# BOUNDS FOR CHARACTERISTIC FUNCTIONS OF POWERS OF ASYMPTOTICALLY NORMAL VARIABLES

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Abstract. We prove a partial extension of Vinogradovs estimates of trigonometric sums to the case of random variables. These generalizations are useful in limit theorems in probability theory and mathematical statistics.

## 1. Introduction and main results

The results in this paper extend the results in Götze et al. (1994) where upper bounds for $|\mathbb{E}\exp\{itf(S_n)\}|$ for polynomials $f$ and normed sums $S_n$ of i. i. d. random vectors $X_1, \dots, X_n$ were proved assuming that the distribution of the summands had either a discrete or an absolutely continuous component without requiring moment conditions.

Under those assumptions it was possible to apply Vinogradov's estimates of trigonometric sums directly to a reduction of the original problem to a sum of Bernoulli random variables, respectively, to a sum of random variables which admit a Lebesgue density.

In this paper we will prove Vinogradov type estimates for the characteristic function of the $k$th power of sums of i. i. d. random variables with 'regular' distribution. This means that the random variables have a positive variance and a finite third absolute moment.

More generally the estimates hold for c. f. of the $k$th power of random variables $S_n$ which are sums of two independent r. v. $U_n$ and $V_n$ which satisfy local upper concentration inequalities compared with Lebesgue measure and some uniformity condition. (see the Remarks following Theorem 1).

¿From now on we denote by $X_1, X_2, \dots$ a sequence of i. i. d. real random variables with $\beta = \mathbb{E}|X_1|^3 < \infty$.

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Define $\sigma^2 = DX_1$ and

$$S_n = n^{-1/2}\sigma^{-1}\sum_{i=1}^{n}(X_i - \mathbb{E}X_1).$$

Let $m, k$ be integers, $k > 1$, $m > 1$.

Let $B > 0$ and $\overline{x} \triangleq (x_1, \ldots, x_m, x_1', \ldots, x_m')$. We define a set in $\mathbb{R}^{2m}$ by the formula

$$\begin{aligned} &A(m,k,\overline{x},B) \\ &\quad\triangleq \left\{ \overline{x} \in \mathbb{R}^{2m} : -0.5 < \sum_{j=1}^{m}(x_j^i - (x_j')^i) \leq 0.5, \right. \\ &\quad \left. -B < x_i \leq B, -B < x_i' \leq B, i = \overline{1, \min(m;k)} \right\}. \end{aligned}$$

We say that a random variable $X$ satisfies the condition $CF(k)$ if for all $m = 1, 2, \ldots k, 2k, 3k, \ldots$ and for independent copies $X_1, \ldots, X_m, X_1^1, \ldots, X_m^1$ of $X$ there exists a constant $c(k,m)$ depending only on $k$ and $m$ such that for every $B > 0$ and all $0 < a \leq B^2$ we have

$$\begin{aligned} &\mathbb{P}(\mathcal{A}(m,k,\bar{X}_a,B)) \\ &\quad \leq c(k,m) \sup_{|L|\leq B} \mathbb{P}(\mathcal{A}(m,k,\bar{X}_L,B)), \end{aligned}$$

where $\bar{X}_a$ is a "shifted" random vector $\bar{X}$, that is

$$\bar{X}_a \stackrel{d}{=} (X_1 - a, \ldots, X_m - a, X_1' - a, \ldots, X_m' - a).$$

**Theorem 1.** Let $X_1, X_2, \ldots$ be i. i. d. random variables with $\beta = \mathbb{E}|X_1|^3 < \infty$. Then for any positive integer $k$ there exist absolute constants $c_1, c_2, c_3$ and a constant $c_4$ depending on $\beta$ such that for all $n \geq 1$

$$\begin{aligned} &|\mathbb{E}\exp\{itS_n^k\}| \\ &\quad \leq c_4 n^{-c_1/(k^3 ln(k+1))}(1 + \sqrt{n}/|t|)^{c_2/(k^2 ln(k+1))} \end{aligned} \tag{0-1}$$

provided that

$$0 < |t| < c_3 n^{(k-0.5)/2} \tag{0-2}$$

and $S_n(\sqrt{n})$ satisfies the condition $CF(k)$, where

$$S_n(\sqrt{n}) = n^{-1/2}\sigma^{-1}\sum_{1}^{n}(X_i - \mathbb{E}X_i)\chi_{i,\sqrt{n}}$$

and $\chi_{i,u}$ denotes the indicator function of $\{|X_i - \mathbb{E}X_i| < \sigma u\}$.

*Remark.* The range for $t$ can be enlarged to $c_3 n^{k/2-\varepsilon}$ for any positive $\varepsilon$. However, in this case the constants $c_1$ and $c_4$ will additionally depend on $\varepsilon$.

*Remark.* The bound in Theorem 1 actually depends only on the following properties of the distribution of the r.v. $S_n$.

(1) The existence of arbitrary high moments of $S_n$ or of an approximation $\widetilde{S}_n$ of $S_n$ with error $\mathcal{O}(n^{-1/2})$.
(2) A decomposition into two independent parts $S_n \stackrel{d}{=} S_{n1} + S_{n2}$ such that $Var(S_{n2})$ can be arbitrary small (as $n \to \infty$). (c.f. 8-2)
(3) Concentration inequalities $\mathbb{P}(|S_{nj}| < \varepsilon Var(S_{nj})^{1/2}) = \mathcal{O}(\varepsilon + n_j^{-1/2})$ for $j = 1, 2$.
(4) $S_{n1}$ should satisfy the condition $CF(k)$.

Thus the results may be generalized to r.v. with such properties.

**Theorem 2.** (cf. Theorem 1, ch. 4 in Vinogradov (1985) and Theorem 1, §1, ch. III in Arkhipov et al. (1987). Let $S$ denote a random variable, let $P, k, \tau, m$ denote positive integers, $k > 10$. Define $f(x) = \alpha_k x^k + \cdots + \alpha_1 x$ and

$$\mathcal{I}_m(P) = \int_{-1}^{1} \cdots \int_{-1}^{1} |\mathbb{E}\exp\{if(S)\} \cdot 1_{\{-P<S\le P\}}|^{2m} d\alpha_k \ldots d\alpha_1, \tag{0-3}$$

where $1_{\{A\}}$ denotes the indicator function of an event $A$.

Assume that $S$ satisfies the condition $CF(k)$.

Then for all integers $m = k\tau$ there exists a constant $D = D(k, \tau)$ depending only on $k$ and $\tau$ such that

$$\mathcal{I}_m(P) \le DP^{2m-\triangle(\tau)} \left(\sup_I \mathbb{P}(S \in I)\right)^{2m}, \tag{0-4}$$

where

$$\triangle(\tau) = 0.5k(k+1)(1 - (1 - 1/k)^\tau)$$

and the supremum is taken over all intervals $\{x : a < x \le a + 1\}$ for arbitrary values of $a \in \mathbb{R}$.

**Theorem 3.** Let the assumptions of Theorem 2 be satisfied. Then for all $m, 1 \le m \le k$,

$$\mathcal{I}_m(P) \le MP^{(3m-1)/2}(\sup_I \mathbb{P}(S \in I))^{2m}, \tag{0-5}$$

where $M$ is a constant depending only on $k$ and $m$.
The proofs of Theorems 1, 2 and 3 are given in sections 3, 4 and 5 respectively. They are based on the Lemmas of section 2.

## 2. LEMMAS

In the following the phrase "an interval of length $l$" means that we consider a set of the form $\{x \in \mathbb{R}^1 : a < x \leq a + l\}$ for an arbitrary $a \in \mathbb{R}$.

**Lemma 1.** *Let $k > 2$, $H = 2(16.48\pi^2)^k(k-1)^{k-2}/(k-2)!$, $P > H^k$ and let $R$ be the smallest integer such that $HR \geq P$. Assume that for some $\omega$ with $0 \leq \omega < P$ there are points $X_i, Y_i \triangleq X_i + Rl$, $i = 1, 2, \ldots, k$, satisfying for some $l \geq 1$ the conditions*

$$-\omega l < X_1, Y_1 + Rl \leq X_2, \ldots, Y_k \leq (P - \omega)l.$$

*Cover every interval $(X_i, Y_i]$ by $\bigcup\limits_{j=1}^{N}(a_{ij}, b_{ij}]$, where*

$$b_{ij} \triangleq a_{ij+1}, b_{ij} - a_{ij} = l, a_{i1} \leq X_i, b_{iN} \geq Y_i, N \leq R + 2.$$

*We say that the vectors $v = (v_1, \ldots, v_k)$ and $v' = (v_1', \ldots, v_k')$ with $X_i < v_i \leq Y_i$, $X_i < v_i' \leq Y_i$, $i = 1, 2, \ldots, k$, belong to the same class $V$ if for every $i = 1, 2, \ldots, k$ there exists an index $j_0 = j_0(i)$ such that*

$$v_i \in (a_{ij_0}, b_{ij_0}] \ \text{and} \ v_i' \in (a_{ij_0}, b_{ij_0}].$$

*Then the number $E_1$ of classes $V$ for which there exists a vector $v \in V$ such that the sums*

$$V_1 = v_1 + \cdots + v_k, \ldots, V_k = v_1^k + \cdots + v_k^k$$

*lie in some intervals of lengths, respectively*

$$l^1 P^0, \ldots, l^k P^{k-1}, \tag{1-1}$$

*satisfies the inequality*

$$E_1 \leq c_1(k), \tag{1-2}$$

*where $c_1(k)$ is a constant depending only on $k$.*
*Let*

$$V_1' = v_1' + \cdots + v_k', \ldots, V_k' = (v_1')^k + \cdots + (v_k')^k$$

*where the vector $(v_1', \ldots, v_k')$ belongs to some class $V'$. Then the number $E$ of classes $V$ and $V'$ such that for some vectors $v \in V$ and $v' \in V'$ the differences*

$$V_1 - V_1', \ldots, V_k - V_k'$$

*lie in some intervals of lengths, respectively*

$$lP^{1-1/k}, \ldots, (lP^{1-1/k})^k, \tag{1-3}$$

*satisfies the inequality*

$$E \leq c_2(k) P^{(3k-1)/2} \tag{1-4}$$

*with some constant $c_2(k)$ depending only on $k$.*

**Remark.** Lemma 1 is a generalization of the corresponding Vinogradov Lemma (see Lemma 3, ch. 4 in Vinogradov (1985)). In the proof of Lemma 1 we use essentially the same arguments as Vinogradov's proof together with some additional arguments (see also Lemma 6, §1, ch. 3 in Arkhipov et al. (1987)).

**Proof.** In the proof we denote by $c$ with indices and without them the constants which may depend only on $k$. The same symbol can be used for different constants. First we shall evaluate $E_1$. Let $s$ be an integer such that $1 < s \leq k$. If for given $v_{s+1}, \dots, v_n$ the sums $V_1, \dots, V_n$ lie in certain intervals of lengths (1-1) then the sums

$$v_1 + \cdots + v_s, \dots, v_1^s + \cdots + v_s^s$$

lie in certain intervals of lengths

$$c_1 l^1 P^0, \dots, c_s l^s P^{s-1}.$$

Let $(\eta_1, \dots, \eta_s)$ and $(\eta_1 + \xi_1, \dots, \eta_s + \xi_s)$ denote two vectors with this property and such that

$$\eta_s \in (a_{sj_0}, b_{sj_o}], \qquad \eta_s + \xi_s \in (a_{sj_1}, b_{sj_1}],$$

where $j_1 > j_0$, i.e. $\xi_s > 0$, and the vectors $\eta$ and $\eta + \xi$ belong to different classes of vectors.

According to our assumptions we have

$$\frac{(\eta_1+\xi_1)-\eta_1}{\xi_1} \cdot \xi_1 \quad + \cdots + \quad \frac{(\eta_s+\xi_s)-\eta_s}{\xi_s} \cdot \xi_s = c_0 l$$

$$\cdots$$

$$\frac{(\eta_1+\xi_1)^s-\eta_1^s}{s\xi_1} \cdot \xi_1 \quad + \cdots + \quad \frac{(\eta_s+\xi_s)^s-\eta_s^s}{s\xi_s} \cdot \xi_s = c_s l^s P^{s-1},$$

Hence we find

$$\triangle \xi_s - \triangle' = 0, \tag{1-5}$$

where

$$\triangle = \begin{vmatrix} \frac{(\eta_1+\xi_1)-\eta_1}{\xi_1} & \cdots & \frac{(\eta_s+\xi_s)-\eta_s}{\xi_s} \\ & \cdots & \\ \frac{(\eta_1+\xi_1)^s-\eta_1^s}{s\xi_1} & \cdots & \frac{(\eta_s+\xi_s)^s-\eta_s^s}{s\xi_s} \end{vmatrix}$$

and

$$\triangle' = \begin{vmatrix} \frac{(\eta_1+\xi_1)-\eta_1}{\xi_1} & \cdots & \frac{(\eta_{s-1}+\xi_{s-1})-\eta_{s-1}}{\xi_{s-1}} & c_0 l \\ & \cdots & & \\ \frac{(\eta_1+\xi_1)^s-\eta_1^s}{s\xi_1} & \cdots & \frac{(\eta_{s-1}+\xi_{s-1})^s-\eta_{s-1}^s}{s\xi_{s-1}} & c_s l^s P^{s-1} \end{vmatrix}.$$

Now we apply the following transformation to (1-5). Expand both determinants in terms of the elements of the first column using Lagrange's formula und multiply the equation by $\xi_1$. The result can be written as a difference of a certain function of $v_1$ evaluated at the points $v_1 = \eta_1 + \xi_1$ and $v_1 = \eta_1$. By the mean value theorem we obtain a new equality for the derivative of that function at some intermediate point $X_1 < x_1 < Y_1$. This leads to a new equation like (1-5) where the elements of the first column of each determinant are replaced by $1, \dots, x_1^{s-1}$. By further transformations with respect to subsequent columns up to the $(s-1)$-th column and finally the last column where only the first determinant is transformed, we obtain

$$\triangle_s \xi_s - \triangle_s' = 0,$$

where

$$\triangle_s = \begin{vmatrix} 1 & \dots & 1 \\ & \dots & \\ x_1^{s-1} & \dots & x_s^{s-1} \end{vmatrix},$$

$$\triangle_s' = \begin{vmatrix} 1 & \dots & 1 & c_0 l \\ & \dots & & \\ x_1^{s-1} & \dots & x_{s-1}^{s-1} & c_s l^s P^{s-1} \end{vmatrix}$$

and $X_i < x_i < Y_i, \quad i = 1, 2, \dots, s.$

Thus we obtain by the well-known formula for Vandermonde's determinant

$$\triangle_s = \prod_{1 \le i < j \le s} (x_i - x_j)$$

the equality

$$\triangle_s' = c \sum_{r=0}^{s-1} l^{r+1} P^r U_r,$$

where $U_r$ is the coefficient of $x_s^r$ in the expansion of

$$\triangle_s = (x_s - x_1) \dots (x_s - x_{s-1}) \triangle_{s-1}$$

in powers of $x_s$ and hence equals the product of $\triangle_{s-1}$ and the sum of all products of $(s-1-r)$ different numbers chosen from $-x_1, \dots, -x_{s-1}$.

Our assumptions about the the location of the numbers $x_j$ imply

$$U_r \le c \triangle_{s-1} P^{s-1-r},$$
$$\xi_s \le c \frac{l^s P^{s-1}}{(x_s - x_1) \dots (x_s - x_{s-1})},$$

where $x_i \in (X_i, Y_i]$. Since for $t \ge 1$ we have

$$x_{j+t} - x_j \ge (2t-1) R l,$$

we get

$$\xi_s < cl.$$

Thus, we found that $v_s$ ( for $s > 1$ and given $v_{s+1}, \ldots, v_k$) cannot be an element of more than $c$ different classes of vectors. Since $v_1$ for given $v_2, \ldots, v_k$ lies in an interval of length $l$ (recall that we consider the intervals of type $(a, b], a < b$ ), $v_1$ cannot be element of more than 2 classes of vectors $V$. Thus we get (1-2)
Let $E'$ denote the number of classes $X$ such that the sums $V_1, \ldots, V_k$ lie in some intervals of lengths (1-3) respectively. In order to derive a bound for $E'$ from (1-2) it is sufficient to note that

$$\left(\frac{lP^{1-1/k}}{l} + 1\right) \cdot \ldots \cdot \left(\frac{(lP^{1-1/k})^{k-1}}{l^{k-1}P^{k-2}} + 1\right) < eP^{(k-1)/2}.$$

Thus,

$$E' < cP^{(k-1)/2}. \tag{1-6}$$

Finally, note that the number of all classes $V$ is not larger than

$$(R+2)^k < (P/H+3)^k < cP^k$$

This together with (1-6) yields the desired inequality (1-4) which completes the proof of Lemma 1.

Let $\mu$ be a nonnegative finite Borel measure on $\mathbb{R}^k$. Define

$$Q_\mu(l) = \sup \mu(I(l)),$$

where supremum is taken over all rectangles of the form

$$I(l) = I(l; a_1, \ldots, a_k) = \{x \in \mathbb{R}^k : a_i - l/2 < x_i \leq a_i + l/2; i = 1, 2, \ldots, k\}$$

with any real $a_i$.

**Lemma 2.** (cf. Lemma 6.1 in Esseen (1968) and Lemmas 2.5 and 2.6 in Zaitsev (1989)) *Define* $(t, x) \triangleq \sum_{j=1}^k t_j x_j$.*Assume that the function*

$$\Psi_\mu(t) = \int\limits_{\mathbb{R}^k} \exp\{i(t, x)\}\mu(dx)$$

*is nonnegative. Then for any* $l > 0$, $b > 0$ *and* $a$ *such that* $0 < al \leq 1$ *we have*

$$[\pi^{-2}l(2+bl)^{-1}]^k \int\limits_{|t_j| \leq b/2} \Psi_\mu(t)dt \leq Q_\mu(l) \leq (1.03a^{-1})^k \int\limits_{|t_j| \leq a} \Psi_\mu(t)dt. \tag{2-1}$$

*Here the expression*

$$\int_{|t_j|\le a} \dots dt$$

*means the integration over the region* $\{t \in \mathbb{R}^k : |t_j| \le a, j = \overline{1,k}\}$.

**Remark.** Lemma 2 is similar to Lemma 2.2.1 in Hengartner and Theodorescu (1973) proved in the onedimensional case.

**Proof.** Define $U(x) = \left(\frac{\sin(x/2)}{x/2}\right)^2$ and $u(t) = 1 - |t|$, if $|t| \le 1$, and $u(t) = 0$ otherwise. It is easy to check that

$$\int_{\mathbb{R}} U(x)dx = 2\pi$$

and

$$u(t) = (2\pi)^{-1} \int_{\mathbb{R}} e^{-itx} U(x) \qquad dx.$$

Similar as in the proof of Lemma 2.2.1 in Hengartner and Theodorescu (1973) we shall use the following equalities

$$\int_{\mathbb{R}^k} \prod_{j=1}^{k} U(a(x_j - z_j))d\mu(x)$$

$$= a^{-k} \int_{|t_j|\le a} \Psi_\mu(t) \prod_{j=1}^{k} u(t_j/a) \exp\{-i(t,z)\}dt \tag{2-2}$$

and for any $b > 0$

$$\nu(I_1(b)) = (\sqrt{2}\pi b)^{-k} \int_{|t_j|\le b} \Psi_\mu(t) \prod_{j=1}^{k} (1 - |t_j|/b) \frac{\sin(t_j/b)}{t_j/b} dt, \tag{2-3}$$

where $I_1(b) = \{x \in \mathbb{R}^k : -b^{-1} < x_j \le b^{-1},\ j = 1, 2, \dots, k\}$ and $\nu$ is a measure with Fourier transform

$$\Psi_\mu(t) \prod_{j=1}^{k} u(t_j/b).$$

In order to derive the upper bound in (2-1) for $Q_\mu(l)$ we shall use the equality (2-2). Note that $\sin x/x \ge \sqrt{0.978}$ for $|x| \le 1/4$. Therefore, under the assumptions of Lemma 2 we get for any fixed $z_j, j = 1, 2, \dots, k$,

$$\int_{\mathbb{R}^k} \prod_{j=1}^{k} U(a(x_j - z_j))d\mu(x) \ge 0.978^k \mu(I_2(z)), \tag{2-4}$$

where

$$I_2(z) = \{x \in \mathbb{R}^k : z_j - l/2 < x_j \le z_j + l/2, \ \ j = 1, \dots, k\}.$$

The upper bound (2-1) for $Q_\mu(l)$ follows from (2-2) and (2-4).

We now prove the lower bound for $Q_\mu(l)$. Since $\sin x/x \ge 2/\pi$ for $|x| \le \pi/2$, the equality (2-3) implies

$$Q_\nu(2b^{-1}) \ge (\pi^2 b)^{-k} \int\limits_{|t_j| \le b/2} \Psi_\mu(t)dt. \tag{2-5}$$

Furthermore, for any $l > 0$

$$Q_\nu(2b^{-1}) \le (2b^{-1}l^{-1} + 1)^k Q_\nu(l). \tag{2-6}$$

Since $Q_\nu(l) \le Q_\mu(l)$, (2-5) and (2-6) together imply the lower bound in (2-1) for $Q_\mu(l)$. This completes the proof of Lemma 2.

**Corollary of Lemma 2.** *Under the assumptions of Lemma 2 we have*

$$\left(\frac{l}{4\pi^2}\right)^k \int\limits_{|t_j| \le l^{-1}} \Psi_\mu(t)dt \le Q_\mu(l) \le (1.03l)^k \int\limits_{|t_j| \le l^{-1}} \Psi_\mu(t)dt. \tag{2-7}$$

**Proof.** Choose $b = 2/l$ and $a = 1/l$ in (2-1).

**Lemma 3.** *Let $X$ and $Y$ denote independent random variables and assume $\mathbb{E}X = 0$. For any real $t$ we have*

$$|\mathbb{E}(\exp\{it(X+Y)\} - \exp\{itY\}| \le t^2 \mathbb{E}X^2/2.$$

**Proof** The result follows by a Taylor expansion.

Recall that

$$S_n = n^{-1/2} \ \ \sigma^{-1} \sum_1^n (X_i - \mathbb{E}X_i).$$

Without loss of generality we will assume in the following that $\mathbb{E}X_1 = 0$, $\mathbb{E}X_1^2 = 1$ since we may in general use the transform $(X_i - \mathbb{E}X_i)/\sigma$ instead of $X_i, i = 1, 2, \dots$.

Let $\chi_{j,L}$ denote the indicator function of $\{|X_j| < L\}$. For any $u > 0$ define

$$S_n(u) = n^{-1/2} \sum_1^n X_i \chi_{i,u}.$$

**Lemma 4.** (see Lemma 7 in Sazonov and Ulyanov (1993)) *For any real measurable function $f$, any $u_1, u_2$ such that $0 \le u_2 \le u_1 \le \infty$ and any random variable $Z$ we have*

$$\begin{aligned}\triangle := &|\mathbb{E}[f(S_n(u_1)+Z) - f(S_n(u_2)+Z)]| \\ &\le n\mathbb{E}|f(S_n(u_1)+Z) - f(S_n(u_2)+Z)|(1-\chi_{1,u_2})\end{aligned}$$

*provided that the expectations exist.*

**Proof.** Below $\mathbf{1}_A$ denotes the indicator function of an event $A$. We have

$$\begin{aligned}\triangle &= |\mathbb{E}[f(S_n(u_1)+Z) - f(S_n(u_2)+Z)]\left(\mathbf{1}_{\cap\{|X_i|<u_2\}} + \mathbf{1}_{\cup\{|X_i|\ge u_2\}}\right)| \\ &\le \mathbb{E}|f(S_n(u_1)+Z) - f(S_n(u_2)+Z)|\sum_1^n (1-\chi_{i,u_2}) \\ &= n\mathbb{E}|f(S_n(u_1)+Z) - f(S_n(u_2)+Z)|(1-\chi_{1,u_2}).\end{aligned}$$

**Corollary of Lemma 4.** *Let $\bar{S}_n \triangleq n^{-1/2}\sum_1^n X_j\chi_{j,\sqrt{n}}$. Then for any $m > 0$*

$$|\mathbb{E}\exp\{itF(S_n)\} - \mathbb{E}\exp\{itF(\bar{S}_n)\}| \le 2n\mathbb{E}(1-\chi_{1,\sqrt{n}}).$$

**Lemma 5.** *Let $\bar{S}_n \triangleq n^{-1/2}\sum_1^n X_j\chi_{j\sqrt{n}}$. Then for any $\alpha < 3/4$ we have*

$$\mathbb{E}\exp\{\alpha|\bar{S}_n|\} \le c(\alpha) < \infty,$$

*where $c(\alpha)$ is a constant depending on $\alpha$ and $\mathbb{E}|X_1|^3$.*

**Proof.** By Bernstein's inequality (see e. g. Bernstein (1946)) for i.i.d. random variables $Z_1, Z_2, \dots$ such that $\mathbb{E}Z_1 = 0$, $\mathbb{E}Z_1^2 = \sigma^2$ and $|Z_j| \le L < \infty$ we have

$$\mathbb{P}(|Z_1 + \cdots + Z_n| > t\sigma\sqrt{n}) \le 2\exp\left(-\frac{t^2}{2(1+Lt/(3\sigma\sqrt{n})}\right). \tag{5-1}$$

This together with the same arguments as those used in the proof of Lemma 2.1 in Bentkus (1984) yields the result of Lemma 5.

**Lemma 6.** *Let $X$ and $Y$ be independent random vectors taking values in $\mathbb{R}^l$ and $\mathbb{R}^m$ resp. Let $f = (f_1, \ldots, f_k) : \mathbb{R}^l \to \mathbb{R}^k$ and $g = (g_1, \ldots, g_k) : \mathbb{R}^m \to \mathbb{R}^k$ denote Borel measurable functions. Fix a measurable sets $A \subset \mathbb{R}^l$, $B \subset \mathbb{R}^m$. and let $d_i(x), D_i(x)$, $i = \overline{1,k}$ denote some measurable functions of $x \in A$.*

*Then for any real $a_i, i = \overline{1,k}$, we get*

$$\begin{aligned} &\mathbb{P}(a_i < f_i(X) + g_i(Y) \le a_i + 1, d_i(X) < g_i(Y) \le D_i(X), \\ &\quad i = \overline{1,k}, X \in A, Y \in B) \\ &\le \sup_1 \mathbb{P}\left(b_i < g_i(Y) \le b_i + 1, \ \ i = \overline{1,k}, Y \in B\right) \\ &\quad \times \mathbb{P}\left(a_i - D_i(X) < f_i(X) \le a_i - d_i(X) + 1, i = \overline{1,k}, X \in A\right), \end{aligned} \tag{6-1}$$

*where $\sup_1$ denotes the supremum over all numbers $b_i, i = \overline{1,k}$.*

**Proof.** Put

$$A_i \triangleq \{x : \ \ a_i - D_i(x) < f_i(x) \le a_i - d_i(x) + 1\},$$

We have

$$\begin{aligned} &\mathbb{P}(a_i < f_i(X) + g_i(Y) \le a_i + 1, d_i(X) < g_i(Y) \le D_i(X), \\ &\quad i = \overline{1,k}, \ \ X \in A, Y \in B) \\ &= \int \mathbb{P}\Big(a_i < f_i(x) + g_i(Y) \le a_i + 1, \\ &\qquad d_i(x) < g_i(Y) \le D_i(x), i = \overline{1,k}, Y \in B\Big) \mathbf{1}_A(x) \mathbb{P}_X(dx) \\ &\le \sup_1 \mathbb{P}\left(b_i < g_i(Y) \le b_i + 1, i = \overline{1,k}, Y \in B\right) \\ &\qquad \times \int \mathbf{1}_A(x) \prod_{i=1}^{k} \mathbf{1}_{A_i}(x) \mathbb{P}_X(dx). \end{aligned} \tag{6-2}$$

Thus (6-1) follows from (6-2).

**Lemma 7.** *Let $k$ and $m$ be natural integers. Denote by $I_i(b), i = \overline{1,k}$, an interval of length $b > 0$. The same symbol may be used in the following for different intervals of the same length. Assume that there exist real numbers $x_i, y_j, \bar{x}_i, \bar{y}_j$ and $a$ such that*

$$\begin{aligned} &x_1 - \bar{x}_1 + \cdots + x_k - \bar{x}_k \\ &\quad + (y_1 + a) - (\bar{y}_1 + a) + \cdots + (y_m + a) - (\bar{y}_m + a) \in I_1(1), \end{aligned} \tag{7-1}$$

$$\cdots\cdots$$

$$\begin{aligned} &x_1^k - \bar{x}_1^k + \cdots + x_k^k - \bar{x}_k^k \\ &\quad + (y_1 + a)^k - (\bar{y}_1 + a)^k + \cdots + (y_m + a)^k - (\bar{y}_m + a)^k \in I_k(1) \end{aligned} \tag{7-k}$$

*and for some* $P \geq 1$

$$|y_j| \leq P^{1-1/k}, \;\; |\bar{y}_j| \leq P^{1-1/k}, \;\; j = \overline{1, m}, \qquad |a| \leq P.$$

*Then*

$$(x_1 - a) - (\bar{x}_1 - a) + \cdots + (x_k - a) - (\bar{x}_k - a) \in I_1(cP^{1-1/k}), \tag{7-k+1}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$(x_1 - a)^k - (\bar{x}_1 - a)^k + \cdots + (x_k - a)^k - (\bar{x}_k - a)^k \in I_k(cP^{k(1-1/k)}) \tag{7-2k}$$

*and for* $M = \overline{2, k}$

$$\sum_{i=1}^{k} \sum_{l=1}^{M-1} (-1)^{l+1} a^l \left(x_i^{M-l} - \bar{x}_i^{M-l}\right) \cdot \binom{M}{l} + \sum_{j=1}^{m} \sum_{l=1}^{M-1} \binom{M}{l} a^l \left(y_j^{M-l} - \bar{y}_j^{M-l}\right) \in I_M(cP^{M-1}), \tag{7-2k+1}$$

*where* $c$ *is a constant depending only on* $k$ *and* $m$.

**Proof.** It suffices to prove the following result. For any real numbers $x, \bar{x}, y, \bar{y}$ and $a$ such that

$$x - \bar{x} + (y + a) - (\bar{y} + a) \in I_1(1) \tag{7-1'}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$x^k - \bar{x}^k + (y + a)^k - (\bar{y} + a)^k \in I_k(1) \tag{7-k'}$$

and

$$|y| \leq P^{1-1/k}, \;\; |\bar{y}| \leq P^{1-1/k}, \;\; |a| \leq P \tag{7-1''}$$

we have

$$(x - a) - (\bar{x} - a) \in I_1(cP^{1-1/k}) \tag{7-(k+1)'}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$(x - a)^k - (\bar{x} - a)^k \in I_k(cP^{k(1-1/k)}) \tag{7-(2k)'}$$

and for $M = \overline{2, k}$

$$\sum_{l=1}^{M-1} (-1)^{l+1} a^l (x^{M-l} - \bar{x}^{M-l}) \binom{M}{l} + \sum_{l=1}^{M-1} \binom{M}{l} a^l (y^{M-l} - \bar{y}^{M-l}) \in I_M(cP^{M-l}). \tag{7-(2k+1)'}$$

The relations (7-(k+1)$'$)-(7-(2k)$'$) follow immediately from (7-(2k+1)$'$), (7-1$''$) together with the fact that for any $M = \overline{1,k}$.

$$P^{M(1-1/k)} \geq P^{M-1}.$$

We now prove (7-(2k+1)$'$). For $M = 2$ it follows from (7-1$'$) and (7-1$''$).

In general, the proof of (7-(2k+1)$'$) is a consequence of (7-1$'$)-(7-(M-1)$'$) and the inequality $|a| \leq P$.

## 3. Proof of Theorem 1

By Lemma 4 we have

$$|\mathbb{E}\exp\{itS_n^k\}| \leq |\mathbb{E}\exp\{itS_n^k(\sqrt{n})\}| + 2\beta n^{-1/2}. \tag{8-1}$$

We have for any $\mu$ with $0 < \mu < n$

$$S_n(\sqrt{n}) \stackrel{d}{=} \sqrt{\frac{n-\mu}{n}} S_{n-\mu}\sqrt{n} + b_\mu S'_\mu(\sqrt{n}), \tag{8-2}$$

where $b_\mu \triangleq (\mu/n)^{1/2}$, $S'_\mu(\sqrt{n})$ and $S_{n-\mu}(\sqrt{n})$ are independent and $S'_\mu(\sqrt{n})$ has the same distribution as $S_\mu(\sqrt{n})$. Since we will assume that $\mu < n/2$ it is sufficient to estimate $|\mathbb{E}\exp\{it(S_n(\sqrt{n}) + b_\mu S'_\mu(\sqrt{n}))^k\}|$ instead of $|\mathbb{E}\exp\{itS_n^k(\sqrt{n})\}|$. Using the Hölder inequality we obtain for an integer $\mu > 0$

$$\begin{aligned}&|\mathbb{E}\exp\{it\left(S_n(\sqrt{n}) + b_\mu S'_\mu(\sqrt{n})\right)^k\}|^{2u} \\ \leq &\mathbb{E}_{S'_\mu(\sqrt{n})}|\mathbb{E}_{S_n(\sqrt{n})}\exp\{it(S_n(\sqrt{n}) + b_\mu S'_\mu(\sqrt{n}))^k\}|^{2u} = S^{2u},\end{aligned}$$

where

$$S^{2u} := \mathbb{E}\exp\{it\sum_{j=1}^{k}\binom{k}{j} f_j(u) b_\mu^{k-j} S'_\mu(\sqrt{n}))^{k-j}\}, \tag{8-3}$$

$$f_j(u) = S^j_{n,1} + \cdots + S^j_{n,u} - S^j_{n,u+1} - \cdots - S^j_{n,2u}$$

and $S_{n,l}, l = \overline{1,2u}$, denote independent copies of $S_n(\sqrt{n})$.

Furthermore, let $\eta_j, j = \overline{1,k}$, be independent random variables with zero mean and bounded second moment such that

$$g_j(t) := \mathbb{E}\exp\{it\eta_j\} = 0 \quad \text{for} \quad |t| \geq 1. \tag{8-4}$$

Using Lemma 3 we get (writing $S_\mu$ for short instead of $S'_\mu(\sqrt{n})$)

$$\left| S^{2u} - \mathbb{E}\exp\left\{ it \sum_{j=1}^{k} \binom{k}{j} (f_j(u) + n^{-j/2}\eta_j) b_\mu^{k-j} S_\mu^{k-j} \right\} \right|$$
$$\leq c|t|^2 \sum_{j=1}^{k} b_\mu^{2(k-j)} n^{-j} \leq c|t|^2 \mu^{(k-1)} n^{-k}, \tag{8-5}$$

where $c$ only depends on the second moments of $\eta_j$ and on the moments of $S_n(\sqrt{n})$.

We denote by $p(r_1, \ldots, r_k)$ the Lebesgue density of the random vector

$$\left( f_1(u) + n^{-1/2}\eta_1, \ldots, f_k(u) + n^{-1/2}\eta_k \right).$$

Using the Hölder inequality and the Plancherel equality we get

$$\begin{aligned}
\triangle_1 :&= \left| \mathbb{E}\exp\left\{ it \sum_{j=1}^{k} \binom{k}{j} \left( f_j(u) + n^{-j/2}\eta_j \right) b_\mu^{k-j} S_\mu^{k-j} \right\} \right| \\
&= \int \left| \mathbb{E}\exp\left\{ it \sum_{j=1}^{k} \binom{k}{j} r_j b_\mu^{k-j} S_\mu^{k-j} \right\} \right| p(r)dr \\
&\leq c \left( \int \left| \mathbb{E}\exp\left\{ it \sum_{j=1}^{k} \binom{k}{j} r_j b_\mu^{k-j} S_\mu^{k-j} \right\} \right|^2 dr \right)^{1/2} \\
&\quad \times \left( \left| \mathbb{E}\exp\left\{ i \sum_{j=1}^{k} z_j \left( f_j(u) + n^{-j/2}\eta_j \right) \right\} \right|^2 dz \right)^{1/2}.
\end{aligned} \tag{8-6}$$

Since $S_n(\sqrt{n})$ has finite exponential moments (see Lemma 5) we may restrict (up to an error of the order $\mathcal{O}(n^{-1/2})$) the region of integration of the first integral on the right-hand side of (8-6) to

$$\{|r_j| \leq c \ln^j n, \ \ j = \overline{1,k}\},$$

where $c$ depends on $u, k$ and moments of $S_n(\sqrt{n})$.

Therefore, by Lemma 2 we have

$$\int_{|r_j| \leq c\ln^j n} \left| \mathbb{E}\exp\left\{ it \sum_{j=1}^{k} \binom{k}{j} r_j b_\mu^{k-j} S_\mu^{k-j} \right\} \right|^2 dr$$
$$\leq c \ln^c n \sup \mathbb{P}\left( |t| b_\mu (S_\mu - \bar{S}_\mu) \in I \right), \tag{8-7}$$

where supremum extends over all intervals of length 1. Here $\bar{S}_\mu$ is an independent copy of $S_\mu$ and $c$ is a constant depending on $k$ only. By the central limit theorem in $\mathbb{R}^1$ and the properties of normal distribution (8-7) implies

$$\int_{|r_j|\le c\ln^j n} \left|\mathbb{E}\exp\left\{ it\sum_{j=1}^{k}\binom{k}{j} r_j b_\mu^{k-j} S_\mu^{k-j}\right\}\right|^2 dr$$
$$\le c\ln^c n\left(\mu^{-1/2}+|t|^{-1}(n/\mu)^{1/2}\right). \tag{8-8}$$

Consider now the second integral on the right-hand side of (8-6). Using (8-4) we obtain

$$\int \left|\mathbb{E}\exp\left\{ i\sum_{j=1}^{k} z_j\left(f_j(u)+n^{-j/2}\eta_j\right)\right\}\right|^2 dz$$
$$\le \int_{|z_j|\le n^{j/2}} \left|\mathbb{E}\exp\left\{ i\sum_{j=1}^{k} z_j f_j(u)\right\}\right|^2 dz$$
$$= n^{k(k+1)/4}\int_{|y_j|\le 1} \mathbb{E}\exp\left\{ i\sum_{j=1}^{k} y_j n^{j/2} f_j(2u)\right\} dy_j. \tag{8-9}$$

Choosing $u=kl$ and using Theorem 2 we obtain that the *integral* on the right-hand side of (8-6) is not larger than

$$c_l n_o^{4kl-0.5k(k+1)(1-(1-1/k)^l)}(\sup_I \mathbb{P}(\sqrt{n}S_n(\sqrt{n})\in I))^{4kl}, \tag{8-10}$$

where $n_0 = c\sqrt{n}\ln n$ and the supremum is taken over all intervals of unit length.

Applying the CLT in $\mathbb{R}^1$ and combining (8-6), (8-8)-(8-10) we get

$$\triangle_1 \le c\ln^c n(\mu^{-1/4}+|t|^{-1/2}(n/\mu)^{1/4})n^{k(k+1)(1-1/k)^l/8}.$$

Thus, choosing $l = 4kln(k+1)$ we finally obtain

$$S\le c\left[t^2\mu^{(k-1)}n^{-k}+n^{-1/2}+\ln^c n\left(\mu^{-1/4}+|t|^{-1/2}(n/\mu)^{1/4}\right)n^{1/(8k^2)}\right]^{1/(8k^2 ln(k+1))}. \tag{8-8}$$

By an appropriate choice of $\mu$ the result (0-1) follows from (8-8) provided that $t$ satisfies (0-2) and $k\ge 11$. For $k=1$ the estimate (0-1) follows from classical results. For $k=2,3,\ldots,10$ it can be proved using the so-called symmetrization inequality (see e. g. Götze (1979)).

This proves the theorem.

## 4. Proof of Theorem 2

Let $\underline{S}_1, \underline{S}'_1, \underline{S}_2, \underline{S}'_2 \dots$ denote independent copies of $S$.
Note, that the Corollary of Lemma 2 implies

$$c_1(k) Q_{P,m} \le \mathcal{I}_m(P) \le c_2(k) Q_{P,m}, \tag{9-1}$$

where

$$c_1(k) = (1.03)^{-k}, c_2(k) = (4\pi^2)^k$$

and

$$Q_{P,m} = \sup \mathbb{P} \begin{pmatrix} \underline{S}_1 - \underline{S}'_1 + \dots + \underline{S}_m - \underline{S}'_m \in I_1 \\ \dots \\ \underline{S}_1^k - (\underline{S}'_1)^k + \dots + \underline{S}_m^k - (\underline{S}'_m)^k \in I_k, \\ \\ \underline{S}_j \in (-P; P], \underline{S}'_j \in (-P; P], j = \overline{1, m} \end{pmatrix} \tag{9-2}$$

where the supremum extends over all intervals $I_1, I_2, \dots, I_k$ of length one.

Therefore, we have for $P' > P$

$$\mathcal{I}_m(P) < c_2(k)/c_1(k) \mathcal{I}_m(P'). \tag{9-3}$$

By the definition (0-3) of $\mathcal{I}_m(P)$ we obtain

$$\mathcal{I}_m(P) \le 2^k (2P)^{2m} (\sup_I P(\underline{S}_1 \in I))^{2m}. \tag{9-4}$$

Thus, (0-4) obviously holds for all $\tau \ge 1$ if

$$DP^{-\triangle(\tau)} \ge 2^{k+2m},$$

that is

$$P \le \left(D \cdot 2^{-k-2m}\right)^{1/\triangle(\tau)}. \tag{9-5}$$

Moreover, for $m = k$ and any $P \ge 1$ the result (0-4) holds too.
It follows from Theorem 3. Hence in view of (9-4) and (9-5) it suffices to consider the case $\tau > 1$ and $P > (2^{-k-2m} \cdot D)^{1/\triangleq(r)}$ only. Let $\tau_0$ and $P_0$ denote natural numbers and assume that the theorem holds for $\tau \le \tau_0$ and $P \le P_0$ as well as for $\tau \le \tau_0 + 1$ and $P < P_0$. Under these assumptions we shall prove that the theorem will hold for $\tau \le \tau_0 + 1$ and $P = P_0$ as well. Thus by induction the inequality (0-4) will hold for all $\tau$ and $P$.

In the proof we shall use the following notations

$$b = k(\tau_0 + 1), \ \ H = 2(16.48\pi^2)^k (k-1)^k/(k-2)!, \ \ R = [PH^{-1} + 1].$$

Thus, we have $P \le RH$ and by (9-3)

$$\mathcal{I}_b(P) \le (4.12\pi^2)^k \mathcal{I}_b(RH). \tag{9-6}$$

In what follows we shall denote by $c$ with indices and without them the constants which can depend only on $k$ and $\tau_0$.

By the definition (0-3) it follows that the expectation in $\mathcal{I}_b(RH)$ can be decomposed into $2RH$ intervals of length one for the values of $S$. Define the function $K(x)$ on $(-RH, RH]$ by the formula

$$\begin{aligned} K(x) &= 1, \quad \text{if } x \in (-RH; -RH+1]; \\ K(x) &= 2, \quad \text{if } x \in (-RH+1; -RH+2]; \\ & \quad \cdots\cdots \\ K(x) &= 2RH, \text{ if } x \in (RH-1; RH]. \end{aligned}$$

Write

$$\begin{aligned} G: &= \mathbb{E}\exp\{if\ \ (S\})\mathbf{1}_{\{-RH<S\le RH\}} \\ &= \sum_{x=1}^{2RH} \mathbb{E}\exp\{if(S\}\mathbf{1}_{\{S\in K^{-1}(x)\}} \\ &= \sum_{y=0}^{H-1} \mathbb{E}G(y,S), \end{aligned}$$

where $G(y,S) \triangleq \sum_{x=1}^{2R} \exp\{if(S)\}\mathbf{1}_{\{S\in K^{-1}(x+2Ry)\}}$and $K^{-1}$ is the mapping inverse to $K$.

Thus,

$$G^b = \sum_{y_1=0}^{H-1} \cdots \sum_{y_b=0}^{H-1} \mathbb{E}G(y_1, \underline{S}_1)\cdot\ldots\cdot G(y_b, \underline{S}_b).$$

A sequence of type $y_1, \ldots, y_b$ and its associated product $G(y_1, \underline{S}_1)\cdot\ldots\cdot G(y_b, \underline{S}_b)$ are called *regular* if there exists an increasing subsequence consisting of $k$ of the numbers $y_1, \ldots, y_b$ such that the difference between any two consecutive terms is larger than 1. A sequence and its associated product are called *irregular*, if such a sequence does not exist.

We now decompose $G^b = W_1 + W_2$, where $W_1$ contains regular products $G(y_1, \underline{S}_1)\cdot \ldots\cdot G(y_b, \underline{S}_b)$ only while $W_2$ contains the irregular products. Obviously, we have

$$\mathcal{I}_b(RH) \le 2\mathcal{I}_1 + 2\mathcal{I}_2, \tag{9-7}$$

where

$$\mathcal{I}_l = \int_{-1}^{\mathbf{1}} \cdots \int_{-1}^{\mathbf{1}} |W_l|^2 d\alpha_1 \ldots d\alpha_k, \quad l = 1, 2.$$

We shall first evaluate $\mathcal{I}_1$. It is clear that

$$\mathcal{I}_1 \le H^{2b} \max_{y_1,\ldots,y_b} \int_{-1}^{\mathbf{1}} \cdots \int_{-1}^{\mathbf{1}} |\mathbb{E}G(y_1, \underline{S}_1)\cdot\ldots\cdot G(y_b, \underline{S}_b)|^2 d\alpha.$$

Without loss of generality we may assume that $y_1 < \cdots < y_k$ and $y_{\nu+1} - y_\nu > 1$ $\nu = \overline{1, k-1}$. Each sum $G(y_\nu, \underline{S}_\nu)$ with $\nu \ge k+1$ will be divided into at most $t$

smaller sums, where $t = [RP^{-1+1/k} + 1]$, and these small sums consist of $2P^{1-1/k}$ summands or possibly less for the last sum. Thus $G(y_{k+1}, \underline{S}_{k+1}) \cdot \ldots \cdot G(y_b, \underline{S}_b)$ can be written as the sum of at most $t^{b-k}$ terms of the type $G'(y_{k+1}, \underline{S}_{k+1}) \cdot \ldots \cdot G'(y_b, \underline{S}_b)$, where $G'(y_j, \underline{S}_j)$ denotes the sums obtained in the subdivision of $G(y_j, \underline{S}_j)$. By the geometric-arithmetic mean inequality for nonnegative numbers we obtain

$$\begin{aligned} &|\mathbb{E}G'(y_{k+1}, \underline{S}_{k+1})|^2 \cdot \ldots \cdot |\mathbb{E}G'(y_b, \underline{S}_b)|^2 \\ &\quad \le \frac{|\mathbb{E}G'(y_{k+1}, \underline{S}_{k+1})|^{2(b-k)} + \cdots + |\mathbb{E}G'(y_b, \underline{S}_b)|^{2(b-k)}}{b-k}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{I}_1 \le \quad & t^{2(b-k)} H^{2k} \int_{-1}^{1} \cdots \int_{-1}^{1} |\mathbb{E}G(y_1, \underline{S}_1) \cdot \ldots \cdot G(y_k, \underline{S}_k)|^2 \\ & \times |\mathbb{E}G'(y, \underline{S}_{k+1})|^{2(b-k)} d\alpha, \end{aligned}$$

where $y$ is one of the numbers $y_{k+1}, \ldots, y_b$. By the Corollary of Lemma 2 and Lemmas 6 and 7 we get

$$\begin{aligned} &\mathcal{I}_1 \le c t^{2(b-k)} H^{2k} \\ &\times \sup\nolimits_1 \mathbb{P} \begin{pmatrix} (\underline{S}_{k+1} - a) - (\underline{S'}_{k+1} - a) + \cdots + (\underline{S}_b - a) - (\underline{S'}_b - a) \in I_1, \\ \cdots\cdots \\ (\underline{S}_{k+1} - a)^k - (\underline{S'}_{k+1} - a)^k + \cdots + (\underline{S}_b - a)^k - (\underline{S'}_b) - a)^k \in I_k, \\ |\underline{S}_j| \le P^{1-1/k}, |\underline{S'}_j| \le P^{1-1/k}, j = \overline{k+1, b} \end{pmatrix} \\ &\times \sup\nolimits_2 \mathbb{P} \begin{pmatrix} (\underline{S}_1 - a) - (\underline{S'}_1 - a) + \cdots + (\underline{S}_k - a) - (\underline{S'}_k - a) \in I^{(1)}(c_1 P^{1-1/k}), \\ \cdots\cdots \\ (\underline{S}_1 - a)^k - (\underline{S'}_1 - a)^k + \cdots + (\underline{S}_k - a)^k - (\underline{S'}_k - a)^k \in I^{(k)}(c_k P^{k(1-1/k)}), \\ \underline{S}_1 \in \bigcup_{x=1}^{2R} K^{-1}(x + 2Ry_1), \underline{S'}_1 \in \bigcup_{x=1}^{2R} K^{-1}(x + 2Ry_1), \\ \cdots\cdots \\ \underline{S}_k \in \bigcup_{x=1}^{2R} K^{-1}(x + 2Ry_k), \underline{S'}_k \in \bigcup_{x=1}^{2R} K^{-1}(x + 2Ry_k), \end{pmatrix} \end{aligned}$$

where $|a| < P$, $\sup_1$ denotes the supremum over all intervals $I_1, \ldots, I_k$ of length one, and $\sup_2$ denotes the supremum over all intervals $I^{(1)}(c_1 P^{1-1/k}), \ldots, I^{(k)}(c_k P^{k(1-1/k)})$ of lengths $c_1 P^{1-1/k}, \ldots, c_k P^{k(1-1/k)}$ respectively.

Applying the Corollary of Lemma 2 to $\sup_1 P(\ldots)$ and Lemma 1 to $\sup_2 P(\ldots)$ we obtain under the assumptions of Theorem 2 (with a different constant $c(b)$ as that for $\mathcal{I}_1$)

$$\mathcal{I}_1 \le c t^{2(b-k)} \mathcal{I}_{b-k}(P^{1-1/k}) P^{(3k-1)/2} (\sup\nolimits_3 \mathbb{P}(\underline{S} \in I))^{2k},$$

where $\sup_3$ denotes the supremum over all intervals of length one. Finally, using the inductions hypothesis we have

$$\begin{aligned}\mathcal{I}_1 &\le cD_{\tau_0} t^{2(b-k)} P^{(2(b-k)-\triangle(\tau_0))(1-1/k)+(3k-1)/2} (\sup_I \mathbb{P}(\underline{S} \in I))^{2b} \\ &\le c(b) D_{\tau_0} (RP^{-1+1/k} + 2)^{2(b-k)} \cdot P^{(2b-k)-\triangle(\tau_0))(1-1/k)+(3k-1)/2} (\sup_I \mathbb{P}(\underline{S} \in I))^{2b} \\ &\le c(b) D_{\tau_0} (\sup_I \mathbb{P}(\underline{S} \in I))^{2b} \cdot P^{2(b-k)/k+(2(b-k)-\triangle(\tau_0))(1-1/k)+(3k-1)/2}.\end{aligned}$$

Note that

$$\begin{aligned}&2(b-k)/k + (2(b-k) - \triangle(\tau_0))(1-1/k) + (3k-1)/2 \\ &= 2b - \triangle(\tau_0 + 1).\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{I}_1 &\le cD_{\tau_0} P^{2b-\triangle(\tau_0+1)} (\sup_I \mathbb{P}(\underline{S} \in I))^{2b} \\ &\le 0.25(4.12\pi^2)^{-k} D_{\tau_0+1} P^{2b-\triangle(\tau_0+1)} (\sup_I \mathbb{P}(\underline{S} \in I))^{2b}, \qquad (9\text{-}8)\end{aligned}$$

provided that $D_\tau$ satisfies the following recursion

$$D_{\tau+1} \ge 4(4.12\pi^2)^k c D_\tau, \qquad (9\text{-}9)$$

where $c$ is the constant from (9-8).
The estimation of $\mathcal{I}_2$ is completely similar to the proof of the original Vinogradov mean theorem (see Theorem 1, ch. 4 in Vinogradov (1985) and Theorem 1, §1, ch. 3 in Arkhipov et al. (1987)). That is, the total number of irregular sequences $y_1, \dots, y_b$ is not larger than

$$B := \frac{H^{k-1}}{(k-1)!}(2k-2)^k.$$

Therefore, the integral $\mathcal{I}_2$ does not exceed

$$B^2 \int_{-1}^{1} \cdots \int_{-1}^{1} \left| \mathbb{E} G(y_1, \underline{S}_1) \cdot \ldots \cdot G(y_b, \underline{S}_b) \right|^2 d\alpha,$$

where $y_1, \dots, y_b$ is the sequence for which the last integral achieves its maximal value. Again using the geometric-arithmetic mean inequality and our choice of $R$ and $H$, we obtain by induction

$$\begin{aligned}\mathcal{I}_2 &\le B^2 \int_{-1}^{1} \cdots \int_{-1}^{1} \left| \mathbb{E} G(y, \underline{S}) \right|^{2b} d\alpha \\ &\le B^2 \mathcal{I}_b(R) \\ &\le B^2 D_{\tau_0+1} R^{2b-\triangle(\tau_0+1)} (\sup_I \mathbb{P}(\underline{S} \in I))^{2b} \\ &\le 0.25(4.12\pi^2)^{-k} D_{\tau_0+1} P^{2b-\triangle(\tau_0+1)} (\sup_I \mathbb{P}(\underline{S} \in I))^{2b}. \qquad (9\text{-}10)\end{aligned}$$

Thus, (9-6)–(9-8) and (9-10) together imply (0-4), if we choose the function $D_\tau$ such that the inequalities (9-9) and

$$D_\tau \geq 2^{k+2\tau} H^{k^2(k+1)/2}$$

hold. Obviously this choice of $D_\tau$ is possible.

This completes the proof of Theorem 2.

## 5. Proof of Theorem 3

We use similar arguments as in the proof of Theorem 2.
Let $\underline{S}_1, \underline{S'}_1, \underline{S}_2, \underline{S'}_2, \dots$ denote independent copies of $S$.
It follows from the Corollary of Lemma 2 that for any real $a$

$$(1.03)^{-k} Q_{P,m}(a) \leq \mathcal{I}_m(P,a) \leq (4\pi^2)^k Q_{P,m}(a), \tag{10-1}$$

where

$$Q_{P,m}(a) = \sup \mathbb{P} \begin{pmatrix} (\underline{S}_1 - a) - (\underline{S}'_1 - a) + \dots + (\underline{S}_m - a) - (\underline{S}'_m - a) \in I_1 \\ \dots \\ (\underline{S}_1 - a)^k - (\underline{S}'_1 - a)^k + \dots + (\underline{S}_m - a)^k - (\underline{S}'_m - a)^k \in I_k \\ \underline{S}_j \in (-P; P], \underline{S}'_j \in (-P; P], j = \overline{1, m} \end{pmatrix}$$

and

$$\mathcal{I}_m(P,a) = \int_{-1}^{1} \cdots \int_{-1}^{1} |\mathbb{E} \exp\{if(S-a)\} \cdot \mathbf{1}_{\{-P<S\leq P\}}|^{2m} d\alpha_k \dots d\alpha_1.$$

We consider $S' = S - a$. Obviously,

$$\sup_I \mathbb{P}(S \in I) = \sup_I \mathbb{P}(S' \in I). \tag{10-2}$$

Hence, we obtain by (0-5) and (10-1) that

$$\begin{aligned} \mathcal{I}_m(P,a) &\leq (4.12\pi^2)^k \cdot 2^{(3m-1)/2} \cdot M \\ &\quad \times P^{(3m-1)/2} \cdot (\sup_I \mathbb{P}(S \in I))^{2m}. \end{aligned} \tag{10-3}$$

Similar to the proof of Theorem 2 we can show that (0-5) holds for all $m, 1 \leq m \leq k$, when

$$P \leq (2^{-3k} \cdot M)^{2/(k+1)}. \tag{10-4}$$

Moreover, by (10-1) we have

$$\begin{aligned} \mathcal{I}_1(P) &= \mathcal{I}_1(P,0) \\ &\leq (4\pi^2)^k \sup P \begin{pmatrix} \underline{S}_1 - \underline{S}'_1 \in I_1 \\ \underline{S}_1 \in (-P; P], \underline{S}'_1 \in (-P; P] \end{pmatrix} \\ &\leq 2 \cdot (4\pi^2)^k P (\sup_I \mathbb{P}(S \in I))^2. \end{aligned} \tag{10-5}$$

Thus we get (0-5) for $m = 1$ and any $P$.

Furthermore, by simple geometric arguments we obtain

$$\mathcal{I}_2(P) \le (4\pi^2)^k P^2 (\sup_I \mathbb{P}(S \in I))^2 \\ \times \mathbb{P}\begin{pmatrix} \underline{S}_1 + \underline{S}_2 \in I_1 \\ \underline{S}_1^2 + \underline{S}_2^2 \in I_2 \\ \underline{S}_j \in (-P; P], j = 1, 2 \end{pmatrix} \\ \le c \cdot (4\pi^2)^k P^{5/2} (\sup_I \mathbb{P}(S \in I))^4. \tag{10-6}$$

Thus, (10-5) and (10-6) together show that (0-5) holds for $m = 1$ and $m = 2$ for any $P$. Therefore (0-5) holds for all $m$, $1 \le m \le k$, when $P$ satisfies (10-4). ¿From here on we can apply the induction arguments with respect to $m$ and $P$ almost in the same way as in the proof of Theorem 2. The difference is that now we call sequences either regular or irregular depending on the existence of a special increasing subsequence consisting of

$$T = [(m+1)/2] + 1$$

numbers.

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