Properly discontinuous groups on certain matrix homogeneous spaces

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Abstract. We characterize discrete groups $\Gamma \subset GL(n, \mathbf{R})$ which act properly discontinuously on the homogeneous space $GL(m, \mathbf{R}) \backslash GL(n, \mathbf{R})$.

§1. Introduction

A manifold M is called a complete locally homogeneous if $M = J \backslash H / \Gamma$. Here H is a finite dimensional Lie group, $J \subset H$ its closed Lie subgroup and $\Gamma \subset H$ is a discrete subgroup which acts freely and properly discontinuously on $J \backslash H$. See [**Gol**]. In the last thirty years there was a lot of activity in the case where J is noncompact (the nonclassical case). Consult our list of references.

The first basic problem in this area is to characterize all discrete subgroups $\Gamma \subset H$ which act freely and properly discontinously on $J \backslash H$. The most known example is Calabi-Markus phenomenon. That is, only finite groups can act freely and properly discontinuously on $J \backslash H$. See [**C-M**], [**Wol1**] and [**Kob**]. The second basic problem is to characterize all Γ for which M is a compact manifolds (cocompact lattices). There are examples in which cocompact lattices do not exist [**B-L**] and [**Zim2**]. The most known problem in this area is the Auslander's conjecture. Here $H = Aff(\mathbf{R}^n)$ - the Lie group of all affine transformations of $\mathbf{R}^n$, $J = GL(n, \mathbf{R})$. Then Auslander's conjecture claims that if M is compact then Γ is virtually solvable. See [**Mil**], [**Mar1-2**], [**G-K**], [**Tom**] and [**D-G**].

The aim of this paper to consider the following specific problems of the above type. Let $H = GL(n, \mathbf{R}), J = GL(m, \mathbf{R}), 1 < m < n$, where J is standardly embedded in the upper left corner in H. Note that $J \backslash H = SL(m, \mathbf{R}) \backslash SL(n, \mathbf{R})$. We characterize all discrete subgroups $\Gamma \subset GL(n, \mathbf{R})$ which act properly discontinously on $J \backslash H$. Let $M = J \backslash H / \Gamma$ be a locally homogeneous manifold. We then show that $GL(n-m, \mathbf{R})$, standardly embedded in the lower right corner of $GL(n, \mathbf{R})$, acts naturally from the left on $J \backslash H$ and this action projects to M. In particular, any one parameter subgroup of $G(n-m, \mathbf{R})$ induces a flow on M.

In [**Zim2**] it was shown that for $2 \le m < \frac{n}{2}, n > 4$ $J \backslash H$ do not admit cocompact lattices. We make the following observation. Assume that $M = J \backslash H / \Gamma$ is compact. Then Γ, considered as a group of transformation on certain compactification of $J \backslash H$, does not have an invariant probability measure. It was our plan to prove the existence of such an invariant probability measure using the flow induced by the action of corresponding subgroups of $GL(n-m, \mathbf{R})$ and the compactness of M (some generalized version of the Raghunathan's measure conjecture [**Rat**]) but we failed to do so.

We now outline briefly the contents of the paper. In §2 we show that $GL(m, \mathbf{R}) \backslash GL(n, \mathbf{R})$ is essentially the Grassmanian G_{mn} times $M^0_{(n-m)n}$ - the set of all $(n-m) \times n$ matrices of rank $n-m$. We then give necessary and sufficient conditions for a discrete group $\Gamma \subset GL(n,\mathbf{R})$ to act properly discontinously (from the right) on M^0_{kn}. In Section 3 we characterize discrete groups $\Gamma \subset GL(n,\mathbf{R})$ which act properly discontinuously on $G(m,\mathbf{R})\backslash G(n,\mathbf{R})$. §4 characterizes subgroups $\Gamma \subset GL(n,\mathbf{R})$, considered as groups of continuous transformations of S_{kn}, which have an invariant probability measure. In the last section we show that if $M = GL(m,\mathbf{R})\backslash GL(n,\mathbf{R})/\Gamma$ is a compact manifold then Γ, considered as a group of continuous transformations of $G_{mn} \times S_{(n-m)m}$, does not have an invariant probability measure on $G_{mn} \times S_{(n-m)m}$.

§2. The action of discrete groups on the frame spaces

Let M_{mn} be the space of $m \times n$ real valued matrices. We view each $X = (x^i_j)^{i=m,j=n}_{i=j=1}$ as composed of m-row vectors $x^i = (x^i_1, ..., x^i_n) \in \mathbf{R}^n, i = 1, ..., m$. Recall that for any $1 \le k \le \min(m,n)$ the matrix $C_k(X)$, the $k-th$ compound of X, is an $\binom{m}{k} \times \binom{n}{k}$ matrix whose entries are $k \times k$ minors of X arranged in a lexigraphical order. See for example [**Gan**] for the standard properties of compound matrices. In what follows we assume that $1 \le m < n$ unless stated otherwise. Then $C_m(X) \in \mathbf{R}^N, N = \binom{n}{m}$. In that case $C_m(X)$ is identified with the wedge product $x^1 \wedge \cdots \wedge x^m$ of the m rows of X. Clearly, $C_m(X) = 0 \iff rank(X) < m$. Let M^0_{mn} be the manifold of $m \times n$ matrices of rank m. This manifold is called the manifold of m- frames in $\mathbf{R}^n$. It is known that for $X, Y \in M^0_{mn}$ $C(X), C(Y)$ are proportional iff the row spaces of the matrices X, Y are identical, e.g. [**Boo**, p'65]. Let $\pi : \mathbf{R}^N\backslash\{0\} \to \mathbf{RP}^{N-1}$ be the canonical projection. Then $\pi(C_m(M_{mn}))$ is the Grassmanian G_{mn}. Set $\phi : M^0_{mn} \to G_{mn}$ to be given by $\phi(X) = \pi(C_m(X))$. Let $X \in M_{mn}, Y \in M_{(n-m)n}$. Note that $C_m(X), C_{n-m}(Y) \in \mathbf{R}^N, N = \binom{n}{m}$. Let $A(X,Y) \in M_{nn}$ be the matrix whose first m rows is the matrix X and and the last $n-m$ rows is the matrix Y. Expanding $det(A(X,Y))$ by the first m row we deduce that

$$det(A(X,Y)) = < C_m(X), C_{n-m}(Y) > .$$

It is not difficult to see that for $u, v \in \mathbf{R}^N$ we can define a "product" $< u, v >$ such that the above equality holds. Moreover, $< v, u > = (-1)^{m(n-m)} < u, v >$.

Let $S \subset M_{nn}$ be the subvariety of singular matrices. That is, $GL(n,\mathbf{R}) = M_{nn}\backslash S$. We view $M^0_{mn} \times M^0_{(n-m)n}$ as a subset of M_{nn}. Set $S' = S \cap M^0_{mn} \times M^0_{(n-m)n}$. Thus, $GL(n,\mathbf{R}) \equiv M^0_{mn} \times M^0_{(n-m)n}\backslash S'$. Note that S' is the variety of "perpendicular" matrices

$$S' = \{A(X,Y) : \ X \in M^0_{mn}, Y \in M^0_{(n-m)n}, < C_m(X), C_{n-m}(Y) >= 0\}.$$

Let

$$\psi : M^0_{mn} \times M^0_{(n-m)n} \to G_{mn} \times M^0_{(n-m)m}, \ \psi((X,Y)) = (\phi(X), Y).$$

Set

$$\psi(GL(n,\mathbf{R})) = \psi(M^0_{mn} \times M^0_{(n-m)m}\backslash S') =$$
$$T_{mn} = \{(Z,Y), \ Z \in G_{mn}, Y \in M^0_{(n-m)n}, < Z, Y > \ne 0\}.$$

(2.1) Theorem. *Let $1 \leq m < n$. Then*

$$SL(m,\mathbf{R})\backslash SL(n,\mathbf{R}) = GL(m,\mathbf{R})\backslash GL(n,\mathbf{R}) \sim T_{mn}.$$

Proof. Observe that $GL(m,\mathbf{R})$ acts on M_{mn}^0 from the left by matrix multiplication. Clearly, the orbit $orb(X), X \in M_{mn}^0$ represents the subspace spanned by the rows of X. Hence, $GL(m,\mathbf{R})\backslash M_{mn}^0 \sim G_{mn}$. Thus, the orbit of $A(X,Y), X \in M_{mn}^0, Y \in M_{(n-m)n}^0$ under the action of $GL(m,\mathbf{R})$ represents the point $(\phi(X),Y)$. So

$$GL(m,\mathbf{R})\backslash M_{mn}^0 \times M_{(n-m)n}^0 \sim G_{mn} \times M_{(n-m)n}^0.$$

Hence, $GL(m,\mathbf{R})\backslash GL(n,\mathbf{R}) = T_{mn}$. Clearly, $GL(m,\mathbf{R})\backslash GL(n,\mathbf{R}) = SL(m,\mathbf{R})\backslash SL(n,\mathbf{R})$ and the theorem follows. $\diamond$

Let $H' \subset GL(m,\mathbf{R})$ be a closed subgroup. Then H' acts from the left on M_{mn}^0. The quotient manifold $H'\backslash M_{mn}^0$ has $\binom{n}{m}$ charts and each chart is isomorphic to $H'\backslash GL(m,\mathbf{R}) \times M_{m(n-m)}$. (Each chart is obtained by considering a corresponding $m \times m$ nonsingular submatrix in $X \in M_{mn}^0$.) Thus H' acts on $M_{mn}^0 \times M_{kn}$ where the action on the second factor is a trivial action. In particular, H' acts from the left on $GL(n,R)$ for any $n > m$ and $H'\backslash GL(n,\mathbf{R})$ is a manifold. Suppose furthermore that H' is homogeneous, i.e. $\mathbf{R}^* H' = H'$. Let $H = H' \cap SL(n,\mathbf{R})$. Then $H\backslash SL(n,\mathbf{R}) = H'\backslash GL(n,\mathbf{R})$. Let $\Gamma \subset GL(n,\mathbf{R})$ be a discrete group. Then Γ acts from the right on $H'\backslash GL(n,\mathbf{R})$. Note that this action consists of two separate actions. Γ acts on the quotient $H'\backslash M_{mn}^0$ and on $M_{(n-m)n}^0$. Recall that a group G is said to act properly discontinuously on a manifold M if for any compact set $K \subset M$ the set of $g \in G$ so that $K \cap Kg \neq \emptyset$ is a finite set. We now give necessary and sufficient conditions for a properly discontinuous action of a discrete group $\Gamma \subset GL(n,\mathbf{R})$ on M_{kn}^0. To do that we need to recall a few standard facts. On $\mathbf{R}^p, p = 1, ...,$ we let $\|\cdot\|$ to be the Euclidean norm. Let

$$\begin{aligned} K(R,\tau) = \{X : X = (x_j^i)_{i=j=1}^{i=k,j=n} \in M_{kn},\\ \|(x_1^i, ..., x_n^i)\| \leq R, i = 1, ..., k, \tau \leq \|C_k(X)\|, 0 \leq \tau, 0 \leq R.\}\end{aligned}$$

It then follows that any compact set $K \subset M_{kn}^0, 1 \leq k \leq n$ is a closed subset of $K(R,\tau)$ for some $0 < \tau, 0 < R$. Let $A \in M_{mn}, B \in M_{np}, 1 \leq k \leq min(m,n,p)$. Then $C_k(AB) = C_k(A)C_k(B)$. Moreover, the $k-th$ compound of the identity $I_n \in M_{nn}$ is also the identity matrix. Thus, for $A \in GL(n,\mathbf{R})$ we have $C_k(X)^{-1} = C_k(X^{-1})$. Furthermore,

$$X \in O(n,\mathbf{R}) \Rightarrow C_k(X) \in O(\binom{n}{k}, \mathbf{R}).$$

Hence, $K(R,\tau)X = K(R,\tau)$ for any $X \in O(n,\mathbf{R})$. Recall that each $A \in GL(n,\mathbf{R})$ can be written in the form

$$A = U\Sigma V, U, V \in O(n,\mathbf{R}), \Sigma = diag(\sigma_1, ..., \sigma_n), \sigma_1 \geq \sigma_2 \geq \cdots \sigma_n > 0, det(\Sigma) = |det(A)|.$$

The above form is called usually the singular value decomposition of A. See for example [**H-J**]. The quanitities $\sigma_i = \sigma_i(A), i = 1, ..., n$, are called the singular values of A. For $A \in SL(n, \mathbf{R})$ it follows that $\Sigma \in SL(n, \mathbf{R})$. Furthermore one can choose $U, V \in SO(n, \mathbf{R})$.

(2.2) Theorem. *Let $\Gamma \subset GL(n, \mathbf{R})$ be a discrete group. Let $1 \leq k < n$. Then Γ acts properly discontinuously on M^0_{kn} iff the following condition hold. For each $1 > \epsilon > 0$ all but a finite number of $\gamma \in \Gamma$ satisfy either $\sigma_{n-k+1}(\gamma) \geq \frac{1}{\epsilon}$ or $\epsilon \geq \sigma_k(\gamma)$.*

Proof. As $K(R, \tau)O = K(R, \tau)$ for any $O \in O(n, \mathbf{R})$ it is enough to consider the sets $K(R, \tau) \cap K(R, \tau)\Sigma(\gamma), \gamma \in \Gamma$. We first show that if for each $1 > \epsilon > 0$ all but a finite number of $\gamma \in \Gamma$ satisfy either $\sigma_{n-k+1}(\gamma) \geq \frac{1}{\epsilon}$ or $\epsilon \geq \sigma_k(\gamma)$ then Γ acts properly discontinuously on M^0_{kn}. Let

$$\Gamma(k, \epsilon) = \{\gamma : \gamma \in \Gamma, \sigma_k(\gamma) \leq \epsilon\}.$$

Then for any $X \in K(R, 0)$ each of the last $n-k+1$ columns of the matrix $X\Sigma(\gamma), \gamma \in \Gamma(k, \epsilon)$ has a norm at most $\sqrt{k}R\epsilon$. Hence for any $Y \in K(R, 0) \cap K(R, 0)\gamma, \gamma \in \Gamma(k, \epsilon)$ we have the estimate $\|C_k(Y)\| \leq C(R)\epsilon$ for some positive constant $C(R)$ and $0 \leq \epsilon \leq 1$. Given $\tau > 0$ there exists $\epsilon(R, \tau)$ so that for any $\gamma \in \Gamma(k, \epsilon(R, \tau))$ and any $Y \in K(R, 0) \cap K(R, 0)\gamma$ we have the inequality $\|C_k(Y)\| < \tau$. It then follows that

$$K(R, \tau) \cap K(R, \tau)\gamma = \emptyset, \gamma \in \Gamma(k, \epsilon(R, \tau)).$$

Observe next that the set of all $\gamma \in \Gamma, \sigma_{n-k+1}(\gamma) > \frac{1}{\epsilon}$ is exactly the set $\Gamma(k, \epsilon)^{-1}$. Thus

$$K(R, \tau) \cap K(R, \tau)\gamma = \emptyset, \gamma^{-1} \in \Gamma(k, \epsilon(R, \tau)).$$

It then follows that Γ acts properly discontinuously on M^0_{kn}.

Assume now that there exists $0 < \epsilon < 1$ and an infinite subsequence of elements of Γ for which $\epsilon < \sigma_k(\gamma), \sigma_{n-k+1}(\gamma) < \frac{1}{\epsilon}$. Pick up a subsequence $\gamma^i, i = 1, ...,$ of this subsequence so that

$$\lim_{i \to \infty} \sigma_j(\gamma^i) = \sigma_j, j = 1, ..., n, \infty \geq \sigma_1 \geq \cdots \geq \sigma_n \geq 0, \epsilon \leq \sigma_k, \sigma_{n-k+1} \leq \frac{1}{\epsilon}.$$

By considering the sequence $(\gamma^i)^{-1}, i = 1, ...,$ if necessary, we may assume that

$$\sigma_1 = ... = \sigma_p = \infty, \infty > \sigma_{p+1} \geq \cdots \sigma_{p+q} > 0, \sigma_{p+q+1} = \cdots = \sigma_n = 0,$$
$$1 \leq p \leq n-k, 0 \leq n-(p+q) \leq \min(p, n-k).$$

Assume first that $q < k$. Let $X_i \in M^0_{kn}$ be the following matrix. The columns $p+1, ..., p+k$ of X_i form a $k \times k$ identity matrix I_k. The column $p+1-j$ of X_i is the $k+1-j$ column of the matrix I_k divided by $\sigma_{p+1-j}(\gamma^i)$ for $j = 1, ..., k-q$. All other columns of X_i are equal to zero. It now follows that there exists $K(R, \tau)$ for some $0 < \tau, R$ so that

$X_i, X_i\Sigma(\gamma^i) \in K(R,\tau), i = 1, ...,$. Hence Γ does not act properly discontinuously on M^0_{kn}. Suppose now that $q \geq k$. Set $Y = (\delta_{p+i,j})_{i=j=1}^{i=k,j=n}$. It now follows that

$$\lim_{i\to\infty} Y\Sigma(\gamma^i) = Z \in M^0_{kn}.$$

Hence, Γ does not act properly discontinuously on M^0_{kn}. $\diamond$

Assume that $\Gamma \subset SL(n,\mathbf{R})$ is a discrete infinite group. As $\sigma_1(\gamma)$ is the spectral norm of γ it follows that the sequence $\sigma_1(\gamma), \gamma \in \Gamma$ converges to ∞. Since $\Sigma(\gamma) \in SL(n,\mathbf{R}), \gamma \in \Gamma$ we deduce that sequence $\sigma_n(\gamma), \gamma \in \Gamma$ converges to zero. According to Theorem 2.3 Γ does not act properly discontinuously on M^0_{1n}.

(2.3) Corollary. *Let $\Gamma \subset SL(n,\mathbf{R})$ be a discrete group. Then Γ acts properly discontinuously on M^0_{1n} iff Γ is a finite group.*

§3. Properly discontinuous groups on $GL(m,\mathbf{R})\backslash GL(n,\mathbf{R})$

(3.1) Theorem. *Let $\Gamma \subset GL(n,\mathbf{R})$ be a discrete group. Assume that $1 \leq m < n$. Then Γ acts properly discontinuously on $GL(m,\mathbf{R})\backslash GL(n,\mathbf{R})$ (from the right) iff for any $0 < \epsilon < 1$ there is only a finite number of elements of Γ which have at least $n-m$ singular values in the interval $(\epsilon, \frac{1}{\epsilon})$.*

Proof. Assume first that for some $\epsilon > 0$ there is an infinite sequence of elements of Γ so that the interval $(\epsilon, \frac{1}{\epsilon})$ contains at least $n-m$ singular values of each element of this sequence. Pick a subsequence of this sequence to obtain the following sequence $\gamma_i \in \Gamma, i = 1, ...,$ such that

$$\lim_{i\to\infty} \sigma_j(\gamma^i) = \sigma_j, j = 1, ..., n, \infty > \sigma_{p+1} \geq ... \geq \sigma_{p+n-m} > 0, 0 \leq p \leq m.$$

As $O(n,\mathbf{R})$ is compact it is enough to show that there is $x \in GL(m,\mathbf{R})\backslash GL(n,\mathbf{R})$ so that the sequence $x\Sigma(\gamma^i)$ has a convergent subsequence. Assume that the last $n-m$ rows of x have zero columns numbered $1, ..., p$ and $p+n-m+1, ..., n$. Assume next that the first m rows of x form the unique subspace spanned by any m linearly independent rows which have zero coordinates in the columns $p+1, ..., p+n-m$. It then follows that the first m rows of $x\Sigma(\gamma^i)$ span the same subspace as x. Also, the last $n-m$ rows of $x\Sigma(\gamma^i)$ converge to $z \in M^0_{n-m}$. Thus,

$$lim_{i\to\infty} x\Sigma(\gamma^i) = y \in GL(m,\mathbf{R})\backslash GL(n,\mathbf{R}).$$

Hence, Γ does not act discontinuously on $GL(m,\mathbf{R})\backslash GL(n,\mathbf{R})$.

Assume now that Γ satisfies the conditions of the theorem. Let $0 \le k \le m+1$ and $\epsilon > 0$ fixed. Set

$$\Gamma(k,\epsilon) = \{\gamma : \ \gamma \in \Gamma, \sigma_k \ge \frac{1}{\epsilon}, \sigma_{k+n-m} \le \epsilon\}.$$

It now follows that $\Gamma \backslash \cup_{k=0}^{m+1} \Gamma(k,\epsilon)$ is a finite set for any $\epsilon > 0$. We next characterize a compact set in $SL(m,\mathbf{R})\backslash SL(n,\mathbf{R})$. To do that we shall view any $\xi \in G_{mn}$ a vector $C_m(X)$ of a unit length. Here, $X \in M_{mn}^0$ is representing the linear subspace generated by the m rows of X. There is an ambiguity up to a sign ± 1. The choice of the sign will not be relevant. We next observe that any compact set in $SL(m,\mathbf{R})\backslash SL(n,\mathbf{R})$ is a closed subset of a set of the following type

$$C(R,\tau) = \{(\xi, Y) : \xi = C_m(X), X \in M_{mn}^0, \|C_m(X)\| = 1, Y \in M_{(n-m)n}^0,$$
$$\|C_{n-m}(Y)\| \le R, | < C_m(X), C_{n-m}(Y) > | \ge \tau\}.$$

Here R,τ are two given positive numbers. Essentially, $R >> 1$ and $0 < \tau << 1$. As in the proof of Theorem 2.3 $C(R,\tau) = C(R,\tau)O$ for any $O \in O(n,\mathbf{R})$. We claim that there exists $\epsilon(R,\tau)$ so that for any $\epsilon < \epsilon(R,\tau)$ and any $0 \le k \le m+1$ one has $C(R,\tau)\Gamma(k,\epsilon) \cap C(R,\tau) = \emptyset$. Assume to the contrary that this assertion is false. Then there exists $0 \le k \le m+1$ and a sequence $\gamma^i, i = 1, ..,$ with the following properties

$$\exists x_i, y_i \in C(R,\tau), y_i = x_i \Sigma(\gamma^i), \lim_{i\to\infty} x_i = x_\infty, \lim_{i\to\infty} y_i = y_\infty,$$
$$\gamma^i \in \Gamma(k,\epsilon_i), \lim_{i\to\infty} \epsilon_i = 0, \lim_{i\to\infty} \sigma_j(\gamma^i) = \sigma_j, j = 1, ..., n,$$
$$\sigma_1 = ... = \sigma_p = \infty, \infty > \sigma_{p+1} \ge \cdots \sigma_{p+q} > 0, \sigma_{p+q+1} = \cdots = \sigma_n = 0.$$

Our assumptions yield that $p \ge k, p+q+1 \le k+n-m$. Set $p' = n-(p+q)$. Then $p' \ge m-k+1$. In particular, $p+p' > m$. As $x_\infty, y_\infty \in C(R,\tau)$ it follows that x_∞ and y_∞ have zero submatrices situated in the last $n-m$ rows and the first p columns and in the last p' columns respectively. Hence, $p, p' \le m$. In particular, $p, p' \ge 1$. Consider the elements $x_i, y_i, i = 1, ..., \infty$. Assume that they are represented by the matrices

$$A(A_i, B_i), A(C_i, D_i) \in M_{m,n}^0 \times M_{(n-m)n}^0, \|C_m(A_i)\| = 1, \|C_m(C_i)\| = 1, i = 1, ..., \infty,$$
$$\lim_{i\to\infty} A(A_i, B_i) = A(A_\infty, B_\infty), \lim_{i\to\infty} A(C_i, D_i) = A(C_\infty, D_\infty).$$

As $x_\infty \in C(R,\tau)$, by expanding the determinant of $A(A_\infty, B_\infty)$ by the first p columns, we deduce that there exists at least one $p \times p$ subdeterminant of A_i based on the first p columns which is nonzero. By interchanging the first p rows of A_∞ if necessary we may assume that the $p \times p$ minor based on the first p rows and columns is nonzero. By choosing the appropriate basis in the subspace spanned by the m rows of $A_i, 1 << i \le \infty$ we may assume in addition to our assumptions that $p \times p$ submatrix of A_i is a diagonal matrix while the submatrix of A_i based on the last $m-p$ rows and the first p columns is zero. Moreover, all the diagonal elements of $p \times p$ submatrix of $A_i, i >> 1$ are equal and bounded above and below by some constants dependending on R, τ. (If $p < m$ we can assume that all the diagonal elements are equal to 1.) Consider the matrices $A_i \Sigma(\gamma^i), i >> 1$. by dividing row

j of A_i by σ_j^i for $j = 1, ..., p$ we deduce that the subspace spanned by the m rows of C_∞ contains the first p rows of the identity matrix in $SL(n, \mathbf{R})$. Without loss of generality we may assume in addition to the above conditions that the first p rows of C_∞ are the first p rows of the identity matrix. As $x_i = y_i \Sigma(\gamma^i)^{-1}$ it follows that the last p' columns of D_∞ are equal to zero. Hence, C_∞ has a nonzero $p' \times p'$ minor different from zero which is based on the last p' columns of C_∞. This minor must be based on the last $m - p$ rows of C_∞. This is imposssible as $m - p < p'$. This contradiction proves the theorem. $\diamond$

§4. Invariant probability measures for certain groups of automorphisms

For $1 \leq k$ set

$$S_{kn} = \{X : X = (x_j^i)_{i=j=1}^{i=k,j=n} \in M_{kn}, \max_{1 \leq i \leq k} \|(x_1^i, ..., x_n^i)\| = 1\}.$$

Note that $S_{1n} = S^{n-1}$ is an $n-1$ dimensional sphere. We then have the natural projection

$$\pi : M_{kn} \backslash \{0\} \to S_{kn}, \pi((x_j^i)_{i=j=1}^{i=k,j=n}) = \frac{(x_j^i)_{i=j=1}^{i=k,j=n}}{\max_{1 \leq i \leq k} \|(x_1^i, ..., x_n^i)\|}.$$

Let $\{0\} \neq U \subset \mathbf{R}^n$. Denote by U_k the k product $U \times \cdots \times U$ and let $U_k' = \pi(U_k \backslash \{0\}) \subset S_{kn}$. Observe next that every $A \in GL(n, \mathbf{R})$ acts on S_{kn} as follows:

$$(x^1, ..., x^k) \mapsto \frac{(x^1, ..., x^k)A}{\max_{1 \leq i \leq k} \|x^i A\|}.$$

Let $R(A)$ be the recurrent set of the above automorphism. In what follows we characterize $R(A)$. In the case $k = 1$, i.e. $S_{1n} = S^{n-1}$, this characterization was given in [**Fri**] and is a more precise version of some Furstenberg's results [**Fur**].

(4.1) Theorem. *Let $A \in GL(n, \mathbf{R})$. Assume that the spectrum of A is located on q circles $|z| = \rho_i(A), i = 1, ..., q$. Let $L_i(A) \subset \mathbf{R}^n$ be the left invariant subspace of A spanned by all eigenvectors of A corresponding to all eigenvalues of A located on the circle $|z| = \rho_i(A)$. Assume that A acts on S_{kn} as above. Then the recurrent set of A is equal to*

$$R(A) = \cup_1^q \pi(L_i(A) \times \cdots \times L_i(A) \backslash \{0\}).$$

Proof. Let $L_{i,k}' = (L_i(A))_k'$. We first show that any point $X = (x^1, ..., x^k) \in L_{i,k}$ is in $R(A)$. For $x^i \neq 0$ set $x^i = e_1^i + ... + e_{p_i}^i$. Here $e_1^i, ..., e_{p_i}^i \in L_i(A)$ are p_i linearly independent eigenvectors of A. Let $I \subset \{1, ..., k\}$ be the set of indices for which $x^i \neq 0$. Set $\hat{A} = \frac{1}{\rho_i(A)} A$. Clearly $R(A) = R(\hat{A})$. Then $e_j^i \hat{A} = \zeta_{ij} e_j^i, |\zeta_{ij}| = 1$. Hence, $X \hat{A}^m$ is given by the coordinates $(\zeta_{ij})^m, j = 1, ..., p_i, i \in I$. Thus, the orbit $X \hat{A}^m, m \in \mathbf{Z}$ is isomorphic to a subgroup of $S^1 \times \cdots \times S^1 = (S^1)^N$ generated by one element g corresponding to $X\hat{A}$. The closure of this group is a compact abelian subgroup of $(S^1)^N$. Clearly, there exists an infinite sequence $0 < n_1 < ...$, of integers so that $\lim_{l\to\infty} g^{n_l} = g^0$. As the identity element g^0 corresponds to X we deduce that $X \in R(\hat{A})$.

We now prove the containment $\cup_{i=1}^q L'_{i,k} \supset R(A)$. Fix $0 \neq z \in \mathbf{C}^n$ and consider the sequence $\{zA^j\}_0^\infty$. Note that all vectors in this sequence lie in the cyclic space $W = span\{z, zA, ..., zA^{n-1}\}$. Assume that $dimW = m$ and let $B = A_{|W}$. Choose a basis $e_1, ..., e_m$ so that so that B is represented in this basis as a Jordan matrix, i.e. is a basis composed of generalized eigenvectors of B. That is, each e_i satisifies the equality $e_i(B - \lambda_i I)^{l_i} = 0$. Assume that m_i is the minimal integer for which the above equality holds. Then m_i is called the index of e_i and denoted by $index(e_i)$. If $index(e_i) = 1$ then e_i is an eigenvector of B with corresponding eigenvalue λ_i. If $index(e_i) > 1$ then e_i is called a generalized eigenvalue corresponding to the eigenvalue λ_i. As usual, let $spec(B)$ denote the spectrum of B. Assume $\lambda \in spec(B)$, i.e. λ is an eigenvalue of B. Then $j = index(\lambda)$ is the maximal index of all generalized eigenvectors cooresponding to λ. Let $z = \sum_1^m \xi_i e_i$. Assume that $index(e_i) = index(\lambda_i)$. As $e_1, ..., e_m$ is a Jordan basis and the dimension of the cyclic space generated by x is m it follows that $\xi_i \neq 0$, e.g. [**Gan**]. Let $\rho(B)$ be the spectral radius of B. Denote by $domspec(B) \subset spec(B)$ the dominant spectrum of B. That is, it is the set of all eigenvalues $\lambda \in spec(B)$ which lie on the maximal circle $|\zeta| = \rho(B)$ and which have the maximal index τ among all eigenvalues on the maximal circle. Equivalently, $domspec(B)$ is the set of all eigenvalues of B lying on the maximal circle to which correspond the maximal Jordan blocks of length τ. Assume that the number of these blocks is β. (Here, $domspec(B)$ is counted with multiplicites, according to the number of maximal Jordan blocks. That is, $domspec(B)$ has exactly $\tau\beta$ eigenvalues.) It is straightforward to show, e.g. use the explicit formulas for B^j in [**Gan**, Ch. 5], that the sequence $\frac{B^j x}{j^{\tau-1}\rho(B)^j}$ is bounded. Furthermore, all the accumulation points of this sequence correspond to a compact abelian group $\mathcal{A}' \subset \mathbf{C}^\beta$ in the subspace whose basis consists of β eigenvectors corresponding to β maximal Jordan blocks of the β eigenvalues in $domspec(B)$. (Note that this eigenvectors are determined uniquely.)

Consider now the sequence $XA^j = (x^1A^j, ..., x^kA^j), X \in S_{kn}$. To each x^i we correspond the matrix B_i its spectral radius $\rho(B_i)$ and its index τ_i for $i = 1, ..., k$. Let $\rho = \max_{1\le i\le k} \rho(B_i) > 0$. Denote by τ the maximal index corresponding to all $\rho(B_i) = \rho$. Let $\rho = \rho_l(A)$. Consider the sequence $\frac{XA^j}{j^{\tau-1}\rho^j}$. Pick up any convergent subsequence. It then follows that every row of the limit matrix Y is either a zero row or a nonzero vector lying in the $L_l(A)$. By the construction $Y \neq 0$. Hence, $\cup_{i=1}^q L'_{i,k} \supset R(A)$. $\diamond$

Let $A \in GL(n, \mathbf{R})$. Then A acts (from the right) on G_{mn}. As in §2 the double cover $\tilde{G}_{mn}$ of G_{mn} can be identified with all $X \in M^0_{nm}, \|C_m(X)\| = 1$. Let $N = \binom{n}{m}$. Then $\tilde{G}_{mn}$ corresponds to all decomposable vectors $x^1 \wedge \cdots \wedge x^m \in \mathbf{R}^n \wedge \cdots \wedge \mathbf{R}^n \cap S^{N-1}$. Thus the action of A on G_{mn} is induced by the action of $C_m(A)$ on S^{N-1}. Theorem 4.1 yields.

(4.2) Theorem. *Let $B \in GL(n, \mathbf{R})$. Assume that $1 \le m < n$. Set $A = C_m(B)$. Let $L_{m,1}(B), ..., L_{m,q}(B)$ be the invariant subspaces given in Theorem 4.1. Assume that B acts on G_{mn}. Then the recurrent set of B is the variety of all decomposable vectors in the set*

$\cup_{i=1}^{q}\pi(L_{m,i}(B))$.

(4.3) Theorem. *Let $\Gamma \subset GL(n, \mathbf{R})$ be a subgroup. Assume that Γ acts on $S_{kn}, 1 \le k$. Suppose furthermore Γ has an invariant probability measure μ on S_{kn}. Then there exists a normal subgroup $\Gamma_0 \subset \Gamma$ of a finite index such that the following conditions hold. There exist $q \ge 1$ nontrivial maximal Γ_0-invariant subspaces $L_1, ..., L_q \subset \mathbf{R}^n$ with the following properties. For each $\gamma \in \Gamma_0$ each L_i is an invariant subspace γ spanned by eigenvectors of γ whose corresponding eigenvalues lie on some circle $|z| = \rho_i(\gamma)$. Let*

$$\phi : \Gamma_0 \to \Gamma_{0,i} \subset GL(L_i), \gamma \mapsto \frac{\gamma}{\rho_i(\gamma)}\Big| L_i, i = 1, ..., q.$$

Then $\Gamma_{0,i}$ is a bounded group for $i = 1, ..., q$. Moreover, the support of μ lies in $\cup_{i=1}^{q}(L_i)'_k$.

Proof. Let $A \in GL(n, \mathbf{R})$. Assume that A acts on S_{kn}. Suppose that μ is a probability measure on S_{kn} which is invariant under the action of A. It is well known that μ is supported on $R(A)$, e.g. [**Wal**, §6.4]. According to Theorem 4.1 $R(A) = \cup_1^{q(A)}(L_i(A))'_k$. Assume that Γ has an invariant probability measure μ on S_{kn}. It then follows that

$$\emptyset \neq \cap_{\gamma \in \Gamma}(\cup_1^{q(\gamma)} L_i(\gamma)\backslash\{0\}) = \cup_1^{\tilde{q}} \tilde{L}_i\backslash\{0\}.$$

Here, for each $\gamma \in \Gamma$ each $\tilde{L}_i$ is a subspace of some $L_j(\gamma)$. As

$$L_i(\gamma)\beta = L_i(\beta^{-1}\gamma\beta), \beta, \gamma \in \Gamma,$$

it then follows that Γ acts on the collection $\tilde{L}_1, ..., \tilde{L}_{\tilde{q}}$ as a subgroup of permutation. Let Γ_0 be the stabilizer of the set $\tilde{L}_1, ..., \tilde{L}_{\tilde{q}}$, i.e. $\tilde{L}_i\Gamma_0 = \tilde{L}_i, i = 1, ..., \tilde{q}$. Then Γ_0 is a normal subgroup of Γ of a finite index. Clearly, μ is supported on $\cup_1^{\tilde{q}}(\tilde{L}_i)'_k$. Assume that μ has a nontrivial restriction μ_i to $(\tilde{L}_i)'_k$. Let $U \subset \tilde{L}_i$ be the minimal subspace so that the set U'_k supports μ_i. As μ is Γ_0 invariant it follows that $U\Gamma_0 = U$. Let $m = dim(U)$. Denote by $\Gamma_0(U)$ the projection of Γ_0 in $GL(U)$ given by the map $\gamma \mapsto \frac{\gamma}{\rho_i(\gamma)}|U$. We claim that $\Gamma_0(U)$ is bounded. We prove this claim by the induction on m. For $m = 1$ $\Gamma_0(U) \subset \{\pm 1\}$ and the claim trivially holds. Assume that $\Gamma_0(U)$ is bounded for all U such that $m \le p - 1$. Suppose that $m = p$. Assume to the contrary that $\Gamma_0(U)$ is unbounded. Hence, there exists a sequence $A_i \in \Gamma_0(U) \subset GL(m, \mathbf{R})$ with the following properties.

$$A_i = P_i\Sigma_iQ_i, P_i, Q_i \in O(m, \mathbf{R}), \Sigma_i = diag(\sigma_1(A_i), ..., \sigma_m(A_i)), i = 1, ...,$$
$$lim_{i\to\infty}P_i = P, lim_{i\to\infty}Q_i = Q, lim_{i\to\infty}\sigma_j(A_i) = \sigma_j, j = 1, ..., m,$$
$$\sigma_1 = \infty, lim_{i\to\infty}\frac{A_i}{\sigma_1(A_i)} = T.$$

As $1 = |det(A_i)| = \prod_{j=1}^{m}\sigma_j(A_i)$ we deduce that $\sigma_m = 0$. Hence, $rank(T) < m$. Observe next that $\lim_{i\to\infty}\frac{xA_i}{\sigma_1(A_i)} = xT$. Let

$$V = \{y : y = xP^{-1}, x = (x_1, ..., x_m), x_1 = 0\}, W = UT, U''_k = U'_k\backslash V'_k.$$

The minimality of U yields that $\mu_i(U''_k) > 0$. Note that for any compact set $C \subset U''_k$ the sets CA_i converge to a subset of W'_k. It then follows that $\mu_i(U''_k) = \mu_i(W_k) > 0$. Let $\mu_i(V'_k) = a$. We claim that $a > 0$. Otherwise the support of μ_i lies on W'_k contrary to our assumptions. It now follows that $\mu_i((V\gamma)'_k) = a, \forall \gamma \in \Gamma_0$. Let $\bar{\mu}_i$ be the restriction of μ_i to V'_k. Denote by $\hat{\mu}_i$ the finite Γ_0 invariant measure generated by $\bar{\mu}_i$.

There are two possible cases. In the first case, the family of subspaces $V\gamma, \gamma \in \Gamma_0$ is finite. Let

$$\Gamma_1 = \{\gamma' : \gamma' \in \Gamma_0, (V\gamma)\gamma' = V\gamma, \forall \gamma \in \Gamma_0\}.$$

Then Γ_1 is a normal subgroup of Γ_0 of a finite index. It then follows that $\bar{\mu}_i$ is Γ_1 nontrivial invariant measure. Let $\bar{U} \subset W$ be the minimal Γ_1 invariant subspace so that $\bar{\mu}_i$ is supported on $\bar{U}'_k$. As $dim(\bar{U}) \leq dim(V) < m$ we can use the induction hypothesis. That is $\Gamma_1(\bar{U})$ is bounded. Then $\hat{\mu}_i$ is supported on the finite union of the sets $(V\gamma)'_k, \gamma \in \Gamma_0/\Gamma_1$. Let

$$\hat{U} = \sum_{\gamma \in \Gamma_0/\Gamma_1} \bar{U}\gamma.$$

As Γ_0/Γ_1 is finite we deduce that $\Gamma_0(\hat{U})$ is a bounded group. Suppose first that $\mu_i = \hat{\mu}_i$. Then $U = \hat{U}$ and $\Gamma_0(U)$ is bounded. Assume that $\nu = \mu_i - \hat{\mu}_i$ is a nonzero Γ_0 invariant measure. As $\nu(V'_k) = 0$ our argument shows that ν is supported on W'_k. Let $U^1 \subset W$ be the minimal Γ_0 invariant subspace so that ν is supported on $(U^1)'_k$. As $dim(U^1) \leq dim(W) < m$ the induction hypothesis yields that $\Gamma_0(U^1)$ is bounded. Then $U = U^1 + \hat{U}$ and $\Gamma_0(U) = \Gamma_0(U^1) + \Gamma(\hat{U})$ is bounded.

To this end assume that the family of subspaces $V\gamma, \gamma \in \Gamma_0$ is infinite. Let $D \subset V$ be a Borel set such that $0 \notin D$. Suppose that there exists an infinite sequence of $\gamma_j \in \Gamma_0, j = 1, ...,$ so that $D\gamma_j \cap D\gamma_l = \emptyset$ for any $j \neq l$. Since μ_i is finite and Γ_0 invariant, it follows that $\mu_i(D'_k) = 0$. Thus, there exists a nontrivial proper subspace $\bar{V} \subset V$ so that the family of subspaces $\bar{V}\gamma, \gamma \in \Gamma_0$ is finite. It then follows that support of $\hat{\mu}_i$ lies on all subspaces $(\bar{V}\gamma)'_k, \gamma \in \Gamma_0$ for which $\bar{V}\gamma, \gamma \in \Gamma_0$ is a finite collection of subspaces. Let $\bar{V}$ be as above. Set $\mathcal{G} \subset GL(m, \mathbf{R})$ to be the algebraic closure of Γ_0 and assume that $\mathcal{G}_0 \subset \mathcal{G}$ is its ireducible component containing the identity. Let $\Gamma_1 \subset \Gamma_0$ be the normal subgroup of a finite index which fixes all subspaces $\bar{V}\gamma, \gamma \in \Gamma_0$. It then follows that the algebraic closure of Γ_1 contains $\mathcal{G}_0$. Set $\tilde{V} = \cap_{g \in \mathcal{G}_0} Vg$. Thus, $\bar{V} \subset \tilde{V}$. It now follows that $\tilde{V}\gamma, \gamma \in \Gamma_0$ is a finite collection of subspaces. Moreover, $\hat{\mu}_i$ is supported on this finite collection of subspaces. We now conclude as above that $\Gamma_0(U)$ is bounded. In both cases we contradict the assumption that $\Gamma_0(U)$ is not bounded.

Suppose that $Z \subset \tilde{L}_i$ be another nontrivial Γ_0 invariant subspace of $\tilde{L}_i$ so that $\Gamma_0(Z)$ is bounded. It then follows that $\Gamma_0(U + Z)$ is also bounded. Hence, there exists a maximal Γ_0 invariant subspace $L_i \subset \tilde{L}_i$ so that $\Gamma_0(L_i)$ is bounded. Moreover, any Γ invariant probability measure is supported on $\cup_{i=1}^q (L_i)'_k$. $\diamond$

(4.4) Corollary. *Let $\Gamma \subset GL(n, \mathbf{R})$. Assume that Γ acts on $G_{mn} \times S_{kn}, 1 \leq m < n, 1 \leq k$. Suppose furthermore that Γ has an invariant probability measure μ. Then there exists a normal subgroup $\Gamma_0 \subset \Gamma$ of a finite index such that the following conditions hold. There exist $q \geq 1$ nontrivial maximal Γ_0-invariant subspaces $L_1, ..., L_q \subset \mathbf{R}^n$ with the following* *properties. For each $\gamma \in \Gamma_0$ each L_i is an invariant subspace γ spanned by eigenvectors of γ whose corresponding eigenvalues lie on some circle $|z| = \rho_i(\gamma)$. Let*

$$\phi : \Gamma_0 \to \Gamma_{0,i} \subset GL(L_i), \gamma \mapsto \frac{\gamma}{\rho_i(\gamma)} \big| L_i, i = 1, ..., q.$$

Then $\Gamma_{0,i}$ is a bounded group for $i = 1, ..., q$. Moreover, the support of μ lies in $G_{mn} \times (\cup_{i=1}^{q} (L_i)'_k)$.

Suppose that $\Gamma \subset GL(n, \mathbf{R})$ satisfies the conclusions of Theorem 4.3. We then claim that the action of Γ on S_{kn} and on $G_{mn} \times S_{kn}$ has an invariant measure. Consider the subgroup Γ_0 acting on $(L_i)'_k \subset S_{kn}$. This action is equivalent to the action of $\Gamma_{0,i}$. As $\Gamma_{0,i}$ is bounded its topological closure is a compact group. Hence, it is amenable, e.g. [**Zim1**]. Thus, $\Gamma_{0,i}$ has an invariant probability measure μ on $(L_i)'_k$. Hence, μ is Γ_0 invariant. Since Γ_0 is a normal subgroup of Γ of a finite index it easily follow that the action of Γ on μ generates a finite invariant measure ν. Normalize this invariant measure to obtain a Γ invariant probability measure. Similar arguments apply to the case $G_{mn} \times S_{kn}$.

§5. Cocompact lattices

(5.1) Theorem. *Let $\Gamma \subset GL(n, \mathbf{R})$ be a discrete group. Assume that Γ acts freely and properly discontinuously on $GL(m, \mathbf{R}) \backslash GL(n, \mathbf{R})$ for $1 \le m < n$. If*

$$M = GL(m, \mathbf{R}) \backslash GL(n, \mathbf{R}) / \Gamma$$

is compact then Γ does not have an invariant probability measure on $G_{mn} \times S_{(n-m)n}$.

Proof. Assume to the contrary that ν is an invariant probability measure under the action of Γ on $G_{mn} \times S_{(n-m)n}$. Apply now Corollary 4.4 to ν. Since μ is supported on M it follows that there exists at least one Γ_0 invariant subspace, say L_1, whose dimension is not less than $n - m$. Set

$$Q = T_{mn} \cap (G_{mn} \times (L_1 \times \cdots \times L_1)).$$

Then $Q\gamma = Q, \forall \gamma \in \Gamma_0$. Hence, Γ_0 acts freely and properly discontinuously on Q. Let $M' = Q/\Gamma_0$ a compact submanifold of a compact manifold $M_0 = SL(m, \mathbf{R}) \backslash SL(n, \mathbf{R}) / \Gamma_0$. Without loss of generality we assume that the bounded group $\Gamma_0(L_1)$ is a subgroup of an orthogonal group. Let $\Gamma'_0 = \Gamma_0 \big| L_1$. That is, all singular values of any $\gamma' \in \Gamma'_0$ are equal. As Γ_0 acts properly discontinuously on Q we can apply the arguments of Theorem 3.1. In particular, for any $\epsilon > 0$ there is only a finite number of elements of Γ'_0 which have all singular values in $(\epsilon, \frac{1}{\epsilon})$. As each element in Γ'_0 is a scalar multiple of an orthogonal matrix it follows that for any $\epsilon > 0$ all but a finite number of elements of Γ'_0 satisfy either $\sigma_{n'}(\gamma') \ge \frac{1}{\epsilon}$ or $\sigma_1(\gamma') \le \epsilon$. (Here, $n' = dim(L_1)$.) According to Theorem 2.2 Γ'_0 acts properly discontinuously on $W = L_1 \times \cdots \times L_1 \cap M^0_{(n-m)n}$. We claim that M' is not compact. Indeed, pick up any compact set $K \subset W$ which contains an open set of W. As

Γ_0' acts properly discontinously on W, we have that $\gamma(K) \cap K = \emptyset$ except for a finite set $\Delta \subset \Gamma_0'$. Consider the set $K' = T_{mn} \cap (G_{mn} \times K)$. Clearly, K' is not compact. On the other hand K' is a subset of a finite cover of M' induced by Δ. Hence, M' is not compact. This contradicts the assumption that M is compact. Therefore, such a ν does not exists. $\diamond$

Let $GL(n-m, \mathbf{R}) \subset GL(n, \mathbf{R})$ be embedded in the lower right corner of $GL(n, \mathbf{R})$. It then follows that the actions of $GL(m, \mathbf{R})$ commutes with the action of $GL(n-m, \mathbf{R})$ on $GL(n, \mathbf{R})$ from the left. Hence, $GL(n-m, \mathbf{R})$ acts from the left on $GL(m, \mathbf{R})\backslash GL(n, \mathbf{R})$. Furthermore, this action projects to the action on the manifold $M = GL(m, \mathbf{R})\backslash GL(n, \mathbf{R})/\Gamma$. Let $U \subset GL(n-m, \mathbf{R})$ be a one parameter subgroup. Assume that M is compact. Then the flow induced by U has an invariant probability measure μ on M. We believe that μ induces an invariant probability measure ν under the action of Γ on $G_{mn} \times S_{(n-m)n}$. More precisely, assume that $n - m > 1$ and U is unipotent subgroup. Does there exists an analog of the Raghunathan conjecture?

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