Eisenstein series, Green functions, and knots

by

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Introduction

In these notes we study a certain type of functions on three dimensional hyperbolic space $\mathbb{H}^3$ automorphic with respect to a cocompact group Γ acting on $\mathbb{H}^3$ which, from their origin, show a close relation to the classical real analytic Eisenstein series. These functions are not of Poincaré series type. We understand their existence as some analytic counterpart of the geometric manipulation which produces, via Dehn surgery, compact hyperbolic manifolds from those with finite volume. A cusp of a lattice Γ_0 in $\mathrm{Iso}^+(\mathbb{H}^3)$ will, when geometrically deforming the hyperbolic structure of $\Gamma_0\backslash\mathbb{H}^3$ become a one dimensional closed geodesic in $\Gamma\backslash\mathbb{H}^3$, so a knot with possibly self intersection. Here, Γ is the cocompact lattice which has been produced by deformation of Γ_0. The functions in question, when regarded as functions on the manifold $\Gamma\backslash\mathbb{H}^3$, will be real analytic complex valued outside such a knot, and they will carry typical singularities along these knots: They will serve, inside the space of smooth functions on $\Gamma\backslash\mathbb{H}^3$, as Green functions in a non classical understanding. Traditionally, Green functions do have singularities concentrated in isolated points, and they, when in a typical way, put as kernel into an integral operator, achieve the evaluation of smooth functions at these singularities. One may look at them as linear operators on smooth function spaces which come from integration along (discrete) zero dimensional subsets of $\Gamma\backslash\mathbb{H}^3$. The functions studied here will do something similar: They may be looked at as functionals on smooth function spaces on $\Gamma\backslash\mathbb{H}^3$ which are created by integrating along geodesic knots in $\Gamma\backslash\mathbb{H}^3$.

This point of view explains why there is no satisfying analogy in dimension 2; Dehn surgery in dimension 2 simply amounts to application of Riemann's theorem on removability of singularities. Analogies in dimension 4 or higher may exist. They have not been studied, yet. We have, instead, been attracted by the idea that a new light might be shed upon knot theory, of course, upon knot theory in the three sphere as well as in general closed three dimensional manifolds. This light might back the philosophy that knot theory is not just algebra but largely geometry and analysis.

We freely make use of standard notations and formalisms in three dimensional (Lobačevskij) hyperbolic geometry: $\mathbb{H}^3 = \mathbb{C}\times\mathbb{R}_{>0}$ is three dimensional upper half space with boundary $\partial\mathbb{H}^3 = P^1\mathbb{C}$ the Riemann sphere. $PSL_2(\mathbb{C})$ is the group $\mathrm{Iso}^+(\mathbb{H}^3)$ of orientation preserving isometries of $\mathbb{H}^3$; the explicit formula

$$(z,r) \xrightarrow{S} \left(\frac{(az+b)(\overline{cz+d}) + a\bar{c}r^2}{|cz+d|^2+|c|^2r^2}, \frac{r}{|cz+d|^2+|c|^2r^2} \right)$$

for $(z,r) \in \mathbb{H}^3, S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{C})$, will be used only rarely. The invariant volume element is normalized to

$$dv(z,r) = \frac{dx \wedge dy \wedge dr}{r^3}$$

where $x = \text{Re} z, y = Im z$.

$$\begin{aligned} \Delta &= r^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2} \right) - r \frac{\partial}{\partial r} \\ &= 4r^2 \frac{\partial^2}{\partial z \partial \bar{z}} + 4r^4 \frac{\partial^2}{\partial (r^2)^2} \end{aligned}$$

is the Laplacian on $\mathbb{H}^3$; in addition to this second order linear differential operator we shall use the first order quadratic differential operator H defined by

$$\begin{aligned} H(f) &= r^2 \left(\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 + \left(\frac{\partial f}{\partial r} \right)^2 \right) \\ &= 4r^2 \frac{\partial f}{\partial z} \frac{\partial f}{\partial \bar{z}} + 4r^4 \left(\frac{\partial f}{\partial (r^2)} \right)^2 . \end{aligned}$$

Both operators are $PSL_2(\mathbb{C})$-invariant:

$$\Delta(f \circ S) = (\Delta f) \circ S$$

and

$$H(f \circ S) = (H(f)) \circ S$$

and related by the chain rule for Δ:

$$\Delta(h \circ f) = H(f) \cdot h'' \circ f + (\Delta f) \cdot h' \circ f$$

for f smooth and complex valued on $\mathbb{H}^3$ and h holomorphic.

When applying Stokes' theorem the following formula will play a rôle: Let, for f, g smooth on $\mathbb{H}^3$

$$\begin{aligned}\omega = \ & \left(g\frac{\partial f}{\partial r} - f\frac{\partial g}{\partial r}\right)\frac{dx\wedge dy}{r}\\ & + \left(g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}\right)\frac{dy\wedge dr}{r}\\ & + \left(g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}\right)\frac{dr\wedge dx}{r}\,.\end{aligned}$$

Then ω is $PSL_2(\mathbb{C})$-invariant, and

$$d\omega = (g\Delta f - f\Delta g)\,\frac{dx\wedge dy\wedge dr}{r^3}\,.$$

Point line invariants

To each nonsingular complex matrix $L = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ which is not central one associates its fixed point polynomial

$$\varphi_L(z) = \gamma z^2 - (\alpha - \delta)z - \beta \in \mathbb{C}[z] \tag{1.1}$$

whose zeros are the finite fixed points of L as a Möbius transformation on $P^1\mathbb{C}$. We define

$$d_L(P) = \frac{1}{r^2}\left(|\varphi_L(z)|^2 + \frac{1}{2}|\varphi_L'(z)|^2 r^2 + \frac{1}{4}|\varphi_L''(z)|^2 r^4\right) \tag{1.2}$$

as a function on $\mathbb{H}^3$ with coordinates (z, r) for $P \in \mathbb{H}^3$. Here φ_L' and φ_L'' denote the derivatives of φ_L.

The function d_L is positive on $\mathbb{H}^3$, and $\varphi_{\lambda L} = |\lambda|^2\varphi_L$ for $\lambda \in \mathbb{C}^*$, trivially. A basic property of d_L is that, for $S \in GL_2(\mathbb{C})$,

$$d_{SLS^{-1}}(S(P)) = d_L(P)\,. \tag{1.3}$$

This is a special case of a very general phenomenon which will be treated in details, below. If L and L' have the same set of fixed points on $P^1\mathbb{C}$ the functions d_L and $d_{L'}$ will differ by a constant nonzero factor, only. For a typical not semisimple L as $L = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha \end{smallmatrix}\right)$ where $\beta \neq 0$ we compute

$$d_L(P) = \frac{|\beta|^2}{r^2} \tag{1.4}$$

and for a typical semisimple one as $L = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right)$, where $\alpha \neq \delta$,

$$d_L(P) = \frac{1}{2}|\alpha - \delta|^2 \left(1 + 2\frac{z\bar{z}}{r^2}\right) . \tag{1.5}$$

In the second case d_L turns out to be constant on the hyperbolic geodesic line whose feet on $P^1\mathbb{C} = \partial\mathbb{H}^3$ are the two fixed points of L there. We shall throughout this paper normalize the matrix L by a scalar factor in a way such that this constant is unity. This means to scale L in such a way that the absolute value of the discriminant of φ_L is 2,

$$\left|(\alpha - \delta)^2 + 4\beta\gamma\right| = 2 . \tag{1.6}$$

If L is normalized this way $d_L(P)$ is a point-line-invariant which measures the hyperbolic distance between P and the point on the axis of L as an isometry of $\mathbb{H}^3$ which is closest to P. From (1.5) we see that this distance is $ln\left(\sqrt{\frac{d+1}{2}} + \sqrt{\frac{d-1}{2}}\right)$.

The function d_L as defined in (1.2) is an eigenfunction of the hypberbolic Laplacian Δ, with eigenvalue 8:

$$\Delta d_L = 8 d_L . \tag{1.7}$$

This, again, is a special case of a general statement; see below. On the other hand, this fact can be checked easily in the cases (1.4) and (1.5) which, by equivariance (1.3), proves it for arbitrary L.

We shall now concentrate on the case when L is semisimple and noncentral, assuming that d_L is unity on the hyperbolic geodesic line whose feet on the boundary $P^1\mathbb{C}$ of $\mathbb{H}^3$ are the fixed points of L. So

$$d_L(P) = 1 + 2\frac{z\bar{z}}{r^2} \tag{1.8}$$

if these points are 0 and ∞. Note that

$$d_L > 1 \tag{1.9}$$

on the complement of the axis of L in $\mathbb{H}^3$. One easily computes

$$H(d_L) = 4\left(d_L^2 - 1\right) \tag{1.10}$$

where H is defined as in the introduction.

Let Γ be a lattice in $\mathbb{H}^3$ with a cusp c. The stabilizer of c in Γ is, up to a finite (small) index, a free abelian group Γ_c of rank 2. The construction of Eisenstein series with respect to c requires to choose an arbitrary nontrivial $L \in \Gamma_c$, then to compute d_L as in (1.2) and, finally to compose this function with some function h such that first the fact that d_L is an eigenfunction of Δ will not be destroyed and, second, the series

$$\sum_{S \in \Gamma_c \backslash \Gamma} h \circ d_L(SP) \tag{1.11}$$

converges. The independence of each summand of the choice of the representative S in a coset $\Gamma_c\backslash\Gamma$ is guaranteed by the equivariance (1.3) and the commutativity of Γ_c. The appropriate choice for the function h in this context is exponentiation with some suitable complex exponent.

We mimic this procedure for a hyperbolic line l stabilized by an infinite subgroup Γ_l of Γ which we require to be cyclic. Γ_l will have finite index in the stabilizer of l in Γ. So let $L \in GL_2(\mathbb{C})$ be loxodromic which means semisimple, not central, normalized in a way such that $d_L = 1$ on the axis l of L. A function h for which $h \circ d_L$ is still an eigenfunction of the Laplacian has, by the chain rule for the Laplacian, to satisfy

$$H(d_L) \cdot h'' \circ d_L + \Delta(d_L) \cdot h' \circ d_L = \lambda \cdot h \circ d_L \tag{1.12}$$

for some $\lambda \in \mathbb{C}$. This, by (1.7) and (1.10), is a differential equation for h, namely:

$$4(w^2 - 1)h''(w) + 8wh'(w) = \lambda h(w)\,. \tag{1.13}$$

This is a hypergeometric differential equation to be solved on the half line $w \geq 1$. We put

$$\lambda = 4s(s-1)\,. \tag{1.14}$$

with $s \in \mathbb{C}$. Then, by a standard device, the solutions of

$$(w^2 - 1)h''(w) + 2wh'(w) - s(s-1)h(w) = 0 \tag{1.15}$$

can be explicitly constructed; they are linear combinations of $h_s(w)$ and $h_{1-s}(w)$ where

$$h_s(w) = \frac{1}{w^s}\,{}_2F_1\left(\frac{1}{2}s, \frac{1}{2}(s+1), \frac{1}{2}(2s+1); \frac{1}{w^2}\right), \tag{1.16}$$

defined as a holomorphic function on the domain

$$\{w \in \mathbb{C} \,|\, |w| > 1,\ |\arg w| < \pi\} \tag{1.17}$$

which is the exterior of the unit circle slit along the negative real axis.

Here ${}_2F_1(a, b, c; T)$ denotes the classical hypergeometric series (see [1])

$${}_2F_1(a, b, c; T) = \sum_{j \geq 0} \frac{(a)_j (b)_j}{(c)_j} \frac{T^j}{j!} \tag{1.18}$$

which is convergent for $|T| < 1$. In order (1.16) to make sense one has to require

$$s \notin -\frac{1}{2} - \mathbb{N}_0 \tag{1.19}$$

or, equivalently, λ not being of the form $(2k+1)(2k+2)$ with some $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

We collect some results in the following

Proposition 1 *Let $L \in GL_2(\mathbb{C})$ be loxodromic with axis l, L being normalized in a way such that $d_L = 1$ on l. Then, for each complex s such that $-\left(s + \frac{1}{2}\right) \neq 0, 1, 2, \ldots$, the function*

$$g_s = g_s(L, P) = h_s(d_L(P))$$

is real analytic on $\mathbb{H}^3$ outside l and satisfies there

$$\Delta g_s = 4s(s-1) g_s\,. \tag{1.20}$$

For further considerations it is important to know the behaviour of $g_s(L, P)$ when P approaches the axis l of L. For this the behaviour of the hypergeometric function ${}_2F_1\left(\frac{1}{2}s, \frac{1}{2}(s+1), \frac{1}{2}(2s+1); T\right)$ is responsible when $T \in \mathbb{R}$ tends to 1 from below. One notices the relation

$$\frac{1}{2}(2s+1) = \frac{1}{2}s + \frac{1}{2}(s+1) \tag{1.21}$$

among the parameters entering into the hypergeometric function. The following information is taken from [1]:

$$
\begin{aligned}
(1.22)\qquad & {}_2F_1\left(\frac{s}{2}, \frac{s+1}{2}, \frac{2s+1}{2}; T\right) \\
&= \frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)} \sum_{j\geq 0} \frac{\left(\frac{s}{2}\right)_j \left(\frac{s+1}{2}\right)_j}{j!^2} k_j(s)(1-T)^j \\
&- \frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)} ln(1-T) \sum_{j\geq 0} \frac{\left(\frac{s}{2}\right)_j \left(\frac{s+1}{2}\right)_j}{j!^2} (1-T)^j
\end{aligned}
$$

for $|T| < 1$, again under the restriction (1.19). Here,

$$
(1.23)\qquad k_j(s) = 2\frac{\Gamma'}{\Gamma}(j+1) - \frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+j\right) - \frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}+j\right).
$$

We note that both terms on the right hand side of (1.22) make sense for $\frac{s}{2} \in -\mathbb{N}_0$ or $\frac{s+1}{2} \in -\mathbb{N}_0$: The second is zero for such s, and the first reduces to ${}_2F_1\left(\frac{s}{2}, \frac{s+1}{2}, \frac{2s+1}{2}; T\right)$ which then will be polynomial in T.

For later purposes we also need the derivative of ${}_2F_1\left(\frac{s}{2}, \frac{s+1}{2}, \frac{2s+1}{2}; T\right)$ with respect to T:

(1.24)

$$
\frac{d}{dT}\, {}_2F_1\left(\frac{s}{2}, \frac{s+1}{2}, \frac{2s+1}{2}; T\right) =
$$

$$
\begin{aligned}
&\frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)} \left(\frac{1}{1-T} + \sum_{j\geq 0} \frac{\left(\frac{s}{2}\right)_{j+1} \left(\frac{s+1}{2}\right)_{j+1}}{(j+1)!^2} \left(1-(j+1)k_{j+1}(s)\right)(1-T)^j \right. \\
&\left. + ln(1-T) \sum_{j\geq 0} \frac{\left(\frac{s}{2}\right)_{j+1} \left(\frac{s+1}{2}\right)_{j+1}}{j!(j+1)!} (1-T)^j \right)
\end{aligned}
$$

for $|T| < 1, s \in \mathbb{C}, s + \frac{1}{2} \notin -\mathbb{N}_0$.

Automorphic functions of Green's type

Theorem 1 *Let $L \in GL_2(\mathbb{C})$ be loxodromic with axis l normalized such that $d_L = 1$ on l. Let $s \in \mathbb{C}$ be not of the form $-\frac{1}{2} - k, k = 0, 1, 2, \ldots$. Then for $f \in \mathcal{C}_c^2(\mathbb{H}^3)$ we have*

$$\int_{\mathbb{H}^3} g_s(L,P)\,(\Delta - 4s(s-1))\,f(P)dv(P)$$
$$= -4\pi \frac{\Gamma\left(\frac{2s+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)} \int_l f(P)ds(P)\,.$$

Proof: We let, with no loss of generality, L be a diagonal matrix, so l is the r-axis and

$$d_L(z,r) = 1 + 2\frac{z\bar{z}}{r^2}\,. \tag{2.1}$$

Let $\varepsilon > 0$ and

$$\mathbb{H}^3_\varepsilon = \left\{ (z,r) \in \mathbb{H}^3 \mid \frac{z\bar{z}}{r^2} \geq \varepsilon^2 \right\}\,.$$

$\mathbb{H}^3_\varepsilon$ is the closed exterior of a cone with tip at $0 \in \mathbb{C}$ and rotational symmetry about the r-axis. It may be described by the condition

$$d_L(z,r) \geq 1 + 2\varepsilon^2\,, \tag{2.2}$$

as well. We have

$$\bigcup_{\varepsilon>0} \mathbb{H}^3_\varepsilon = \mathbb{H}^3 \backslash l\,. \tag{2.3}$$

$\mathbb{H}^3_\varepsilon$ will exhaust $\mathbb{H}^3\backslash l$ when letting ε tend to 0.

Let f be smooth on $\mathbb{H}^3$ with compact support. We compute

$$\begin{aligned} I_\varepsilon(f) : &= \int_{\mathbb{H}^3_\varepsilon} g_s(L,P)\,(\Delta - 4s(s-1))\,f(P)dv(P) \\ &= \int_{\mathbb{H}^3_\varepsilon} \left(g_s(L,P)\Delta f(P) - f(P)\Delta g_s(L,P)\right) dv(P)\,. \end{aligned} \tag{2.4}$$

By Stokes' theorem we get

$$\begin{aligned} I_\varepsilon(f) &= \int_{\partial\mathbb{H}^3_\varepsilon} \frac{1}{r} g_s(L,P) \left(\frac{\partial f}{\partial r} dx \wedge dy + \frac{\partial f}{\partial x} dy \wedge dr + \frac{\partial f}{\partial y} dr \wedge dx \right) \\ &\quad - \int_{\partial\mathbb{H}^3_\varepsilon} \frac{1}{r} f(P) \left(\frac{\partial g_s}{\partial r} dx \wedge dy + \frac{\partial g_s}{\partial x} dy \wedge dr + \frac{\partial g_s}{\partial y} dr \wedge dx \right) \end{aligned} \tag{2.5}$$

Here

$$\partial\mathbb{H}^3_\varepsilon = \left\{(z,r)\in\mathbb{H}^3 \mid d_L(z,r) = 1+2\varepsilon^2\right\} \tag{2.6}$$

gets its orientation from the complex z-plane punctured at 0 which parametrizes it. We introduce polar coordinates on $\partial\mathbb{H}^3_\varepsilon$:

$$z = \varepsilon r e^{i\varphi}, \qquad 0\le\varphi\le 2\pi\,, \tag{2.7}$$

so

$$\begin{aligned} dx &= \varepsilon(\cos\varphi dr - r\sin\varphi d\varphi) \\ dy &= \varepsilon(\sin\varphi dr + r\cos\varphi d\varphi) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} I_\varepsilon(f) &= \varepsilon\int g_s(L,P)\left(\varepsilon\frac{\partial f}{\partial r} - \cos\varphi\frac{\partial f}{\partial x} - \sin\varphi\frac{\partial f}{\partial y}\right) dr d\varphi \\ &\quad -\varepsilon\int f(P)\left(\varepsilon\frac{\partial g_s}{\partial r} - \cos\varphi\frac{\partial g_s}{\partial x} - \sin\varphi\frac{\partial g_s}{\partial y}\right) dr d\varphi \end{aligned} \tag{2.9}$$

where the domain of integration is $\mathbb{R}_{>0}\times[0,2\pi]$.

Now, $g_s(L,P)$ is constant on $\partial\mathbb{H}^3_\varepsilon$:

$$g_s(L,P) = \frac{1}{(1+2\varepsilon^2)^s}\,{}_2F_1\left(\frac{s}{2},\frac{s+1}{2},\frac{2s+1}{2};\frac{1}{(1+2\varepsilon^2)^2}\right) \tag{2.10}$$

for $P\in\partial\mathbb{H}^3_\varepsilon$. So

$$\begin{aligned} I^{(1)}_\varepsilon(f) &:= \varepsilon\int g_s(L,P)\left(\varepsilon\frac{\partial f}{\partial r} - \cos\varphi\frac{\partial f}{\partial x} - \sin\varphi\frac{\partial f}{\partial y}\right) dr d\varphi \\ &= \varepsilon g_s(L,P)\cdot\int\left(\varepsilon\frac{\partial f}{\partial r} - \cos\varphi\frac{\partial f}{\partial x} - \sin\varphi\frac{\partial f}{\partial y}\right) dr d\varphi\,. \end{aligned} \tag{2.11}$$

We compute

$$1-\frac{1}{(1+2\varepsilon^2)^2} = 4\varepsilon^2\frac{1+\varepsilon^2}{(1+2\varepsilon^2)^2}\,, \tag{2.12}$$

so, by (1.22) we conclude

$$\lim_{\varepsilon\to 0}\varepsilon g_s(L,P) = 0 \tag{2.13}$$

for $P \in \partial\mathbb{H}^3_\varepsilon$, and, therefore

$$\lim_{\varepsilon \to 0} I^{(1)}_\varepsilon(f) = 0 \tag{2.14}$$

by the smoothness of f.

The evaluation of the second integral in (2.5) for $\varepsilon \to 0$ requires to differentiate $h_s(w)$, see (1.16). We compute

$$\begin{aligned} \frac{d}{dw} h_s(w)|_{w=d_L} &= h_s'(d_L) \\ &= -\frac{s}{d_L^{s+1}}\, {}_2F_1\left(\frac{s}{2}, \frac{s+1}{2}, \frac{2s+1}{2}; \frac{1}{d_L^2}\right) \\ &\quad -\frac{2}{d_L^{s+3}}\, {}_2F_1'\left(\frac{s}{2}, \frac{s+1}{2}, \frac{2s+1}{2}; \frac{1}{d_L^2}\right), \end{aligned} \tag{2.15}$$

which is, again, constant. So

(2.16)

$$\begin{aligned} I^{(2)}_\varepsilon(f) &:= -\int\limits_{\partial\mathbb{H}^3_\varepsilon} \frac{1}{r} f(P) \left(\frac{\partial g_s}{\partial r} dx \wedge dy + \frac{\partial g_s}{\partial x} dy \wedge dr + \frac{\partial g_s}{dy} dr \wedge dx\right) \\ &= 4 \cdot h_s'(d_L) \int\limits_{\partial\mathbb{H}^3_\varepsilon} \frac{1}{r} f(P) \left(\frac{z\bar{z}}{r^3} dx \wedge dy - \frac{x}{r^2} dy \wedge dr - \frac{y}{r^2} dr \wedge dx\right) \end{aligned}$$

where we have made use of

$$\begin{aligned} \frac{\partial d_L}{\partial r} &= -4\frac{z\bar{z}}{r^3} \\ \frac{\partial d_L}{\partial x} &= 4\frac{x}{r^2} \\ \frac{\partial d_L}{\partial y} &= 4\frac{y}{r^2}\,. \end{aligned} \tag{2.17}$$

Parametrization of $\partial\mathbb{H}^3_\varepsilon$ as in (2.7) yields

$$I^{(2)}_\varepsilon(f) = 8\pi\varepsilon^2(\varepsilon^2+1) h_s'(d_L) \int\limits_0^\infty f(P) \frac{dr}{r}\,. \tag{2.18}$$

We have

$$\varepsilon^2 \cdot h_s'(d_L) = -\varepsilon^2 \cdot \frac{s}{d_L^{s+1}}\ {}_2F_1\left(\frac{s}{2}, \frac{s+1}{2}, \frac{2s+1}{2}; \frac{1}{d_L^2}\right) - \frac{2\varepsilon^2}{d_L^{s+3}}\ {}_2F_1'\left(\frac{s}{2}, \frac{s+1}{2}. \frac{2s+1}{2}; \frac{1}{d_L^2}\right) . \tag{2.19}$$

The first summand, as in the case (2.13) of $I_\varepsilon^{(1)}(f)$, tends to zero when $\varepsilon \to 0$. The second, by (1.24) and (2.12), has limit

$$-\frac{1}{2}\frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)},$$

so

$$\lim_{\varepsilon\to 0} I_\varepsilon^{(2)}(f) = -4\pi\frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty f(P)\frac{dr}{r} . \tag{2.20}$$

This is the statement of the theorem.

Eisenstein Green type functions

We, now, discuss the implication of theorem 1 for analysis on three dimensional closed hyperbolic manifolds. Let $\Gamma < SL_2(\mathbb{C})$ be a cocompact lattice. We fix an infinite cyclic subgroup Γ_l of Γ; l stands for the line stabilized by Γ_l. The functions $d_L(P)$ of (1.2) will be, up to nonzero constant factors, the same for all nontrivial $L \in \Gamma_l$. Let $d_L(P)$ as in (1.2) with $L \in GL_2(\mathbb{C})$ loxodromic which, as an isometry of $\mathbb{H}^3$, is contained in Γ_l as a group of isometries of $\mathbb{H}^3$, such that $d_L = 1$ on the axis of L. This axis, of course, will be stabilized by Γ_l.

The aim of this section is the following

Theorem 2 *Let, for $s \in \mathbb{C}, s \notin -\frac{1}{2} - \mathbb{N}_0$,*

$$h_s(w) = \frac{1}{w^s}\ {}_2F_1\left(\frac{s}{2}, \frac{s+1}{2}, \frac{2s+1}{2}; \frac{1}{w^2}\right)$$

as in (1.16), and

$$g_s(L, P) = h_s(d_L(P))$$

as in proposition 1. Then,

$$G_s(\Gamma_l, P) := \sum_{S \in \Gamma_l \backslash \Gamma} g_s(L, SP)$$

will converge uniformly on $\mathbb{H}^3$ *if Re* $s > 1$. G_s *will be*

a) *automorphic with respect to* Γ,

b) *square integrable over a fundamental domain of* Γ,

c) *real analytic on* $\mathbb{H}^3$ *outside the axis of* Γ_l *(equivalently* L*) and its images under motions by* Γ,

d) *an eigenfunction of the Laplacian there:*

$$\Delta G_s(\Gamma_l, P) = 4s(s-1) G_s(\Gamma_l, P)\,.$$

In order to prove convergence of $G_s(\Gamma_l, P)$ one needs a simple geometric

Lemma 1 *Let for* $\rho > 1, 0 < \alpha < \frac{\pi}{2}$

$$C(\rho, \alpha) = \left\{(z, r) \in \mathbb{H}^3 \mid 1 \le z\bar{z} + r^2 \le \rho^2, z\bar{z} \le r^2 tg^2 \alpha \right\}\,.$$

The volume of $C(\rho, \alpha)$ *is*

$$\mathrm{vol}(C(\rho, \alpha)) = \pi \cdot tg^2\alpha \cdot ln\rho\,.$$

An invariant description of $C(\rho, \alpha)$ is the following: If l is a hyperbolic geodesic line stabilized by some loxodromic $L \in GL_2(\mathbb{C})$ such that $d_L = 1$ on l, then $C(\rho, \alpha)$ is that part of hyperbolic space which consists of points between to hyperbolic planes perpendicular to l with distance ρ, whose distance from l is bounded by $d_L(P) \le 1 + 2tg^2\alpha$. The computation of the volume of $C(\rho, \alpha)$ is straightforward.

Introducing polar coordinates on $C(\rho, \alpha)$

$$(z, r) = \left(te^{i\varphi} \sin\psi, t\cos\psi\right) \tag{3.1}$$

where $1 \leq t \leq \rho, 0 \leq \psi \leq \alpha, 0 \leq \varphi \leq 2\pi$ leads to

$$
\begin{aligned}
(3.2) \qquad \mathrm{vol}(C(\rho,\alpha)) &= \int\limits_{C(\rho,\alpha)} \frac{dx \wedge dy \wedge dr}{r^3} \\
&= \int \frac{\sin\psi}{t\cos^3\psi} dt d\varphi d\psi \\
&= \pi \cdot ln\rho \cdot tg^2\alpha\,.
\end{aligned}
$$

In order to prove the theorem we assume the axis l of some loxodromic element of Γ to be the r-axis in $\mathbb{H}^3$. The stabilizer of l in Γ will be isomorphic to $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}$ with some integer $n \geq 1, n > 1$ meaning that $\Gamma\backslash\mathbb{H}^3$ will have orbifold structure of ramification index n along the projection of l into it. (The stabilizer is understood to contain only those elements of Γ which fix l including its oritentation.) Let $L \in GL_2(\mathbb{C})$ such that, as an isometry of $\mathbb{H}^3$, it is primitive of infinite order in the stabilizer; we assume 0 to be the repelling and ∞ to be the attracting fixed point of L. As before L should be normalized to $d_L = 1$ on l. Its action on $\mathbb{H}^3$ is then described by

$$
(3.3) \qquad L(z,r) = (\lambda z, |\lambda| r)
$$

where $\lambda \in \mathbb{C}, |\lambda| > 1$.

By v we denote the volume of $\Gamma\backslash\mathbb{H}^3$. We define α_k, for $k \geq 0$ integral, by requiring $\mathrm{vol}(C(|\lambda|, \alpha_k)) = k \cdot \mathrm{vol}(\Gamma\backslash\mathbb{H}^3)$, so

$$
(3.4) \qquad \pi \cdot ln|\lambda| \cdot tg^2\alpha_k = k \cdot v\,.
$$

We then order the set of coset representatives S of $\Gamma_l\backslash\Gamma$ by $k = 0, 1, 2, \ldots$, requiring that $d_L(S_k(0,1))$ should increase. This ordering will mean, by standard arguments, that

$$
(3.5) \qquad d_L(S_k(0,1)) \sim 1 + 2tg^2\alpha_k
$$

and that the convergence of G_S will be equivalent to the convergence of

$$
\sum_{k=0}^{\infty} \frac{1}{(1+2tg^2\alpha_k)^s}\ {}_2F_1\left(\frac{s}{2}, \frac{s+1}{2}, \frac{2s+1}{2}; \frac{1}{(1+2tg^2\alpha_k)^2}\right)\,.
$$

We have, by (3.4)

$$
(3.6) \qquad tg^2\alpha_k = \frac{v \cdot k}{\pi \cdot ln|\lambda|}
$$

which tends to infinity of k does.

So, for large k, ${}_2F_1\left(\frac{s}{2}.\frac{s+1}{2},\frac{2s+1}{2};\frac{1}{(1+2tg^2\alpha_k)^2}\right)$ is practically one, and the (absolute) convergence of the sum is decided by the convergence of

$$\sum_{k=0}^{\infty}\frac{1}{(1+2tg^2\alpha_k)^s}=\sum_{k=0}^{\infty}\frac{1}{\left(1+\frac{2v}{\pi ln|\lambda|}k\right)^s} \tag{3.7}$$

which is ensured if and only if Re$s>1$. So the first part of the theorem is proven.

The remaining statements of the theorem, except the square integrability, are formal consequences, only. Concerning b) we note that for small $d_L(z,r)<1$ by (1.22) the behaviour of $g_s(L,P)$ is described by

$$g_s(L,P)=f_1\left(\frac{z\bar{z}}{r^2}\right)+ln\frac{z\bar{z}}{r^2}\cdot f_2\left(\frac{z\bar{z}}{r^2}\right) \tag{3.8}$$

with two power series f_1 and f_2.

The square integrability of $g_s(L,P)$ now follows from the (square) integrability of $ln\frac{z\bar{z}}{r^2}$ over a domain $1\le r\le|\lambda|,\frac{z\bar{z}}{r^2}\le\varepsilon$, which is, of course, equivalent to the (square) integrability of lnx on any intevall $0\le x\le\varepsilon$.

The geometric-analytic significance of the two preceding theorems for closed hyperbolic three dimensional manifolds or orbifolds is stated in the following theorem. Before stating it it seems natural to renormalize some ingredients: Define, for $|w|<1$

$$\begin{aligned}\widetilde{{}_2F_1}\left(\frac{s}{2},\frac{s+1}{2},\frac{2s+1}{2};w\right)&:\\ =\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{2s+1}{2}\right)}&\,{}_2F_1\left(\frac{s}{2},\frac{s+1}{2},\frac{2s+1}{2};w\right)\\ =\sum_{j\ge0}&\frac{\Gamma\left(\frac{s}{2}+j\right)\Gamma\left(\frac{s+1}{2}+j\right)}{\Gamma\left(\frac{2s+1}{2}+j\right)}\,\frac{w^j}{j!}\end{aligned} \tag{3.9}$$

Here $s\in\mathbb{C},\notin-\mathbb{N}_0$. Similarly

$$\tilde{h}_s(w)\;:=\;\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{2s+1}{2}\right)}h_s(w)\,, \tag{3.10}$$

$$
\begin{aligned}
\tilde{g}_s(L,P) &:= \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{2s+1}{2}\right)}\, g_s(L,P)\,,\\
\tilde{G}_s(\Gamma_l,P) &:= \sum_{s\in\Gamma_l\backslash\Gamma} \tilde{g}_s(L,SP)\\
&= \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{2s+1}{2}\right)} G_s(\Gamma_l,P)\,.
\end{aligned}
$$

We look at l when projected into $\Gamma\backslash\mathbb{H}^3$ as a (geodesic) knot (which may have index >1 and be, at the same time, part of the branching locus of the orbifold $\Gamma\backslash\mathbb{H}^3$). We then state

Theorem 3 *For $f\in\mathcal{C}^2(\Gamma\backslash\mathbb{H}^3)$ and $s\in\mathbb{C},\mathrm{Re}s>1$, we have*

$$
\int\limits_{\Gamma\backslash\mathbb{H}^3} \tilde{G}_s(\Gamma_l,P)(-\Delta-4s(1-s))f(P)dv(P) = 4\pi\int\limits_{\Gamma_l\backslash l} f(P)ds(P)\,.
$$

Here the second integral means integration along a hyperbolic segment with invariant linear measure ds, there. The theorem then may be looked at in the following way: The integral on the left represents a functional defined on $\mathcal{C}^2(\Gamma\backslash\mathbb{H}^3)$, and its evaluation means "evaluation", rather integration along the knot specified by l. So when recalling the corresponding formula for the Selberg-Maaß series $\Phi_s(Q,P)$ [7] instead of $\tilde{G}_s$ which identifies it as a Green function on $\Gamma\backslash\mathbb{H}^3$:

$$
\int\limits_{\Gamma\backslash\mathbb{H}^3} \Phi_s(Q,P)\left(-\Delta-\left(1-s^2\right)\right)f(P)dv(P) = 4\pi f(Q) \tag{3.11}
$$

one may adapt the following point of view: $\tilde{G}_s(\Gamma_l,P)$ is a Green function with one dimensional singularity on the manifold $\Gamma\backslash\mathbb{H}^3$; and the elementary interpretation of the functional with a Green kernel as simply an evaluation (at the singularity of the kernel) has to be replaced by evaluation, i.e. integration, along this singularity. We shall make some comments on still more complicated or higher dimensional singularities of Eisenstein type automorphic forms which lead to an integration as Green type functions further down.

The proof of the theorem requires only a procedure which consists of replacing the integration over $\Gamma\backslash\mathbb{H}^3$ by summation over $\Gamma_l\backslash\Gamma$, this being standard in connection with Eisenstein series:

$$
\begin{aligned}
(3.12)\qquad & \int\limits_{\Gamma\backslash\mathbb{H}^3} \tilde{G}_s(\Gamma_l, P)(-\Delta - 4s(1-s))f(P)dv(P) \\
&= \sum_{S\in\Gamma_l\backslash\Gamma}\ \int\limits_{\Gamma\backslash\mathbb{H}^3} \tilde{g}_s(L, SP)(-\Delta - 4s(1-s))f(P)dv(P) \\
&= \int\limits_{\Gamma_l\backslash\mathbb{H}^3} \tilde{g}_s(L, P)(-\Delta - 4s(1-s))f(P)dv(P) \\
&= 4\pi \int\limits_{\Gamma_l\backslash l} f(P)ds(P)
\end{aligned}
$$

by theorem 2.

Dependence on the parameter

An immediately arising question concerns the behaviour of $\tilde{G}_s(\Gamma_l, P)$ as a function of s which, until now, is restricted to the half plane $\mathrm{Re} s > 1$. We have the following

Theorem 4 *Let $\Gamma < SL_2(\mathbb{C})$ a cocompact lattice and Γ_l an infinite cyclic subgroup of Γ. Then $\tilde{G}_s(\Gamma_l, P)$, as a function of s, may be extended to a meromorphic function on the whole complex plane. Its poles will lie on $\frac{1}{2}+i\mathbb{R}$, except for two simple poles at $s = 1$ and $s = 0$ and possibly finitely many more in the interval $[0, 1]$. $\tilde{G}_s$ will satisfy the functional equation*

$$\tilde{G}_{1-s}(\Gamma_l, P) = \tilde{G}_s(\Gamma_l, P)\,.$$

The proof will require an expansion of $\tilde{G}_s(\Gamma_l, P)$ in a series with respect to a complete orthonormal system of eigenfunctions e_n with respect to Δ on $\Gamma\backslash\mathbb{H}^3$, only:

$$(4.1)\qquad \tilde{G}_s(\Gamma_l, P) = \sum_{n\geq 0} c_n(s)e_n(P)$$

allows to compute

$$c_n(s) = \int\limits_{\Gamma\backslash\mathbb{H}^3} \tilde{G}_s(\Gamma_l, P)\overline{e_n(P)}dv(P)\,. \tag{4.2}$$

Let

$$\Delta e_n = \lambda_n e_n \tag{4.3}$$

where λ_n is real (and ≤ 0). We then have

(4.4)

$$\begin{aligned} c_n(s) &= \frac{1}{\lambda_n - 4s(s-1)} \int\limits_{\Gamma\backslash\mathbb{H}^3} \tilde{G}_s(\Gamma_l, P)(\Delta - 4s(s-1))\overline{e_n(P)} \cdot dv(P) \\ &= -\frac{4\pi}{\lambda_n - 4s(s-1)} \int\limits_{\Gamma_l\backslash l} \overline{e_n(P)}ds(P)\,. \end{aligned}$$

Obviously $c_n(1-s) = c_n(s)$. Especially, for the constant eigenfunction which we give the index 0 we have

$$e_0(P) = \frac{1}{v} \tag{4.5}$$

with $v = \mathrm{vol}(\Gamma\backslash\mathbb{H}^3)$ as above. We get

$$\begin{aligned} c_0(s) &= \frac{4\pi}{s(s-1)}\frac{\rho}{v} \\ &= \frac{4\pi\rho}{v}\left(\frac{1}{s-1} - \frac{1}{s}\right) \end{aligned} \tag{4.6}$$

where ρ stands for the hyperbolic length of the knot $\Gamma_l\backslash l$ in $\Gamma\backslash\mathbb{H}^3$. In particular, $\tilde{G}_s(\Gamma_l, P)$ has a simple pole at $s = 1$, and

$$\operatorname*{Res}_{s=1} \tilde{G}_s(\Gamma_l, P) = \frac{4\pi\rho}{v}\,. \tag{4.7}$$

It should be noted that, if the expansion of $\tilde{G}_s(\Gamma_l, P)$ in (4.1) converges for one s only, it will converge for all $s \in \mathbb{C}$ (except for those which lead to $4s(s-1) = \lambda_n$ with some n). The poles of $\tilde{G}_s$ are located at

$$s_n = \frac{1}{2}\left(1 + \sqrt{1 + \lambda_n}\right)\,, \tag{4.8}$$

each n giving rise to two singularities according to the two possible choices of the sign of the square root. So all poles are simple except possibly one located at $\frac{1}{2}$. If $s_n \neq \frac{1}{2}$ we have

$$\operatorname*{Res}_{s=s_n} \tilde{G}_s(\Gamma_l, P) = \frac{\pi}{\sqrt{1+\lambda_n}} \int\limits_{\Gamma_l \backslash l} \overline{e_n(P)} ds(P) \,. \tag{4.9}$$

In case -1 is an eigenvalue of Δ with index k, say, the corresponding term in (4.1) has coefficient

$$c_k = \frac{\pi}{\left(s-\frac{1}{2}\right)^2} \int\limits_{\Gamma_l \backslash l} \overline{e_k(x)} ds(P) \,, \tag{4.10}$$

so we have a second order pole of $\tilde{G}_s(\Gamma_l, P)$, if any at all, with explicitely known principal part.

It may be appropriate to look at theorem 3 as a motivation to studying geodesic knots or links in hyperbolic manifolds, with self intersection being allowed. Once an orthonormal eigenbasis $\{e_n\}$ on this manifold with respect to the Laplacian Δ chosen the countable set of complex numbers

$$4\pi \int \overline{e_n(P)} ds(P)$$

where integration is performed along the geodesic may be viewed as an invariant of this object. This invariant may very well have significance for knot theory in the sphere S^3 as a hypberbolic threedimensional manifold quite commonly arises as result of Dehn surgery along a knot in S^3. We recall that, e.g., the knot 4_1 has been intensively studied in this context, see [4], [8].

Some general formalism

In this paragraph we collect some material of analytic geometrical nature parts of which have been used, already, e.g. when discussing the function $d_L(P)$ of (1.2). The motivation comes from representation theory of $\mathrm{Iso}^+(\mathbb{H}^3)$. The theory exists in every dimension d instead of 3, though the case $d = 3$ seems to be the most interesting.

Let $n \geq 0$ an integer. We define $\mathcal{P}_n(\mathbb{C})$ to be the complex vector space consisting of all polynomials in z and $\bar{z}$ as indeterminates with complex coefficients and degree in both z and $\bar{z}$ bounded by n. So

$$\dim \mathcal{P}_n(\mathbb{C}) = (n+1)^2 \,. \tag{5.1}$$

By $\mathcal{P}_n(\mathbb{H}^3)$ we mean the space of all polynomials in $z, \bar{z}$, and $t = z\bar{z} + r^2$ with complex coefficients and total degree $\leq n$. We have

$$\dim \mathcal{P}_n(\mathbb{H}^3) = \binom{n+3}{3} \,, \tag{5.2}$$

and the elements of $\mathcal{P}_n(\mathbb{H}^3)$ will be viewed as (polynomial) complex valued functions, defined on $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_{>0}$. We have a natural homomorphism

$$\eta_n : \mathcal{P}_n(\mathbb{H}^3) \longrightarrow \mathcal{P}_n(\mathbb{C}) \tag{5.3}$$

defined by

$$\eta_n\left(f(z, \bar{z}, t)\right) = f(z, \bar{z}, z\bar{z})$$

which simply means restriction of functions on $\mathbb{C} \times \mathbb{R}$ to $\mathbb{C}$. The homomorphism η_n is surjective, trivially. Both $\mathcal{P}_n(\mathbb{C})$ and $\mathcal{P}_n(\mathbb{H}^3)$ are $SL_2(\mathbb{C})$-modules: For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{C})$ put

$$\begin{aligned} j_2(S; z) &= |cz + d|^2 \,, \\ j_3(S; z, r) &= |cz + d|^2 + |c|^2 r^2 \,. \end{aligned} \tag{5.4}$$

The formulae

$$\begin{aligned} \mathcal{P}_n(\mathbb{C}) \ni \varphi &\longrightarrow \varphi|_n S \in \mathcal{P}_n(\mathbb{C}) \,, \\ \mathcal{P}_n(\mathbb{H}^3) \ni f &\longrightarrow f|_n S \in \mathcal{P}_n(\mathbb{H}^3) \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} (\varphi|_n S)(z) &= j_2^n(S; z)(\varphi \circ S)(z) \,, \\ (f|_n S)(z, r) &= j_3^n(S; z, r)(f \circ S)(z, r) \end{aligned}$$

define an $SL_2(\mathbb{C})$ action on $\mathcal{P}_n(\mathbb{C})$ and $\mathcal{P}_n(\mathbb{H}^3)$, respectively. It is easily observed that the homomorphism η_n from (5.3) is $SL_2(\mathbb{C})$-equivariant. The reason, of course, is simply the following: $PSL_2(\mathbb{C})$ is, as $\mathrm{Iso}^+(\mathbb{H}^3)$ the subgroup of the Möbius group on S^3 which leaves invariant S^2, embedded into

S^3 as $P^1\mathbb{C}$, and the two connected components of $S^3\backslash S^1$, and so is the group of orientation preserving Möbius transformations on $\partial\mathbb{H}^3 = S^2 = P^1\mathbb{C}$.

By general representation theoretic reasons there must be an equivariant $SL_2(\mathbb{C})$-section

$$
\begin{array}{rcl}
h_n : \mathcal{P}_n(\mathbb{C}) & \longrightarrow & \mathcal{P}_n(\mathbb{H}^3)\,, \\
\eta_n \circ h_n & = & id\mathcal{P}_n(\mathbb{C})\,.
\end{array}
\tag{5.6}
$$

Theorem 5 *Let, for* $\varphi \in \mathcal{P}_n(\mathbb{C})$,

$$[\varphi]_n(z,r) = \sum_{j=0}^{n} \frac{1}{j!^2\binom{n}{j}} \frac{\partial^{2j}\varphi(z)}{\partial z^j \partial \bar{z}^j} r^{2j}\,.$$

Then $[\varphi]_n \in \mathcal{P}_n(\mathbb{H}^3)$, *and the homomorphism* h_n *defined by*

$$\mathcal{P}_n(\mathbb{C}) \ni \varphi \longrightarrow [\varphi]_n \in \mathcal{P}_n(\mathbb{H}^3)$$

is $SL_2(\mathbb{C})$*-equivariant.*

As a consequence of this theorem a decomposition of $\mathcal{P}_n(\mathbb{H}^3)$ into irreducible $PSL_2(\mathbb{C})$ modules drops out: We find

$$\mathcal{P}_n(\mathbb{H}^3) \cong \mathcal{P}_n(\mathbb{C}) \oplus r^2\mathcal{P}_{n-2}(\mathbb{H}^3) \tag{5.7}$$

if $n \geq 2$, and so

$$\mathcal{P}_n(\mathbb{H}^3) \cong \bigoplus_{2j\leq n} \mathcal{P}_{n-2j}(\mathbb{C})\,, \tag{5.8}$$

the direct summands constituting $\mathcal{P}_n(\mathbb{H}^3)$ being well known to be irreducible.

It is not intended to give a proof of the theorem, here. (See, e.g. [3]). We, instead shall study the analytic version of the construction $[\varphi]_n$.

Let φ be complex valued and real analytic on some domain Ω in the complex plane looked at as the affine part of the boundary of $\mathbb{H}^3$. Let γ be a complex number, not integral ≥ 0. We define

$$[\varphi]_\gamma(z,r) := \sum_{j\geq 0} \frac{1}{j!^2\binom{\gamma}{j}} \cdot \frac{\partial^{2j}\varphi(z,\bar{z})}{\partial z^j \partial \bar{z}^j} r^{2j}\,. \tag{5.9}$$

The infinite series defining $[\varphi]_\gamma$ will converge in some open neighborhood of Ω in $\mathbb{C}\times\mathbb{R}$, this being guaranteed by the analyticity of φ. By construction it is even in r, so we may look at $[\varphi]_\gamma$ as a function on (Ω and) some region of $\mathbb{H}^3$. Observe that $[\varphi]_\gamma(z,0) = \varphi(z)$ wherever it is defined.

In analogy to (5.5) we put, for $S \in SL_2(\mathbb{C})$:

$$(5.10) \qquad \begin{aligned} (\varphi|_\gamma S)(z) &= j_2^\gamma(S;z)(\varphi \circ S)(z)\,, \\ (f|_\gamma S)(z,r) &= j_3^\gamma(S;z,r)(f \circ S)(z,r)\,. \end{aligned}$$

Here φ and f are complex valued real analytic functions on some domain in $\mathbb{C}$ or $\mathbb{H}^3$, respectively; the objects $\varphi|_\gamma S$ and $f|_\gamma S$ will be of the same nature.

We note

$$(5.11) \qquad \begin{aligned} \varphi|_\gamma(S_1S_2) &= (\varphi|_\gamma S_1)|_\gamma S_2\,, \\ f|_\gamma(S_1S_2) &= (f|_\gamma S_1)|_\gamma S_2 \end{aligned}$$

for φ and f as above and $S_1, S_2 \in SL_2(\mathbb{C})$. This is, of course, due to the fact that $j_2(S;z)$ and $j_3(S;z,r)$ are, up to some exponent, the Jacobians of S acting on the smooth real manifolds $\partial\mathbb{H}^3$ and $\mathbb{H}^3$ of dimension 2 and 3, respectively.

Let f be complex valued on some domain $\tilde{\Omega} \subset \mathbb{C} \times \mathbb{R}$ and even in the second variable; let φ be the restriction of f to that part of $\mathbb{C}$ where f is defined. For $S \in SL_2(\mathbb{C})$ we have

$$(5.12) \qquad (f|_\gamma S)(z,0) = (\varphi|_\gamma S)(z)$$

wherever both sides are defined. The main statement in this context is the following:

Theorem 6 *Let $\gamma \in \mathbb{C}$; let φ be complex valued and real analytic on some domain $\Omega \subset \mathbb{C}$ with the restriction that, if $\gamma = n \in \mathbb{N}_0$ is a nonnegative integer then φ should be a polynomial in z and $\bar{z}$ with degree in z and $\bar{z}$ bounded by n. Then, for $S \in SL_2(\mathbb{C})$ we have*

$$[\varphi|_\gamma S]_\gamma = [\varphi]_\gamma|_\gamma S\,.$$

The proof of the theorem requires to compute $[\varphi]_\gamma$ for

$$(5.13) \qquad \varphi(z,\bar{z}) = \varphi_{\alpha,\beta}(z,\bar{z}) = z^\alpha \bar{z}^\beta\,,$$

only, and due to the multiplicativity of the statement with respect to S to check it on a suitable set of generators of $SL_2(\mathbb{C})$. In (5.13) α and β are allowed to be any pair of complex numbers, and $\varphi_{\alpha,\beta}$ will be defined in the complex plane slit from 0 to ∞ by selecting some branch of the logarithm. In case $\alpha \equiv \beta \bmod \mathbb{Z}$

$$\varphi_{\alpha,\beta}(z,\bar{z}) = (z\bar{z})^{\beta} z^{\alpha-\beta} \tag{5.14}$$

is well defined and real analytic on the complex plane punctured at 0 or even on all of $\mathbb{C}$ ($\alpha, \beta \in \mathbb{N}_0$).

In any case it is an easy exercise to compute

$$[\varphi_{\alpha,\beta}]_\gamma(z,r) = \varphi_{\alpha,\beta}(z,\bar{z})\, {}_2F_1\left(-\alpha,-\beta,-\gamma;-\frac{r^2}{z\bar{z}}\right) \tag{5.15}$$

where ${}_2F_1$ again is given by its powerseries inside the unit circle. We note that (5.15) makes sense at first for z in the slit or punctured plane and $r^2 < z\bar{z}$, only. This last restriction will be removed after havaing analytically continued the hypergeometric function to $\mathbb{C}\backslash\{x \geq 1\}$, so, especially to the negative real axis. (5.15) also makes sense for $z \in \mathbb{C}$ unrestricted in case $\gamma = n \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0, \alpha \leq n, \beta \leq n$.

In order to prove the theorem we restrict ourselves to $S = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ with $b \in \mathbb{C}$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. For the Euclidean translations the statement is trivial. For $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ we note

$$\begin{aligned} j_2(S;z) &= z\bar{z} \\ j_3(S;z,r) &= z\bar{z} + r^2 = t \end{aligned} \tag{5.16}$$

and, for α, β nonnegative integers:

$$\begin{aligned} (\varphi_{\alpha,\beta}|_\gamma S)(z) &= (-1)^{\alpha+\beta} z^{\gamma-\alpha} \bar{z}^{\gamma-\beta} \\ &= (-1)^{\alpha+\beta} |z|^{2\gamma} z^{-\alpha} \bar{z}^{-\beta} \end{aligned} \tag{5.17}$$

on $\mathbb{C}\backslash\{0\}$. So

$$\begin{aligned} &[\varphi_{\alpha,\beta}|_\gamma S]_\gamma(z,r) = \\ &(-1)^{\alpha+\beta} \varphi_{\gamma-\alpha,\gamma-\beta}(z,\bar{z})\, {}_2F_1\left(\alpha-\gamma, \beta-\gamma, -\gamma; -\frac{r^2}{z\bar{z}}\right) . \end{aligned} \tag{5.18}$$

On the other hand, as

$$S(z,r) = \left(\frac{-\bar{z}}{t}, \frac{r}{t}\right) \tag{5.19}$$

with $t = z\bar{z} + r^2$ as above we find

(5.20)

$$\begin{aligned}
&\left([\varphi_{\alpha,\beta}]_\gamma|_\gamma S\right)(z,r)\\
&= (-1)^{\alpha+\beta} t^\gamma \frac{\bar{z}^\alpha}{t^\alpha}\frac{z^\beta}{t^\beta}\ {}_2F_1\left(-\alpha,-\beta,-\gamma;-\frac{r^2}{z\bar{z}}\right)\\
&= (-1)^{\alpha+\beta} t^{\gamma-\alpha-\beta}\bar{z}^\alpha z^\beta\ {}_2F_1\left(-\alpha,-\beta,-\gamma;-\frac{r^2}{z\bar{z}}\right)\\
&= (-1)^{\alpha+\beta} t^{\gamma-\alpha-\beta} z^\beta \bar{z}^\alpha \left(1+\frac{r^2}{z\bar{z}}\right)^{\alpha+\beta-\gamma} \cdot {}_2F_1\left(\alpha-\gamma,\beta-\gamma,-\gamma;-\frac{r^2}{z\bar{z}}\right)
\end{aligned}$$

by a standard identity for hypergeometric functions (see [1, vol 1, pg. 64]) often referred to as Pfaff-Saalschütz-identity, [3]. So we get

$$\begin{aligned}
&\left([\varphi_{\alpha,\beta}]_\gamma S|_\gamma\right)(z,r)\\
&= (-1)^{\alpha+\beta} z^{\gamma-\alpha}\bar{z}^{\gamma-\beta}\ {}_2F_1\left(\alpha-\gamma,\beta-\gamma,-\gamma;-\frac{r^2}{z\bar{z}}\right)
\end{aligned} \tag{5.21}$$

which coincides with (5.18). So the theorem is proven.

The real analytic function

$$f : \mathbb{H}^3 \longrightarrow \mathbb{C} \tag{5.22}$$

defined by

$$f(z,r) = r^\gamma$$

shows, $\gamma \in \mathbb{C}$ given, the following behaviour:

$$\begin{aligned}
(f|_\gamma S)(z,r) &= j_3^\gamma(S;z,r)\cdot\left(\frac{r}{j_3(S;z,r)}\right)^\gamma\\
&= r^\gamma\,,
\end{aligned} \tag{5.23}$$

equivalently

$$(f\circ S)(z,r) = j_3^{-\gamma}(S;z,r)\cdot(f|_\gamma S)(z,r)\,.$$

So we have the following

Corollary 1 *Let $\gamma \in \mathbb{C}$ and φ real analytic, complex valued on some domain $\Omega \subset \mathbb{C}$, restricted to being a polynomial of degree at most n in z and $\bar{z}$ if $\gamma = n \in \mathbb{N}_0$. Then, for $S \in SL_2(\mathbb{C})$:*

$$\frac{[\varphi]_\gamma}{r^\gamma} \circ S = \frac{[\varphi|_\gamma S]_\gamma}{r^\gamma}\,.$$

The proof requires a formal computation, only:

$$\begin{aligned} \frac{[\varphi|_\gamma S]}{r^\gamma} &= \frac{[\varphi]_\gamma|_\gamma S}{r^\gamma} \\ &= j_3^\gamma(S;z,r)\frac{[\varphi]_\gamma \circ S}{r^\gamma} \\ &= \frac{[\varphi]_\gamma}{r^\gamma} \circ S \end{aligned} \tag{5.24}$$

This corollary contains the equivariance of $d_L(P) = d_L(z,r)$ above: For $L = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in GL_2(\mathbb{C})$ with square (or linear) fixed point polynomial $\gamma z^2 - (\alpha - \delta)z - \beta$ we compute, for

$$\begin{aligned} \varphi_L(z,\bar{z}) &= |\gamma z^2 - (\alpha-\delta)z - \beta|^2 : \\ \frac{1}{r^2}[\varphi_L]_2(z,r) &= d_L(z,r)\,. \end{aligned} \tag{5.25}$$

(The complex parameter called γ previously is chosen here to be 2.) We observe that

$$\varphi_L|_2 S = \varphi_{S^{-1}LS} \tag{5.26}$$

for $S \in SL_2(\mathbb{C})$, so

$$\begin{aligned} d_{SLS^{-1}} \circ S(z,r) &= \frac{[\varphi_{SLS^{-1}}]_2 \circ S}{(r \circ S)^2}(z,r) \\ &= \frac{[\varphi_L|_2 S^{-1}]_2 \circ S}{(r \circ S)^2}(z,r) \\ &= \frac{([\varphi_L]_2|_2 S^{-1}) \circ S}{(r \circ S)^2}(z,r) \\ &= \frac{([\varphi_L]_2|_2 S^{-1})\,|_2 S}{(r \circ S)^2} j^{-2}(S;z,r) \\ &= \frac{[\varphi_L]_2}{r^2}(z,r) = d_L(z,r)\,. \end{aligned} \tag{5.27}$$

The definition of $\varphi|_\gamma S$ and $f|_\gamma S$, as given in (5.10), means, in case $\gamma = 0$, simply $\varphi|_0 S = \varphi \circ S, f|_0 S = f \circ S$.

The following lemma may be viewed as a generalization of the well known statement saying that the action of $\mathrm{Iso}^+(\mathbb{H}^3)$ commutes with differention by the hyperbolic Laplacian:

Lemma 2 *For $\gamma \in \mathbb{C}$ let*

$$\Delta_\gamma := 4r^2 \left(\frac{\partial^2}{\partial z \partial \bar{z}} + r^2 \frac{\partial^2}{\partial (r^2)^2} \right) - 4\gamma r^2 \frac{\partial}{\partial (r^2)} .$$

Then, under the appropriate conditions about differentiability we have

$$\Delta_\gamma (f|_\gamma S) = (\Delta_\gamma f)|_\gamma S$$

for all $S \in SL_2(\mathbb{C})$.

The proof requires, after having observed the multiplicativity of the statement with respect to S, some elementary but lengthy computation, only. For $S = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right), b \in \mathbb{C}$, it is trivial; for $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ it is straightforward.

We shall, in the sequel, omit the index γ at Δ_γ if this number is 0, $\Delta = \Delta_0$ then being the ordinary hypberbolic Laplacian.

Lemma 3 *Let $\varphi : \Omega \to \mathbb{C}$ real analytic on some domain Ω in the complex plane. Let*

$$\Delta^e = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$$

be the Laplacian there. We have

$$\Delta[\varphi]_\gamma = r^2 [\Delta^e \varphi]_{\gamma - 1} .$$

Here, $\gamma \in \mathbb{C}$ is arbitrary; if $\gamma = n \in \mathbb{N}_0, \varphi$ again is restricted to be a polynomial in z and $\bar{z}$ of degree at most n.

The proof is done by elementary term by term differentiation. The situation is similar with respect to the following

Lemma 4 *Let $\varphi : \Omega \to \mathbb{C}$ be real analytic on some domain Ω in the complex plane. Let $\gamma \in \mathbb{C}$. Then*

$$\Delta \frac{[\varphi]_\gamma}{r^\gamma} = \gamma(\gamma + 2) \frac{[\varphi]_\gamma}{r^\gamma}$$

(φ again a polynomial of degree at most n in z and $\bar{z}$ if $\gamma = n \in \mathbb{N}_0$.)

This lemma explains the fact that $d_L(z,r)$ as in (1.8) or, more generally, by equivariance with respect to $\mathrm{Iso}^+(\mathbb{H}^3)$, $d_L(z,r)$ in (1.2) is an eigenfunction of Δ with eigenvalue $2 \cdot 4 = 8$.

At this point we want to demonstrate that the construction of $[\varphi]_\gamma$ at nonnegative integral $\gamma = n$ allows a natural modification. The function

$$\gamma \longrightarrow \frac{1}{\binom{\gamma}{j}} \tag{5.28}$$

defined for fixed $j \in \mathbb{N}_0$ is meromrophic on the whole γ-plane with simple poles at the integers $n, 0 \leq n \leq j-1$, the residues there being

$$\begin{aligned} \mathrm{Res}_{\gamma=n} \frac{1}{\binom{\gamma}{j}} &= \lim_{\gamma \to n} (\gamma - n) \cdot \frac{1}{\binom{\gamma}{j}} \\ &= (-1)^{j-n-1} \frac{j!}{n!(j-n-1)!} \,. \end{aligned} \tag{5.29}$$

We define, for $\varphi : \Omega \to \mathbb{C}$ real analytic

$$\begin{aligned} [\varphi]_n^*(z,r) &= \mathrm{Res}_{\gamma=n} [\varphi]_\gamma(z,r) \\ &= \frac{1}{n!} \sum_{j \geq n+1} \frac{(-1)^{j-n-1}}{j!(j-n-1)!} \frac{\partial^{2j}\varphi}{\partial z^j \partial \bar{z}^j} r^{2j} \,. \end{aligned} \tag{5.30}$$

Here, $n = 0, 1, 2, \ldots$ is any nonnegative integer.

By construction, for $\varphi \in \mathcal{P}_n(\mathbb{C})$ we have $[\varphi]_n^* = 0$.

In analogy to the corresponding statements on $[\varphi]_\gamma$ we have

Theorem 7 *Let $\varphi : \Omega \to \mathbb{C}$ be real analytic on a domain Ω in $\mathbb{C}$; let $n \geq 0$ be integral, $S \in SL_2(\mathbb{C})$. Then*

$$[\varphi|_n S]_n^* = [\varphi]_n^*|_n S \,.$$

The proof requires only the observation that the pole of $\gamma \to [\varphi]_\gamma$ at $\gamma = n$ is of order 1 (if there is any at all). So

$$[\varphi]_n^* = \lim_{\gamma \to n} (\gamma - n)[\varphi]_\gamma \,, \tag{5.31}$$

and the equivariance of $[\varphi]_\gamma$ then extends trivially to the equivariance of $[\varphi]_n^*$ expressed in the theorem.

Theorem 8 *For $\varphi : \Omega \to \mathbb{C}$ real analytic on the domain $\Omega \subset \mathbb{C}$ and $n \geq 1$ integral we have*

$$\begin{aligned} \Delta[\varphi]^*_n &= r^2[\Delta^e\varphi]^*_{n-1} = 4nr^2\tfrac{\partial}{\partial(r^2)}[\varphi]^*_n\,, \\ \Delta[\varphi]^*_0 &= 0\,. \end{aligned}$$

Proof: by elementary calculation

Theorem 9 *For real analytic $\varphi : \Omega \to \mathbb{C}$ where Ω is some domain in $\mathbb{C}$ we have, if $n \geq 0$ is integral,*

$$\Delta\frac{[\varphi]^*_n}{r^n} = n(n+2)\frac{[\varphi]^*_n}{r^n}\,.$$

The proof uses the corresponding statement for $[\varphi]_\gamma$ instead of $[\varphi]^*_n$ and (5.31).

We shall present an example of some interest in this context:

Proposition 2 *Let*

$$g_0(z) = -1 + ln\left(1 + \frac{1}{z\bar{z}}\right)$$

be Green's function on $\mathbb{C}$ ([5]).

Then

$$\begin{aligned} [g_0]^*_0(z,r) &= \tfrac{1}{2}\left(\tfrac{r^2+z\bar{z}+1}{\sqrt{r^4+2(z\bar{z}-1)r^2+(z\bar{z}+1)^2}} - 1\right) \\ &= \tfrac{1}{\rho^2(z,r)-1}\,. \end{aligned}$$

Here $\rho(z,r) \geq 1$ is the radial coordinate arising when introducing hyperbolic polar coordinates around (0,1):

$$(z,r) = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}(0, \rho(z,r))$$

with $\left(\begin{smallmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{smallmatrix}\right) \in SU(2)$.

This proposition shows some relation to [5], [6]. The proof, again, requires some lengthy computations. They are simplified by making use of peculiarities which show up when dealing with a rotationally symmetric φ, i.e. $\varphi(z,\bar{z}) = \varphi(z\bar{z})$ for some real analytic function φ of one (real positive) variable. One proves easily by induction:

Lemma 5 *Let $\varphi : I \to \mathbb{C}$ be twice differentiable on $I \subset \mathbb{R}$. Then, for $n \geq 0$ integral, we have*

$$\frac{\partial^2}{\partial z \partial \bar{z}} \left((z\bar{z})^n \varphi(z\bar{z}) \right) = n^2 (z\bar{z})^{n-1} \varphi(z\bar{z}) + (2n+1)(z\bar{z})^n \varphi'(z\bar{z}) + (z\bar{z})^{n+1} \varphi''(z\bar{z}).$$

Furthermore:

Lemma 6 *For $\varphi : I \to \mathbb{C}$ sufficiently smooth on $I \subset \mathbb{R}$ we have*

$$\begin{aligned} \left(\tfrac{\partial^2}{\partial z \partial \bar{z}}\right)^{(n)} \varphi(z\bar{z}) &= \tfrac{\partial^{2n}}{\partial z^n \partial \bar{z}^n} \varphi(z\bar{z}) \\ &= \sum_{0 \leq j \leq n} \binom{n}{j}^2 (n-j)(z\bar{z})^j \varphi^{(n+j)}(z\bar{z}). \end{aligned}$$

($n \geq 0$ integral).

$\varphi^{(k)}$ denotes the k^{th} derivative of φ.

Hypergeometric differential equations

The analysis developped so far allows to present the origin of various types of functions such as Eisenstein series, Green functions, and others under a unified view point. We show, first:

Lemma 7 *Let $\varphi = \mathcal{P}_n(\mathbb{C})$ with $n \geq 0$ integral. Let*

$$f(z, r) = \frac{[\varphi]_n(z, r)}{r^n}.$$

If $H(f)$ can be expressed as a complex polynomial in f, then this polynomial is (at most) quadratic. More precisely:

$$H(f) = n^2 f^2 + \lambda f + \mu$$

with $\lambda, \mu \in \mathbb{C}$.

Proof: The case $n = 0$ allows only constant φ and constant f, so, in that situation, $H(f) = 0$, and the lemma is true with $\lambda = \mu = 0$. Let us assume $n \geq 1$. We write

$$\varphi(z, \bar{z}) = \sum_{0 \leq j \leq n} p_j(z) \bar{z}^j \tag{6.1}$$

with polynomials $p_j(z) \in \mathbb{C}[z]$ of degree at most n. Let, for $k, l \geq 0$

$$\varphi_{kl}(z, \bar{z}) = \frac{\partial^{k+l} \varphi(z, \bar{z})}{\partial z^k \partial \bar{z}^l}\,, \tag{6.2}$$

so $\varphi_{00} = \varphi$. We compute, for

$$f = \frac{1}{r^n}[\varphi]_n = \sum_{0 \leq j \leq n} \frac{1}{j!^2 \binom{n}{j}} \varphi_{jj} r^{2j-n} : \tag{6.3}$$

$$\begin{aligned} H(f) = 4r^2 & \sum_{0 \leq j,k \leq n-1} \frac{\varphi_{j+1,j} \varphi_{k,k+1}}{j!^2 k!^2 \binom{n}{j} \binom{n}{k}} r^{2(j+k-n)} \\ + & \sum_{0 \leq j,k \leq n} \frac{(2j-n)(2k-n)}{j!^2 k!^2 \binom{n}{j} \binom{n}{k}} \varphi_{jj} \varphi_{kk} r^{2(j+k-n)}\,. \end{aligned} \tag{6.4}$$

The lowest power of r^2 here occurs in the second sum, the corresponding term being $n^2 \varphi^2 r^{-2n}$. This summand will be compensated by $n^2 f^2$, only, under the assumption that $H(f)$ be a polynomial in f. So, in that situation, $H(f) - n^2 f^2 = \lambda f + \mu$ with $\lambda, \mu \in \mathbb{C}$ as stated in the lemma. For further purposes we compute

$$\begin{aligned} H(f) - n^2 f^2 \;=\; & 4r^2 \sum_{0 \leq j,k \leq n-1} \frac{\varphi_{j+1,j} \varphi_{k,k+1}}{j!^2 k!^2 \binom{n}{j} \binom{n}{k}} r^{2(j+k-n)} \\ & -2 \sum_{0 \leq j,k \leq n} \frac{n(j+k) - 2jk}{j!^2 k!^2 \binom{n}{j} \binom{n}{k}} \varphi_{jj} \varphi_{kk} r^{2(j+k-n)}\,. \end{aligned} \tag{6.5}$$

Note that the two summands in the second sum belonging to $(j, k) = (0, 0)$ and $(j, k) = (n, n)$ actually are zero.

Lemma 8 *Let $n \geq 0$ integral, $n \neq 1, 2$, and let $\varphi = \varphi(z, \bar{z}) \in \mathcal{P}_n(\mathbb{C})$. Then in order $f = \frac{1}{r^n}[\varphi]_n$ satisfy*

$$H(f) = n^2 f^2 + \lambda f + \mu$$

with $\lambda, \mu \in \mathbb{C}$ we must have $\varphi(z, \bar{z}) = p(z)\overline{q(z)}$ with $p, q \in \mathbb{C}[z]$ of degree at most n.

Proof: Again $n = 0$ is trivial, so we assume $n \geq 3$, which is equivalent to

$$-2(n-1) < -n \,. \tag{6.6}$$

The lowest r^2-term in (6.5) is

$$4(\varphi_{10}\varphi_{01} - \varphi_{00}\varphi_{11})r^{-2(n-1)} \,.$$

This, in view of (6.6) cannot be cancelled by $\lambda f + \mu$, so

$$\varphi_{01}\varphi_{10} = \varphi_{00}\varphi_{11} \,. \tag{6.7}$$

From this we conclude, following the notation of (6.1):

$$\sum_{0 \leq j \leq k} (k - 2j) p_j(z) p'_{k-j}(z) = 0 \tag{6.8}$$

for all k. Assuming $\varphi \neq 0$ let $l \geq 0$ be maximal such that $p_l \neq 0$. Then (6.8) has the consequence

$$p_j(z) = \alpha_j p_l(z) \tag{6.9}$$

for all $j, 0 \leq j \leq n$, with $\alpha_j \in \mathbb{C}$. So we have $\varphi(z, \bar{z}) = p(z)\overline{q(z)}$ as stated in the lemma.

Lemma 9 *Let $n \geq 0$ be integral, $n \neq 1, 2, 4$. Let $\varphi \in \mathcal{P}_n(\mathbb{C})$ such that, for $f = \frac{1}{r^n}[\varphi]_n, H(f) = n^2 f^2 + \lambda f + \mu$ with $\lambda, \mu \in \mathbb{C}$. Write $\varphi(z, \bar{z}) = P(z)\overline{Q(z)}$ with $P, Q \in \mathbb{C}[z]$ of degree at most n. Then P or Q is the n^{th} power of some polynomial p respectively q in $\mathbb{C}[z]$ which, consequently, can have degree at most unity.*

Proof: Again $n = 0$ being trivial assume $n \geq 3, n \neq 4$. After having obeyed (6.7) the next lowest r^2-term in (6.5) is

$$4\left(\frac{\varphi_{10}\varphi_{12}}{n} + \frac{\varphi_{01}\varphi_{21}}{n} - \frac{\varphi_{00}\varphi_{22}}{n-1} - \frac{n-1}{n^2}\varphi_{11}^2\right) r^{-2(n-2)} \,.$$

This cannot be cancelled by $\lambda f + \mu$: If $n = 3$ or any odd integer ≥ 5 from (6.3) it is seen that f starts out with $\varphi \cdot r^{-n}$. This odd exponent term cannot cancel against any term in $H(f) - n^2 f^2$ as it contains only even exponent terms with respect to r. So, $\lambda = 0$ whenever $n \geq 3$ is odd. $H(f) - n^2 f^2 = \mu$

is constand and the above noted lowest term can actually not be there. For even $n \geq 6$ we have

$$-2(n-2) < -n$$

which, again says that this term has no counterpart in $\lambda f + \mu$. So

$$\varphi_{11}^2 - \frac{n}{n-1}\varphi_{10}\varphi_{12} - \frac{n}{n-1}\varphi_{01}\varphi_{21} + \frac{n^2}{(n-1)^2}\varphi_{00}\varphi_{22} = 0 \tag{6.10}$$

for all $n \geq 3, n \neq 4$. This, in view of the previous lemma implies

$$\begin{aligned} & P'^2(z)\overline{Q'^2(z)} - \frac{n}{n-1}P'^2(z)\overline{Q(z)Q''(z)} \\ & -\frac{n}{n-1}P(z)P''(z)\overline{Q'^2(z)} + \frac{n^2}{(n-1)^2}P(z)P''(z)\overline{Q(z)Q''(z)} \\ = & \left(P'^2(z) - \frac{n}{n-1}P(z)P''(z)\right)\overline{\left(Q'^2(z) - \frac{n}{n-1}Q(z)Q''(z)\right)} = 0\,. \end{aligned} \tag{6.11}$$

The differential equation

$$P'^2(z) - \frac{n}{n-1}P(z)P''(z) = 0 \tag{6.12}$$

for $p \in \mathbb{C}[z]$ of degree at most n will then force $P(z)$ to be of the for $p(z)^n$ with some polynomial p which then can at most be linear. Alternatively the same statement will hold true for Q, so the lemma is proven.

Proposition 3 *Let $n \geq 0$ be integral, $n \neq 1, 2, 4$. Let $\varphi \in \mathcal{P}_n(\mathbb{C})$ such that, for $f = \frac{1}{r^n}[\varphi]_n$ we have*

$$H(f) = n^2 f^2 + \lambda f + \mu$$

with $\lambda, \mu \in \mathbb{C}$. Then $\lambda = \mu = 0$.

Proof: We write $\varphi(z,\bar{z}) = p(z) \cdot \overline{q(z)}^n$ with $p, q \in \mathbb{C}[z]$, with degrees bounded by n and 1, respectively. The statement of the proposition being invariant under the action of $PSL_2(\mathbb{C})$ which means replacement of f by $f \circ S$ with any $S \in SL_2(\mathbb{C})$ we may assume q to be constant. So $\varphi(z,\bar{z}) = p(z)$ is a polynomial in z alone of degree at most n. We have

$$f(z,r) = \frac{1}{r^n}[\varphi]_n(z,r) = \frac{p(z)}{r^n} \tag{6.13}$$

and, consequently,

$$
\begin{aligned}
(6.14) \qquad H(f) &= 4r^4p^2(z)\cdot\frac{n^2}{4}\frac{1}{r^{2n}} = n^2\left(\frac{p(z)}{r^n}\right)^2 \\
&= n^2f^2\,.
\end{aligned}
$$

This proves the proposition.

It appears that the cases $n = 1, 2, 4$ which have largely been excluded behave quite exceptionally and that they are the most interesting ones. The analysis in these cases can be given some geometric interpretation. We have the feeling that this point especially in the case $n = 4$, deserves further advertance.

Proposition 4 *(the case $n = 1$). For*

$$\varphi(z,\bar z) = a + b_1z + b_2\bar z + cz\bar z \in \mathcal{P}_1(\mathbb{C})$$

and

$$f(z,r) = \frac{[\varphi]_1(z,r)}{r}$$

we have

$$H(f) = f^2 - 4(ac - b_1b_2)\,.$$

The proof requires an easy computation, only. It should be noted that $f = \frac{[\varphi]_1}{r}$, for suitable φ, is the basic ingredient for measuring distances between points in $\mathbb{H}^3$: Taking, for fixed $(z_0,r_0)\in\mathbb{H}^3$

$$(6.15) \qquad \varphi(z,r) = \frac{1}{2r_0}\left(|z-z_0|^2 + r_0^2\right)$$

we get

$$(6.16) \qquad \frac{[\varphi]_1(z,r)}{r} = \frac{|z-z_0|^2 + r_0^2 + r^2}{2r_0r}\,,$$

and this is the hyperbolic cosine of the distance between (z_0,r_0) and (z,r). The existence of certain point pair invariant functions on $\mathbb{H}^3$, the so called Selberg-Maaß functions is based upon this proposition. We shall not go into the details of this construction but refer to [7]. One should note that Selberg-Maaß series are functions on three dimensional hyperbolic manifolds which are real analytic outside an isolated singularity. They play the rôle of a Green function on such a manifold.

Proposition 5 *(the case $n=2$). Let $\varphi \in \mathcal{P}_2(\mathbb{C})$ and*

$$f(z,r) = \frac{[\varphi]_2(z,r)}{r^2}\,.$$

In order that

$$H(f) = 4f^2 + \lambda f + \mu$$

with $\lambda, \mu \in \mathbb{C}$ it is necessary and sufficient that φ be of one of the following types:

a)
$$\varphi(z,\bar{z}) = p_1(z)\cdot\overline{p_2(z)}$$

or

b)
$$\varphi(z,\bar{z}) = \psi^2(z,\bar{z})$$

where $\psi \in \mathcal{P}_1(\mathbb{C})$.

In the first case we have

$$\lambda = 0,\ \ \mu = -d_1\overline{d_2}$$

where $d_1 = b_1^2 - 4a_1c_1, d_2 = b_2^2 - 4a_2c_2$ are the discriminants of $p_1(z) = a_1z^2 + b_1z + c_1$ and $p_2(z) = a_2z^2 + b_2z + c_2$, respectively. In the second case we have

$$\lambda = 8(b_1b_2 - ac),\ \ \mu = -12(b_1b_2 - ac)^2$$

where

$$\psi(z,\bar{z}) = a + b_1z + b_2\bar{z} + cz\bar{z}$$

is a square root of φ.

The proof requires some elementary though lengthy computation: We write

$$\begin{aligned} \varphi(z,\bar{z}) \;=\;& \alpha_1 + \alpha_2 z\bar{z} + \alpha_3 z^2\bar{z}^2 \\ &+\beta_1 z + \beta_2\bar{z} \\ &+\gamma_1 z^2 + \gamma_2\bar{z}^2 \\ &+\delta_1 z^2\bar{z} + \delta_2 z\bar{z}^2\,, \end{aligned} \tag{6.17}$$

compute $f = \frac{1}{r^2}[\varphi]_2$ as in (6.3) and compare coefficients in $H(f) - 4f^2$ as in (6.5) on the one hand with those of $\lambda f + \mu$ on the other hand. The result is the following set of relations:

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \beta_2 & \alpha_2 & \delta_1 \\ \gamma_1 & \delta_2 & \alpha_3 \end{pmatrix}^{*} = -\frac{\lambda}{16} \begin{pmatrix} 4\alpha_3 & -2\delta_1 & 4\gamma_1 \\ -2\delta_2 & \alpha_2 & -2\beta_1 \\ 4\gamma_2 & -2\beta_2 & 4\alpha_1 \end{pmatrix} \tag{6.18}$$

and, additionally,

$$4\beta_1\delta_2 + 4\beta_2\delta_1 - 16\alpha_1\alpha_3 = \alpha_2^2 + \frac{1}{2}\lambda\alpha_2 + \mu\,. \tag{6.19}$$

Here, for the 3×3-matrix

$$\mathfrak{M} = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \beta_2 & \alpha_2 & \delta_1 \\ \gamma_2 & \delta_2 & \alpha_3 \end{pmatrix} \tag{6.20}$$

$\mathfrak{M}^*$ denotes the matrix of the algebraic complements of $\mathfrak{M}$:

$$\mathfrak{M} \cdot \mathfrak{M}^* = \mathfrak{M}^*\mathfrak{M} = (\det \mathfrak{M})\mathcal{E} \tag{6.21}$$

where $\mathcal{E}$ is the 3×3 unit matrix. We have to distinguish between the two cases $\mathfrak{M}^* = 0$ and $\mathfrak{M}^* \neq 0$. If $\mathfrak{M}^* = 0$ we get from (6.18) that $\lambda = 0$, after excluding the case $\varphi = 0$. In that case $\mathfrak{M}$ from (6.20) will have rank exactly one. So the rows of $\mathfrak{M}$ are scalar multiples of each other, and this yields

$$\varphi(z, \bar{z}) = \left(a_1 z^2 + b_1 z + c_1\right) \overline{(a_2 z^2 + b_2 z + c_2)} \tag{6.22}$$

with certain complex coefficients $a_1, b_1, c_1, a_2, b_2, c_2$. In this situation μ is then computed from (6.19).

In the second case, $\mathfrak{M}^* \neq 0$, the matrix $\mathfrak{M}$ has rank at least 2. Using (6.18) and (6.21) we conclude

$$\begin{gathered} \alpha_1 = a^2, \alpha_2 = 2(ac + b_1 b_2), \alpha_3 = c^2 \\ \beta_1 = 2ab_1, \beta_2 = 2ab_2\,, \\ \gamma_1 = b_1^2, \gamma_2 = b_2^2 \\ \delta_1 = 2cb_1, \delta_2 = 2cb_2 \end{gathered} \tag{6.23}$$

in terms of certain complex numbers a, b_1, b_2, c. This leads to the desired form

$$\varphi(z,\bar z) = \psi^2(z,\bar z) \tag{6.24}$$

with

$$\psi(z,\bar z) = a + b_1 z + b_2 \bar z + cz\bar z\,. \tag{6.25}$$

λ and μ, in this case, are then easily computed from (6.18) and (6.19). So, the proposition is proven.

Behind the analysis of this proposition one notices at least three different geometric configurations:

Take, first, $\varphi \in \mathcal{P}_2(\mathbb{C})$ to be constant, $\varphi(z,\bar z) = 1$. Then $f(z,r) = \frac{1}{r^2}$. This function, in some sense, measures the distance of (z,r) from the point at infinity, and it is responsible for the existence of Eisenstein series (see below).

Second, let $\varphi \in \mathcal{P}_2(\mathbb{C})$ be the square of the absolute value of some (non constant, at most) quadratic polynomial in z, say $\varphi(z,\bar z) = z\bar z$. Then $f(z,r) = \frac{z\bar z}{r^2} + \frac{1}{2} = \frac{1}{2}d_L$, see (1.8), and this situation has been the main topic of this paper: f plays the rôle of a point line invariant, and, as has been shown above, leads to functions on three dimensional hyperbolic manifolds which are analytic outside certain closed geodesics (knots, links, or graphs).

These one dimensional objects carry singularities which give the functions a Green function type character: Integration versus an $\mathcal{L}^2$-function means evaluation, i.e. an integration, along the singularity.

Third, it is expected that the case $\varphi(z,\bar z) = \psi^2(z,\bar z)$ with $\psi \in \mathcal{P}_1(\mathbb{C})$ will have some geometric significance at least if $\psi(z,\bar z) = 0$ is the equation of a circle on $P^1\mathbb{C}$ which is the case if in (6.25) we have $a, c \in \mathbb{R}, b_2 = \bar b_1$, and $|b_1|^2 - ac > 0$.

We notice that $f(z,r) = \frac{1}{r^2}[\psi^2]_2(z,r)$ will measure the hyperbolic distance between the point (z,r) and the hyperbolic plane which meets $\partial\mathbb{H}^3 = P^1\mathbb{C}$ along $\psi(z,\bar z) = 0$. To see this we may, making use of the equivariance of f, without loss of generality assume that $\psi(z,\bar z) = i(z-\bar z)$; so the circle in question is $P^1\mathbb{R}$. We compute

$$\begin{aligned} \frac{1}{r^2}[\psi^2]_2(z,r) &= -\frac{(z-\bar z)^2}{r^2} + 1 \\ &= \frac{|z-\bar z|^2 + 2r^2}{r^2} - 1\,. \end{aligned} \tag{6.26}$$

The geometric meaning of this expression now is transparent: $\frac{|z-\bar{z}|^2+2r^2}{2r^2}$ is the hyperbolic cosine of the distance between (z,r) and its mirror image $(\bar{z},r)$ with respect to the hyperbolic plane above the real axis. So, $f = \frac{1}{r^2}[\psi^2]_2$ plays the rôle of a point plane invariant. We expect this to have implications analoguous to the main topic of this paper in case the lattice Γ in $SL_2(\mathbb{C})$ contains some Fuchsian group Φ. The Eisenstein type summation would, in this case, be a summation over the cosets $\Phi\backslash\Gamma$ and presumably lead to Green type functions whose singularities in $\Gamma\backslash\mathbb{H}^3$ are distributed along geodesic surfaces. The above integration along knots would have to be replaced by integration along such surfaces. This and other subjects will be treated in a subsequent paper.

The geometric significance of the third exceptional case $n = 4$ is not so clear, yet. We note that as geometric objects behind the analysis come up so called double crosses which have been studied in [2] and which play a basic rôle in [6]. A double cross is a pair of hyperbolic lines in $\mathbb{H}^3$, joined by their common perpendicular. (We leave aside degenerate cases such as the two lines coinciding.) There are (at least) two normal forms of double crosses which, for different purposes, serve differently well:

(a) the two lines have the feet $\{\pm 1\}$ and $\{\pm\omega\}$, respectively ($\omega \in \mathbb{C}, \neq 0$). Then the common perpendicular will be the r axis, or, (b) the feet of the two lines are $\{1,\omega\}$ and $\{-1,-\omega\}$, respectively. Then the r axis is the axis of symmetry of the geometric configuration. We first state and prove

Proposition 6 *(the case $n = 4$). Let $\varphi \in \mathcal{P}_4(\mathbb{C})$ and $f = \frac{1}{r^4}[\varphi]_4$. Then $H(f)$ is a square polynomial in f if and only if*

$$\varphi(z,\bar{z}) = p_1^2(z) \cdot \overline{p_2^2(z)}$$

with $p_1, p_2 \in \mathbb{C}[z]$ of degree at most 2. Writing

$$\begin{aligned} p_1(z) &= a_1 z^2 + b_1 z + c_1\,, \\ p_2(z) &= a_2 z^2 + b_2 z + c_2\,, \end{aligned}$$

with discriminants

$$d_1 = b_1^2 - 4a_1c_1, d_2 = b_2^2 - 4a_2c_2$$

the following holds true:

$$H(f) = 16\left(f - \frac{d_1\bar{d}_2}{6}\right)\left(f + \frac{d_1\bar{d}_2}{12}\right)\,.$$

Proof: From lemma 8 we know that $\varphi(z,\bar z) = P_1(z)\overline{P_2(z)}$ with $P_1, P_2 \in \mathbb{C}[z]$ at most biquadratic. The term carrying r^{-4} in $H(f) - 16f^2$ of (6.5) now has to cancel the term carrying r^{-4} in $\lambda f + \mu$, see (6.3). We so get

$$\varphi_{10}\varphi_{12} + \varphi_{01}\varphi_{21} - \frac{4}{3}\varphi_{00}\varphi_{22} - \frac{3}{4}\varphi_{11}^2 = \lambda\varphi_{00} \tag{6.27}$$

which, in terms of P_1 and P_2 reads

$$\left(P'^2_1 - \frac{4}{3}P_1P_1''\right)\overline{\left(P'^2_2 - \frac{4}{3}P_2P_2''\right)} = -\frac{4}{3}\lambda P_1\overline{P_2}\,. \tag{6.28}$$

So we have two differential equations for P_1 and P_2 separately:

$$\begin{aligned} P'^2_1 - \frac{4}{3}P_1P_1'' &= \alpha_1 P_1\,, \\ P'^2_2 - \frac{4}{3}P_2P_2'' &= \alpha_2 P_2 \end{aligned} \tag{6.29}$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1\bar\alpha_2 = -\frac{4}{3}\lambda$. It is easily observed that a polynomial solution of a differential equation of type (6.29) cannot have a zero of order 1 or 3, so P_1 and P_2 must be squares:

$$P_1 = p_1^2\ ,\ P_2 = p_2^2 \tag{6.30}$$

with

$$\begin{aligned} p_1(z) &= a_1z^2 + b_1z + c_1\,, \\ p_2(z) &= a_2z^2 + b_2z + c_2 \end{aligned}$$

being at most quadratic. The differential equations (6.29) rewritten for p_1 and p_2 are

$$\begin{aligned} p'^2_1 - 2p_1p_1'' &= \frac{3}{4}\alpha_1 = b_1^2 - 4a_1c_1 = d_1 \\ p'^2_2 - 2p_2p_2'' &= \frac{3}{4}\alpha_2 = b_2^2 - 4a_2c_2 = d_2 \end{aligned} \tag{6.31}$$

(excluding the trivial situation $\varphi = 0$), so

$$\lambda = -\frac{3}{4}\alpha_1\bar\alpha_2 = -\frac{4}{3}d_1\bar d_2\,. \tag{6.32}$$

We see that the terms carrying r^{-2} in $H(f) - 16f^2$ and, on the other hand, in λf, coincide; see, again (6.5) and (6.3) for $n = 4$: The contribution from (6.5) is

$$\left(\frac{1}{6}\varphi_{10}\varphi_{23} + \frac{1}{6}\varphi_{01}\varphi_{32} + \frac{1}{4}\varphi_{12}\varphi_{21} - \frac{1}{3}\varphi_{00}\varphi_{33} - \frac{1}{3}\varphi_{11}\varphi_{22}\right) r^{-2}$$

which, by some elementary computation is shown to be $-\frac{4}{3}p_1p_1'\overline{p_2p_2'}r^{-2}$; and this equals the corresponding term in $\lambda f + \mu$.

The constant μ is determined from the summand in (6.5) and (6.3) which are independant of r, in a similar manner. We get

$$\mu = -\frac{2}{9}\left(d_1\bar{d}_2\right)^2 . \tag{6.33}$$

It remains to check that, with λ and μ from (6.32) and (6.33), the terms in $\lambda f + \mu$ carrying positive powers r^2, r^4, r^6, r^8 are identical with those arising in $H(f) - 16f^2$. Theses computations are elementary, again. So the proposition is proven.

The geometry alluded to above lies in the fact that a polynomial $\varphi \in \mathcal{P}_4(\mathbb{C})$ of the form $\varphi(z,\bar{z}) = p_1^2(z)\overline{p_2^2(z)}$ with $p_1, p_2 \in \mathbb{C}[z]$ corresponds, if p_1 and p_2 have non varishing discriminants, to a (non ordered) pair of (possibly coinciding) hyperbolic lines, namely those, whose feet on $\partial\mathbb{H}^3$ are the zeros of p_1 and p_2, respectively, with the standard understanding that ∞ is a zero of p_1 or p_2 if its degree is unity. We explicitely note down the function $\frac{1}{r^4}[\varphi]_4$ in the two geometrically normalized positions mentioned above:

(a) Let $\omega \in \mathbb{C}, \omega \neq 0$. Let

$$\begin{aligned} &p_1(z) = z^2 - \omega^2\,,\ p_2(z) = \omega(z^2-1)\,,\\ &\varphi(z,\bar{z}) = p_1^2(z)\overline{p_2^2(z)}\,. \end{aligned}$$

(We have arranged p_1 and p_2 to have equal discriminant $d_1 = d_2 = 4\omega^2$.) The two hyperbolic lines corresponding to φ have feet $\{\pm 1\}$ and $\{\pm\omega\}$, respectively. We compute

$$\begin{aligned} \frac{1}{r^4}[\varphi]_4(z,r) = \;& \bar{\omega}^2(z^2-\omega^2)\overline{(z^2-1)^2}r^{-4}\\ &+4\bar{\omega}^2 z(z^2-\omega^2)\overline{z(z^2-1)}r^{-2}\\ &+\frac{2}{3}\bar{\omega}^2(3z^2-\omega^2)\overline{(3z^2-1)}\\ &4\bar{\omega}^2 z\bar{z}r^2 + \bar{\omega}^2 r^4\,. \end{aligned}$$

We point out that this function is constant on the two lines in question, these constants being $\omega^2\bar{\omega}^2\left(\bar{\omega}^2+\bar{\omega}^{-2}+\frac{2}{3}\right)$ on the line with feet $\{\pm\omega\}$ and $\omega^2\bar{\omega}^2\left(\omega^2+\omega^{-2}+\frac{2}{3}\right)$ on the line with feet $\{\pm 1\}$.

(b) Let $\omega \in \mathbb{C}, \omega \neq 1$. Let

$$\begin{aligned}
&p_1(z) = (z-\omega)(z-1), p_2(z) = (z+\omega)(z+1)\,,\\
&\varphi(z,\bar{z}) = p_1^2(z)\overline{p_2^2(z)}\,.
\end{aligned}$$

The two hyperbolic lines corresponding to φ now have feet $\{1,\omega\}$ and $\{-1,-\omega\}$, respectively. We compute

$$\begin{aligned}
&\frac{1}{r^4}[\varphi]_4(z,r) = (z-\omega)^2(z-1)^2\overline{(z+\omega)^2(z+1)^2}r^{-4}\\
&+(2z-(\omega+1))(z^2-(\omega+1)z+\omega)\overline{(2z+(\omega+1))(z^2+(\omega+1)z+\omega)}r^{-2}\\
&+\frac{1}{6}(6z^2-6(\omega+1)z+(\omega^2+4\omega+1))\overline{(6z^2+6(\omega+1)z+(\omega^2+4\omega+1))}\\
&+(2z-(\omega+1))\,\overline{(2z+(\omega+1))}r^2\\
&+r^4\,.
\end{aligned}$$

We, finally, collect the differential equations which have to be satisfied by functions $h(w)$ for which $h\left(\frac{1}{r^n}[\varphi]_n\right)$ is an eigenfunction of Δ under the assumption that $H\left(\frac{1}{r^n}[\varphi]_n\right)$ is a polynomial in $\frac{1}{r^n}[\varphi]_n$. They are for $n \geq 0, n \neq 1,2,4$:

$$n^2w^2h''(w) + n(n+2)wh'(w) = \lambda h(w)\,,$$

for $n=1$:

$$(w^2+\alpha)h''(w) + 3wh'(w) = \lambda h(w)\,,$$

for $n=2$:

$$4(w^2+\alpha)h''(w) + 8wh'(w) = \lambda h(w)$$

or

$$4(w-\alpha)(w+3\alpha)h''(w) + 8wh'(w) = \lambda h(w)\,,$$

and, for $n=4$:

$$16(w-\alpha)(w+2\alpha)h''(w) + 24wh'(w) = \lambda h(w)\,.$$

Here, α in each case is a constant determined by the individual function φ as described above. In the cases $n = 1, 2, 4$ the differential equations arising are of hypergeometric type.

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