)\cdot 1\Box C\Box(n-i-2)\cdot 1$. This functor maps the braid generators b_1', b_2' of $\pi_1(B\mathcal{F}_3) = B_3$ to b_i, b_{i+1} respectively. Since b_1', b_2' satisfy the braid relation, so do their images b_i, b_{i+1}.

To show that the b_i's generate $\pi_1(B\mathcal{F}_n)$, we first choose morphisms $\lambda_A : n \cdot 1 \to A$ for each object $A \in \mathcal{F}_n$. We do this by induction on n. Of course if $n = 0$ or $n = 1$, there is only one choice. Now let C be an object of $\mathcal{F}_n$. Then there is a nontrivial decomposition of either the form $C = A \Box B$ or $C = A \boxdot B$, with A, B objects of $\mathcal{F}_p$, $\mathcal{F}_q$, respectively. By induction λ_A and λ_B are already defined. In the first case we define $\lambda_C = \lambda_A \Box \lambda_B$. In the second case we define λ_C to be the composite

$$n \cdot 1 = p \cdot 1 \Box q \cdot 1 \xrightarrow{\eta_{p \cdot 1, E, E, q \cdot 1}} p \cdot 1 \boxdot q \cdot 1 \xrightarrow{\lambda_A \boxdot \lambda_B} A \boxdot B = C.$$

It is easy to check that this definition doesn't depend on the choice of decomposition of C. (In the particular case $C = A_i$ as defined above, we have $\lambda_{A_i} = g_i$.) Thus we associate to every morphism $f : A \to B$ in $\mathcal{F}_n$, the element of $\pi_1(B\mathcal{F}_n)$ represented by the diagram

$$n \cdot 1 \xrightarrow{\lambda_A} A \xrightarrow{f} B \xleftarrow{\lambda_B} n \cdot 1$$

It suffices to show that every such element can be expressed in terms of the b_i's. A standard argument shows that if the elements corresponding to composable morphisms f and g can be so expressed, so can the element corresponding to the composite. Thus it suffices to consider firstly morphisms which decompose as $f \Box g$ (or $f \boxdot g$) and secondly morphisms of the form $\eta_{A,B,C,D}$. The first case is handled by an argument similar to the argument of the previous paragraph showing that nonadjacent b_i's commute with each other. In the second case, we can use the associativity diagrams for η to reduce to the first case. One for instance can combine the internal associativity diagram for $W = V = E$ with the derived triangles of the Yang Baxter Hexagon.

Remark 4: This result also follows from the work of Joyal and Street [6] mentioned previously. For it is known (cf. [12]) that inverting all the morphisms in a category preserves the fundamental group of the nerve while killing all the higher homotopy groups. By inverting the morphisms in $\mathcal{F}_n$ we obtain the corresponding component of a free 2-fold monoidal category in the sense of Joyal and Street with $\eta_{A,B,C,D}$ required to be an isomorphism. Their results then show that this in turn is equivalent to a free braided monoidal category, and the resulting fundamental groups of the nerves of the components are tautologically braid groups. Note however that inverting the morphisms in a category automatically destroys the finiteness of the nerve.

n-fold Monoidal Categories

It is now easy to determine what the appropriate notion of a 2-fold monoidal functor between 2-fold monoidal categories is. This defines a category **2-**$MonCat$ of 2-fold monoidal categories. Then following the analogy between monoids and loop spaces, it is reasonable to define a 3-fold monoidal category as a monoid in **2-**$MonCat$. Iterating this process leads us to the following definition.

Definition 6: An *n-fold monoidal category* is a category $\mathcal{C}$ with the following structure.

1. There are n distinct multiplications

$$\Box_1, \Box_2, \ldots, \Box_n : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

   which are strictly associative and $\mathcal{C}$ has an object E which is a strict unit for all the multiplications.

2. For each pair (i, j) such that $1 \le i < j \le n$ there is a natural transformation

$$\eta^{ij}_{A,B,C,D} : (A \Box_j B) \Box_i (C \Box_j D) \to (A \Box_i C) \Box_j (B \Box_i D).$$

These natural transformations η^{ij} are subject to the following conditions:

(a) Internal unit condition: $\eta^{ij}_{A,B,E,E} = \eta^{ij}_{E,E,A,B} = id_{A \Box_j B}$

(b) External unit condition: $\eta^{ij}_{A,E,B,E} = \eta^{ij}_{E,A,E,B} = id_{A \Box_i B}$

(c) Internal associativity condition: The following diagram commutes

$$\begin{tikzcd}
(U \Box_j V) \Box_i (W \Box_j X) \Box_i (Y \Box_j Z) \arrow[r, "\eta^{ij}_{U,V,W,X} \Box_i id_{Y \Box_j Z}"] \arrow[d, "id_{U \Box_j V} \Box_i \eta^{ij}_{W,X,Y,Z}"'] & \big((U \Box_i W) \Box_j (V \Box_i X)\big) \Box_i (Y \Box_j Z) \arrow[d, "\eta^{ij}_{U \Box_i W, V \Box_i X, Y Z}"] \\
(U \Box_j V) \Box_i \big((W \Box_i Y) \Box_j (X \Box_i Z)\big) \arrow[r, "\eta^{ij}_{U,V,W \Box_i Y, X \Box_i Z}"] & (U \Box_i W \Box_i Y) \Box_j (V \Box_i X \Box_i Z)
\end{tikzcd}$$

(d) External associativity condition: The following diagram commutes

$$\begin{tikzcd}
(U\Box_j V\Box_j W)\Box_i(X\Box_j Y\Box_j Z) \arrow[r, "\eta^{ij}_{U\Box_j V,W,X\Box_j Y,Z}"] \arrow[d, "\eta^{ij}_{U,V\Box_j W,X,Y\Box_j Z}"'] & \big((U\Box_j V)\Box_i(X\Box_j Y)\big)\Box_j(W\Box_i Z) \arrow[d, "\eta^{ij}_{U,V,X,Y}\Box_j id_{W\Box_i Z}"] \\
(U\Box_i X)\Box_j\big((V\Box_j W)\Box_i(Y\Box_j Z)\big) \arrow[r, "id_{U\Box_i X}\Box_j\eta^{ij}_{V,W,Y,Z}"] & (U\Box_i X)\Box_j(V\Box_i Y)\Box_j(W\Box_i Z)
\end{tikzcd}$$

Finally it is required that for each triple (i,j,k) satisfying $1\le i<j<k\le n$ the (big!) hexagonal interchange diagram commutes.

$$\begin{tikzcd}
& ((A_1\Box_k A_2)\Box_j(B_1\Box_k B_2))\Box_i((C_1\Box_k C_2)\Box_j(D_1\Box_k D_2)) \arrow[dl, "\eta^{jk}_{A_1,A_2,B_1,B_2}\Box_i\eta^{jk}_{C_1,C_2,D_1,D_2}"'] \arrow[dr, "\eta^{ij}_{A_1\Box_k A_2,B_1\Box_k B_2,C_1\Box_k C_2,D_1\Box_k D_2}"] & \\
((A_1\Box_j B_1)\Box_k(A_2\Box_j B_2))\Box_i((C_1\Box_j D_1)\Box_k(C_2\Box_j D_2)) \arrow[d, "\eta^{ik}_{A_1\Box_j B_1,A_2\Box_j B_2,C_1\Box_j D_1,C_2\Box_j D_2}"] & & ((A_1\Box_k A_2)\Box_i(C_1\Box_k C_2))\Box_j((B_1\Box_k B_2)\Box_i((D_1\Box_k D_2)) \arrow[d, "\eta^{ik}_{A_1,A_2,C_1,C_2}\Box_j\eta^{ik}_{B_1,B_2,D_1,D_2}"'] \\
((A_1\Box_j B_1)\Box_i(C_1\Box_j D_1))\Box_k((A_2\Box_j B_2)\Box_i(C_2\Box_j D_2)) \arrow[dr, "\eta^{ij}_{A_1,B_1,C_1,D_1}\Box_k\eta^{ij}_{A_2,B_2,C_2,D_2}"] & & ((A_1\Box_i C_1)\Box_k(A_2\Box_j C_2))\Box_j((B_1\Box_i D_1)\Box_k(B_2\Box_j D_2)) \arrow[dl, "\eta^{jk}_{A_1\Box_i C_1,A_2\Box_j C_2,B_1\Box_i D_1,B_2\Box_j D_2}"'] \\
& ((A_1\Box_i C_1)\Box_j(B_1\Box_i D_1))\Box_k((A_2\Box_i C_2)\Box_j(B_2\Box_i D_2)) &
\end{tikzcd}$$

The corresponding notion of n-fold monoidal functor between such categories is

Definition 7: An *n-fold monoidal functor* $(F, \lambda^1, \dots, \lambda^n) : \mathcal{C} \to \mathcal{D}$ between n-fold monoidal categories consists of a functor F such that $F(E) = E$ together with natural transformations

$$\lambda^i_{A,B} : F(A)\Box_i F(B) \to F(A\Box_i B) \quad i = 1, 2, \dots, n$$

satisfying the same associativity and unit conditions as monoidal functors. In addition the following interchange diagram commutes:

$$\begin{tikzcd}
(F(A)\Box_j F(B))\Box_i(F(C)\Box_j F(D)) \arrow[r, "\eta^{ij}_{F(A),F(B),F(C),F(D)}"] \arrow[d, "\lambda^j_{A,B}\Box_i\lambda^j_{C,D}"'] & (F(A)\Box_i F(C))\Box_j(F(B)\Box_i F(D)) \arrow[d, "\lambda^i_{A,C}\Box_j\lambda^i_{B,D}"] \\
F(A\Box_j B)\Box_i F(C\Box_j D) \arrow[d, "\lambda^i_{A\Box_j B, C\Box_j D}"'] & F(A\Box_i C)\Box_j F(B\Box_i D) \arrow[d, "\lambda^j_{A\Box_i C, B\Box_i D}"] \\
F((A\Box_j B)\Box_i(C\Box_j D)) \arrow[r, "F(\eta^{ij}_{A,B,C,D})"] & F((A\Box_i C)\Box_j(B\Box_i D))
\end{tikzcd}$$

Composition of n-fold monoidal functors is defined in exactly the same way as for monoidal functors. However there is an additional exercise to check that the resulting composite satisfies the hexagonal interchange diagram.

It is pretty straightforward to check that an $(n+1)$-fold monoidal category is exactly the same thing as a monoid in $\mathbf{n}$-$MonCat$, the category of n-fold monoidal categories and functors.

Theorem 3: The group completion of the nerve of an n-fold monoidal category is an n-fold loop space.

Proof Sketch: We proceed as in the 2-fold case. Given an n-fold monoidal category $\mathcal{C}$, we first construct a lax functor $\mathcal{C}_{**\dots*} : (\Delta^{op})^n \to \mathbf{C}at$, with $\mathcal{C}_{m_1,m_2,\dots,m_n} = \mathcal{C}^{m_1 m_2 \dots m_n}$. Then apply Street rectification to obtain a genuine functor $\widehat{\mathcal{C}}_{**\dots*} : (\Delta^{op})^n \to \mathbf{C}at$, with $\widehat{\mathcal{C}}_{m_1,m_2,\dots,m_n} \simeq \mathcal{C}^{m_1 m_2 \dots m_n}$. Taking nerves we obtain a functor $B\widehat{\mathcal{C}}_{**\dots*} : (\Delta^{op})^n \to \mathbf{T}op$, with $B\widehat{\mathcal{C}}_{m_1,m_2,\dots,m_n} \simeq (B\mathcal{C})^{m_1 m_2 \dots m_n}$. The result now follows from Segal's work.

It is our feeling that all n-fold loop spaces can be obtained as group completions of nerves of n-fold monoidal categories. Again the natural starting point is to analyze the homotopy type of the nerve of the free n-fold monoidal category $\mathcal{F}^{(n)}$ on one object. As in the 2-fold case $\mathcal{F}^{(n)}$ breaks up into a disjoint union of subcategories $\amalg_{k\geq 0}\mathcal{F}^{(n)}{}_k$, where $\mathcal{F}^{(n)}{}_k$ is the full subcategory whose objects are k-fold products of the generating object using the n available multiplications.

It follows from work of Boardman and Vogt [1] and of May [9] that the topological analog of $\mathcal{F}^{(n)}$ is $\amalg_{k\geq 0}\mathcal{C}_n(k)/\Sigma_k$, where $\mathcal{C}_n$ denotes the little n-cubes operad. They also noted that $\mathcal{C}_n(k)/\Sigma_k$ has the homotopy type of $\widetilde{F}(\mathbb{R}^n; k)$ the reduced configuration space k points in $\mathbb{R}^n$. Thus we conjecture

Conjecture 3: The nerve $B\mathcal{F}^{(n)}{}_k$ is homotopy equivalent to the reduced configuration space $\widetilde{F}(\mathbb{R}^n; k)$.

So far for $n \geq 3$ we have checked that this holds in the first nontrivial case $k = 2$ where we have verified that

$$B\mathcal{F}^{(n)}{}_2 \cong \mathbb{R}P^{n-1} \simeq \widetilde{F}(\mathbb{R}^n; 2).$$

As we previously mentioned Joyal and Street [6] have shown that if require η^{ij} to be isomorphisms, then the resulting notion of n-fold monoidal category is equivalent to the notion of symmetric monoidal category for $n \geq 3$. This implies that $\pi_1 B\mathcal{F}^{(n)}{}_k \cong \Sigma_k$ which is consistent with our conjecture.

Remark 5: One approach in the study of braided monoidal categories and their generalizations has been to view them as "weak 3-categories" (cf. [7]). Similarly one can view our notion of n-fold monoidal category as a special type of weak $(n+1)$-category. Briefly one takes the set of i-morphisms, $0 \leq i \leq n-1$ to consist of one element. One takes the n-morphisms to be the objects of the original n-fold monoidal category and takes the n compositions to be the given products $\Box_1, \Box_2, \dots, \Box_n$. The $(n+1)$-morphisms are taken to be the morphisms of the original n-fold monoidal category and the $(n+1)$ compositions are taken to be $\Box_1, \Box_2, \dots, \Box_n$ together with the original composition in the n-fold monoidal category. The "weakness" in the structure comes from the fact that the interchange law between the compositions $\Box_i$ and $\Box_j$ does not hold strictly but only up to the $(n+1)$-morphism η^{ij}.

Literatur

[1] J. M. Boardman and R. M. Vogt, Homotopy invariant algebraic structures on topological spaces, Lecture Notes in Mathematics, Vol. 347, Springer, 1973.

[2] J. M. Boardman and R. M. Vogt, Tensor products of theories, applications to infinite loop spaces, J. Pure Appl. Algebra 14(1979), 117-129.

[3] G. Dunn, Tensor product of operads and iterated loop spaces, J. Pure Appl. Algebra 50(1988), 237-258.

[4] Z. Fiedorowicz, The symmetric bar construction, to appear in J. Pure Appl. Algebra.

[5] Z. Fiedorowicz, R. Schwänzl, and R. M. Vogt, Tensor products of A_∞ operads, in preparation.

[6] A. Joyal and R. Street, Braided tensor categories, preprint.

[7] M. Kapranov and V. Voevodsky, Braided monoidal 2-categories, 2-vector spaces and Zamolodchikov's tetrahedral equations, preprint.

[8] S. MacLane, Categorical algebra, Bull. A. M. S. 71(1965), 40-106.

[9] J. P. May, The geometry of iterated loop spaces, Lecture Notes in Mathematics, Vol. 271, Springer, 1972

[10] D. McDuff and G. Segal, Homology fibrations and the "group completion" theorem, Invent. Math. 31(1976), 279-284.

[11] R. J. Milgram, Iterated loop spaces, Ann. Math. 84(1966), 386-403.

[12] D. Quillen, Higher algebraic K-theory I, Lecture Notes in Math. 341 (Springer-Verlag 1973), 85-147.

[13] M. Salvetti, Topology of the complement of of real hyperplanes in $\mathbb{C}^n$, Invent. Math. 88(1987), 603-618.

[14] G. Segal, Categories and cohomology theories, Topology 13(1974), 293-312.

[15] J. D. Stasheff, Homotopy associativity of H-spaces I, Trans. A. M. S. 108(1963), 275-292.

[16] R. Street, Two constructions on lax functors, Cahiers Topologie Geometrie Differentielle 13(1972), 217-264.

[17] K. Tatsuoka, A finite $K(\pi,1)$ for Artin groups of finite type, preprint.

[18] R. W. Thomason, Homotopy colimits in the category of small categories, Math. Proc. Cambridge Phil. Soc. 85(1979), 91-109.

Z. Fiedorowicz
Dept. of Mathematics
Ohio State University
Columbus, OH

R. Schwänzl R. Vogt
Fb. Mathematik/Informatik
Universität Osnabrück
Osnabrück, NdS