The approximation theorem and the K-theory of generalized free products

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1 Introduction

One of the main results of [8] is a decomposition of the K-theory of a ring which is a generalized free product in terms of the K-theories of the constituent rings, and one other object, which is generally called the Nil-term. We recall this in the fibration square described below in Theorem 4. Underlying the derivation of the square given in [8] is the construction of a suitable fibration. Besides a combinatorial analysis of a generalized free product, the construction depends on an *ad hoc* development of some parts of abstract algebraic K-theory, later systematized and generalized in [9]. It is the purpose of this paper to show how to derive the main theorems of [8] as applications of results and methods of [9]. Using different methods, Pierre Vogel [7] has also reconsidered results of [8].

We first develop language so that essentially the same fibration used in [8] may be derived from a general fibration theorem (Theorem 1.6.4 of [9, page 354]) developed as part of the overall approach to abstract algebraic K-theory described in [9]. This is Proposition 2.1 below. Our main contribution is the description of how the approximation theorem (Theorem 1.6.7 of [9, page 354]) may then be used to interpret terms in the fibration. These results are Theorems 2 and 3 stated at the end of the section. We are able to replace the technical manouvering required in the original proofs with arguments that follow a standard pattern and are more conceptual. The paper also provides an introduction to a few of the ideas we will use in [5], where we generalize the situation to the case of simplicial rings. One of the goals of [5] is to set up a framework which will also allow us to handle decomposition problems in the K-theory of A_∞-rings (informally, rings-up-to-homotopy), including partial analysis of the Nil terms. To these ends, several parts of the program in the present paper have to be treated differently in [5], with the result that the specialization of those proofs to the case of ordinary rings does not immediately make the underlying principles clear.

We will concern ourselves only with the case of a generalized free product of rings, but it will be clear that the methods also reprove the results for Laurent extensions and for polynomial extensions which are found in sections 12 and 13 of [8], respectively.

We start with a diagram

$$A \xleftarrow{\alpha} C \xrightarrow{\beta} B$$

of discrete rings satisfying the freeness and purity conditions of [8]. That is, we require for purity that $\alpha : C \longrightarrow A$ be injective and that there be a splitting $A = \alpha(C) \oplus A'$ of C-bimodules. Though the splitting is not part of the data and only its existence is required, it is convenient to refer to a fixed complement A' of $\alpha(C)$ in A. The freeness condition we will impose is that A' shall be free as a left C-module. We impose the same conditions on $\beta : C \longrightarrow B$.

Let

$$R = A *_C B$$

be the free product as displayed in [8]. As a C-bimodule, then

$$R = C \oplus A' \oplus B' \oplus A' \otimes_C B' \oplus B' \otimes_C A' \oplus A' \otimes_C B' \otimes_C A' \oplus B' \otimes_C A' \otimes_C B' \oplus \cdots$$

and the problem is to describe the K-theory of R in terms of the K-theories of A, B, and C.

In this paper K-theory is defined in terms of the $S_\bullet$ construction of [9], which from a category with cofibrations and weak equivalences produces its K-theory space. We consider certain categories of simplicial modules over the rings A, B, C, and R. These categories, whose definitions we recall in Section 2, will be denoted by $\mathcal{M}_f(A)$, $\mathcal{M}_f(B)$, $\mathcal{M}_f(C)$, and $\mathcal{M}_f(R)$, respectively. Each of these categories then supports a notion of cofibration, which we also recall below, and a weak equivalence in each case will be a homotopy equivalence, or, briefly, an h-map. Then the K-theory of $\mathcal{M}_f(A)$ with respect to these notions of cofibration and weak equivalence (generically indicated by the presence of an h somewhere in the symbol) is defined in [9] as

$$K(\mathcal{M}_f(A); h) = \Omega|hS_\bullet\mathcal{M}_f(A)|.$$

According to Theorem 2.3.2 in [9, page 394] the relation with Quillen's plus construction definition of K-theory is

$$\Omega|hS_\bullet\mathcal{M}_f(A)| \simeq K_0'(A) \times BGL(A)^+,$$

where $K'_0(A)$ is the subgroup of the usual Grothendieck group of isomorphism classes of projective modules $K_0(A)$ generated by the free modules. Therefore, we will abbreviate

$$K(\mathcal{M}_f(A); h) = K(A)$$

and use similar notations for the other rings B, C, and R.

We next describe the auxiliary categories which are essential to the proofs. First of all, we have the category of Mayer-Vietoris presentations $\mathcal{MV}$, in which an object is a sextuple $(M, M_A, M_B, M_C, \iota, \kappa)$, where M is an object of $\mathcal{M}_f(R)$, and M_A, M_B, and M_C are similarly objects of $\mathcal{M}_f(A)$, $\mathcal{M}_f(B)$, and $\mathcal{M}_f(C)$, respectively. Also, ι and κ are maps of R-modules such that

$$0 \longrightarrow M \xrightarrow{\iota} M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R \longrightarrow 0$$

is a fibration sequence of simplicial R-modules and

$$\kappa(M_A) \subset M_C \otimes_C A \text{ and } \kappa(M_B) \subset M_C \otimes_C B.$$

These are the same conditions imposed in [8, page 142], but now taken degreewise. We will also find it useful to write κ as $\kappa = \kappa_\alpha - \kappa_\beta$, where $\kappa_\alpha(M_B) = 0$ and $\kappa_\beta(M_A) = 0$. A map of Mayer-Vietoris presentations is a quadruple of maps (f, f_A, f_B, f_C) in the respective module categories, such that the resulting ladder diagram commutes. By means of the forgetful functors to the respective module categories $\mathcal{M}_f(A)$, $\mathcal{M}_f(B)$, $\mathcal{M}_f(C)$, and $\mathcal{M}_f(R)$, one may define cofibrations in $\mathcal{MV}$. We discuss this in more detail in Section 2.

We will be working with two notions of weak equivalence in the category $\mathcal{MV}$. A map of Mayer-Vietoris presentations (f, f_A, f_B, f_C) is a coarse weak equivalence if the map f is a homotopy equivalence. The subcategory of these maps will be denoted by $w\mathcal{MV}$, and such a map will be called a w-map for short. A map of Mayer-Vietoris presentations is a fine weak equivalence, or a v-map, for short, if the maps f_A, f_B, and f_C are homotopy equivalences. We denote the subcategory of $\mathcal{MV}$ where the arrows are the v-equivalences by $v\mathcal{MV}$. Notice that it follows from the five lemma that a v-map is also a w-map, so that $v\mathcal{MV} \subset w\mathcal{MV}$.

In Section 2, Proposition 2.1, we observe that Theorem 1.6.4 of [9, page 350] provides the following fibration-up-to-homotopy

$$vS_\bullet\mathcal{MV}^w \longrightarrow vS_\bullet\mathcal{MV} \longrightarrow wS_\bullet\mathcal{MV},$$

where $\mathcal{MV}^w$ denotes the subcategory of $\mathcal{MV}$ consisting of Mayer-Vietoris presentations in which the R-module is contractible. After taking loop spaces of the realizations, from left to right these terms are by definition the K-theory of the category of Mayer-Vietoris presentations of contractible R-modules, the K-theory of the category of Mayer-Vietoris presentations with respect to the v-equivalences (the fine equivalences) and the K-theory of the category of Mayer-Vietoris presentations with respect to the w-equivalences (the coarse equivalences).

In Section 2 we prove this first interpretative theorem.

Theorem 1 *The forgetful functors from $\mathcal{MV}$ to the module categories $\mathcal{M}_f(A)$, $\mathcal{M}_f(B)$, and $\mathcal{M}_f(C)$ induce a homotopy equivalence*

$$u_* : vS_\bullet\mathcal{MV} \longrightarrow hS_\bullet\mathcal{M}_f(A) \times hS_\bullet\mathcal{M}_f(B) \times hS_\bullet\mathcal{M}_f(C).$$

In Section 4 we make our first use of the approximation theorem and prove this result.

Theorem 2 *The forgetful functor*

$$u_R : \mathcal{MV} \longrightarrow \mathcal{M}_f(R)$$

induces a homotopy equivalence

$$wS_\bullet\mathcal{MV} \longrightarrow hS_\bullet\mathcal{M}_f(R).$$

Finally, we define $\mathcal{S}$, the category of split modules. This subcategory of $\mathcal{MV}$ consists of the Mayer-Vietoris presentations

$$0 \longrightarrow 0 \xrightarrow{\iota} M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R \longrightarrow 0.$$

Alternatively, one characterizes objects of this category as Mayer-Vietoris presentations in which the map κ is an isomorphism. This subcategory of $\mathcal{MV}$ inherits all the structures which are on $\mathcal{MV}$, but, of course, only the v-equivalences are relevant. Obviously it is a subcategory of $\mathcal{MV}^w$.

In section 5 we prove our second main result.

Theorem 3 *The inclusion functor*

$$i : \mathcal{S} \longrightarrow \mathcal{MV}^w$$

induces a homotopy equivalence

$$vS_\bullet\mathcal{S} \longrightarrow vS_\bullet\mathcal{MV}^w.$$

Following the pattern of manipulations of cartesian squares given in section 11 of [8, pages 210–218] one arrives at the following theorem on the structure of $K(R)$.

Theorem 4 *There is a homotopy-commutative square*

$$\begin{tikzcd} vS_\bullet\mathcal{S} \arrow[r] \arrow[d] & hS_\bullet\mathcal{M}_A \times hS_\bullet\mathcal{M}_B \arrow[d] \\ hS_\bullet\mathcal{M}_C \arrow[r] & hS_\bullet\mathcal{M}_R \end{tikzcd}$$

which is homotopy cartesian with respect to the simplicial homotopy which to any object of $\mathcal{S}$ associates its structure map κ.

In general, after geometric realization one can produce a splitting

$$|vS_\bullet\mathcal{S}| \simeq |hS_\bullet\mathcal{M}_C| \times |hS_\bullet\mathcal{M}_C| \times \mathrm{Nil}(C; A', B'),$$

where the third factor remains obscure in general. In case the ring C is regular coherent in the sense of [8, page 160], one may also use the techniques of this paper to prove an analogue of Theorem 11.2 of [8] which identifies the homotopy type of $vS_\bullet\mathcal{S}$ with that of $hS_\bullet\mathcal{M}_C \times hS_\bullet\mathcal{M}_C$. Then a little more manipulation leads to the familiar Mayer-Vietoris type sequence of algebraic K-groups:

$$\cdots \longrightarrow K_{n+1}(R) \longrightarrow K_n(C) \longrightarrow K_n(A) \oplus K_n(B) \longrightarrow K_n(R) \longrightarrow \cdots.$$

Going beyond what we mention here, the proof of the equivalence of stable K-theory with topological Hochschild homology which we have outlined in [4] requires a partial analysis of the Nil-term for certain diagrams of A_∞-rings. We establish a framework for doing this. In [5], where, to warm up, we will return to the result of [8] just mentioned. With these remarks out of the way, we proceed to collect some background material, and then we move on to the proofs.

2 Basics

The categories of simplicial modules we use in this paper are defined in Section 2.3 of [9]. We start this section with a few amplifications of the ideas introduced there. Categories of simplicial modules without extra finiteness conditions have also been introduced in Chapter II, Sections four and six, of [3].

Let A be a ring with 1. A right module over A is a simplicial abelian group M, together with a unital and associative action of A from the right. This action may be codified in terms of a homomorphism of abelian groups $M \otimes A \longrightarrow M$ where the tensor product is taken degreewise, such that certain diagrams commute. $\mathcal{M}(A)$ denotes the category of right A-modules and their A-linear maps.

A simplicial set Y gives rise to a module $[Y]A$ if we let

$$([Y]A)_n = [Y_n]A,$$

the free right A-module generated by Y_n. We may, occasionally, need to construct left modules in a similar manner. Notice that if $X = \Delta^q/\partial\Delta^q$ is a standard model for the simplicial q-sphere, then $[X]A$ is isomorphic to a wedge of simplicial Eilenberg-MacLane spaces $K(A,q) \vee K(A,0)$. If Y is a simplicial set with a basepoint $*$, then one reduces $[Y]A$ by dividing out by the submodule $[*]A$. For example, if we view the sphere above as a pointed simplicial set, the construction produces a simplicial Eilenberg-MacLane space $K(A,q)$.

To attach an n-cell to an A-module M means to form a pushout of the kind

$$M \longleftarrow [\partial\Delta^n]A \longrightarrow [\Delta^n]A.$$

A module N is obtained from M by attaching cells if it can be constructed from M by means of this process together with taking a direct limit; we also observe that in this case $M \longrightarrow N$ is a free map in the sense of [3, page 6.3].

We will also be using the following notations. A pair of simplicial sets (Y, Y') gives rise to a pair of modules denoted by $[Y, Y']A$. More generally, if M is any A-module, then we define

$$[Y, Y']M = [Y, Y']\mathbf{Z} \otimes M,$$

where $\mathbf{Z}$ denotes the integers and where we take the tensor product degreewise.

The category of modules we are interested in is $\mathcal{M}_f(A)$, the full subcategory $\mathcal{M}(A)$ whose objects are the modules obtainable from the zero module by attaching finitely many cells. In this category the cofibrations are the free maps and the weak equivalences are the maps which become homotopy equivalences after realization. However, simplicial abelian groups are Kan sets, so it would suffice to use simplicial homotopy equivalences. Recall that the crucial axiom for cofibrations is that a cofibration admits cobase changes. That is, if $c : M \longrightarrow N$ is a cofibration and $f : M \longrightarrow M'$ is arbitrary, then the pushout square

$$\begin{array}{ccc} M & \xrightarrow{c} & N \\ {\scriptstyle f}\downarrow & & \downarrow \\ M' & \longrightarrow & N' \end{array}$$

exists in $\mathcal{M}_f(A)$ and, most importantly, $M' \longrightarrow N'$ is also a cofibration. That this latter condition is satisfied may be verified by induction using the cell attachment characterization of a cofibration.

We make explicit the notion of cylinder functor [9, page 348] admitted by each of our categories of modules. In the category $\mathcal{M}_f(A)$, for instance, the mapping cylinder of a homomorphism $f : M \longrightarrow N$ is constructed by taking the following pushout:

$$\begin{array}{ccc} M \oplus M \;\cong [\partial\Delta^1]\mathbf{Z} \otimes M & \longrightarrow & [\Delta^1]\mathbf{Z} \otimes M \\ \downarrow {\scriptstyle 1 \oplus f} & & \downarrow \\ M \oplus N & \longrightarrow & T(f) \end{array}$$

where the upper horizontal arrow is the cofibration induced by the inclusion $\partial\Delta^1 \longrightarrow \Delta^1$. We also note that the weak equivalences satisfy the extension and saturation axioms of [9, page 327], as well as the cylinder axiom [9, page 349].

As mentioned in the introduction, the category $\mathcal{MV}$ inherits notions of cofibration and mapping cylinder from the categories $\mathcal{M}_f(A)$, $\mathcal{M}_f(B)$, $\mathcal{M}_f(C)$, and $\mathcal{M}_f(R)$. A map

$$(c, c_A, c_B, c_C) : (M, M_A, M_B, M_C, \iota, \kappa) \longrightarrow (N, N_A, N_B, N_C, \iota, \kappa)$$

is a cofibration if and only if each of the maps c, c_A, c_B, and c_C is a cofibration in its module category. To see that the cobase-change axiom holds, let

$$(f, f_A, f_B, f_C) : (M, M_A, M_B, M_C, \iota, \kappa) \longrightarrow (M', M'_A, M'_B, M'_C, \iota, \kappa)$$

be an arbitrary map of Mayer-Vietoris presentations and define N', N'_A, N'_B, and N'_C via the pushout squares

$$\begin{tikzcd} M_? \arrow[r, "c_?"] \arrow[d, "f_?"'] & N_? \arrow[d] \\ M'_? \arrow[r] & N'_? \end{tikzcd}$$

in the appropriate categories. Then the cofibration condition in the axiom is satisfied, but we need to know that the given maps ι and κ induce an exact sequence

$$0 \longrightarrow N' \xrightarrow{\iota} N'_A \otimes_A R \oplus N'_B \otimes_B R \xrightarrow{\kappa} N'_C \otimes_C R \longrightarrow 0.$$

For this, consider the following diagram:

$$\begin{tikzcd} 0 & N \arrow[l] \arrow[r, "\iota"] & N_A \otimes_A R \oplus N_B \otimes_B R \\ 0 \arrow[u] \arrow[d] & M \arrow[l] \arrow[r, "\iota"] \arrow[u] \arrow[d] & M_A \otimes_A R \oplus M_B \otimes_B R \arrow[u] \arrow[d] \\ 0 & M' \arrow[l] \arrow[r, "\iota"] & M'_A \otimes_A R \oplus M'_B \otimes_B R \end{tikzcd}$$

Now calculate the colimit iteratively in two ways, and recall that tensoring with R is exact, by our freeness hypothesis, so that tensoring with R then preserves colimits. Calculating the column colimits first, one finds that one expression for the colimit of the diagram is $(N'_A \otimes_A R \oplus N'_B \otimes_B R)/\iota(N')$, and, calculating the row colimits first, one finds that another expression for the colimit of the diagram is $N'_C \otimes_C R$. It follows that

$$0 \longrightarrow N' \xrightarrow{\iota} N'_A \otimes_A R \oplus N'_B \otimes_B R \xrightarrow{\kappa} N'_C \otimes_C R \longrightarrow 0$$

is exact.

Similarly, we may introduce mapping cylinders on $\mathcal{MV}$ by pulling them back from the mapping cylinders on $\mathcal{M}_f(A)$, $\mathcal{M}_f(B)$, $\mathcal{M}_f(C)$, and $\mathcal{M}_f(R)$. One first observes that

$$T(f_A \otimes_A R) = T(f_A) \otimes_A R$$

and so on for the other cases. Then, since we have defined the mapping cylinders on our module categories in terms of pushouts, we derive from the argument above that taking mapping cylinders induces a Mayer-Vietoris presentation

$$(T(f), T(f_A), T(f_B), T(f_C), \iota, \kappa)$$

if (f, f_A, f_B, f_C) is a map of Mayer-Vietoris presentations, and that the cofibration conditions [8, page 348] put on the front and back inclusions are satisfied.

Recall that we defined a map (g, g_A, g_B, g_C) in $\mathcal{MV}$ to be a w-equivalence if g is a weak homotopy equivalence and to be a v-equivalence if the three maps g_A, g_B, and g_C are all weak homotopy equivalences. Since the weak homotopy equivalences in the module categories satisfy the extension, saturation, and cylinder axioms, it follows that both the v-equivalences and the w-equivalences satisfy these three axioms, too. Therefore Theorem 1.6.4 of [9, page 350] applies to the category $\mathcal{MV}$ with these notions of weak equivalence and we immediately derive the following proposition.

Proposition 2.1 *Let $\mathcal{MV}^w$ be the subcategory of $\mathcal{MV}$ consisting of Mayer-Vietoris presentations in which the R-module is contractible. Then there is a fibration-up-to-homotopy*

$$vS_\bullet\mathcal{MV}^w \longrightarrow vS_\bullet\mathcal{MV} \longrightarrow wS_\bullet\mathcal{MV}.$$

□

We now prove the first of the identification theorems. It will be seen that the argument is basically formal and depends on another fundamental result of K-theory called the additivity theorem. See [9, page 336] and [1] for proofs and various formulations of the result and [6] for reductions of some other standard theorems in K-theory to applications of the additivity theorem.

Theorem 1 *The forgetful functors from $\mathcal{MV}$ to the module categories $\mathcal{M}_f(A)$, $\mathcal{M}_f(B)$, and $\mathcal{M}_f(C)$ induce a homotopy equivalence*

$$u_* : vS_\bullet\mathcal{MV} \longrightarrow hS_\bullet\mathcal{M}_f(A) \times hS_\bullet\mathcal{M}_f(B) \times hS_\bullet\mathcal{M}_f(C).$$

Proof: Let

$$u_A : \mathcal{MV} \longrightarrow \mathcal{M}_f(A)$$

denote the functor which selects the module M_A from a Mayer-Vietoris presentation

$$0 \longrightarrow M \xrightarrow{\iota} M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R \longrightarrow 0,$$

and define u_B and u_C similarly. Now the functor u_A has an obvious section s_A which associates to an A-module M_A the Mayer-Vietoris presentation

$$0 \longrightarrow (M_A \otimes_A R) \xrightarrow{=} M_A \otimes_A R \oplus 0 \longrightarrow 0 \longrightarrow 0.$$

Define a section s_B of u_B similarly.

Define a section s_C of u_C by associating to the C-module M_C the Mayer-Vietoris presentation

$$0 \longrightarrow (M_C \otimes_C R) \xrightarrow{\Delta} (M_C \otimes_C A) \otimes_A R \oplus (M_C \otimes_C B) \otimes_B R \xrightarrow{\tilde{\Delta}} M_C \otimes_C R \longrightarrow 0,$$

where Δ is the diagonal and $\tilde{\Delta}$ is the skew codiagonal. The composites $u_C \circ s_A$, $u_B \circ s_A$, $u_C \circ s_B$, and $u_A \circ s_A$ are all trivial, the composite $u_A \circ s_C$ is the functor

$$\alpha : \mathcal{M}_f(C) \longrightarrow \mathcal{M}_f(A)$$

whose value on M_C is $M_C \otimes_C A$, and the composite $u_B \circ s_C$ is a similar functor

$$\beta : \mathcal{M}_f(C) \longrightarrow \mathcal{M}_f(B).$$

It follows that if we write

$$s_* = (s_A)_* \vee (s_B)_* \vee (s_C)_* : hS_\bullet\mathcal{M}_f(A) \times hS_\bullet\mathcal{M}_f(B) \times hS_\bullet\mathcal{M}_f(C) \longrightarrow vS_\bullet\mathcal{MV},$$

where $\vee$ indicates the H-space addition on $vS_\bullet\mathcal{MV}$ induced by the summing of Mayer-Vietoris presentations, then $u_* \circ s_*$ is a self-map of $hS_\bullet\mathcal{M}_f(A) \times hS_\bullet\mathcal{M}_f(B) \times hS_\bullet\mathcal{M}_f(C)$ which has the matrix representation

$$\begin{pmatrix} \mathrm{Id}_* & 0 & \alpha_* \\ 0 & \mathrm{Id}_* & \beta_* \\ 0 & 0 & \mathrm{Id}_* \end{pmatrix}$$

and is therefore a self-equivalence. It follows that u_* has a right homotopy inverse.

To complete the proof, we have to prove that u_* has a left homotopy inverse. This we will do using the additivity theorem. First note that, given a Mayer-Vietoris presentation

$$0 \longrightarrow M \xrightarrow{\iota} M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R \longrightarrow 0,$$

the arrow ι defines a natural transformation

$$\iota : \mathrm{Id} \longrightarrow s_A \circ u_A \oplus s_B \circ u_B .$$

Explicitly, this is represented by the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\iota"] \arrow[d, "\iota"'] & M_A \otimes_A R \oplus M_B \otimes_B R \arrow[r, "\kappa"] \arrow[d, "="'] & M_C \otimes_C R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_A \otimes_A R \oplus M_B \otimes_B R \arrow[r, "="] & M_A \otimes_A R \oplus M_B \otimes_B R \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Unfortunately, the natural transformation is not a cofibration in the category $\mathcal{MV}$, but we take mapping cylinders to correct this, obtaining an exact functor F whose value on our typical Mayer-Vietoris presentation is

$$0 \longrightarrow T(\iota) \longrightarrow ([\Delta^1]\mathbf{Z} \otimes M_A) \otimes_A R \oplus ([\Delta^1]\mathbf{Z} \otimes M_B) \otimes_B R \longrightarrow C(M_C) \otimes_C R \longrightarrow 0$$

where $C(M_C) = T(M_C \longrightarrow 0)$ is the cone on M_C. Now we have a factorization of ι through F,

$$\mathrm{Id} \longrightarrow F \xrightarrow{\sim} s_A \circ u_A \oplus s_B \circ u_B ,$$

in which the transformation $\mathrm{Id} \longrightarrow F$ is a cofibration, because it is the front inclusion of a mapping cylinder, and the transformation $F \longrightarrow s_A \circ u_A \oplus s_B \circ u_B$ is a v-equivalence, because the mapping cylinders for Mayer-Vietoris presentations satisfy the cylinder axiom. Applying Proposition 1.3.1 of [9, page 330], which is just an application to the $S_\bullet$ construction of the principle that a natural transformation of functors induces a simplicial homotopy of maps of nerves of categories, it follows that F and $s_A \circ u_A \oplus s_B \circ u_B$ induce homotopic maps on $vS_\bullet\mathcal{MV}$:

$$F_* \simeq (s_A \circ u_A)_* \vee (s_B \circ u_B)_* .$$

Now we apply the additivity theorem [9, page 336] to the cofibration sequence of functors

$$\mathrm{Id} \longrightarrow F \longrightarrow F/\mathrm{Id} = F',$$

obtaining a homotopy

$$F_* \simeq Id_* \vee F'_*$$

of self maps of $vS_\bullet\mathcal{MV}$.

One sees that the value of F' on a Mayer-Vietoris presentation

$$0 \longrightarrow M \xrightarrow{\iota} M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R \longrightarrow 0$$

is the Mayer-Vietoris presentation

$$0 \longrightarrow T(\iota)/M \longrightarrow \left(\frac{[\Delta^1]\mathbf{Z} \otimes M_A}{[0]\mathbf{Z} \otimes M_A}\right) \otimes_A R \oplus \left(\frac{[\Delta^1]\mathbf{Z} \otimes M_B}{[0]\mathbf{Z} \otimes M_B}\right) \otimes_B R \longrightarrow B(M_C) \otimes_C R \longrightarrow 0,$$

where $B(M_C) = C(M_C)/M_C$ is the classifying space, or reduced suspension of M_C, and the modules in the middle are contractible.

Now we want to relate F' to $s_C \circ u_C$. Recall from Section 1 that we have written the map κ in the original Mayer-Vietoris presentation as $\kappa = \kappa_\alpha - \kappa_\beta$ where $\kappa_\alpha(M_A) \subset M_C \otimes_C A$ and $\kappa_\beta(M_B) \subset M_C \otimes_C B$. By exactness $\kappa_\alpha \circ \iota = \kappa_\beta \circ \iota$, so we may use these two maps to define a natural transformation

$$\mathrm{Id} \longrightarrow s_C \circ u_C .$$

If we apply F' to this natural transformation, we obtain a ladder diagram

$$
\begin{tikzcd}
0 \arrow[d] & \\
T(\iota)/M \arrow[d] \arrow[dr] & 0 \arrow[d] \\
\left(\dfrac{[\Delta^1]\mathbf{Z}\otimes M_A}{[0]\mathbf{Z}\otimes M_A}\right)\otimes_A R \oplus \left(\dfrac{[\Delta^1]\mathbf{Z}\otimes M_B}{[0]\mathbf{Z}\otimes M_B}\right)\otimes_B R \arrow[d] \arrow[dr] & T(\Delta)/M_C \arrow[d] \\
B(M_C)\otimes_C R \arrow[d] \arrow[dr, "="] & \left(\dfrac{[\Delta^1]\mathbf{Z}\otimes M_C}{[0]\mathbf{Z}\otimes M_C}\right)\otimes_C A\otimes_A R \oplus \left(\dfrac{[\Delta^1]\mathbf{Z}\otimes M_C}{[0]\mathbf{Z}\otimes M_C}\right)\otimes_C B\otimes_B R \arrow[d] \\
0 & B(M_C)\otimes_C R \arrow[d] \\
 & 0
\end{tikzcd}
$$

Applying the five lemma to the ladder, we see that the diagram displays a v-equivalence of Mayer-Vietoris presentations, so that on the space $|vS_\bullet\mathcal{MV}|$ we have $F'_* \simeq (F' \circ s_C \circ u_C)*$.

To summarize what we have learned so far, we have

$$
\begin{aligned}
(s_A \circ u_A)_* \vee (s_B \circ u_B)_* &\simeq F_* \\
&\simeq \mathrm{Id}_* \vee F'_* \\
&\simeq \mathrm{Id}_* \vee (F' \circ s_C \circ u_C)_*
\end{aligned}
$$

Recall that $hS_\bullet\mathcal{M}(C)_f$ is a connected H-space with respect to $\vee$, and that suspension induces a homotopy inverse with respect to the H-space operation [9, page 349]. Therefore, we may shift $(F' \circ s_C \circ u_C)_*$ to the other side of the homotopy, obtaining finally

$$(s_A \circ u_A)_* \vee (s_B \circ u_B)_* \vee (F' \circ s_C \circ (-1) \circ u_C)_* \simeq \mathrm{Id}_*$$

It follows that u_* has a left homotopy inverse, as required.
□

Concerning this proof we make two remarks. First, we invite the reader who likes to play with functors to verify the formula

$$(F' \circ s_C)_* \simeq (s_A \circ \alpha)_* \vee (s_B \circ \beta)_*.$$

Second, we call attention to the fact that this proof is purely K-theoretical, making use only of functors built into the definitions of the diagram categories we are using, mapping cylinders, and the additivity theorem. We will see in [5] that essentially the same argument proves a parallel result for simplicial rings, once the appropriate definitions are made.

3 Approximation theorem

Now we recall the approximation theorem (Theorem 1.6.7 on page 354 of [9]). This theorem gives conditions under which an exact functor between two categories with cofibrations and weak equivalences induces an equivalence at the level of the K-theories of the two categories. Roughly speaking, if the functor passes to an equivalence of homotopy categories, then it induces an equivalence of the K-theories.

Specifically, let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor of categories with cofibrations and weak equivalences. One says that F has the *approximation property* if it satisfies these two conditions:

> *App* 1: An arrow in $\mathcal{A}$ is a weak equivalence if and only if its image in $\mathcal{B}$ is a weak equivalence.
>
> *App* 2: Given any object A in $\mathcal{A}$ and any map $x : F(A) \longrightarrow B$ in $\mathcal{B}$, there exist a cofibration $a : A \longrightarrow A'$ and a weak equivalence $x' : F(A') \longrightarrow B$ in $\mathcal{B}$ such that the following triangle commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(a)"] \arrow[d, "x"'] & F(A') \arrow[dl, "x'"] \\ B & \end{tikzcd}$$

The approximation theorem (Compare [9, page 354].) reads as follows:

Theorem 5 *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with cofibrations and weak equivalences. Suppose that the weak equivalences in $\mathcal{A}$ and $\mathcal{B}$ satisfy the saturation axiom. Suppose further that $\mathcal{A}$ has a cylinder functor and that weak equivalences in $\mathcal{A}$ satisfy the cylinder axiom. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor and suppose that F has the approximation property. Then the induced maps $w\mathcal{A} \longrightarrow w\mathcal{B}$ and $wS_\bullet\mathcal{A} \longrightarrow wS_\bullet\mathcal{B}$ are homotopy equivalences.*

Before we can get to our applications of the theorem, we state the technical results which are used to verify the difficult condition *App* 2. To prove Theorem 2 concerning the forgetful functor $u_R : \mathcal{MV} \longrightarrow \mathcal{M}_f(R)$, we use the following fact.

Proposition 3.1 *Given a Mayer-Vietoris presentation pair*

$$0 \longrightarrow (M, M') \xrightarrow{\iota} (M_A, M'_A) \otimes_A R \oplus (M_B, M'_B) \otimes_B R \xrightarrow{\kappa} (M_C, M'_C) \otimes_C R \longrightarrow 0$$

and a pair of finite simplicial sets (X, X') together with a pair of maps of simplicial R-modules

$$(f, f') : [X, X']R \longrightarrow (M, M')$$

then there is a Mayer-Vietoris presentation

$$0 \longrightarrow R \xrightarrow{\iota} F_A \otimes_A R \oplus F_B \otimes_B R \xrightarrow{\kappa} F_C \otimes_C R \longrightarrow 0$$

of the free R-module of rank one, where F_A, F_B, and F_C are also free over their respective rings, and an extension of (f, f') to a pair of maps of Mayer-Vietoris presentations, as follows.

$$\begin{array}{ccccccccc} 0 \longrightarrow & [X, X']R & \longrightarrow & [X, X']F_A \otimes_A R \oplus [X, X']F_B \otimes_B R & \longrightarrow & [X, X']F_C \otimes_C R & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & (M, M') & \xrightarrow{\iota} & (M_A, M'_A) \otimes_A R \oplus (M_B, M'_B) \otimes_B R & \xrightarrow{\kappa} & (M_C, M'_C) \otimes_C R & \longrightarrow 0 \end{array}$$

Actually, in all our applications f' will be the zero map, but a little extra generality doesn't hurt.

Proof: The result is easily established by induction up the skeleta of the pair, taking direct sums of Mayer-Vietoris presentations constructed using Lemma 3.3 stated below. Note that this lemma is available only for genuine rings, so it is at this point a restriction in the range of our results appears.
□

To state Lemma 3.3, we need to refer the reader to parts of section one of [8]. First recall the recipe

$$R = C \oplus A' \oplus B' \oplus A' \otimes_C B' \oplus B' \otimes_C A' \oplus A' \otimes_C B' \otimes_C A' \oplus B' \otimes_C A' \otimes_C B' \oplus \cdots$$

from the introduction and write A'_n for the summand which has n factors and A' on the left. Define B'_n similarly, so that

$$R = C \oplus (\oplus_{n\geq 1} A'_n) \oplus (\oplus_{n\geq 1} B'_n).$$

If we define $A'_0 = B'_0 = C$, then we have $A' \otimes_C B'_n = A'_{n+1}$ and $B' \otimes_C A'_n = B'_{n+1}$ for $n \geq 0$. Writing $A'' = \oplus_{n\geq 1} A'_n$ and $B'' = \oplus_{n\geq 1} B'_n$, we can put the recipe in the form

$$R = C \oplus A'' \oplus B''.$$

Using the splittings $A = C \oplus A'$ and $B = C \oplus B'$ and collecting differently, we see that we can also write

$$R = A \oplus A \otimes_C B'' \text{ and } R = B \oplus B \otimes_C A''.$$

If we now choose 1 as the basis element for C and C-bases for A' and B', for which we write $\langle C\rangle$, $\langle A'\rangle$, and $\langle B'\rangle$, respectively, then these choices induce C-bases $\langle A'_n\rangle$, $\langle A''\rangle$, $\langle B'_n\rangle$, and $\langle B''\rangle$, respectively. Then a C-basis for R may be written

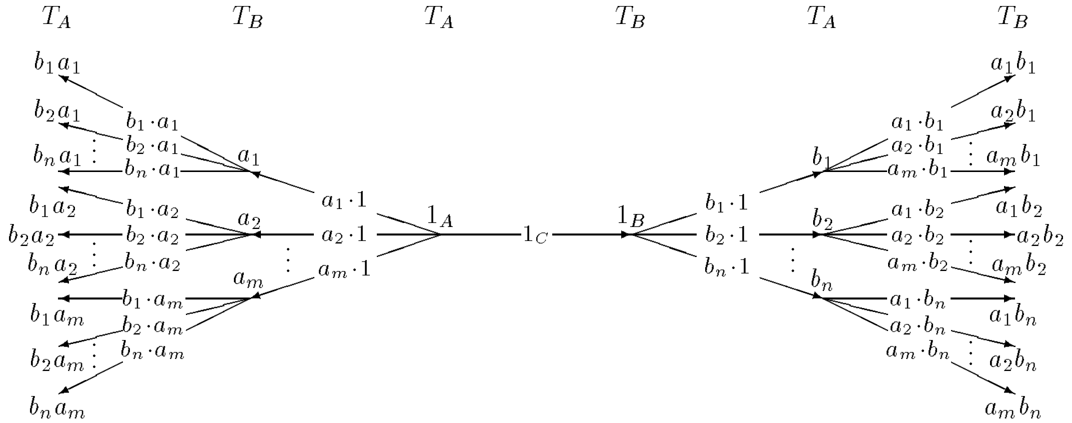
$$\langle R\rangle = \langle C\rangle \cup \langle A''\rangle \cup \langle B''\rangle.$$

Looking at the description $R = A \oplus A \otimes_C B''$, we find a left A-basis for R, which we denote T_A and which is $\langle C\rangle \cup \langle B''\rangle$ when viewed as a subset of R. Similarly we find and identify a left B-basis T_B. In [8] the three sets T_A, T_B, and $\langle R\rangle$ are organized into a directed tree T with a basepoint. The disjoint union of the A-basis T_A and the B-basis T_B forms the set of vertices and the C-basis $\langle R\rangle$ forms the set of edges of T. Notice that the multiplicative identity in R appears in three rôles: as $1_C \in \langle R\rangle$, as $1_A \in T_A$, and as $1_B \in T_B$. We have defined the bases in such a way that there are a bijection $\langle A\rangle \times T_A \longrightarrow \langle R\rangle$, written $(a, x) \mapsto a \cdot x$ on an element of the cartesian product, and a similar bijection $\langle B\rangle \times T_B \longrightarrow \langle R\rangle$. Both of these are used in [8] to define incidence and to give orientation to the edges.

An alternative inductive description of the tree T as a tree filtered by subtrees follows. Let T_0 consist of the vertices $1_A \in T_A$ and $1_B \in T_B$ connected by the edge $1_C \in \langle R\rangle$. We make the arbitrary choice that the edge is oriented from 1_A to 1_B. This then fixes the vertex 1_A as the basepoint, or root. Proceeding inductively, suppose that we have defined an increasing union

$$T_0 \subset T_1 \subset \cdots \subset T_n,$$

so that the extremal vertices of T_n are the disjoint union of $\langle A'_n\rangle \subset T_B$ and $\langle B'_n\rangle \subset T_A$. We now want to extend T_n by adding and connecting the vertices $\langle A'_{n+1}\rangle$ and $\langle B'_{n+1}\rangle$. To see how to do this, we use the formulas $A' \otimes_C B'_n = A'_{n+1}$ and $B' \otimes_C A'_n = B'_{n+1}$ for $n \geq 0$ given above. For $a \in \langle A'\rangle$ and $x \in \langle B'_n\rangle$ we connect the vertex x to the vertex $ax \in \langle A'_{n+1}\rangle$ by a segment $a \cdot x \in \langle R\rangle$. The segment is oriented so as to run from x to ax. The following figure illustrates the inclusions $T_0 \subset T_1 \subset T_2$.



In the context of Mayer-Vietoris presentations of ordinary, i.e., not simplicial, modules we have the following version of Proposition 1.1 of [8].

Proposition 3.2 *Let N be the free right R-module on the basis element n. Let Δ be a finite subtree of T, containing the basepoint. Then Δ determines a canonical Mayer-Vietoris presentation of N,*

$$0 \longrightarrow N \xrightarrow{\iota} N_A \otimes_A R \oplus N_B \otimes_B R \xrightarrow{\kappa} N_C \otimes_C R \longrightarrow 0,$$

in which the modules N_A, N_B, and N_C are also free. We denote this presentation by $\langle N, n, \Delta \rangle$. Also, given $m \in M$, there exists a map of $\langle N, n, \Delta \rangle$ into the Mayer-Vietoris presentation

$$0 \longrightarrow M \xrightarrow{\iota} M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R \longrightarrow 0$$

inducing $n \to m$ if and only if Δ contains a certain finite tree $\Delta(m)$. The entire map is uniquely determined by m.

One sees immediately that if one defines a category of such Mayer-Vietoris presentations $\langle N, n, \Delta \rangle$ over a fixed Mayer-Vietoris presentation pointed by $m \in M$, then the category has a terminal object, corresponding to the tree $\Delta(m)$, and that finite limits in the category may be constructed by taking the Mayer-Vietoris presentation corresponding to the union of the subtrees of T occuring in the constituents of a diagram. Using these remarks, it is now easy to prove the following lemma.

Lemma 3.3 *Suppose that*

$$0 \longrightarrow M \xrightarrow{\iota} M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R \longrightarrow 0$$

is a given Mayer-Vietoris presentation of ordinary modules and that $\langle N_i, n_i, \Delta_i \rangle$, $1 \leq i \leq p$ is a family of Mayer-Vietoris presentations mapping to the given presentation, with $n_i \to m_i$. Then there is a single Mayer-Vietoris presentation of the free module on the generator n $\langle N, n, \Delta \rangle$ admitting p maps to the given presentation, such that the ith map carries n to m_i.

Proof: We put $\Delta = \cup_{i=1}^{p} \Delta_i$, a subtree of T, and construct the corresponding Mayer-Vietoris presentation. By construction Δ contains $\Delta(m_i)$ for each i, so by Proposition 3.2 the desired maps exist.
□

Consequently, there is a standard Mayer-Vietoris presentation of the free R-module of rank p mapping to the given presentation such that preferred generators are mapped respectively to the elements m_1, …, m_p of M.

In section 5 we verify *App* 2 for the inclusion functor $i : \mathcal{S} \longrightarrow \mathcal{MV}^w$ using the following result.

Proposition 3.4 *Let*

$$(0, M_A, M_B, M_C) \longrightarrow (N, N_A, N_B, N_C)$$

be a cofibration of a split module into a Mayer-Vietoris presentation in which N is contractible. Let $n \geq 0$ and let

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{f} & M_C \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow{g} & N_C \end{array}$$

be a commutative diagram of simplicial maps. Then there is a homotopy of g relative to the boundary to a map g', a free split (ordinary) module $(0, F_A, F_B, F_C)$, and maps of Mayer-Vietoris presentations

$$\begin{array}{ccc} [\partial\Delta^n](0, F_A, F_B, F_C) & \longrightarrow & (0, M_A, M_B, M_C) \\ \Big\downarrow & & \Big\downarrow \\ [\Delta^n](0, F_A, F_B, F_C) & \longrightarrow & (N, N_A, N_B, N_C) \end{array}$$

such that the diagram commutes and $g'(\Delta^n) \subset \mathrm{Im}(F_C \otimes_C R)$.

Also, if f maps a horn $\Lambda \subset \partial\Delta^n$ to zero, then we may assume that the restriction to $[\Lambda](0, F_A, F_B, F_C)$ of the upper horizontal map in the diagram is also zero.

The tools for proving the lemma are some simplicial homotopy theory and the following version of Proposition 1.2 of [8, page 144].

First we recall the based augmented trees ${}^+\Delta$ of [8, page 144]. One way to think of these objects is to consider the directed graph $T - \{1_C\}$, which consists of two trees. We may define an A-tree to be a finite subtree of $T - \{1_C\}$ with its base point at 1_A. A B- tree is defined similarly. If Δ is an A- or a B-tree, then we call ${}^+\Delta = \Delta \cup \{1_C\}$ a based augmented tree, with the extra edge 1_C being called the augmentation segment. We write ${}_A\Delta$ for a typical based augmented A-tree and ${}_B\Delta$ for a based augmented B-tree.

Proposition 3.5 *To any finite based augmented tree ${}^+\Delta$ there is canonically associated a split module $\langle {}^+\Delta\rangle$ of the form*

$$0 \longrightarrow 0 \longrightarrow F_A \otimes_A R \oplus F_B \otimes_B R \xrightarrow{\cong} F_C \otimes_C R \longrightarrow 0$$

in which F_A, F_B, and F_C are free. Given a Mayer-Vietoris presentation of ordinary modules

$$0 \longrightarrow M \xrightarrow{\iota} M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R \longrightarrow 0$$

and an element $m' \in M_C$, then there exist finite based augmented trees ${}_A\Delta$ and ${}_B\Delta$ and a map from the direct sum of the split modules $\langle {}_A\Delta\rangle \oplus \langle {}_B\Delta\rangle$ to the given Mayer-Vietoris presentation such that m' is contained in the image.

□

Although the argument in [8] for this result is not presented as such, it is actually an algorithm for constructing from an equation

$$\kappa(\sum_{x \in T_A} m_x \otimes x, \sum_{y \in T_B} m_y \otimes y) = m'$$

based augmented trees ${}_A\Delta$ and ${}_B\Delta$ and the required map into the Mayer-Vietoris presentation. By means of this algorithm we are provided with just enough naturality to produce the simplicial map we are looking for. That all our basic data is generated by one simplex is essential. To prove that the restriction to the horn of the map we construct is trivial also requires an analysis based on the algorithm. We therefore recommend that the reader review the proof of Proposition 3.5 in [8] with these points in mind. The reader may also find it helpful to refer to our picture of part of the tree T above while doing so.

We also remark that Proposition 3.5 concerns maps into Mayer-Vietoris presentations, whereas Proposition 1.2 of [8] addresses a more general situation.

Proof of 3.4: The case $n = 0$ of 3.4 is a trivial consequence of the statement of Proposition 3.5 and no homotopy is required; we explain this in the course of the proof of Theorem 3.

Now suppose $n \geq 1$. First use the isomorphism $M_A \otimes_A R \oplus M_B \otimes_B R \cong M_C \otimes_C R$ and the given map of the split module into the Mayer-Vietoris presentation to produce

$$f_1 : \partial\Delta^n \longrightarrow N_A \otimes_A R \oplus N_B \otimes_B R$$

lifting the composite map

$$\partial\Delta^n \longrightarrow M_C \otimes_C R \longrightarrow N_C \otimes_C R.$$

On the other hand,

$$0 \longrightarrow N \xrightarrow{\iota} N_A \otimes_A R \oplus N_B \otimes_B R \xrightarrow{\kappa} N_C \otimes_C R \longrightarrow 0$$

is a fibration and $\Lambda \subset \Delta^n$ is a trivial cofibration, so we may pick a lifting

$$F_0 : \Delta^n \longrightarrow N_A \otimes_A R \oplus N_B \otimes_B R$$

of the map $\Delta^n \longrightarrow N_C \otimes_C R$ extending $f_1|\Lambda$. Put $f_0 = F_0|\partial\Delta^n$. By construction $f_1 - f_0$ is a map to the fibre N, which is contractible and a Kan set, so that $f_1 - f_0$ is simplicially null homotopic inside of N by a homotopy which is zero on Λ. Translating the homotopy, f_1 and f_0 are simplicially fibrewise homotopic relative to Λ. Paste the homotopy onto Δ^n to obtain

$$\Delta^n \cup (\partial\Delta^n \times \Delta^1) \longrightarrow N_A \otimes_A R \oplus N_B \otimes_B R$$

extending F_0 and extend this map to

$$H : \Delta^n \times \Delta^1 \longrightarrow N_A \otimes_A R \oplus N_B \otimes_B R$$

by the Kan extension property. Let $F_1 = H|\Delta^n \times 1$ and put $g' = \kappa \circ F_1$. We have a commutative diagram

$$\begin{tikzcd}
& \partial\Delta^n \arrow[dl, "f_1"'] \arrow[dr, "g'|"] \arrow[dd] & \\
M_A \otimes_A R \oplus M_B \otimes_B R \arrow[rr] \arrow[dd] & & M_C \otimes_C R \arrow[dd] \\
& \Delta^n \arrow[dl, "F_1"'] \arrow[dr, "g'"] & \\
N_A \otimes_A R \oplus N_B \otimes_B R \arrow[rr, "\kappa"] & & N_C \otimes_C R
\end{tikzcd}$$

and, by construction, g' is homotopic relative to the boundary to the original map g. Moreover, if $g'|\Lambda = g|\Lambda$ is the zero map, then $F_1|\Lambda$ is also zero, since $M_A \otimes_A R \oplus M_B \otimes_B R \longrightarrow M_C \otimes_C R$ is an isomorphism.

Now we can apply the algorithm described on page 145 of [8] to the lifting

$$F_1(\sigma^n) = (\sum_{x \in T_A} n_x \otimes x, \sum_{y \in T_B} n_y \otimes y)$$

of $g'(\sigma^n)$, σ^n being the generating simplex of Δ^n. We obtain a split module $(0, F_A, F_B, F_C)$ and a map to the n-simplices of (N, N_A, N_B, N_C) with $g'(\sigma^n)$ in the image. We must now extend the split module $(0, F_A, F_B, F_C)$ to a simplicial split module $[\Delta^n](0, F_A, F_B, F_C)$, and we must also extend the map. For this we require the naturality implied by using the proof of Proposition 3.5 as described in [8] as an algorithm. Now, for any order-preserving map $\alpha : [k] \longrightarrow [n]$,

$$\alpha^* F_1(\sigma^n) = (\sum \alpha^*(n_x) \otimes x, \sum \alpha^*(n_y) \otimes y)$$

lifts $\alpha^* g'(\sigma^n)$. Application of naturality to these liftings yields a simplicial map of Mayer-Vietoris presentations

$$[\Delta^n](0, F_A, F_B, F_C) = \oplus_k \oplus_{\alpha:[k] \longrightarrow [n]} (0, F_A, F_B, F_C) \longrightarrow (N, N_A, N_B, N_C)$$

with $g'(\Delta^n)$ in the image. We see that the restriction to $[\partial\Delta^n](0, F_A, F_B, F_C)$ factors through the split module $(0, M_A, M_B, M_C)$ by making use of the preceding diagram.

To conclude, we need to see that if $g'(\Lambda) = g(\Lambda) = 0$, then $[\Lambda](0, F_A, F_B, F_C)$ also maps to zero. This actually requires a look at the algorithm itself. Recall that we have seen earlier that the hypothesis implies $F_1(\Lambda) = 0$, so that $[\Lambda]F_A$ and $[\Lambda]F_B$ map to zero, by following the algorithm. To see that $[\Lambda]F_C$ maps to zero, let $(0, \gamma_A, \gamma_B, \gamma_C)$ be the components of the map $[\Delta](0, F_A, F_B, F_C) \longrightarrow (N, N_A, N_B, N_C)$. Now let s be an edge of one of the trees ${}_A\Delta$ or ${}_B\Delta$, and suppose s is extremal, incident with a terminal vertex $x \in T_A$, say. By the definition of the incidence relation in T and the orientation of the tree T, $s = 1 \otimes x$. Consequently, by definition of κ, $\kappa(n_x) = n_s$, so that

$$\begin{aligned}
\gamma_C(\Lambda \cdot n_s) &= \gamma_C(\kappa\Lambda \cdot n_x) \\
&= \kappa(\gamma_A \Lambda \cdot n_x) \\
&= \kappa(0) = 0.
\end{aligned}$$

Now suppose that s is no longer an extremal edge, but that we know that $\gamma_C(\Lambda \cdot n_{s'}) = 0$ for all edges s' closer to a terminal vertex. Suppose again that the terminal vertex is $x \in T_A$, and let x be the initial vertex of edges $s_i = a_i \otimes x$. By the algorithm $\kappa(n_x) = n_s + \sum n_{s_i} \otimes a_i$, so that

$$\begin{aligned} 0 &= \kappa\gamma_A(\Lambda \cdot n_x) \\ &= (\gamma_C \otimes A)\kappa(\Lambda \cdot n_x) \\ &= (\gamma_C \otimes A)(\Lambda \cdot n_s \otimes 1 + \sum \Lambda \cdot n_{s_i} \otimes a_i) \\ &= \gamma_C(\Lambda \cdot n_s) \otimes 1 + \sum \gamma_C(\Lambda \cdot n_{s_i}) \otimes a_i \\ &= \gamma_C(\Lambda \cdot n_s) \otimes 1, \end{aligned}$$

by inductive hypothesis. Of course, similar arguments go through if we have vertices in T_B involved.
□

In both of the following sections we make use of the following well known lemma. A convenient reference is [2, page 369].

Lemma 3.6 *Let*

$$0 \to Z_k \to C_k \to C_{k-1} \to \cdots \to C_0 \to 0$$

be an exact sequence of modules in which C_j is free and finitely generated for all j. Then Z_k is stably-free and finitely generated.

□

4 Coarse equivalences and R-modules

In this section we take up the identification of the K-theory of the Mayer-Vietoris presentations with respect to the coarse notion of weak equivalences.

Theorem 2 *The forgetful functor*

$$u_R : \mathcal{MV} \longrightarrow \mathcal{M}_f(R)$$

induces a homotopy equivalence

$$wS_\bullet\mathcal{MV} \longrightarrow hS_\bullet\mathcal{M}_f(R).$$

Proof : All the categorical hypotheses of Theorem 5 (the approximation theorem) are satisfied by $\mathcal{MV}$ and $\mathcal{M}_f(R)$, where we take w-equivalences in $\mathcal{MV}$. We have to verify that u_R has the approximation property. Condition *App* 1 is obviously satisfied so we must concern ourselves with the factorization condition *App* 2.

Here we are given an arbitrary Mayer-Vietoris presentation

$$0 \longrightarrow M \xrightarrow{\iota} M_A \otimes_A R \oplus M_B \otimes_B R \xrightarrow{\kappa} M_C \otimes_C R \longrightarrow 0$$

and an arbitrary R-map

$$f : M \longrightarrow N,$$

and we must construct a cofibration of Mayer-Vietoris presentations

$$(c, \ldots) : (M, M_A, M_B, M_C, \iota, \kappa) \longrightarrow (M', M'_A, M'_B, M'_C, \iota, \kappa)$$

and a homotopy equivalence

$$f' : M' \longrightarrow N$$

so that

$$f' \circ c = f.$$

If the map

$$f_* : \pi_0 M \longrightarrow \pi_0 N$$

is not onto, then we may find a constant simplicial R-module F, such that F_0 is free and finitely generated over R, and a map

$$f^0 : M \oplus F \longrightarrow N$$

such that

$$f^0_* : \pi_0(M \oplus F) \longrightarrow \pi_0 N$$

is onto. Put $M^0 = M \oplus F$ and let $N^0 = T(f^0)$. Also let $g^0 : M^0 \longrightarrow N^0$ be the front inclusion and let $p^0 : N^0 \longrightarrow N$ be the back projection. At this point we have a commutative square

$$\begin{tikzcd} M \arrow[r, "c^0"] \arrow[d, "f"'] & M^0 \arrow[d, "g^0"] \arrow[dl, "f^0"'] \\ N & N^0 \arrow[l, "p^0"'] \end{tikzcd}$$

in which c^0 and g^0 are cofibrations and p^0 is a homotopy equivalence. It remains to extend the Mayer-Vietoris presentation of M to a Mayer-Vietoris presentation of M^0. For this, simply note that F_0 itself may be presented in many ways as

$$0 \longrightarrow F_0 \xrightarrow{\iota} (F_A)_0 \otimes_A R \oplus (F_B)_0 \otimes_B R \xrightarrow{\kappa} (F_C)_0 \otimes_C R \longrightarrow 0$$

Extend the modules $(F_A)_0$, $(F_B)_0$, and $(F_C)_0$ to constant simplicial modules F_A, F_B, and F_C, and the maps ι and κ to constant simplicial maps, and put $M^0_? = M_? \oplus F_?$ for $? = A$, B, or C, and let $c^0_? : M_? \longrightarrow M^0_?$ be the inclusion of the first summand. Obviously

$$0 \longrightarrow M^0 \xrightarrow{\iota} M^0_A \otimes_A R \oplus M^0_B \otimes_B R \xrightarrow{\kappa} M^0_C \otimes_C R \longrightarrow 0$$

is a Mayer-Vietoris presentation, and the quadruple $(c^0, c^0_A, c^0_B, c^0_C)$ is a cofibration of Mayer-Vietoris presentations

$$(c^0, \ldots) : (M, M_A, M_B, M_C, \iota, \kappa) \longrightarrow (M^0, M^0_A, M^0_B, M^0_C, \iota, \kappa).$$

Now let $n \geq 0$ and assume that we have constructed a diagram

$$\begin{tikzcd} M \arrow[r, "c^n"] \arrow[d, "f"'] & M^n \arrow[d, "g^n"] \arrow[dl, "f^n"'] \\ N & N^n \arrow[l, "p^n"'] \end{tikzcd}$$

in which $f^n : M^n \longrightarrow N$ is an n-connected map and where c^n is part of a cofibration of Mayer-Vietoris presentations

$$(c^n, \ldots) : (M, M_A, M_B, M_C, \iota, \kappa) \longrightarrow (M^n, M^n_A, M^n_B, M^n_C, \iota, \kappa).$$

We take $N^n = T(f^n)$ to be the mapping cylinder of f^n, and we let g^n and p^n be the front inclusion and back projection of the mapping cylinder, respectively. We say that f^n is n-connected if the pair $(N^n, g^n(M^n))$ is n-connected. This means, in turn, that every component of N^n meets a component of $g^n(M^n)$ and, for any basepoint in M^n, the relative homotopy groups $\pi_k(N^n, g^n(M^n)) = 0$ for $1 \leq k \leq n$. However, the homotopy groups of a simplicial abelian group relative to an arbitrary base point are isomorphic to those relative to the basepoint zero. Therefore, in this section and in the next, all homotopy groups are defined with respect to the zero basepoint.

To return to the argument, we have

$$0 \longrightarrow M^n \xrightarrow{g^n} N^n \longrightarrow N^n/M^n \longrightarrow 0$$

a simplicial fibration of R-modules over a connected base N^n/M^n and $\pi_k(N^n/M^n) = 0$ for $0 \leq k \leq n$. We find that the R-module $\pi_{n+1}(N^n/M^n)$ is a quotient of a finitely generated stably free R-module by 3.6, so that finitely many simplicial maps

$$\Delta^{n+1}/\partial\Delta^{n+1} \longrightarrow N^n/M^n$$

suffice to represent a set of R-module generators for the group. Consider any one of these maps as a map of pairs

$$(\Delta^{n+1}, \partial\Delta^{n+1}) \longrightarrow (N^n/M^n, 0)$$

and let Λ denote the horn of the simplex formed by taking all faces except $d_0(\sigma_{n+1})$. By the lifting property for simplicial fibrations we may lift such an arrow to the diagonal one in the following diagram

$$\begin{tikzcd}
\Lambda \arrow[r] \arrow[d] & \{0\} \subset N^n \arrow[d] \\
\Delta^{n+1} \arrow[r] \arrow[ur] & N^n/M^n.
\end{tikzcd}$$

This procedure transforms a single map into a diagram of maps

$$\begin{array}{ccc}
(\partial\Delta^{n+1}, \Lambda) & \longrightarrow & (M^n, 0) \\
\downarrow & & \downarrow \\
(\Delta^{n+1}, \Lambda) & \longrightarrow & (N^n, 0)
\end{array}$$

Since $\pi_{n+1}(N^n/M^n)$ is finitely generated as an R-module, we may choose a finitely generated free module F^n and construct a diagram of simplicial R-modules and homomorphisms

$$\begin{array}{ccc}
(\partial\Delta^{n+1}, \Lambda)F^n & \longrightarrow & (M^n, 0) \\
\downarrow & & \downarrow \\
(\Delta^{n+1}, \Lambda)F^n & \longrightarrow & (N^n, 0).
\end{array}$$

such that the induced map on relative homotopy groups is surjective. Then the usual coning-off process displayed in the following pushout diagram

$$\begin{array}{ccc}
\left[\partial\Delta^{n+1}\right] F^n / [\Lambda]\, F^n & \xrightarrow{\varphi} & M^n \\
\downarrow & & \downarrow \\
\left[\Delta^{n+1}\right] F^n / [\Lambda]\, F^n & \xrightarrow{\psi} & M^{n+1},
\end{array}$$

defines the R-module M^{n+1} and a cofibration $M^n \longrightarrow M^{n+1}$. The universal property of the pushout yields an $(n+1)$-connected map $f^{n+1} : M^{n+1} \longrightarrow N$ which may be factored over its mapping cylinder $T(g^{n+1}) = N^{n+1}$ as before, so that we now have the commuting diagram

$$\begin{tikzcd}
M \arrow[r, "c^{n+1}"] \arrow[d, "f"'] & M^{n+1} \arrow[d, "g^{n+1}"] \arrow[dl, "f^{n+1}"'] \\
N & N^{n+1} \arrow[l, "p^{n+1}"']
\end{tikzcd}$$

in which the right-hand vertical arrow $g^{n+1} : M^{n+1} \longrightarrow N^{n+1}$ is an $(n+1)$-connected cofibration.

It remains to extend the Mayer-Vietoris presentation $(M^n, M^n_A, M^n_B, M^n_C, \iota, \kappa)$ to a Mayer-Vietoris presentation of M^{n+1}. Since maps into Mayer-Vietoris presentations may be added, it suffices to explain how this works with F^n of rank one. In this case we have a diagram

$$\begin{array}{cccccccc}
 & [\partial\Delta^{n+1}]R/[\Lambda]R & & & & & & \\
 & \downarrow & & & & & & \\
0 \longrightarrow & M^n & \xrightarrow{\iota} & M^n_A \otimes_A R \oplus M^n_B \otimes_B R & \xrightarrow{\kappa} & M^n_C \otimes_C R & \longrightarrow 0 &
\end{array}$$

which we would like to complete to the following diagram

$$\begin{tikzcd}
0 \arrow[r] & {[\Delta^{n+1}]R} \arrow[r] & {[\Delta^{n+1}]F^n_A \otimes_A R \oplus [\Delta^{n+1}]F^n_B \otimes_B R} \arrow[r] & {[\Delta^{n+1}]F^n_C \otimes_C R} \arrow[r] & 0 \\
0 \arrow[r] & {[\partial\Delta^{n+1}]F^n} \arrow[r] \arrow[u] \arrow[d] & {[\partial\Delta^{n+1}]F^n_A \otimes_A R \oplus [\partial\Delta^{n+1}]F^n_B \otimes_B R} \arrow[r] \arrow[u] \arrow[d] & {[\partial\Delta^{n+1}]F^n_C \otimes_C R} \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & M^n \arrow[r] & M^n_A \otimes_A R \oplus M^n_B \otimes_B R \arrow[r] & M^n_C \otimes_C R \arrow[r] & 0
\end{tikzcd}$$

(For simplicity we have omitted the horn $[\Lambda]R$ from the diagram, as it just maps to zero.) To obtain the downwardpointing arrows we just apply Proposition 3.1, and to obtain the upward-pointing arrows we apply the naturality of the construction $[-]$M in the space variable. Then the cofibration

$$(c^{n+1},\ldots):(M,M_A,M_B,M_C,\iota,\kappa)\longrightarrow(M^{n+1},M_A^{n+1},M_B^{n+1},M_C^{n+1},\iota,\kappa)$$

is gotten from the pushout construction in the category $\mathcal{MV}$.

Since M and N are finite dimensional, inductively applying the preceding construction eventually yields a cofibration of Mayer Vietoris presentations

$$(c^d,\ldots):(M,M_A,M_B,M_C,\iota,\kappa)\longrightarrow(M^d,M_A^d,M_B^d,M_C^d,\iota,\kappa)$$

and a situation

$$\begin{tikzcd}
M \arrow[r, "c^d"] \arrow[d, "f"'] & M^d \arrow[d, "g^d"] \arrow[dl, "f^d"'] \\
N & N^d \arrow[l, "p^d"']
\end{tikzcd}$$

as above in which all nondegenerate simplices of M^d and N are of dimension less than or equal to d and f^d is d-connected. Therefore $\pi_n(N)=0$ for $n\geq d+1$, and the only nonvanishing homotopy group of N^d/M^d is $\pi_{d+1}(N^d/M^d)$, arising as a kernel as in Lemma 3.6. From the homotopy sequence of the fibration

$$0\longrightarrow M^d\longrightarrow N^d\longrightarrow N^d/M^d\longrightarrow 0$$

we extract a short exact sequence

$$0\longrightarrow\pi_{d+1}(N^d/M^d)\longrightarrow\pi_d M^d\longrightarrow\pi_d N^d\longrightarrow 0.$$

By Lemma 3.6 the R-module $\pi_{d+1}(N^d/M^d)$ will be stably free, so we may choose a natural number k and an isomorphism

$$R^l\xrightarrow{\cong}\pi_{d+1}(N^d/M^d)\oplus R^k.$$

Construct $\overline{M}^d$ by taking the pushout

$$\begin{array}{ccc}
\left[\partial\Delta^d\right]R^k & \longrightarrow & M^d \\
\downarrow & & \downarrow \\
\left[\Delta^d\right]R^k & \longrightarrow & \overline{M}^d
\end{array}$$

where the upper horizontal map is zero. Extend $M^d\longrightarrow N^d$ to $\overline{M}^d\longrightarrow N^d$ by mapping the newly attached cells to zero. As before, we extend the Mayer-Vietoris presentation of M^d to one of $\overline{M}^d$ and consider the revised situation

$$\begin{tikzcd}
M \arrow[r, "\bar{c}^d"] \arrow[d, "f"'] & \overline{M}^d \arrow[d, "\bar{g}^d"] \arrow[dl, "\bar{f}^d"'] \\
N & \overline{N}^d \arrow[l, "\bar{p}^d"']
\end{tikzcd}.$$

The segment of the homotopy sequence is now

$$\begin{array}{ccccccc} & & R^l & & & & \\ & & \downarrow & & & & \\ 0 \longrightarrow & & \pi_{d+1}(\overline{N}^d/\overline{M}^d) & \longrightarrow \pi_d \overline{M}^d \longrightarrow & \pi_d \overline{N}^d & \longrightarrow 0 \end{array}$$

in which the vertical map is an isomorphism. If we use this isomorphism to pick a minimal number of generators for $\pi_{d+1}(\overline{N}^d/\overline{M}^d)$, add $(d+1)$-cells to $\overline{M}^d$ accordingly, and extend the map to N, then a calculation using the exact homotopy sequence of the triple $(\overline{N}^d, M^{d+1}, \overline{M}^d)$ shows that we kill exactly this last kernel without introducing any homotopy in the next higher dimension, thereby producing an equivalence $M^{d+1} \longrightarrow N$. As before, we extend the Mayer-Vietoris presentation of $\overline{M}^d$ to one of M^{d+1}. At this point the improvement procedure has produced the required data, so we terminate the construction.

The proof that the forgetful functor u_R satisfies *App* 2 is then complete. By application of theorem 5 we conclude that the forgetful functor u_R induces a homotopy equivalence

$$wS_\bullet \mathcal{MV} \longrightarrow hS_\bullet \mathcal{M}_f(R),$$

as claimed.
□

5 K-theory of split modules

Let $\mathcal{S}$ denote the subcategory of Mayer-Vietoris presentations of the trivial R-module, also known as the category of split modules. One of our standard Mayer-Vietoris presentations is in $\mathcal{S}$ if and only if the map κ is an isomorphism. $\mathcal{MV}^w$ is the subcategory of the category of Mayer-Vietoris presentations in which the R-module M is contractible. We are going to prove the following theorem.

Theorem 3 *The inclusion functor*

$$i : \mathcal{S} \longrightarrow \mathcal{MV}^w$$

induces a homotopy equivalence

$$vS_\bullet \mathcal{S} \longrightarrow vS_\bullet \mathcal{MV}^w.$$

Proof: This is also an application of Theorem 5, the approximation theorem. Once again the condition *App* 1 is satisfied by definition of the class of weak equivalences v, so we have to check *App* 2. In this situation we are given a map $(0, f_A, f_B, f_C)$ of Mayer-Vietoris presentations

$$\begin{array}{ccccccc} 0 \longrightarrow & 0 & \longrightarrow & M_A \otimes_A R \oplus M_B \otimes_B R & \longrightarrow & M_C \otimes_C R & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & N & \longrightarrow & N_A \otimes_A R \oplus N_B \otimes_B R & \longrightarrow & N_C \otimes_C R & \longrightarrow 0 \end{array}$$

in which N is contractible, and we seek a cofibration in $\mathcal{S}$

$$(0, c_A, c_B, c_C) : (0, M_A, M_B, M_C) \longrightarrow (0, M'_A, M'_B, M'_C)$$

and a v-equivalence

$$(0, f'_A, f'_B, f'_C) : (0, M'_A, M'_B, M'_C) \longrightarrow (N, N_A, N_B, N_C)$$

such that

$$(0, f_A, f_B, f_C) = (0, f'_A, f'_B, f'_C) \circ (0, c_A, c_B, c_C).$$

This we do stepwise, as before.

Given a finite set of r classes in $\pi_0(N_C)$ which represent generators of $\pi_0(N_C)$ modulo the image of $\pi_0(M_C)$, we may represent each class by a map $\Delta^0 \longrightarrow N_C$. Consider the diagram

$$\begin{array}{ccccccc} & & & & & [\Delta^0]C^r \otimes_C R & \\ & & & & & \downarrow & \\ 0 \longrightarrow & N & \longrightarrow & N_A \otimes_A R \oplus N_B \otimes_B R & \longrightarrow & N_C \otimes_C R & \longrightarrow 0. \end{array}$$

Applying Proposition 3.4 r times in the case $n = 0$, taking direct sums, and then extending the domain to a constant simplicial module, we obtain the following map

$$\begin{array}{ccccccccc} 0 \longrightarrow & 0 & \longrightarrow & F_A \otimes_A R \oplus F_B \otimes_B R & \longrightarrow & F_C \otimes_C R & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & N & \longrightarrow & N_A \otimes_A R \oplus N_B \otimes_B R & \longrightarrow & N_C \otimes_C R & \longrightarrow 0 \end{array}$$

whose domain is a split module, and such that the given set of classes in $\pi_0(N_C)$ is in the image of $F_C \longrightarrow N_C$.

If we now add this map to the original one, putting $M_A^0 = M_A \oplus F_A$, $M_B^0 = M_B \oplus F_B$, and $M_C^0 = M_C \oplus F_C$, then we obtain a diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & 0 & \longrightarrow & M_A^0 \otimes_A R \oplus M_B^0 \otimes_B R & \longrightarrow & M_C^0 \otimes_C R & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & N & \longrightarrow & N_A \otimes_A R \oplus N_B \otimes_B R & \longrightarrow & N_C \otimes_C R & \longrightarrow 0, \end{array}$$

and we have extended our original map of Mayer-Vietoris presentations to one for which $\pi_0(M_C^0) \longrightarrow \pi_0(N_C)$ is surjective.

Consequently,

$$(f_C^0 \otimes_C R)_* : \pi_0(M_C^0 \otimes_C R) \longrightarrow \pi_0(N_C \otimes_C R)$$

is also is also surjective, and from the contractibility of N it follows that the other maps $f_A^0 \otimes_A R$ and $f_B^0 \otimes_B R$ also induce surjections on π_0 Finally, since R is free over A and B, the maps

$$f_A^0 : M_A^0 = M_A \oplus F_A \longrightarrow N_A \text{ and } f_B^0 : M_B^0 = M_B \oplus F_B \longrightarrow N_B$$

also induce surjections on π_0. Also, at this stage we can factor $(0, f_A^0, f_B^0, f_C^0)$ through its mapping cylinder, so that it becomes a cofibration, which is convenient for describing higher relative homotopy groups.

Note that in the preceding paragraph we made application of the freeness hypothesis on the original diagram of rings and its consequences for the ring R. At similar points in the rest of the argument we must again make use of the freeness hypothesis.

Let $n \geq 1$ and suppose that we have factored our original map through a cofibration and an $(n-1)$-connected map:

$$(0, f_A, f_B, f_C) = (0, f_A^{n-1}, f_B^{n-1}, f_C^{n-1}) \circ (0, c_A^{n-1}, c_B^{n-1}, c_C^{n-1}).$$

Since we want to factor $(0, f_A, f_B, f_C)$ through a cofibration and a v-equivalence, saying the map $(0, f_A^{n-1}, f_B^{n-1}, f_C^{n-1})$ is $(n-1)$-connected means that the three maps f_A^{n-1}, f_B^{n-1}, and f_C^{n-1} are all $(n-1)$-connected. We will also have that $(0, f_A^{n-1}, f_B^{n-1}, f_C^{n-1})$ itself is the composition

$$(0, f_A^{n-1}, f_B^{n-1}, f_C^{n-1}) = (\mathrm{Id}, p_A^{n-1}, p_B^{n-1}, p_C^{n-1}) \circ (0, g_A^{n-1}, g_B^{n-1}, g_C^{n-1}),$$

where

$$(0, g_A^{n-1}, g_B^{n-1}, g_C^{n-1}) : (0, M_A^{n-1}, M_B^{n-1}, M_C^{n-1}) \longrightarrow (N, N_A^{n-1}, N_B^{n-1}, N_C^{n-1})$$

is an $(n-1)$-connected cofibration and

$$(\mathrm{Id}, p_A^{n-1}, p_B^{n-1}, p_C^{n-1}) : (N, N_A^{n-1}, N_B^{n-1}, N_C^{n-1}) \longrightarrow (N, N_A, N_B, N_C)$$

is a fine equivalence. This factorization $f = p \circ g$ is obtained using the mapping cylinder construction.

Now we need to improve the connectivity in this situation. Interpreting homotopy elements of the base of the fibration

$$0 \longrightarrow M_C^{n-1} \longrightarrow N_C^{n-1} \longrightarrow N_C^{n-1}/M_C^{n-1} \longrightarrow 0$$

as in the proof of Theorem 2, we find again that there are a finitely generated free C-module L_C and a diagram of simplicial maps

$$\begin{array}{ccc} [\partial\Delta^n, \Lambda]\, L_C & \longrightarrow & M_C^{n-1} \\ \downarrow & & \downarrow \\ [\Delta^n, \Lambda]\, L_C & \longrightarrow & N_C^{n-1}, \end{array}$$

where Λ maps to zero, which one may use to improve the connectivity of f_C^{n-1} by adding cells to M_C^{n-1}. But we are required to add cells to the whole Mayer-Vietoris presentation $(0, M_A^{n-1}, M_B^{n-1}, M_C^{n-1})$, so the argument continues as follows.

For simplicity of notation fix attention on the submodule of $\pi_n(N_C^{n-1}, M_C^{n-1})$ generated by a single element, and fix attention on a single component pointed by m, so that the starting situation simplifies to a diagram

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & M_C^{n-1} \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow{g} & N_C^{n-1} \end{array}$$

where the horn Λ is mapped to zero. Apply Proposition 3.4 to change g by a homotopy relative to $\partial\Delta^n$ and to complete the resulting diagram to a diagram of maps of Mayer-Vietoris presentations

$$\begin{array}{ccc} [\partial\Delta^n](0, F_A, F_B, F_C) & \longrightarrow & (0, M_A^{n-1}, M_B^{n-1}, M_C^{n-1}) \\ \Big\downarrow & & \Big\downarrow \\ [\Delta^n](0, F_A, F_B, F_C) & \longrightarrow & (N, N_A^{n-1}, N_B^{n-1}, N_C^{n-1}), \end{array}$$

where $[\Lambda](0, F_A, F_B, F_C)$ is mapped trivially. Take the pushout of the lefthand cofibration along the upper arrow in the category $\mathcal{S}$ and denote it provisionally by $(0, M_A^n, M_B^n, M_C^n)$. Denote the mapping cylinder of the canonical arrow

$$(0, M_A^n, M_B^n, M_C^n) \longrightarrow (N, N_A^{n-1}, N_B^{n-1}, N_C^{n-1})$$

provisionally by (N, N_A^n, N_B^n, N_C^n), and consider the relative homotopy group $\pi_n(N_C^n, M_C^n)$. One sees that the new group is the quotient of the old group $\pi_n(N_C^{n-1}, M_C^{n-1})$ by a C- submodule containing that submodule generated by our chosen element. Since $\pi_n(N_C^{n-1}, M_C^{n-1})$ is finitely generated, after finitely many applications of the procedure we have extended $(0, M_A^{n-1}, M_B^{n-1}, M_C^{n-1})$ to a split module $(0, M_A^n, M_B^n, M_C^n)$ such that the relative homotopy group $\pi_n(N_C^n, M_C^n)$ is zero. Consequently,

$$0 = \pi_n(N_C^n \otimes_C R, M_C^n \otimes_C R) \cong \pi_n(N_A^n \otimes_A R, M_A^n \otimes_A R) \oplus \pi_n(N_B^n \otimes_B R, M_B^n \otimes_B R)$$

and we use the freeness of R over B and A to conclude that $\pi_n(N_A^n, M_A^n)$ and $\pi_n(N_B^n, M_B^n)$ are also now both zero. This ends the explanation of the inductive step of the improvement procedure.

The finiteness hypothesis on all the modules, together with Lemma 3.6, brings the procedure to a halt. Applying the inductive step and using the finiteness hypothesis, we may assume that the only nonvanishing relative homotopy groups are in dimension $d+1$ and that all of the cells involved in building up the modules M_A^d, M_B^d, M_C^d, N_A^d, N_B^d, and N_C^d are of dimension less than or equal to d. So we have a relative homotopy group

$$\begin{aligned} Z_C &= \pi_{d+1}(N_C^d, M_C^d) \\ &\cong \mathrm{Kernel}(\pi_d(M_C^d) \longrightarrow \pi_d(N_C^d)), \end{aligned}$$

and similarly defined A- and B-modules Z_A and Z_B. By Lemma 3.6 each of these modules is stably free and finitely generated, so we may attach d-cells to $(0, M_A^d, M_B^d, M_C^d)$ by trivial maps and extend the map to (N, N_A^d, N_B^d, N_C^d) by making it zero on the new cells to make first Z_C and then Z_A and Z_B free while they remain finitely generated. Assume that this has been done.

The prodedure for killing these last relative homotopy groups is different from that used in the inductive step, for we are now dealing with kernels. First we have a diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ Z_A \otimes_A R \oplus Z_B \otimes_B R & \xrightarrow{\cong} & Z_C \otimes_C R \\ \downarrow & & \downarrow \\ \pi_d(M_A \otimes_A R) \oplus \pi_d(M_B \otimes_B R) & \xrightarrow{\cong} & \pi_d(M_C \otimes_C R) \\ \downarrow & & \downarrow \\ \pi_d(N_A \otimes_A R) \oplus \pi_d(N_B \otimes_B R) & \xrightarrow{\cong} & \pi_d(N_C \otimes_C R) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

with exact columns, by exactness of all tensor products with R. For any choice of d-simplices $z_1, \ldots,$ z_r of M_C satisfying $d_i z_k = 0$ for all i and k and such that the homotopy classes $[z_1], \ldots, [z_r]$ provide a C-basis for Z_C, we get a map of split modules

$$[\partial\Delta^{d+1}/\Lambda](0, Z_A, Z_B, Z_C) \longrightarrow (0, M_A, M_B, M_C).$$

Besides using the fact that under our freeness hypotheses on A, B, and C a map of split modules has a kernel which is also a split module, we need to know Z_A, Z_B, and Z_C are finitely generated, so that the kernel is indeed in $\mathcal{S}$, but this has been guaranteed by Lemma 3.6. By the lifting argument found at the beginning of the proof of Proposition 3.4, there is a choice $z'_1, \ldots, z'_r$ such that the map just described extends to a diagram

$$\begin{array}{ccc} [\partial\Delta^{d+1}](0, Z_A, Z_B, Z_C) & \longrightarrow & (0, M_A^d, M_B^d, M_C^d) \\ \Big\downarrow & & \Big\downarrow \\ [\Delta^{d+1}](0, Z_A, Z_B, Z_C) & \longrightarrow & (N, N_A^d, N_B^d, N_C^d). \end{array}$$

in which the lefthand vertical arrow is obviously a cofibration. Take the pushout of this arrow along the upper horizontal arrow, obtaining the split module $(0, M_A^{d+1}, M_B^{d+1}, M_C^{d+1})$ as the result. Now we may define $(N, N_A^{d+1}, N_B^{d+1}, N_C^{d+1})$ to be the mapping cylinder of the canonical map of the pushout to the lower right corner.

Clearly $\pi_*(N_C^{d+1}, M_C^{d+1}) = 0$, so the usual argument provides equalities

$$\pi_*(N_A^{d+1}, M_A^{d+1}) = 0 \text{ and } \pi_*(N_B^{d+1}, M_B^{d+1}) = 0.$$

We have therefore completed the verification of *App* 2 for the inclusion $\mathcal{S} \longrightarrow \mathcal{MV}^w$. Then, by Theorem 5

$$vS_\bullet\mathcal{S} \longrightarrow vS_\bullet\mathcal{MV}^w$$

is a homotopy equivalence, as desired.

□

We conclude by remarking that the obstruction to making everything in this paper work directly for particular diagrams of simplicial rings is found here in this section at those points where we want to deduce that a map of A-modules is highly connected, say, given that the map tensored up to R is highly connected. Appropriate modifications of the arguments are discussed in [5].

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