SOME GLUEING FORMULAS FOR SPIN POLYNOMIALS AND A PROOF OF THE VAN DE VEN CONJECTURE.*

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Abstract. We deduce some formulas for the spin invariants of the connected sum of an arbitrary 4-manifold $X, b_2^+(X) > 0$ with $\overline{CP^2}$ in terms of the spin invariants of X and apply this to prove the Van de Ven conjecture.

§0. Introduction

In this article we present some formulas for the spin-polynomial of the connected sum of an arbitrary oriented smooth simply connected four-manifold X with $b_+^2(X) > 0$ with the orientation reversed complex projective plane. These formulas are similar to ones for Donaldson's polynomials presented by Donaldson ([D]) and Morgan-Mrowka ([MM]) in the sense that they do not involve evaluating ν-classes for the polynomial of X. One can show that these formulas also hold in the case of connected sums with arbitrary negative definite manifolds. However the case of $\overline{CP^2}$ is easier from the technical point of view.

As an application one has a proof of the Van de Ven conjecture [VdV]:

Conjecture. *The Kodaira dimension of a simply connected algebraic surface is an invariant of its smooth structure.*

This has been proved to be true for all cases except possibly nonminimal surfaces of general type (cf. [FM], [K], [OVdV], [P], [PT], [Q1], [Q2]). Therefore it suffices to prove that

No surface of general type can be diffeomorphic to a rational one.

The proof is a reduction to the case of minimal surfaces of general type, which already was treated in [PT]. This reduction uses the glueing formulas and results of [FM] describing diffeomorphisms of rational surfaces . In [FQ1] a different approach to the Van de Ven conjecture is announced. The authors use a glueing formula for Donaldson polynomials (and predict our approach) and also explicitly formulate the result on diffeomorphism of rational surfaces which we use.

§1. Connected sum formulas.

We shall start with some notations.

Let X be a Riemannian, simply-connected 4-manifold with a metric g and $E \to X$ some complex hermitian vector bundle of rank 2 with chern classes $c_1(E), c_2(E)$.

*This is the translation of an article submitted to Izvestija of Russian Academy of Sciences, Ser.Math. While this translation was in progress there appeared a paper [FQ2] with the proofs of the results announced in [FQ1].

This vector bundle as well as the corresponding principal $U(2)$ -bundle P is uniquely determined by its chern classes. Denote also by P_{ad} the $PU(2)$-principal bundle $P_{ad} = P/S^1$, and by adE the corresponding vector bundle.

We shall consider the space $\mathcal{A}_E$ of $U(2)$- connections on the bundle E together with the action of the gauge group $\mathcal{G}_{U(2)}$ and the space of orbits of this action $\mathcal{B}_E$. The space of $SO(3)$ -connections for the bundle adE, the gauge group, and the orbit space, we denote by $\mathcal{A}_{adE}, \mathcal{G}_{SO(3)}, \mathcal{B}_{adE}$ resp. We write $\mathcal{A}^*_{adE} \subset \mathcal{A}_{adE}$ and $\mathcal{B}^*_{adE} \subset \mathcal{B}_{adE}$ for the set of irreducible connections on adE and the orbits of irreducible connections.

We shall also need the space of orbits of framed connections $\widetilde{\mathcal{B}_{adE}}$, that is the orbit space of the space of connections by the subgroup $\mathcal{G}^0_G$ of the gauge group acting trivially on the fibre over some fixed point p.

Let
$$\mathcal{A}_\lambda = \{A \in \mathcal{A}_E \mid \frac{1}{2} trF_A = \lambda\}$$
for some closed differential 2-form λ on X. There is an isomorphism:
$$\mathcal{A}_\lambda/\mathcal{G}_{U(2)} = \mathcal{A}_{adE}/\mathcal{G}_{(SO(3)=PU(2))}. \tag{1.1}$$

Let C be some $Spin^{\mathbb{C}}$-structure on the manifold X. This means that one fixes an integral lift C of the second Stiefel-Whitney class $w_2(X)$ of X, and this lift defines a pair of complex hermitian rank 2 vector bundles W^+, W^- subject to the following conditions:
$$T^*_{\mathbb{C}}X = Hom_{\mathbb{C}}(W^+, W^-),\ c_1(\Lambda^2 W^\pm) = C.$$
A choice of a connection ∇ on the line bundle $\Lambda^2 W^+$ gives rise to a Dirac operator on X:
$$\mathcal{D}^{C,\nabla} : \Gamma(W^+) \to \Gamma(W^-).$$

Coupling it with connections $a \in \mathcal{A}_E$ then gives a family of Fredholm operators over $\mathcal{A}_E$:
$$\mathcal{D}^{C,\nabla}_a : \Gamma(W^+ \otimes E) \to \Gamma(W^- \otimes E),$$
which is equivariant with respect to the gauge group action. Using the isomorphism (1.1) we consider it as a family over $\mathcal{B}_{adE}$.

We shall always suppose that
$$ind(\mathcal{D}^{C,\nabla}_a) = dimker(\mathcal{D}^{C,\nabla}_a) - dimcoker(\mathcal{D}^{C,\nabla}_a) \leq 0.$$

Let $\mathbb{S}(\Gamma(W^+ \otimes E))$ denote the unit sphere in the Hilbert space $\Gamma(W^+ \otimes E)$, and $\mathbb{P}(\Gamma(W^+ \otimes E))$ the projectivisation of this space. There is a tautological bundle $\mathcal{O}_{\mathbb{P}(\Gamma(W^+ \otimes E))}(1)$ over this projective space, and the coupled Dirac operator can be interpreted as a section of the bundle
$$\Gamma(W^- \otimes E) \otimes \mathcal{O}_{\mathbb{P}(\Gamma(W^+ \otimes E))}(1).$$

The space
$$\mathcal{P} = \mathcal{A}_\lambda \times_{\mathcal{G}_{U(2)}} \mathbb{S}(\Gamma(W^+ \otimes E)) \tag{1.2}$$

is the base of the following Hilbert bundle

$$\mathcal{H}_C = \mathcal{A}_\lambda \times_{\mathcal{G}_{U(2)}} Tot,$$

where Tot stands for the total space of the bundle $\Gamma(W^- \otimes E) \otimes \mathcal{O}_{\mathbb{P}(\Gamma(W^+ \otimes E)}(1)$. One can see that the family of coupled Dirac operators gives a section s_D of the bundle $\mathcal{H}_C \to \mathcal{P}$ (details in 1.1.26, [PT]). In the same way one gets a section s_{asd} of the Hilbert bundle $\mathcal{H}_+$ associated to the principal bundle $\mathcal{A}^*_{adE} \to \mathcal{B}^*_{adE}$ and a natural representation of the gauge group $\mathcal{G}_{adE}$ in the Hilbert space $\Omega^+_2(adE)$.

Definition. The moduli space of pairs $\mathcal{MP}^{g,\nabla}(X, p_1(E), C + c_1(E))$ is the space of zeroes of the section $s_D \oplus \pi^* s_{asd}$ where $\pi : \mathcal{P}^* \to \mathcal{B}^*$ is the natural projection.

Using (1.1) one can give a point of $\mathcal{MP}^{g,\nabla}(X, p_1(E), C + c_1(E))$ as a pair $(a, \langle s\rangle), a \in \mathcal{M}^g_{asd}(X, p_1(E), w_2(E))$ and $\langle s\rangle$ is a point of the projective space $\mathbb{P}(\Gamma(W^+ \otimes E))$ given by a complex line spanned by a nonzero vector spinor s. There is an exact sequence which describes the tangent bundle of moduli of pairs at a point $(a, \langle s\rangle)$:

$$0 \to T_{a,s}\mathcal{MP}^{g,\nabla} \to (ker\mathcal{D}_a)/\langle s\rangle \oplus H^1_a(adE) \xrightarrow{i} coker\mathcal{D}_a \tag{1.3}$$

$$i(s', \alpha) = \varpi(\alpha * s),$$

(here $*$ is a combination of spinor multiplication and multiplication of a matrix by a vector, ϖ is the orthogonal projection to the space of harmonic vector spinors $coker\mathcal{D}_a$). Therefore $T_{a,s}\mathcal{MP}^{g,\nabla}$ is the kernel of the linearisation of our asd- and Dirac- equations and $coker(i)$ is its cokernel. One can globalise this exact sequence to a fibre of π as an exact sequence of bundles over $\mathbb{P}(ker\mathcal{D}^{C,\nabla}_a)$:

$$0 \to T\mathcal{MP}^{g,\nabla}_{|\pi^{-1}a} \to T\mathbb{P}(ker\mathcal{D}^{C,\nabla}_a) \oplus H^1_a(adE) \xrightarrow{i} coker\mathcal{D}_a \otimes \mathcal{O}_{\mathbb{P}(ker\mathcal{D}^{C,\nabla}_a)}(1) \to 0.$$

An orientation of the moduli space of pairs is defined by the orientation of the moduli space of asd connection via the exact sequence (1.3).

A model for a neighborhood of the point $(a, \langle s\rangle)$ in the space $\mathcal{MP}^{g,\nabla}(X)$ is given by the zero set of some smooth section of the bundle

$$\begin{array}{c} (coker(i)) \oplus H^2_a(adE) \otimes \mathcal{O}_{\mathbb{P}(ker(\mathcal{D}^{C,\nabla}_a))} \\ \downarrow \\ \mathbb{P}(ker(\mathcal{D}^{C,\nabla}_a)) \times H^1_a(adE) \end{array}$$

(cf. [PT], ch.1).

This moduli space is a source of various invariants of the smooth structure of X as described in [PT] (numerical invariants) and [T] (polynomial invariants). It is a smooth manifold provided $b^+_2(X) > 0$ and the metric g and the connection ∇ both are generic. For $b^+_2(X)$ odd we use polynomial invariants

$$\gamma^X_{p_1, C+c_1(E)}(\Sigma_i) = \#\{\cap V_{\Sigma_i} \cap \mathcal{PM}(X, p_1(E), C + c_1(E))\},$$

where $\Sigma_i \in H_2(X)$, and the V_{Σ_i} are codimension 2 submanifolds of $\mathcal{P}$, defined as the zero sets of certain sections s_Σ of certain complex line bundles $\mathcal{L}_{\Sigma_i}$ over $\mathcal{P}$ (cf. [T], [D]). Such a section is defined once we have chosen a smooth Riemann surface representing the class Σ. We shall use the same notation for these representatives: Σ_i. If $b_2^+ > 1$ then these polynomials do not depend on the choice of the metric g and the connection ∇ on the line bundle $\Lambda^2 W^+$. If $b^+ = 1$ and $p_1 \geq -7$ or $p_1 = -8, w_2 = (c_1) \pmod 2 \neq 0$ then there is a chamber structure on the space of periods $\mathcal{P}(X)$ of metrics and the polynomials depends only on the chamber, not on the metric itself (cf. [BP]). We shall use the notation $\gamma^X_{p_1, C+c_1(E), \mathcal{C}}(\Sigma_i)$ to show the dependence on the chamber $\mathcal{C}$. In either case as a function of the cohomology classes $c_2(E), c_1(E), C$ the polynomials depend in fact only on the combinations $C + c_1(E), p_1(adE) = c_1(E)^2 - 4c_2(E)$ of these classes.

The connected sum $X\#\overline{CP^2}$ of a manifold X and a projective plane with reversed orientation $\overline{CP^2}$ can be viewed as a union

$$(X \setminus D_1) \sqcup (S^3 \times [-t/2, t/2]) \sqcup (\overline{CP^2} \setminus D_2)$$

where $D_1 \subset X, D_2 \subset \overline{CP^2}$ are small disks around some points $p \in X, q \in \overline{CP^2}$, $\partial D_i = S^3$. Define a metric g_t on $X\#\overline{CP^2}$ which is generic when restricted to $X - D_1$ and $\overline{CP^2} - D_2$ and a product of S^3 with radius r and a standard metric on the interval $[-t/2, t/2]$ for some interior of the cylinder.

When t tends to infinity one gets two manifolds - X and $\overline{CP^2}$ joined by an "infinite"cylinder or, in a conformally equivalent metric, two open manifolds $(X \setminus p)$ and $(\overline{CP^2} \setminus q)$.

Let e is a generator of $H^2(\overline{CP^2})$. Fix some $U(2)$ vector bundle E and $Spin^{\mathbb{C}}$ structure on the connected sum $X\#\overline{CP^2}$, such that $(C + c_1(E))_{|\overline{CP^2}} = ke$ with $-3 < k < 3$.

It is known that if one takes a sequence of metrics $g_{t_i}, t_i \to \infty$ and a sequence of connections a_i on E such that a_i is antiselfdual (asd) with respect to g_{t_i} then there exist asd-connections $a_X, a_{\overline{CP^2}}$ on $X, \overline{CP^2}$ resp., and a finite collection of points $x_1, ..., x_m$ such that after gauge transformation some subsequence of a_i converges weakly over compact subsets of $X \cup \overline{CP^2} - \{p, q, x_1, ..., x_m\}$ to $a_X, a_{\overline{CP^2}}$ and the curvature densities $|F_{a_i}|$ converge to $F_{a_X} + F_{a_{\overline{CP^2}}} + \sum_i 8\pi^2 \delta_{x_i}$ (cf. ch.7 of [DK]).

The following lemma describes the case when there is a nontrivial element of the kernel of the coupled Dirac operator D_{a_i} (positive harmonic vector-spinor) for any asd-connection a_i.

Lemma 1.1. *Take E, C, a_i as above. If for any asd-connection a_i with $i >> 0$ there exists a nonzero positive harmonic vector-spinor s_i (which we shall suppose to be of unit norm) , then a_X also has nonzero positive harmonic vector-spinor s_X. Moreower a subsequence of s_i converges to s_X on any compact subset of $X - p$ in C^∞ topology.*

Proof. We shall use the norm $L^{8/3}$ on the space $\Gamma(W^+ \otimes E)$ of positive vector spinors which has following property: if $g' = e^{2f} g$ is a conformal change of a metric then

$$s \in ker(\mathcal{D}^g_a) \Leftrightarrow s' = e^{-\frac{3}{2}f} s \in ker(\mathcal{D}^{g'}_a)$$

and

$$\|s\|_{L^{8/3}(X,g)} = \|s'\|_{L^{8/3}(X,g')}.$$

This gives a possibility to derive estimates in an arbitrary conformal model of the connected sum, identifying vector spinors as above.

Let S_t be a scalar cuvature of $X\#\overline{CP^2}, g_t$. It is easy to see that it is uniformly bounded from above for all t. Let ∇_{a_i} be a connection on the bundle $W^+ \otimes E$ built from a_i and ∇. It follows from the Weitzenböck formula

$$(\mathcal{D}_{a_i})^* \mathcal{D}_{a_i} s_i = \nabla^*_{a_i} \nabla_{a_i} s_i - F^+_{a_i} s_i + 1/4 S_t \cdot s_i + F_\nabla s_i,$$

that if σ the maximum of the norm of the sum of curvatures $1/4S_t + F_\nabla$ then there is the pointwise inequality:

$$(\nabla^*_{a_i} \nabla_{a_i} s_i, s_i) \leq (\sigma)(s_i, s_i).$$

Being combined with the standart estimate of the l.h.s. through Laplasian Δ of the function $|s_i|$

$$(\nabla^*_{a_i} \nabla_{a_i} s_i, s_i) \geq |s_i| \cdot \Delta(|s_i|),$$

which follows from the Kato inequality, it gives then

$$\Delta(|s_i|) \leq \sigma |s_i| \quad \text{and} \quad \|\Delta(|s_i|)\|_{L^{8/3}} \leq \sigma \|s_i\|_{L^{8/3}}.$$

Since Δ is an elliptic operator one has an a priori inequality for functions on $X\#\overline{CP^2}$:

$$\|s_i\|_{L_2^{8/3}} \leq const \|\Delta s_i\|_{L^{8/3}} + const \|s_i\|_{L^{8/3}}$$

with $const$ independent of t and a_i. The last two estimates give

$$\|s_i\|_{L_2^{8/3}} \leq const(1+\sigma)\|s_i\|_{L^{8/3}}.$$

Applying the Sobolev embedding theorem for functions one gets

$$\|s_i\|_{L^8} \leq const' \|s_i\|_{L_2^{8/3}} \leq const'' \|s_i\|_{L^{8/3}}$$

with $const''$ independent of t, a_i. With this estimate in mind we now look how far the truncated spinor is from being harmonic.

There are some fixed θ_1, θ_2 with $-1/2 < \theta_1, \theta_2 < 1/2$ such that for a subsequence of a_i restricted to the subcylinder $S^3 \times [\theta_1 t, \theta_2 t]$ the curvature density $|F_{a_i}|$ is small. Thus we can apply the theorem of K. Uhlenbeck on the existence of local Coulomb gauge with small connection matrix over the subcylinder.

Now let ψ_i be some cut-off function with support on $X \cup S^3 \times [-t/2, \theta_2 t]$, $supp(d\psi_i) \subset S^3 \times [\theta_1 t, \theta_2 t]$ and $\|d\psi_i\|_{L^4} \leq \epsilon(t_i)$, where $\epsilon(t_i) \to 0$ when $t_i \to \infty$. We assume for simplicity of notations that there is no bubbling points on X - the modification to this case consists of cutting off also in small neighborhood of the bubbling points and has been treated in [DKr, Lemma 7.1.24]. Therefore we can assume that there is an estimate for the supremum of $(a_i - a_X)$:

$$sup(|a_i - a_X|_{|supp\psi_i}) < \epsilon'(t), \quad \epsilon'(t) \to 0 \text{ as } t \to \infty.$$

This provides the estimates

$$\|\mathcal{D}_{a_X}\psi_i s_i\|_{L^{8/3}(X)} \leq$$

$$\leq \|\mathcal{D}_{a_i}s_i\|_{L^{8/3}(X\#\overline{CP^2})}+\|d\psi_i\cdot s_i\|_{L^{8/3}(X\#\overline{CP^2})}+sup|a_i-a_X|_{|supp\psi_i}\|s_i\|_{L^{8/3}(X\#\overline{CP^2})} \leq$$

$$\leq \|d\psi_i\|_{L^4(X)}\|s_i\|_{L^8(X\#\overline{CP^2})} + sup|a_i - a_X|_{|supp\psi_i}\|s_i\|_{L^{8/3}(X\#\overline{CP^2})} \leq$$

$$\leq (const''\epsilon(t) + \epsilon'(t))\|s_i\|_{L^{8/3}(X\#\overline{CP^2})} = \epsilon_1(t_i)\|s_i\|_{L^{8/3}(X\#\overline{CP^2})} = \epsilon_1(t_i), \quad (1.4)$$

where $\epsilon_1(t_i)$ tends to 0 if t_i tends to infinity.

On the other hand applying these estimates to the cut-off function $(1-\psi_i)$ provides a similar inequality:

$$\|\mathcal{D}_{a_{\overline{CP^2}}}(1-\psi_i)s_i\|_{L^{8/3}} \leq \epsilon_2.$$

But the auxilary Lemma 1.2 shows the first eigenvalue of the operator $\mathcal{D}_{a_{\overline{CP^2}}}$ is positive and uniformly estimated from below by some positive constant c. Therefore there is an inequality

$$\|(1-\psi)s_i\|_{L^{8/3}} \leq \epsilon_2/c.$$

So the norm of $(1-\psi_i)s_i$ is arbitrarily small and hence the norm of $\psi_i s_i$ is arbitrarily close to 1. Now the first statement of the Lemma follows from inequality (1.4) and the variational characterisation of the first eigenvalue. The second statement follows from standart bootstrapping argument using the equation

$$\mathcal{D}_{(a_X)_{|K}}(s_i)_{|K} = (a_i - a_X)_{|K} * (s_i)_{|K}$$

restricted to the compact subset K of $X - p$.

Remark. *If one has*

$$a_i \in V_\Sigma, \Sigma \in H_2(X) \subset H_2(X\#\overline{CP^2})$$

and no bubbling occures at Σ, i.e. none of the x_i lies on Σ then for the limit connection it follows

$$a_X \in V_\Sigma,$$

since this depends only on the restriction of the connection to a neighborhood of the Riemaniann surface Σ where there is C^∞ convergence of connections of our sequence.

Lemma 1.2. *Let F be a $U(2)$ vector bundle on $\overline{CP^2}$ and C a $Spin^{\mathbb{C}}$-structure on $\overline{CP^2}$ such that $C + c_1(E) = ke, -3 < k < 3$, where e is a generator of $H^2(\overline{CP^2})$. There exists a neighborhood U of the Fubini-Study metric on $\overline{CP^2}$ and some connection in the product of the space of metrics on $\overline{CP^2}$ and the space of connections on the line bundle $L = \Lambda^2 W^\pm, c_1(L) = ke$, such that for any asd-connection on adF and any point $(g,\nabla) \in U$, one has*

$$ker\mathcal{D}_a^{g,\nabla} = 0.$$

Moreover, the first eigenvalue μ_a *of the Laplasian* $(\mathcal{D}_a^{g,\nabla})^*\mathcal{D}_a^{g,\nabla}$ *satisfies*

$$\mu_a > c$$

for some positive constant c *independent of the point in* U *and* a.

Proof. The proof is based on the Weitzenböck formula:

$$(\mathcal{D}_a)^*\mathcal{D}_a s = (\nabla_a)^*\nabla_a s - F_a^+ s + 1/4S \cdot s + F_\nabla s, \tag{1.5}$$

where S is the scalar curvature of $\overline{CP^2}$, ∇_a the connection on the bundle $W^+ \otimes F$ built from a and ∇. One has $F_a^+ = 0$, and example 2.1.2 [H] gives that for the Fubini-Study metric, some connection ∇_0 on $\Lambda^2 W^\pm$ and some positive number c there is an estimate of $1/4S + F_{\nabla_0}$ as an endomorphism of vector-spinors:

$$1/4S + F_{\nabla_0} > 2c.$$

Let U be given by the condition $(g, \nabla) \in U$ if $1/4S + F_\nabla > c$. Taking the scalar product of (1.5) with s and integrating over $\overline{CP^2}$ gives the required estimate.

Therefore to each point of the space $\mathcal{MP}^{g_i,\nabla}(X\#\overline{CP^2}, C + c_1, p_1)$ for $i \to \infty$ one can assign a point of the product

$$\mathcal{MP}^{g,\nabla}(X, (C + c_1)_{|X}, k) \times \mathcal{M}^g(\overline{CP^2}, (w_2)_{|\overline{CP^2}}, m),$$

$w_2 = c_1 \pmod 2$, at least for one pair of negative integers $k, m; k + m \geq p_1$. Conversely, there is a procedure for glueing two asd-connections $a_X, a_{\overline{CP^2}}$, defined on the summands, which gives an asd connection on the connected sum and one shows that any asd-connection on a connected sum with long enough tube is obtained in this way (cf. [DKr]). In the following we extend this construction to the case of the moduli space of pairs.

To start with we need glueing data for connections, i.e. isometries

$$\tau : T_p X \to T_q\overline{CP^2}, \rho : (adE_X)_p \to (adE_{\overline{CP^2}})_q,$$

which for given asd-connections $a_X, a_{\overline{CP^2}}$ on $X$and $\overline{CP^2}$ resp. give an $SO(3)$-asd-connection $a = I(a_X, a_{\overline{CP^2}}, \rho)$ on $X\#\overline{CP^2}$ with appropriate choices of some other parameters . Varying $a_X, a_{\overline{CP^2}}$ in some small neighborhood, and varying the glueing parameter ρ in the space of all glueing parameters Gl we get a smooth map to some open set of asd connections on the connected sum.

In order to glue vector-spinors one has to glue the corresponding vector bundles of vector-spinors with a unitary map σ:

$$\sigma : (W_X^+ \otimes E_X)_p \to (W_{\overline{CP^2}} \otimes E_{\overline{CP^2}})_q.$$

The isometries τ and ρ give σ up to a unitary scaling, which may be interpreted as a glueing parameter for glueing connections on the determinant line bundles

$$(det(W_X^+ \otimes E_X))_p \to (det(W_{\overline{CP^2}} \otimes E_{\overline{CP^2}}))_q.$$

This scaling is in fact inessential since different glueing parameters for $U(1)$- connections give gauge equivalent connections provided X is simply connected. Taking a cut-off function ψ as in Lemma 1.1 one considers ψs as a section of the glued bundle $(W^+\otimes E)_{X\#\overline{CP^2}}$ and it turns out that the point $\left(a = I(a_X, a_{\overline{CP^2}}, \rho), \langle\psi s\rangle\right)$ is close to a point of $\mathcal{MP}^{g,\nabla}(X\#\overline{CP^2}, C+c_1, p_1)$. In fact we can deform it to a point of the moduli space. The next lemma describes a neighborhood of this deformation in the space $\mathcal{MP}^{g_i,\nabla}(X\#\overline{CP^2}, C+c_1, p_1)$, as the zero set of some map defined on some neighborhood in the space

$$\widetilde{\mathcal{MP}}^{g,\nabla}(X,(C+c_1)_{|X}, -4k) \times_{PU(2)} \widetilde{\mathcal{M}}^g(\overline{CP^2}, (w_2)_{|\overline{CP^2}}, p_1+4k)$$

of the preimage of the pair

$$((a_X,\langle s\rangle); a_{\overline{CP^2}}) \in \mathcal{MP}^{g,\nabla}(X,(C+c_1)_{|X}, -4k) \times \mathcal{M}^g(\overline{CP^2}, (w_2)_{|\overline{CP^2}}, p_1+4k)$$

under the natural projection

$$\widetilde{\mathcal{MP}}^{g,\nabla}(X,(C+c_1)_{|X}, -4k) \times_{PU(2)} \widetilde{\mathcal{M}}^g(\overline{CP^2}, (w_2)_{|\overline{CP^2}}, p_1+4k) \to$$

$$\mathcal{MP}^{g,\nabla}(X,(C+c_1)_{|X}, -4k) \times \mathcal{M}^g(\overline{CP^2}, (w_2)_{|\overline{CP^2}}, p_1+4k).$$

Lemma 1.3. *For the point $(a_X,\langle s\rangle)$ in the space $\mathcal{MP}^{g,\nabla}(X,(C+c_1)_{|X}, -4k)$ and $a_{\overline{CP^2}} \in \mathcal{M}^g(\overline{CP^2}, (w_2)_{|\overline{CP^2}}, p_1+4k)$ there is a neighborhood in the space $\mathcal{MP}^{g_i,\nabla}(X\#\overline{CP^2}, C+c_1, p_1)$ which is diffeomorphic to the zero set of a smooth map Φ defined on some neighborhood*

$$\mathcal{U} \subset T_{a,s}\mathcal{MP}^{g,C,\nabla} \times H^1_{a_{\overline{CP^2}}}(adE) \times Gl$$

of $\{0\}\times\{0\}\times Gl$:

$$\mathcal{U} \xrightarrow{\ \Phi\ } coker(i) \oplus (H^2_{a_X}(adE) \oplus H^2_{a_{\overline{CP^2}}}(adE)) \oplus coker\mathcal{D}_{a_{\overline{CP^2}}}$$

where $coker(i)$ is defined in (1.3).

Proof. We assume for simplicity that $H^2_{a_X} = H^2_{\overline{CP^2}} = 0$. Fix some glueing parameter ρ' and a connection $a' = I(a_X, a_{\overline{CP^2}}, \rho')$. Consider a neighborhood in $\mathcal{M}^g(X\#\overline{CP^2}, (c_1) \pmod 2, p_1)$ of the subspace $\cup_\rho I(a_X, a_{\overline{CP^2}}, \rho)$ consisting of all connections represented in the form $I(a_X+\delta a, a_{\overline{CP^2}}, \rho)$ subject to an estimate of $\delta I(a_X+\delta a, a_{\overline{CP^2}}, \rho) = I(a_X+\delta a, a_{\overline{CP^2}}, \rho) - I(a_X, a_{\overline{CP^2}}, \rho')$:

$$\|\delta I(a_X+\delta a, a_{\overline{CP^2}}, \rho)\|_{L^4} \leq \epsilon$$

with ϵ to be fixed in the proof. Fix lifts

$$\xi_X : coker(\mathcal{D}_X) \to \Gamma(W^-\otimes E_X),\ \ \xi_{\overline{CP^2}} : coker\mathcal{D}_{a_{\overline{CP^2}}} \to \Gamma(W^-\otimes E_{\overline{CP^2}})$$

of the obvious projections $\pi_X, \pi_{\overline{CP^2}}$ onto the cokernels with all section in the images $im\xi_X, im\xi_{\overline{CP^2}}$ supported away from the points $p \in X, q \in \overline{CP^2}$. There is a linear map

$$\xi_{X\#\overline{CP^2}} = \xi_X \oplus \xi_{\overline{CP^2}} : coker\mathcal{D}_{a_X} \oplus coker\mathcal{D}_{a_{\overline{CP^2}}} \to \Gamma(W^- \otimes E_{X\#\overline{CP^2}}),$$

which is a lift of the projection $\pi(\cdot) = \pi_X(\psi\cdot) + \pi_{\overline{CP^2}}((1-\psi)\cdot)$, where ψ is the cut-off function as in Lemma 1.1. Let s be a non-zero element of the kernel of $\mathcal{D}_{a_X}$. Then we can make $\|\mathcal{D}_{a'}(\psi s)\| \leq \epsilon(t)$ with $\epsilon(t) \to 0$ as $t \to \infty$ for suitable choices as above. Consider the map ϕ, which assignes to a triple consisting of an asd-connection $a_X + \delta a_X$ on X, a vector spinor δs on the connected sum $X\#\overline{CP^2}$ and an element $h \in coker\mathcal{D}_{a_X} \oplus coker\mathcal{D}_{a_{\overline{CP^2}}}$ the vector spinor on the connected sum:

$$\phi(a_X + \delta a_X, h, \delta s) = \mathcal{D}_{I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho)}(\psi s + \delta s) + \xi_{X\#\overline{CP^2}}(h).$$

The tangent map of ϕ at the point $(a_X, 0, 0)$ with respect to $\delta s, h$ is given by

$$(\mathcal{D}_{a'} \oplus \xi)(h, \delta s) = \mathcal{D}_{a'}(\delta s) + \xi_{X\#\overline{CP^2}}(h),$$

and the application of the standart technique shows that it has a right inverse

$$S \oplus \pi : \mathcal{D}_{a'} S(\eta) + \xi_{X\#\overline{CP^2}}\pi(\eta) = \eta$$

with the estimate of the norm of $S \oplus \pi$ independent of t (compare 7.2.14, 7.2.18 of [DKr]). Since by definition $pr_2 = \xi_{X\#\overline{CP^2}}\pi$ is a projector, $pr_1 = \mathcal{D}_{a'}S$ is also a projector and one can also assume that ImS is transversal to $ker\mathcal{D}_{a'}$. This means that $S(\eta) = S(\eta_1)$, where $\eta_i = pr_i(\eta)$. Although pr_i are not orthogonal projectors one can estimate their norm by, say, $\|pr_i\| \leq 2$. Now we shall look for the solution of the equation $\phi(a_X + \delta a_X, h, \delta s) = 0$ in the form $(h, \delta s) = (\pi(\eta), S(\eta)) = (\pi(\eta_2), S(\eta_1))$:

$$\mathcal{D}_{I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho}(\psi s - S(\eta)) + \xi(\pi(\eta)) =$$

$$= \mathcal{D}_{a'}(\psi s) + \eta_1 + \eta_2 + \delta I(a_X + \delta a_X, a_{\overline{CP^2}}, \rho) * S(\eta) + \delta I(a_X + \delta a_X, a_{\overline{CP^2}}, \rho) * (\psi s) = 0, \tag*{((1.6).)}$$

Following system of two equations is equivalent to (1.6):

$$pr_1(\mathcal{D}_{a'}(\psi s)) + \eta_1 + pr_1(\delta I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho)*(\psi s)) + pr_1(\delta I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho)*S(\eta_1)) = 0, \tag{1.7}$$

$$pr_2(\mathcal{D}_{a'}(\psi s)) + \eta_2 + pr_2(\delta I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho)*(\psi s)) + pr_2(\delta I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho)*S(\eta_1)) = 0. \tag{1.8}$$

Equation (1.7) can be rewritten in the form:

$$(1 + A)(\eta_1) = -pr_1(\mathcal{D}_{a'}(\psi s)) - pr_1(\delta I(a_X + \delta a_X, a_{\overline{CP^2}}, \rho) * (\psi s) \tag{1.9}$$

with the norm of $A(\eta_1) = pr_1(\delta I(a_X + \delta a_X, a_{\overline{CP^2}}, \rho) * S(\eta_1))$ estimated by an inequality:

$$\|A(\eta_1)\|_{L^{8/3}} \leq 2\|\delta I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho)\|_{L^4}\|S(\eta_1)\|_{L^8} \leq 2\epsilon\|S\|\|\eta_1\|_{L^{8/3}} \leq \|\eta_1\|_{L^{8/3}}/2$$

provided $\epsilon \leq \|S\|/4$. Therefore the operator $1+A$ is invertible and $\|(1+A)^{-1}\| \leq 2$. This gives the existence and uniqueness of the solution η_1 of (1.9) together with an estimate:

$$\begin{aligned}\|\eta_1\|_{L^{8/3}} &= \|(1+A)^{-1}(-pr_1(\mathcal{D}_{a'}(\psi s)) - pr_1(\delta I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho) * (\psi s))\|_{L^{8/3}} \leq \\ &\leq 2(\|pr_1(\mathcal{D}_{a'}(\psi s))\|_{L^{8/3}} + \|pr_1(\delta I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho) * (\psi s))\|_{L^{8/3}}) \leq \\ &\leq 4(\|\mathcal{D}_{a'}(\psi s)\|_{L^{8/3}} + \|\delta I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho) * (\psi s)\|_{L^{8/3}}) \leq \\ &\leq 4(\epsilon(t) + \|\psi \delta I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho)\|_{L^4}\|s\|_{L^8}) \leq 4(\epsilon(t) + const_1\epsilon) \qquad (1.10)\end{aligned}$$

(we use here the estimate $\|s\|_{L^8} \leq const'\|s\|_{L^{8/3}_2} \leq const"\|s\|_{L^{8/3}}$ from Lemma 1). Let $\mathcal{N}$ be the image of some splitting j of the exact sequence (1.3). Now consider the map $\tau : \mathcal{N} \to coker\mathcal{D}_{a_X} \oplus coker\mathcal{D}_{a_{\overline{CP^2}}}$ given by

$$\delta a_X \mapsto \pi_X(\delta I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho) * (\psi s)) + \pi_X(\delta I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho) * S(\eta_1)).$$

In order to get the proper obstruction space we shall prove that the image of the composition $pr_{im(i)}\tau$ containes an open ball centered at the origin, where $pr_{im(i)}$ is the projection to the image of the map i defined in (1.3). Indeed, this follows from estimates:

$$\begin{aligned}&\|\pi_X(\delta I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho) * (\psi s)) + \pi_X(\delta I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho) * S(\eta_1))\|_{L^{8/3}} \geq \\ &\geq \|\pi_X((\delta a_X) * (\psi s)\|_{L^{8/3}}/2 - \|\delta I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho)\|_{L^4}\|S\| \cdot \|\eta_1\|_{L^{8/3}} \geq \\ &\geq \|\pi_X((\delta a_X) * s)\|_{L^{8/3}}/4 - 2\|\delta a_X\|_{L^4}\|S\| \cdot 4(\epsilon(t) + const_1\epsilon) = \\ &= \frac{1}{4}\|i_{|\mathcal{N}}(\delta a_X))\|_{L^{8/3}} - 8\|S\|(\epsilon(t) + const_1\epsilon)\|(\delta a_X)\|_{L^4} \geq \\ &\geq (const_2 - 8\|S\|(\epsilon(t) + const_1\epsilon))\|(\delta a_X)\|_{L^4}, \qquad (1.11)\end{aligned}$$

where the first inequality uses the estimate

$$\|\delta I(a_X+\delta a_X, a_{\overline{CP^2}}, \rho) - \psi \cdot \delta a_X\|_{L^4} \leq const \cdot e^{-2t} \qquad (1.12)$$

(cf. [DKr, (7.2.37)]). The second inequality in (1.11) uses (1.12), (1.10) and the estimate of the norm of ψs from below given in Lemma 1.1. Since $i_{|\mathcal{N}}$ is a linearisation of $\pi_X((\delta a_X) * s)$ (cf.1.3) and $i_{|\mathcal{N}}$ is a monomorphism by the definition of $\mathcal{N}$, there is a constant $const_2$ such that for small enough ϵ

$$\|\delta a_X\| \leq \epsilon \Rightarrow \|\pi_X((\delta a_X) * s)\| \geq \frac{1}{2}\|i_{|\mathcal{N}}(\delta a_X))\|_{L^{8/3}} \geq const_2\|(\delta a_X)\|_{L^4}$$

(here we identify δa_X with the corresponding element in $H^1_{a_X}$). This gives the last inequality in (1.11), and the statement about the composition $pr_{im(i)}\tau$ follows, provided $\epsilon(t)$ and ϵ are small enough. Therefore the subspace $im(\tau) + coker(i) \subset coker(\mathcal{D}_{a_X})$ contains some metric ball B, and there exist ϵ_0 such that

$$\epsilon(t) \leq \epsilon_0 \Rightarrow \pi(\mathcal{D}_{a'}(\psi s)) \in B \oplus coker\mathcal{D}_{a_{\overline{CP^2}}} \subset coker\mathcal{D}_{a_X} \oplus coker\mathcal{D}_{a_{\overline{CP^2}}}.$$

Therefore the obstruction space can be identified with $coker(i) \oplus coker\mathcal{D}_{a_{\overline{CP^2}}}$. The Lemma follows.

Denote by ϵ_Ω the constant in the generalisation of Uhlenbeck's local Coulomb gauge theorem to any strongly simply connected domain Ω (Prop.4.4.10 of [DKr]).

Corollary 1.4. *There exists a real positive number t_0 such that for $t > t_0$ there exists a covering $\{\mathcal{U}_i\}$ of the moduli space $\mathcal{MP}^{g,\nabla}(X\#\overline{CP^2}, C + c_1, p_1)$ such that for subsets*

$$\widetilde{\mathcal{MP}^{\varrho}}(X, (C + c_1)_{|X}, i) \subset \widetilde{\mathcal{MP}}(X, (C + c_1)_{|X}, i))$$

and

$$\widetilde{\mathcal{M}^{\varrho}}(\overline{CP^2}, (w_2)_{|\overline{CP^2}}, p_1 + 4i) \subset \widetilde{\mathcal{M}}(\overline{CP^2}, (w_2)_{|\overline{CP^2}}, p_1 + 4i)$$

given by the condition

$$\|F_{a_{|D^4}}\|_{L_2(D^4)} \leq \epsilon_{D^4_\varrho}$$

where D^4_ϱ is a small disk around $p \in X$, $q \in \overline{CP^2}$, resp., of radius ϱ, one has

$$\mathcal{U}_i \subset \widetilde{\mathcal{MP}}^{rexp(-t)}(X, (C + c_1)_{|X}, -4i) \times_{PU(2)} \widetilde{\mathcal{M}}^{rexp(-t)}(\overline{CP^2}, (w_2)_{|\overline{CP^2}}, p_1 + 4i)$$

$$\widetilde{\mathcal{MP}^{r}}(X, (C + c_1)_{|X}, -4i) \times_{PU(2)} \widetilde{\mathcal{M}^{r}}(\overline{CP^2}, (w_2)_{|\overline{CP^2}}, p_1 + 4i) \subset \mathcal{U}_i.$$

Proof. Take a subdivision of the tube $S^3 \times [-t/2, t/2]$ into N pieces

$$\Omega_l = S^3 \times [-t/2 + lt/N, -t/2 + (l + 1)t/N], l = 0, ..., N - 1.$$

We shall use a cut off function ψ with support of $d\psi$ in one of Ω_l with an estimate $\|d\psi\|_{L^4} \leq (10(t/N)^3)^{-1/4}$. Take $N - 2 > max\{-p_1/\epsilon_F, 10(\epsilon_0)^{-4}t^{-3}\}$, where $\epsilon_F \leq \epsilon_{\Omega_l}$ and ϵ_0 is the constant of Lemma 1.3. This provides the existence for any asd-connection a with first Pontrjagin class p_1 of at least one $0 < l < N - 1$ for which

$$\|F_{a_{|\Omega_l}}\|_{L_2} \leq \epsilon_F,\ \|\mathcal{D}_a(\psi s)\|_{L^{8/3}(\Omega_l)} \leq \|d\psi\|_{L^4(\Omega_l)}\|s_{|\Omega_l}\| \leq (10(t/N)^3)^{-1/4}\|s_{|\Omega_l}\| \leq$$

$$(10(t/N)^3)^{-1/4}N^{-1} = (10t^3N)^{-1/4} \leq \epsilon_0 \tag{1.13}$$

Now let $\mathcal{U}_i$ be the open set of those connections a in $\mathcal{MP}^{g,\nabla}(X\#\overline{CP^2}, C+c_1, p_1)$ for which for at least one integer l (1.13) is satisfied, and the L_2-norm of the curvature of a restricted to $X_l = X \cup S^3 \times [-t/2, -t/2 + lt/N]$ is close to $8\pi^2 i$. Then by the Uhlenbeck theorem there is a trivialisation of $E_{|\Omega_l}$ in which the connection matrix is small in L^2_k and therefore one can show that norms of $F^+_{\psi a}, F^+_{(1-\psi)a}, \mathcal{D}_a(\psi s)$ are small. By Theorem 7.2.41 of [DKr] $a = I(a_X, a_{\overline{CP^2}}, \rho)$, $\|\psi a - a_X\|_{L^q} \leq const\epsilon_F$. If $l < N - 1$, X_l is a subset of fixed compact subspace in $X - p$ and by Lemma 1.1 one has for some $s_X \in ker\mathcal{D}_{a_X}$

$$\|(s - s_X)_{|X_l}\|_{L_1^{8/3}(X_l)} \to 0$$

as $t \to \infty$. Symmetrically, if $l > 0$ then

$$\|s_{|(X\#\overline{CP^2} - X_l)}\|_{L_1^{8/3}(X\#\overline{CP^2} - X_l)} \to 0.$$

This means that

$$\|s - \psi s_X\|_{L_1^{8/3}(X\#\overline{CP^2})} \leq \|(s - s_X)_{|X_l}\|_{L_1^{8/3}(X_l)} +$$

$$+\|s_{|(X\#\overline{CP^2}-X_l)}\|_{L_1^{8/3}(X\#\overline{CP^2}-X_l)} + \|s\|_{L_1^{8/3}(\Omega_l)} + \|\psi\|_{L_2^2(\Omega_l)}\|s_X\|_{L_1^{8/3}(\Omega_l)} \to 0$$

as $t \to \infty$, so it follows that varying slightly s_X in $ker\mathcal{D}_{a_X}$ (what does not change S_a) we get $s - \psi s_X = S_a(\eta)$ with $\eta = \mathcal{D}_a(s - \psi s_X) = (\mathcal{D}_{a_X} + (\psi a - \psi a_X))(s_X)$ small enough to be in the set of solutions of (1.7 -1.8).

Now we shall consider some particular cases in which Lemma 1.3 will provide sufficient information about the moduli space of the connected sum $X\#\overline{CP^2}$ in order to compute the spin-polynomials of the connected sum in terms of those of X. Consider a spin polynomial of degree d on X:

$$\gamma^X_{k,\omega}(\Sigma_i) = \#\{\cap V_{\Sigma_i} \cap \mathcal{PM}(X, k, \omega)\},$$

which is defined only if

$$2d = -(3/2)k - 3(1 + b_2^+(X)) + (\omega)^2/2 - sign(X)/2 - 2$$

for some negative integer k and some 2-dimensional cohomology class ω on X.

Let e be a generator of the second cohomology group $H_2(\overline{CP^2})$: $e \in H_2(\overline{CP^2}) \subset H_2(X\#\overline{CP^2})$. Take a bundle E and a $Spin^{\mathbb{C}}$ -structure C on the connected sum $X\#\overline{CP^2}$ given by one of the following three choices.

1. $p_1(E) = k, C + c_1(E) = \omega + (\pm e)$;
2. $p_1(E) = k + 1, C + c_1(E) = \omega + (\pm 2e)$;
3. $p_1(E) = k + 1, C + c_1(E) = \omega$.

By a dimension count it follows that in the first and second case one has

$$\dim\Big(\cap V_{\Sigma_i} \cap \mathcal{PM}(X\#\overline{CP^2}, p_1(E), C + c_1(E))\Big) = 0,$$

and in the third -

$$\dim\Big(\cap V_{\Sigma_i} \cap \mathcal{PM}(X\#\overline{CP^2}, p_1(E), C + c_1(E))\Big) = 2.$$

Therefore in the first and second case the polynomial

$$\gamma^{X\#\overline{CP^2}}_{p_1,C+c_1(E)}(\Sigma_i) = \#\{\cap V_{\Sigma_i} \cap \mathcal{PM}(X\#\overline{CP^2}, p_1(E), C + c_1(E))\},$$

is defined and in the third the polynomial

$$\gamma^{X\#\overline{CP^2}}_{p_1,C+c_1(E)}(\Sigma_i, e) = \#\{\cap V_{\Sigma_i} \cap V_e \cap \mathcal{PM}(X\#\overline{CP^2}, p_1(E), C + c_1(E))\}$$

is defined.

Let a_j be a sequence of connections in $\cap V_{\Sigma_i}$ which are asd in the metric g_j resp. In each of the three cases by the remark to Lemma 1.1 the limit connection a_X on X belongs to the moduli space

$$\cap V_{\Sigma_i} \cap \mathcal{PM}(X, k + 4m, \omega),$$

for some $m \geq 0$, of the virtual dimension $-8m$. Since we took the metric on X and the connection on the bundle $\Lambda^2 W^{\pm}$ to be generic, and since $b_2^+(X) > 0$ there are no reducible asd-connections and the moduli spaces of negative virtual dimension are empty (cf. [PT], ch.1, sec.3). Therefore the only remaining case is $m = 0$.

From the inequality

$$p_1(a_X) + p_1(a_{\overline{CP^2}}) \geq p_1(E)$$

it follows that $p_1(a_{\overline{CP^2}}) = 0$ for the first case and $p_1(a_{\overline{CP^2}}) = -1$ for the second and the third. Moduli space of asd-connections on $\overline{CP^2}$ for a generic metric with these Pontrjagin numbers has negative virtual dimension and therefore contain only reducible connections. It follows that $a_{\overline{CP^2}}$ is the trivial connection in the first case, and the only reducible one with $p_1 = -1$ in the second.

In this situation Lemma 1.3 provides a satisfactory description of the (compact) manifold

$$Z = \cap \tilde{V}_{\Sigma_i} \cap \widetilde{\mathcal{PM}}(X, k, \omega) \times_{PU(2)} \tilde{M}(\overline{CP^2}, w_2(a_{\overline{CP^2}}), p_1(a_{\overline{CP^2}}))$$

and the bundle Ξ on Z which is a tensor product of the lift of the $PU(2)$ -equivariant bundle

$$U(2) \times_{S^1 \times S^1} coker \mathcal{D}_{a_{\overline{CP^2}}} \to \tilde{M}(\overline{CP^2}, w_2(a_{\overline{CP^2}}), p_1(a_{\overline{CP^2}})) = U(2)/(S^1 \times S^1)$$

via obvious projection

$$\begin{array}{c} \cap \tilde{V}_{\Sigma_i} \cap \widetilde{\mathcal{PM}}(X, k, \omega) \times_{PU(2)} \tilde{M}(\overline{CP^2}, w_2(a_{\overline{CP^2}}), p_1(a_{\overline{CP^2}})) \\ \downarrow \\ \tilde{M}(\overline{CP^2}, w_2(a_{\overline{CP^2}}), p_1(a_{\overline{CP^2}})) \end{array}$$

and tautological line bundle on $\cap \tilde{V}_{\Sigma_i} \cap \widetilde{\mathcal{PM}}(X, k, \omega)$ such that our, cut down moduli space

$$\cap V_{\Sigma_i} \cap \mathcal{PM}(X \# \overline{CP^2}, p_1(E), C + c_1(E))$$

is embedded in Z as the zero set of some generic section of Ξ. This gives a possibility of computing the polynomial for $X \# \overline{CP^2}$ in topological terms.

In the first case our manifold Z is parametrised as a family of glued connections of the type $a_X \# \theta$ with $(a_X, < s_X >) \in \cap V_{\Sigma_i} \cap \mathcal{PM}(X, p_1(E), C + c_1(E))$. Since a connected sum with the trivial connection θ does not need any glueing parameters, one has $Z = \cap V_{\Sigma_i} \cap \mathcal{PM}(X, k, \omega)$. Index of the coupled Dirac operator for θ is zero and hence $\Xi = 0$.

In the second case the reducible connection on $\overline{CP^2}$ has the stabilizer group S^1 and therefore

$$Z = \cap \tilde{V}_{\Sigma_i} \cap \widetilde{\mathcal{PM}}(X, k, \omega)/S^1 = \cup \mathbb{P}^1.$$

For each component $\mathbb{P}^1$ the restriction to it of the bundle Ξ is

$$\left((U(2) \times U(2)) \times_{S^1 \times (S^1 \times S^1)} (\mathbb{C} \otimes coker \mathcal{D}_{a_{\overline{CP^2}}}) \right) / U(2)$$

where $S^1 \times (S^1 \times S^1)$ is the subgroup given as the product of the center subgroup $S^1 \subset U(2)$ and the maximal torus $(S^1 \times S^1) \subset U(2)$. The representation of this subgroup is the product of the generating linear one of S^1 and the natural representation of the stabiliser subgroup $(S^1 \times S^1)$ of the reducible connection $a_{\overline{CP^2}} = \lambda_1 \oplus \lambda_2$ which is the sum of representations $\oplus_i coker\mathcal{D}_{\lambda_i}$, each S^1 acting on its summand with degree ± 1. For the specified $Spin^{\mathbb{C}}$ -structure on $\overline{CP^2}$ this cokernel is one dimensional and therefore

$$\Xi_{|\mathbb{P}^1} = U(2) \times_{S^1 \times S^1} coker\mathcal{D}_{a_{\overline{CP^2}}} = \mathcal{O}_{\mathbb{P}^1}(1).$$

In the third case the space Z is the same, but the index of the reducible connection on $\overline{CP^2}$ is zero, the Dirac operator coupled to this connection is an isomorphism and the bundle arises as a line bundle $\mathcal{L}_e, c_1(\mathcal{L}_e) = \mu(e)$:

$$\Xi = \mathcal{L}_e = \mathcal{O}_{\mathbb{P}^1}(-1).$$

Therefore the following theorem is proved:

Theorem. *Let $k \in \mathbb{Z}, k \leq 0; \omega \in H^2(X\mathbb{Z}), \Sigma_1, ..., \Sigma_d \in H_2(X, \mathbb{Z})$ such that*

$$2d = -3/2k - 3(1 + b_2^+(X)) + (\omega)^2/2 - sign(X) - 2 \geq 0.$$

Then

$$\gamma_{k,\omega \pm e}^{X\#\overline{CP^2}}(\Sigma_1, ..., \Sigma_d) = \pm\gamma_{k,\omega}^X(\Sigma_1, ..., \Sigma_d),$$

$$\gamma_{k+1,\omega \pm 2e}^{X\#\overline{CP^2}}(\Sigma_1, ..., \Sigma_d) = \pm\gamma_{k,\omega}^X(\Sigma_1, ..., \Sigma_d),$$

$$\gamma_{k+1,\omega}^{X\#\overline{CP^2}}(\Sigma_i, ..., \Sigma_d, e) = \pm\gamma_{k,\omega}^X(\Sigma_1, ..., \Sigma_d).$$

Remarks.

1.Signes in the r.h.s. of the formulas are determined by the choice of orientation.

2.Considering the case $b_2^+(X) = 1$ one has to take into account the chamber structure i.e. the fact that stretching the neck of the connected sum $X\#\overline{CP^2}$ provides periods of our metric tending to the hyperplane of orthogonals to e in the limit. However in the case of Spin-polynomials this hyperplane is not a wall by the Proposition 1.4.2 of [PT] and the negative definiteness of the intersection form on $H^2(\overline{CP^2})$. Therefore any chamber in the period space $\mathcal{P}(X)$ for metrics on X is embedded in a unique chamber in the period space $\mathcal{P}(X\#\overline{CP^2})$.

§2.Application

Let S_r and S_g be a rational surface and a surface of general type with $p_g(S_g) = 0$ resp. We assume the rational surface to be a good generic surface (cf. [FM, 1.2.1]). Denote by $p : S_g \to S_{mg}$ the map to the minimal model of S_g and by $\{L_i\}$ a collection of exceptional fibres of this map. For $e \in H^2(S_r, \mathbb{Z}), (e)^2 = -1$ the wall W^e will be a hyperplane of orthogonals to e in $H^2(S_r, \mathbb{Z})$. One says that the wall is ordinary if the restriction of the intersection form to the wall is odd, and extraordinary otherwise. For $e, (e)^2 = -1$, the reflection R_e in the wall W^e is defined by $R_e(x) = x + 2(x.e)e$. It follows from [FM, Prop.3.2.4] that if e is represented by an embedded (-1)-sphere, i.e. there is an embedded sphere S^2 with the fundamental class Poincare dual to $e, (e)^2 = -1$, there is an orientation preserving self-diffeomorphism of S_r inducing R_e. This is the case in particular for any $e \in \mathcal{E}(S_r)$ where $\mathcal{E}(S_r)$ is a set of all cohomology classes represented by exceptional (-1)-curves on S_r. We shall use the notions of a P-cell and a super P-cell as defined in [FM, ch2].

Lemma 2.1. [1] *Suppose f is a hypothetical diffeomorphism*

$$S_r \to S_g.$$

There exists a self-diffeomorphism ψ of S_r and a collection of exceptional curves E_i on S_r such that for all (-1)-curves L_i on S_g we have

$$\psi^* f^* l_i = e_i$$

with $l_i = [L_i], e_i = [E_i]$ denoting the 2-cohomology classes in $H_2(S_g), H_2(S_r)$ represented by L_i, E_i resp.

Proof. Let $\mathcal{K}$ be the Kähler cone of S_r. Take any polarisation $H \in \mathcal{K}$. Consider the projection $\prod_i \frac{(1+R_{f^*l_i})}{2}$ onto the subspace $\cap_i W^{f^*l_i}$ and consider the point

$$p = \prod_i \frac{(1 + R_{f^*l_i})}{2} H$$

in this subspace. Since the cohomology class f^*l_i is represented by a (-1)-sphere $f^{-1}(L_i)$ the reflection $R_{f^*l_i}$ is induced by a diffeomorphism. Thus does any reflection of the type $R_I = \prod_{i \in I} R_{f^*l_i}$. Therefore by Thm.10 of [FM] it follows that R_I leave the super P-cell $\mathbb{S}(\mathcal{K})$ invariant and $R_I(H) \in \mathbb{S}(\mathcal{K})$. By the convexity of the supercell (cf. Prop. 2.5.6 of [FM]) one has therefore $p \in \mathbb{S}(\mathcal{K})$.

Take any P-cell $\mathcal{P}$ in $\mathbb{S}(\mathcal{K})$ which contains the point $p \in \mathcal{P}$ under consideration. By the definition of the super P-cell there exists an isometry ψ in the subgroup of isometries generated by reflections in the set $\mathcal{F}_{\mathcal{K}}$ of ordinary walls of $\mathcal{K}$ such that $\psi(\mathcal{P}) = \mathcal{K}$. But these latter reflections are induced by diffeomorphisms since all such walls are given by (-1)-curves on S_r (cf. Prop.3.6 and Thm10B of [FM]). Therefore, ψ is induced by a diffeomorphism.

Now, any wall W^e for $e = \psi^* f^* l_i$ contains the point $\psi^*(p)$ of the P-cell $\mathcal{K}$. In fact W^e is a wall of the P-cell $\mathcal{K}$. To establish this fact one needs to show that the intersection $W^e \cap \mathcal{K}$ containes an open subset of W^e. By Proposition 3.4 of [FM]

$$\mathcal{K} = \{y \in \mathbb{H}_+ | y \cdot (-K_{S_r}) \geq 0,\ y \cdot h \geq 0 \text{ for all classes } h \in \mathcal{E}(S_r)\}.$$

Therefore

$$W^e \cap \mathcal{K} = \{y \in \mathbb{H}_+ | y \cdot pr(-K_{S_r}) \geq 0,\ y \cdot pr(h) \geq 0 \text{ for all classes } h \in \mathcal{E}(S_r)\},$$

[1] Proved also in [FQ2] as Theorem 1.7. The arguments of our proof are a minor modification of the arguments in [FM] and are of course similar to those in [FQ2].

where pr is the projection to W^e. Let $E_1, ..., E_k$ be the set of all exceptional curves satisfying $q \in W^{[E_i]}$ and assume that $\pm e \neq E_i$ (otherwise we are done). It follows from Lemma 1.11 of ch.2 of [FM] that $[E_i] = pr[E_i] \in W^e$ is a set of pairwise orthogonal elements. For some neighborhood U_q of $q \in W^e$ one has

$$U_q \cap W^e \cap \mathcal{K} = \{y \in W^e \cap U_q | y \cdot pr(-K_{S_r}) \geq 0,\ y \cdot pr([E_i]) \geq 0\},$$

and this intersection is an open set in W^e if the elements $pr(-K_{S_r}), [E_i] \in W^e$ are linearly independent. But if this is not the case one has $K_{S_r} = \pm ae$. This is impossible since $W^{K_{S_r}}$ is even and e defines an ordinary wall unless S_g is homeomorphic to $(CP^1 \times CP^1)\#\overline{CP^2}$ and in this case $K^2_{S_g} = 7$ is not a complete square.

So $W^{\psi^* f^* l_i}$ is a collection of ordinary walls of the P-cell $\mathcal{K}$ with a nonempty intersection $\cap W^{\psi^* f^* l_i} = \psi^* f^* H^2(S_{gm})$, and for any such collection one has $\psi^* f^* l_i = [E_i]$, as follows from Prop3.6, ch.2 of [FM].

Corollary 2.2. *No surface of general type can be diffeomorphic to a rational one.*

Proof. Assume that there exists a diffeomorphism $f : S_r \to S_g$. By Lemma 2.1 one can also assume that there exists a blow-down $S_r \to \tilde{S}$ such that $f^*(H^2(S_{mg}) = H^2(\tilde{S})$. It follows from [PT] that there are classes $p_1 \in H^4(S_{mg}), \beta \in H^2(S_{mg}), p_1 > -8$ and a chamber $\mathcal{C} \subset \mathcal{P}(S_{mg})$ such that

$$\gamma^{S_{mg}}_{p_1,\beta,\mathcal{C}} \neq 0,\ \gamma^{\tilde{S}}_{p_1,f^*\beta,f^*\mathcal{C}} = 0.$$

Now by Theorem we have a contradiction

$$0 \neq \gamma^{S_{mg}}_{p_1,\beta,\mathcal{C}} = \gamma^{S_g}_{p_1,\beta+\sum l_i,\mathcal{C}} = \pm\gamma^{S_r}_{p_1,f^*\beta+\sum e_i,f^*\mathcal{C}} = \pm\gamma^{\tilde{S}}_{p_1,f^*\beta,f^*\mathcal{C}} = 0.$$

Acknowledgment. The author is grateful to Frank-Olaf Schreyer for the kind invitation and the Mathematical Institute of the University of Bayreuth for their hospitality during research on this project. The author is also grateful to Stefan Bauer for his kind invitation and to the SFB 343 at Bielefeld University for the support during the translation of this text. Thanks to Rogier Brussee and Manfred Lehn for help with the translation of this article and once more to Rogier Brussee for pointing out the wrong integer factor in one of the glueing formulas.

References

[BP] S.Bauer and V.Pidstrigatch, *Spin polynomial invariants for Dolgachev survaces*, alg-geom/9311008.

[D] S.Donaldson, *Polynomial invariants for smooth four-manifolds*, Topology **29** (1990), 257-315.

[DKr] S.Donaldson and P.Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, Oxford University Press, 1990.

[FM] R.Friedman and J.W.Morgan, *On the diffeomerphism types of certain algebraic surfaces I*, J. Diff. Geom. **27** (1988), 297-369.

[FQ1] R.Friedman and Z.Qin, *The smooth invariance of the Kodaira dimension of a complex surface*, Mathematical Research Letters 1 (1994), 369-376.

[FQ2] R.Friedman and Z.Qin, *On complex surfaces diffeomorphic to rational surfaces*, alg-geom/9404010.

[H] N.J.Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974), 1-54.

[K] D.Kotschick, *On manifolds homeomorphic to $CP^2\#8\overline{CP^2}$*, Invent. Math. **95** (1989), 591-600.

[MM] J.Morgan and T.Mrowka, *A note on Donaldson's polynomial invariants*, Int. Math. Research Notices **10** (1992), 222-230.

[OVdV] C.Okonek and A. Van de Ven, *Γ-invariants associated to $\mathbb{P}U(2)$-bundles and the differentiable structure of Barlow surface.*, Invent.Math. **95** (1989), 602-614.

[P] V.Pidstrigatch, *Deformation of instanton surfaces*, Math USSR Izv. **38** (1992), 313-331.

[PT] V.Pidstrigatch and A.Tyurin, *Invariants of the smooth structure of an algebraic surface arising from the Dirac operator*, Russian Acad. Sci. Izv. Math. **40** (1993), 267-351.

[T] A.Tyurin, *The Spin-polynomial invariants of the smooth structures of algebraic surface*, Izv. Russ. Acad. Nauk, Ser. Mat. **57** (1993) (Russian); English transl. in Mathematica Gottingensis **6** (1993); and to appear in Russian Acad. Sci. Izv. Math. (1994).

[Q1] Z.Qin, *Complex structures on certain differentiable 4-manifolds*, Topology **32** (1993), 551-566.

[Q2] Z.Qin, *On smooth structures of potential surfaces of general type homeomorphic to rational surfaces*, Invent.Math. **113** (1993), 163-175.

[VdV] A. Van de Ven, *On the differentiable structure of certain algebraic surfaces*, Sem. Bourbaki **667** (Juin 1986).

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