Regularity of invariant measures on finite and infinite dimensional spaces and applications

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Abstract

In this paper we prove new results on the regularity (i.e., smoothness) of measures μ solving the equation $L^*\mu = 0$ for operators of type $L = \Delta + B \cdot \nabla$ on finite and infinite dimensional state spaces E. In particular, we settle a conjecture of I. Shigekawa in the situation where $\Delta = \Delta_{_H}$ is the Gross-Laplacian, (E, H, γ) is an abstract Wiener space and $B = -\mathrm{id}_E + v$ where v takes values in the Cameron-Martin space H. Using Gross' logarithmic Sobolev-inequality in an essential way we show that μ is always absolutely continuous w.r.t. the Gaussian measure γ and that the square root of the density is in the Malliavin test function space of order 1 in $L^2(\gamma)$. Furthermore, we discuss applications to infinite dimensional stochastic differential equations and prove some new existence results for $L^*\mu = 0$. These include results on the "inverse problem", i.e., we give conditions ensuring that B is the (vector) logarithmic derivative of a measure. We also prove necessary and sufficient conditions for μ to be symmetrizing (i.e., L is symmetric on $L^2(\mu)$). Finally, a substantial part of this work is devoted to the uniqueness of symmetrizing measures for L. We characterize the cases, where we have uniqueness, by the irreducibility of the associated (classical) Dirichlet forms.

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Contents

0 Introduction

The topic of this paper is the study of elliptic equations for measures on infinite dimensional spaces. More precisely, we discuss equations of the following type:

$$L^*\mu = 0, \tag{0.1}$$

where μ is a (finite) measure on a locally convex space X, and the operator L is defined on smooth finitely based functions by

$$Lf = \Delta_H f + {}_{X^*}\langle f', B\rangle_X. \tag{0.2}$$

Here H is a fixed Hilbert space continuously and densely embedded into X, and Δ_H is Gross's Laplacian [40], generated by H. $B: X \to X$ is a Borel map such that

$${}_{X^*}\langle \ell, B(\cdot)\rangle_X \in L^2(\mu) \quad \text{for all} \quad \ell \in X^* \ . \tag{0.3}$$

A measure μ is said to be a solution of (0.1), if for all smooth finitely based functions f

$$\int_X Lf(x)\,\mu(dx) = 0. \tag{0.4}$$

The growing interest in such equations is motivated by numerous applications in mathematical physics, in particular, in quantum field theory, and in infinite dimensional stochastic analysis. A closely related problem (which we also discuss in this paper in Section 4 below), is the study of invariant and symmetrizing measures for infinite dimensional diffusions, more precisely processes $(\xi_t)_{t\geq 0}$ which are weak solutions of stochastic differential equations of type

$$d\xi_t = \sqrt{2}dw_t^H + B(\xi_t)\,dt. \tag{0.5}$$

Here $(w_t^H)_{t\geq 0}$ is a Brownian motion on X starting at 0 with covariance given by the inner product of H. Such measures are then solutions of (0.1).

If an extension of L generates a strongly continuous semigroup $(T_t)_{t\geq 0}$ on $L^2(\mu)$ and μ is an invariant measure for $(T_t)_{t\geq 0}$, then μ solves (0.1), but in general not vice versa (cf. Remark 3.0 below). In this sense solutions to (0.1) could be called *generalized invariant measures*. We, however, shall not use this terminology, but it explains the title of this paper.

Though we also present some new existence results for equation (0.1) in Section 5 below, the main problems we deal with are:

(1) Regularity of solutions to (0.1);

(2) Uniqueness of those solutions μ to (0.1) which are, in particular, symmetrizing for L, i.e., satisfy

$$\int Lu\, v\, d\mu = \int u\, Lv\, d\mu \quad \text{for all smooth finitely based functions } u, v, \tag{0.6}$$

(cf. Section 6).

The latter problem is of fundamental importance for uniquenesss of Gibbs states (cf. Remark 3.11 (ii)).

Let us now briefly describe our main results on (1) and (2).

If $X = \mathbb{R}^n$ and $B: \mathbb{R}^n \to \mathbb{R}^n$ is a smooth mapping, then a classical result, which goes back to H. Weyl, says that any solution of (0.1) is an absolutely continuous measure which has a smooth density with respect to Lebesgue measure dx. Less restrictive conditions, such as local boundedness, are known to be sufficient for μ to possess a density φ^2 with integrable $\nabla\varphi^2$. As the first main result in this paper (cf. Theorem 3.1) we prove that, if merely (0.3) holds, a solution μ of equation (0.1) must be absolutely continuous w.r.t. Lebesgue measure and, if φ^2 is the density, then φ is in the Sobolev space $H^{1,2}(\mathbb{R}^n; dx)$. The major part of our paper is, however, devoted to infinite-dimensional generalizations of this result. We settle a conjecture of I. Shigekawa [65] (and R.v. Vintschger [68]) by proving that any solution of (0.1) is absolutely continuous with respect to the Gaussian measure γ on X corresponding to H, provided $B(x) = -x + v(x)$, where $v: X \to H$ and $\|v\|_H \in L^2(\mu)$. In addition, if φ^2 is the corresponding Radon-Nikodym density then φ is in $H^{1,2}(X;\gamma)$, i.e., in the Sobolev space over (X, H, γ) (cf. Theorem 3.5 below). For the proof it is crucial that the logarithmic Sobolev inequality holds in this case (cf. [41]). Some other regularity results in infinte dimensions are presented in Proposition 3.3 and Corollary 3.14. All these results may be regarded as hypoellipticity of elliptic operators in infinite dimensions (cf. Malliavin's fundamental work [53]).

As there is no Lebesgue measure on infinite dimensional spaces X, one has to make precise what it means here that a measure μ on X is "regular". It turns out that the appropiate notion is that μ has a *(vector) logarithmic derivative.*

In Sections 1 and 2 we present all details on this and only mention here that if $\mu = \varphi^2\, dx$ on $X = \mathbb{R}^n$, then its logarithmic derivative exists (and is then equal to $\nabla(\log\varphi^2)$), if and only if $\varphi \in H^{1,2}(\mathbb{R}^d; dx)$. This is thus exactly the type of regularity proved in Theorem 3.1.

Hence the existence of a vector logarithmic derivative is an infinite-dimensional substitute of the existence of a density with integrable gradient. According to Proposition 2.4 vector logarithmic derivatives exist under wide conditions. However, in infinite dimensions they are not gradients of functions, but, as Theorem 2.8 below says, still "generalized gradients" (i.e., certain limits of gradients of functions).

Vector logarithmic derivatives also arise naturally when one deals with symmetric diffusion processes in infinite dimensions. This connection was discovered by Albeverio and Høegh-Krohn and investigated in [3, 4, 5], [10, 11, 12]. In Section 4 we prove that square-integrable vector logarithmic derivatives are precisely the drifts of symmetric diffusions.

In Section 5 on existence of invariant measures we also derive sufficient conditions for a given map $B: X \to X$ to be the vector logarithmic derivative of a measure μ (cf. Remark 5.5(ii)). This is the so-called "inverse problem"(cf e.g. [15]).

We emphazise that the existence of a vector logarithmic derivative for μ has nothing to do with "Gaussian". There exists such μ on X which is singular with respect to all Gaussian measures on X (cf. Example 2.1 (ii)).

The regularity results described above can be interpreted in the sense that they identify conditions under which the logarithmic derivative of a measure μ satisfying (0.1) exists. In the second half of Section 3 we systematically discuss consequences of the existence of logarithmic derivatives. In particular, we prove that L is symmetric on $L^2(\mu)$ (cf. (0.6)) if and only if B is the logarithmic derivative of μ (cf. Theorem 3.10). This (in connection with Theorems 2.8 and 4.1) extends A. Kolmogorov's classical result (cf. [49]) on symmetrizable diffusions to our infinite dimensional framework. As as consequence we obtain that, if (0.6) is satisfied for two probability measures μ and ν, then their logarithmic derivatives must coincide. This provides a uniqueness criterion for symmetrizing measures.

Section 6 is entirely devoted to this uniqueness criterion, more precisely to the question under what conditions probability measures are uniquely determined by their (vector or even only partial) logarithmic derivatives. Clearly, this is not always the case as Example 6.1 even for $X = \mathrm{R}^1$ shows. Theorem 6.2, however, states that we have uniqueness if $X = \mathrm{R}^d$ and the logarithmic derivative is locally Lebesgue integrable. This is in contrast to the infinite dimensional case, since there one can find two different centred Gaussian measures μ_1 and μ_2 having the same logarithmic derivative, and the latter is linear and continuous (cf. Theorem 6.8; see also Theorem 6.5 for the much easier non centred case, including the *free field of Euclidean quantum field theory* in Example 6.7). This leads to an example of an infinite dimensional Ornstein-Uhlenbeck process with bounded linear drift which has two different centred Gaussian measures as invariant measures and is symmetric w.r.t. both of them (cf. Corollary 5.9).

Our main result on uniqueness is, however, Theorem 6.15 which gives a complete characterization in terms of the associated Dirichlet forms. Crucial for a part of the proof is Proposition 6.11 (in this form taken from [11], [60]) which characterizes weak differentiability on infinite dimensional spaces in terms of absolute continuity on rays. With regard to the length of this paper we do not include any of the numerous applications of this result. These will appear in forthcoming papers. We only indicate here that for finite range interactions on a lattice the (easy) half of Theorem 6.15(ii) in physics terminology says that if there is only one Gibbs state, then the infinite volume Hamiltonian has the constant function 1 as its unique (normalized) ground state (cf. Remark 6.16(i)5.).

Finally, we mention that part of the results in this paper have been announced in [23]. Essentially, all results have been presented in talks at various conferences at Edinburgh, Holzhau, Marseille, Passau, and the International Congress of Mathematicians in Zürich during the spring and summer of 1994.

1 Notation and terminology

Let X be a locally convex (topological vector) space over R with dual X^*.

$\mathcal{F}C^\infty$ resp. $\mathcal{F}C_b^\infty$ is the class of functions $f = g(\ell_1, ..., \ell_n)$, $n \in \mathbb{N}$, $\ell_i \in X^*$, $g \in C^\infty(\mathbb{R}^n)$ resp. $g \in C_b^\infty(\mathbb{R}^n)$. We say that $H \subset X$ is a continuously embedded Hilbert space, if H is a linear subspace equipped with a separable Hilbert norm such that the natural embedding $H \subset X$ is continuous. If H is dense in X, we can define an injection $j_H : X^* \to H$ as follows. For any $k \in X^*$ the restriction of k to H is continuous in the Hilbert norm of H. Hence by the Riesz representation theorem there is an element $j_H(k) \in H$ such that

$$_{X^*}\langle k, h\rangle_X = (j_H(k), h)_H \qquad \forall h \in H.$$

Since H is dense, such an element is unique. Let $R_H : H^* \to H$ denote the Riesz isomorphism. Note that if $X = H$ then $j_H = R_H$. Otherwise, these maps are completely different. If there is no risk of ambiguity j_H is denoted by j. Identifying X^* with $j(X^*)$ we have: $X^* \subset H \subset X$, densely and continuously.

Note that if $\{e_i\}$ is an orthonormal basis in H then

$$j(k) = \sum_{i=1}^{\infty} k(e_i)e_i \qquad \forall k \in X^*,$$

(with the sum converging in H). Note that if X is a Banach space with norm $|\ |_X$, then $j(X^*)$ consists of exactly those $h \in H$ which satisfy

$$\sup\{(h, h')_H : \ h' \in H, \ |h'|_X \leq 1\} < \infty.$$

Example 1.1 (i) Let $X := \mathcal{S}'(\mathbb{R}^n), H := L^2(\mathbb{R}^n) := L^2(\mathbb{R}^n; dx)$ where dx denotes Lebesgue measure on $\mathbb{R}^n$. Then we get the triple $\mathcal{S} \subset H \subset \mathcal{S}'$ with natural embeddings.

(ii) Let $X := \mathbb{R}^\infty, H := \ell^2$. Then $X^* = \mathbb{R}_0^\infty \subset H$. Note that here (as in (i)) we have considered the duals already as subspaces of H hence $j(k)$ is represented by k itself.

(iii) Let X be a separable Hilbert space, T an injective nonnegative definite Hilbert-Schmidt operator in X, and $H := T(X)$ be equipped with the natural Hilbert norm $|h|_H := |T^{-1}h|_X$. Then $j_H(X^*) = T^2(X)$, so that,

$$_{X^*}\langle k, h\rangle_X = (T^{-2} j_H(k), \, h)_X; \ k \in X^*, \ h \in H \ ,$$

i.e., $j_H = T^2 R_X$.

(iv) Let $X = \mathrm{R}^\infty, H = \{(h_n) : \sum \alpha_n^2 h_n^2 < \infty\}$ where $\alpha_n \in \mathrm{R},\ n \in \mathrm{N}$. Then X^* is naturally identified with R_0^∞, but $j(k)$ is represented by the element $(\alpha_n^{-2} k_n)$ and not by k itself (except the case $\alpha_n = 1\ \forall n$).

A finite *nonnegative* Radon measure μ on the σ-field of Borel sets $\mathcal{B}(X)$ of X will be simply called a measure on X. In typical examples, such as $X = \ell^2,\ X = \mathrm{R}^\infty,\ X = \mathcal{S}'(\mathrm{R}^n)$, any finite Borel measure is Radon. Set $L^2(\mu) := L^2(X;\mu)$ (i.e., the space of classes of μ-square integrable real functions on X) and $L^2(\mu, Z) := L^2(X \to Z; \mu)$ if Z is a Banach space.

Let γ be a centred Gaussian measure on a locally convex space X (see [25]). $H(\gamma)$ denotes the *reproducing kernel Hilbert space* of γ, that is

$$H(\gamma) := \{h \in X :\ |h|_{H(\gamma)} < \infty\},$$

where

$$|h|_{H(\gamma)} := \sup\left\{\ell(h) :\ \ell \in X^* \quad \text{with} \quad \int \ell(x)^2\, \gamma\,(dx) \le 1\right\}.$$

Note that for any $\ell \in X^*$ one has that

$$\|\ell\|_{L^2(\gamma)} = |\ell|_{H^*} \left(:= \sup_{|h|_H \le 1} |\ell(h)|\right).$$

Denote the completion of X^* in $L^2(\gamma)$by X_γ^*. It is known that $X_\gamma^{**} = H(\gamma)$. Hence for the canonical linear isometry J between X_γ^* and $H(\gamma)$ we have that $Jk = j_H(k)$ for all $k \in X^*$ and that

$$\int k(x)\, \ell(x)\, \gamma\,(dx) = (Jk, J\ell)_{H_{(\gamma)}} \quad \forall\, k, \ell \in X_\gamma^*.$$

Example 1.2 Let X be a Hilbert space, T a nonnegative injective self-adjoint Hilbert-Schmidt operator in X with eigenvalues t_n and (normed) eigenvectors $e_n,\ n \in \mathrm{N}$. Then the function $\exp(-(Tx, Tx)_X/2)$ is the Fourier transform of a centred Gaussian measure γ on X, such that

$$H(\gamma) = T(X), \quad (h_1, h_2)_{H(\gamma)} = (T^{-1}h_1, T^{-1}h_2)_X.$$

In this case the elements of X_γ^* have the representation $\ell(x) = \sum\limits_{n=1}^{\infty} c_n t_n^{-1} (x, e_n)_X$, where $\sum\limits_{n=1}^{\infty} c_n^2 < \infty$. The operator T^2 is called the *covariance operator of* μ.

We say that a measure μ on a LCS X is *differentiable* along a vector $h \in X$ (in the sense of Fomin) and has the (partial) *logarithmic derivative* β_h^μ along h if β_h^μ is a μ-integrable function such that the following integration by parts formula holds:

$$\int \partial_h f(x)\, \mu\,(dx) = -\int f(x)\, \beta_h^\mu(x)\, \mu\,(dx) \quad \forall\, f \in \mathcal{F}C_b^\infty\,, \tag{1.1}$$

where $\partial_h f = \frac{\partial f}{\partial h} := \frac{d}{ds} f(\cdot + sh)\big|_{s=0}$. In this case (1.1) holds for any function f with μ-integrable partial derivative $\partial_h f$, provided $f\beta_h^\mu \in L^1(\mu)$ (see [22, Theorem 2.6]). For a characterization of such h with $\beta_h^\mu \in L^2(\mu)$ in terms of the corresponding one dimensional conditional measures we refer to [8, Theorem 2.5]. Clearly, β_h^μ is uniquely determined by (1.1).

Note that a measure μ on R is differentiable along the vector $h = 1$ if and only if it has an absolutely continuous density p with respect to Lebesgue measure such that $p' \in L^1(\mathrm{R}^1)$. In this case $\beta_1^\mu = p'/p$. In a similar way, a measure μ on R^n is differentiable along all directions if and only if it has a density p w.r.t. Lebesgue measure such that the partial derivatives $\partial_h p$ in the sense of distributions are integrable functions on R^n.

Remark 1.3 A measure μ on R^n is differentiable along all $h \in \mathrm{R}^n$ with $\beta_h^\mu \in L^2(\mu)\ \forall h \in \mathrm{R}^n$ if and only if $\mu = \varphi^2\, dx$ with $\varphi \in H^{1,2}(\mathrm{R}^n; dx)$, i.e., the classical Sobolev space of order 1 in $L^2(\mathrm{R}^n; dx)$ (cf. e.g. [14]). In this case $\beta_h\mu = 2\frac{\partial\varphi}{\partial h}/\varphi$.

In the infinite dimensional case no nonzero measure can be differentiable along all directions (see [22]).

It is known (see [22]) that μ is differentiable along h if and only if for all Borel sets $B \subset X$

$$\lim_{t\to 0} \frac{\mu(B+th) - \mu(B)}{t}$$

exists. This limit defines a signed measure $D_h\mu$. Moreover, $D_h\mu \ll \mu$ (see [15], [26]). The signed measure $D_h\mu$ is called the derivative of μ along h. The Radon-Nikodym density $D_h\mu/\mu$ is exactly β_h^μ. In this paper we only use the definition of differentiability via (1.1).

The subspace of differentiability $D(\mu)$ of a measure μ is defined (see [16]) as the collection of all vectors h along which μ is differentiable. Clearly, by (1.1) $h \mapsto \beta_h^\mu$ is linear and injective on $D(\mu)$. $D(\mu)$ is a Banach space with the norm $\|\beta_h^\mu\|_{L^1(\mu)}$ (see [22]). For a centred Gaussian measure γ, $D(\gamma)$ coincides with the reproducing kernel Hilbert space $H(\gamma)$. For a general measure μ, the symbol $H(\mu)$ denotes the space of all vectors $h \in D(\mu)$, such that $\beta_h^\mu \in L^2(\mu)$. Equipped with the norm

$$|h|_{H(\mu)} = \|\beta_h^\mu\|_{L^2(\mu)}$$

$H(\mu)$ is a Hilbert space which is continuously embedded into the Banach space $D(\mu)$. By [22, Corollary 3 of Theorem 3.3] the embedding $H(\mu) \subset X$ is a Hilbert-Schmidt map, if X is a Hilbert space.

Note that if γ is a centred Gaussian measure on a locally convex space X and $H := H(\gamma)$, then for any $\ell \in X_\gamma^*$

$$\beta_{J\ell}^\gamma(x) = -\ell(x) \tag{1.2}$$

(cf. e.g. [11, Proposition 5.5]).

To conclude this section we mention the following result to be used later, whose prove can be found in [17, Proposition 3.6] .

Lemma 1.4 *For $n \in \mathrm{N}$ let μ_n be a measure on X differentiable along $h_n \in X$. Assume that $(h_n)_{n\in\mathrm{N}}$ converges weakly to $h \in X$ and that*

$$C := \sup_n \left\|\beta_{h_n}^{\mu_n}\right\|_{L^2(\mu_n)} < \infty .$$

If there exists a measure λ such that the Fourier transforms of μ_n converge pointwise to the Fourier transform of λ, then λ is differentiable along h and

$$\|\beta_h^\mu\|_{L^2(\mu)} \le C.$$

2 Vector logarithmic derivatives

Now we describe a construction which associates to a differentiable measure μ some map - its *vector logarithmic derivative*. In $\mathbb{R}^n$ with $\mu = p\,dx$ this map gives $\nabla p/p$.

In this section let X be a locally convex space, $H \subset X$ a continuously and densely embedded separable Hilbert space, $j_H : X^* \to H$ the injection defined above. If a measure μ on X is differentiable along all directions from $j_H(X^*)$ and there exists a Borel map $\beta_H^\mu : X \to X$ such that

$$ {}_{X^*}\langle k, \beta_H^\mu \rangle_X = \beta_{j_H(k)}^\mu \quad \forall\, k \in X^*, \tag{2.1} $$

then this map β_H^μ is called the *vector logarithmic derivative* of μ associated with H. β_H^μ is uniquely determined by μ (and H).

Note that the definition of β_H^μ can be restated as an integration by parts formula:

$$ \int \partial_{j_H(k)} f(x)\mu(dx) = -\int f(x)\ {}_{X^*}\langle k, \beta_H^\mu(x)\rangle_X\ \mu(dx) \quad \forall\, f \in \mathcal{F}C_b^\infty\ , \quad \forall\, k \in X^*\ . $$

Example 2.1 (i) Assume that H is the reproducing kernel Hilbert space of some Gaussian Radon measure on X. Note that this is e.g. the case if X contains a dense separable Banach space which is the completition of H w.r.t. a *measurable norm* (in the sense of L. Gross, cf. [39] and also [50]). Let A be a self-adjoint operator on H such that for some $c \in]0,\infty[$, $(Ah,h)_H \ge c(h,h)_H$ for all $h \in H$. Let γ^A be the centered Gaussian measure on X with covariance operator A^{-1}, i.e.,

$$ \int_X e^{i\ell(x)}\, \gamma^A\,(dx) = e^{-\frac{1}{2}|A^{-1/2} j_H(\ell)|_H^2} \qquad \text{for all} \quad \ell \in X^*. $$

Define for $h \in H$

$$ X_h := \lim_{n\to\infty} {}_{X^*}\langle \ell_n, \cdot\rangle_X \quad \text{in } L^2(\gamma^A) $$

where $\ell_n \in X^*,\ n \in \mathbb{N}$, such that $\ell_n \underset{n\to\infty}{\longrightarrow} h$ in H. Let $\ell \in X^*$ such that $j_H(\ell) \in D(A)$. Then by direct computation (cf. e.g. [11, Proposition 5.5(i)] and [59, Subsection 6a)])

$$ \beta_{j_H(\ell)}^{\gamma^A} = -X_{Aj_H(\ell)}\ \left(\in L^2(\gamma^A)\right)\ . $$

In particular, if in addition $Aj_H(\ell) \in j_H(X^*)$, then

$$ \beta_{j_H(\ell)}^{\gamma^A} = -\left(j_H^{-1} A j_H\right)(\ell) \qquad (\in X^*). $$

Suppose now that

$$ j_H(X^*) \subset D(A), $$

that

$$ A(j_H(X^*)) \subset j_H(X^*), $$

and that

$$ \Lambda\ :\ \ell \mapsto \left(j_H^{-1} A j_H\right)(\ell) \quad \text{is continuous on } X^* $$

which by the closed graph theorem is e.g. the case if X^* is completely metrizable. Suppose that $X^{**} = X$ and let A^X denote the bounded linear operator on X adjoint to Λ. Then, clearly,

$$\beta_H^{\gamma^A} = -A^X.$$

In particular, if $A = \mathrm{id}_H$ and $\gamma := \gamma^{\mathrm{id}_H}$, then $\beta_H^{\gamma}(x) = -x \ \forall x \in X$.

(ii) The fact that a measure μ is Gaussian is, however, not at all necessary for the existence of β_H^{μ}. This is illustrated by the following example. Let λ be the countable product of the measures λ_n defined on R with probability densities $c_n p(t/M_n)$, where $M_n := \log(n+1)$, $p \in C_0^{\infty}(]-1,1[)$. According to [22, Theorem4.1]

$$D(\lambda) = \left\{ (x_n)_{n\in \mathrm{N}} \in \mathrm{R}^{\infty} : \ \sum_{n=1}^{\infty} M_n^2 x_n^2 < \infty \right\}.$$

Moreover, $D(\lambda) = H(\lambda) =: H$ and $|h|_H = \|\beta_h^{\lambda}\|_{L^2(\mu)}$ for all $h \in H$. Let X be an arbitrary Hilbert space, $X \subset \mathrm{R}^{\infty}$, with $\lambda(X) = 1$. Finally, let

$$\mu := \sum_{n=1}^{\infty} 2^{-n} \lambda_{a_n} \ ,$$

where $\{a_n | \, n \in \mathrm{N}\}$ is dense in H. Then it follows by [22, Theorem3.4] that β_H^{μ} exists, $\|\beta_H^{\mu}\|_X \in L^2(\mu)$ and μ is singular w.r.t. *all* Gaussian measures on X.

(iii) Let $X := \mathrm{R}^{\infty}$, $H := \ell^2$ and let μ be the countable product of probability measures μ_n, defined by smooth densities p_n on the line. Then $\beta_H^{\mu} : \mathrm{R}^{\infty} \to \mathrm{R}^{\infty}$ is given by

$$\left(\beta_H^{\mu}(x)\right)_n = \frac{p_n'(x_n)}{p_n(x_n)} = \frac{d}{dx_n} \log p_n(x_n).$$

In the infinite dimensional case the vector logarithmic derivative of a nonzero measure is not an $H-$ *gradient* $\nabla_H f$ of a Gâteaux differentiable function f where $\nabla_H f(x) := j_H(f'(x))$, $x \in X$. This will be proved (even in more generality) in Proposition 2.7 below. Here we only give an example.

Example 2.2 Let $X := \mathrm{R}^{\infty}$, $H := \ell^2$, and let μ be the countable product of standard Gaussian measures on the line. Then again $\beta_H^{\mu}(x) = -x$, but there is no function f with $\nabla_H f(x) = -x$. Indeed, the only possibility is $f(x) = -\sum x_i^2/2 + C$, but this series diverges μ-a.e.

Remark 2.3 Let V be a finite dimensional subspace of X such that $X = V \oplus Y$, where Y is a closed linear subspace of X. If a measure μ on X is differentiable along V, then one can choose conditional measures μ^y, $y \in Y$, on slices $V + y$, differentiable along V (see [22, Theorem 2.5], or [14, Proposition 4.1]). If the corresponding logarithmic derivatives are exponentially integrable, then μ^y admit densities $\exp(U_y)$ with respect to Lebesgue measure on $V + y$. This observation relates logarithmic derivatives with Gibbs measures (see [30], [31], [32], [35], [37], [36], [44]).

Proposition 2.4 *Assume X is a Banach space and let μ be a measure on X.*

(i) Assume that X has the Radon–Nikodym property and that the natural embedding $H \subset X$ is absolutely summing (see [57]; both assumptions are e.g. fulfilled if X is a Hilbert space and $H \subset X$ is Hilbert–Schmidt). If $H \subset D(\mu)$ then β^μ_H exists and $\|\beta^\mu_H\|_X \in L^1(\mu)$.

(ii) Assume that X is a Hilbert space, that $j_H(X^*) \subset H(\mu)$ (which is e.g. fulfilled if $H = H(\mu)$), and that this embedding is Hilbert-Schmidt. Then β^μ_H exists and $\|\beta^\mu_H\|_X \in L^2(\mu)$.

Proof (i): We start with recalling Pettis' theorem. Let ϕ be a map from a σ-field $\mathcal{B}$ to a Banach space Z. Assume that for each continuous linear functional f on Z the real-valued set function $f(\phi)$ is countably additive. Then according to Pettis' theorem (see [33, Theorem 4.10.1]) ϕ is a countably additive Z-valued measure of bounded semivariation. We apply this theorem to the following situation.

By (1.1) and the closed graph theorem the embedding $H \to D(\mu)$ is continuous. Hence for any Borel set B the functional $h \mapsto D_h\mu(B)$ is continuous on the Hilbert space H by the continuity of this functional on $D(\mu)$. According to the Riesz representation theorem there is a vector $D\mu(B) \in H$, such that for any $h \in H$

$$(D\mu(B), h)_H = D_h\,\mu(B) = \int_B \beta^\mu_H(x)\,\mu(dx). \tag{2.2}$$

Thus, we get an H-valued measure $D\mu$, which is weakly countably additive. By Pettis' theorem the vector measure $D\mu$ is countably additive and has bounded semivariation. Now we shall use the following simple fact from the theory of vector measures: for any Z-valued measure ϕ of bounded semivariation and any absolutely summing map $T : Z \to X$ the measure $T(\phi)$ has bounded variation (cf. [16, Lemma1]). Thus, in our case the same measure $D\mu$ regarded as an X-valued measure has bounded variation since the embedding $H \subset X$ is assumed to be absolutely summing. Since in addition, we suppose that X has the Radon-Nikodym property, $D\mu$ has a vector density Λ with respect to μ, since $D\mu \ll \mu$. This means that there is a strongly μ-integrable map $\Lambda : X \to X$ such that for any Borel set B the vector integral $\int_B \Lambda(x)\mu(dx)$ is an element of H and for any $h \in H$ one has:

$$\left(\int_B \Lambda(x)\,\mu(dx), h\right)_H = D_h\mu(B). \tag{2.3}$$

Now we can set

$$\beta^\mu_H := \Lambda.$$

Indeed, if $h = j(k),\ k \in X^*$, then the left-hand side is equal to

$${}_{X^*}\langle k, \int_B \Lambda(x)\mu(dx)\rangle_X = \int_B {}_{X^*}\langle k, \Lambda(x)\rangle_X\,\mu(dx).$$

Recall that Hilbert spaces have the Radon-Nikodym property and absolutely summing operators between Hilbert spaces are exactly Hilbert-Schmidt operators (see [57]). This proves assertion (i).

(ii): Take an orthonormal basis $\{e_n\}$ in X and let $\{e_n^*\}$ be its dual basis in X^*. Note that the series $\sum_{n=1}^{\infty} \beta^{\mu}_{j_H(e_n^*)}(x)^2$ converges μ-a.e.. Indeed, by the Hilbert-Schmidt character of the embedding $j_H : X^* \to H(\mu)$ we have:

$$\sum_{n=1}^{\infty} |j_H(e_n^*)|^2_{H(\mu)} < \infty.$$

This is precisely the convergence of the series

$$\sum_{n=1}^{\infty} \int \beta^{\mu}_{j_H(e_n^*)}(x)^2 \mu(dx).$$

Hence we get the convergence of the series above μ-a.e.. Therefore, one can define a square integrable map $\beta^{\mu}_{H} : X \to X$ by

$$\beta^{\mu}_{H}(x) := \sum_{n=1}^{\infty} \beta^{\mu}_{j_H(e_n^*)}(x) e_n, \quad x \in X \ ,$$

(where the sum converges both μ-a.e. and in $L^2(\mu, X)$). By construction this is vector logarithmic derivative of μ. □

Example 2.5 Assume that X is a Hilbert space and let γ be a non–degenerate centered Gaussian measure on X with corresponding operator T as in Example 1.2. Then $D(\gamma) = H(\gamma) = T(X)$ and $|\cdot|_{H(\gamma)} = |T{-}1 \cdot|_X$. Let Q be another injective non-negative definite Hilbert–Schmidt operator on X such that $Q(X) \subset T(X)$ and $Q^2(X) \subset T^2(X)$. Then $R := T^{-2}Q^2$ is a bounded operator on X and if $H := Q(X), |\cdot|_H := |Q^{-1}\cdot|_X$, then

$$\beta^{\gamma}_{H} = -R^*$$

(where R^* is the adjoint of R on X).

Proof By the closed graph theorem $T^{-2}Q^2$ is continuous on X. By 1.1 (iii)

$$j_H = Q^2 R_X, \quad j_{H(\gamma)} = T^2 R_X \ .$$

Let $k \in X^*$, then by assumption and 1.1 (iii)

$$j_H(k) \in j_{H(\gamma)}(X^*) \ ,$$

hence by (1.2)

$$\begin{aligned}
\beta^{\gamma}_{j_H(k)}(z) &= - \ {}_{X^*}\langle j^{-1}_{H(\gamma)}(j_H(k)), z\rangle_X \\
&= - \ {}_{X^*}\langle (R_X^{-1} T^{-2} Q^2 R_X)(k), z\rangle_X \\
&= - \ ((T^{-2}Q^2 R_X)(k), z)_X \\
&= - \ (R_X(k), (T^{-2}Q^2)^* z)_X \\
&= - \ {}_{X^*}\langle k, R^* z\rangle_X \ .
\end{aligned}$$
□

Later we shall see that T^2 is not uniquely determined by Q and R. More precisely, the range of R might not be dense in X(cf. Theorem 6.8 below).

It should be noted that in Proposition 2.4(ii), it is not sufficient that the embedding $H \subset X$ is Hilbert-Schmidt as the following example shows.

Example 2.6 Let $H := \ell^2$, $X := \left\{(x_n) : \sum\limits_{n=1}^{\infty} n^{-2}x_n^2 < \infty\right\}$. Then $H \subset X$ is Hilbert–Schmidt and

$$j_{_H}(X^*) = \left\{(x_n) : \sum_{n=1}^{\infty} n^2 x_n^2 < \infty\right\}.$$

For μ take the countable product of the Gaussian measures on R^1 with densities

$$p_n(t) = n(2\pi)^{-1/2}\exp(-n^2t^2/2).$$

In this case $D(\mu) = j_{_H}(X^*)$, and the vector logarithmic derivative $\beta^\mu_{_H}$ does not exist. Indeed, otherwise we would have got $(\beta^\mu_H(x))_n = -n^2x_n$, since for the n-th coordinate functional $\ell_n : x \mapsto x_n$ one has: $j_{_H}(\ell_n) = e_n$, where $e_n = (0, ..., 0, 1, 0, ...)$, and $\beta^\mu_{e_n}(x) = p'_n(x_n)/p_n(x_n) = -n^2x_n$. But the series

$$\sum_{n=1}^{\infty} n^{-2}\left(\beta^\mu_{_H}(x)\right)^2_n = -\sum_{n=1}^{\infty} n^2x_n^2$$

diverges μ-a.e..

The last result of this section describes vector logarithmic derivatives as *generalized $H-$gradients*, that is as L^2-limits of H-gradients of smooth functions. However, as mentioned before, in infinite dimensions vector logarithmic derivatives cannot be gradients of functions. The heuristic idea behind this is that if $\beta^\mu_{_H} = \nabla_{_H} p$, then μ would be of the form $\mu =$ "$e^p dx$", where "dx" does not exist separately. More precisely, we have

Proposition 2.7 *Suppose that X is an infinite dimensional Banach space and that μ is differentiable along all h in a dense set M of H. If there exists a function p with p^{-1} locally μ–integrable having (μ-weak) partial derivatives along all directions in H such that $\beta^\mu_h = \partial_h p/p$ and $p^{-1}\beta^\mu_h$ locally μ–integrable for all $h \in M$. Then $\mu = 0$.*

Proof We may assume that $M = \{e_n |\ n \in \mathrm{N}\}$ is an orthonormal base in H. Let $x_0 \in X$ and $r > 0$ such that p is μ–integrable on $B_r(x_0) := \{x \in X : |x - x_0|_X \leq r\}$. Let g be a Lipschitz continuous function such that $I_{B_{r/2}(x_0)} \leq g \leq I_{B_r(x_0)}$. Consider $\nu := gp^{-1}\mu$. Then obviously $\beta^\nu_{e_n} = -\frac{\partial_{e_n} g}{g}p^{-1}$ on $\{g > 0\}$, hence $\beta^\nu_{e_n} = 0$ on $B_{r/2}(x_o)$. Since for the measure ν_{te_n} obtained from ν by translation by te_n, we have $\beta^{\nu_{te_n}}_{e_n} = \beta^\nu_{e_n}(\cdot - te_n)$, it follows that the ν–measure of a small ball remains invariant when the ball is shifted within $B_{r/2}(x_0)$ in direction e_n for all $n \in \mathrm{N}$. Hence ν must be the zero measure, since it is countably additive. Since x_0 was arbitrary, we obtain $\mu = 0$. $\Box$

Assume that X is a separable Hilbert space. Denote the orthogonal projection onto the closure of the class of H-valued maps

$$\{\nabla_{_H} f : f \in \mathcal{F}C^\infty_b\} \ \text{ in } \ L^2(\mu, X) ,$$

by Γ. Elements in this closure are called *generalized H-gradients.*

Theorem 2.8 *(i) If $H = E = \mathrm{R}^n$ with the Euclidean inner product and $\mu = \varphi^2 \cdot dx$ where $\varphi \in H^{1,2}(\mathrm{R}^n; dx)$, then $\beta^\mu_{_H} = 2\nabla\varphi/\varphi$ is in the closure of $\{\nabla f : \ f \in C^\infty_0(\mathrm{R}^n)\}$ in $L^2(\mu, \mathrm{R}^n)$.*

(ii) Assume that X is a Hilbert space and let μ be a measure on X. Assume that β^μ_H exists and $\beta^\mu_H \in L^2(\mu, X)$. Then β^μ_H is a generalized H-gradient.

Proof (i): Note that by Remark 1.3 indeed

$$\beta^\mu_H = 2\nabla\varphi/\varphi,$$

where ∇ is understood in the Sobolev sense (i.e., the w.r.t. Lebesgue measure weak gradient). We want to approximate β^μ_H by smooth gradients in $L^2(\mu, \mathrm{R}^n)$. For $n \in \mathrm{N}$ let $g_n(t) = t$, if $t \in [-n, n]$, and $g_n(t) = n \cdot sgn(t)$, if $|t| > n$. Then each g_n is Lipschitzian on $[0, \infty)$, and hence the functions $\phi_n = g_n(\log(\varphi + \frac{1}{n})^2)$ belong to the domain $D(\mathcal{E}^+_\varphi)$ of the maximal Dirichlet form $\mathcal{E}^+_\varphi(f, f) = \int(\nabla f, \nabla f)\varphi^2 dx$ (cf. [14, Sections 2,3]. In addition,

$$\nabla\phi_n = g'_n\left(\log\left(\varphi + \frac{1}{n}\right)^2\right) 2\nabla\varphi\Big/\left(\varphi + \frac{1}{n}\right) = I_{\{e^{-n/2}-\frac{1}{n}<\varphi<e^{n/2}-\frac{1}{n}\}} 2\nabla\varphi\Big/\left(\varphi + \frac{1}{n}\right).$$

So, $\nabla\phi_n$ converge to $2\nabla\varphi/\varphi$ in $L^2(\mu, \mathrm{R}^n)$. It remains to notice that by virtue of [61, Proof of Theorem 3.1] one can approximate ϕ_n in the Dirichlet norm by smooth compactly supported functions. This gives L^2-approximations of $2\nabla\varphi/\varphi$ by the gradients of such functions.

(ii): Let $\{e_n\}$ be an orthonormal base in H, such that $e_n = j_H(\ell_n)$ and $\{\ell_n : n \in \mathrm{N}\}$ is dense in X^*. Denote by P_n the projection of X onto the subspace X_n spanned by $e_1, ..., e_n$. Conditional expectations B_n of the map β^μ_H with respect to the σ-fields, generated by P_n, form a square-integrable X–valued martingale converging to β^μ_H in $L^2(\mu, X)$ (see [67, Chapter II, Theorem 4.1]). Therefore, $\{P_n B_n\}$ also converges to β^μ_H in $L^2(\mu, X)$ since

$$\|P_n B_n - \beta^\mu_H\|_{L^2(\mu,X)} \le \|P_n B_n - P_n\beta^\mu_H\|_{L^2(\mu,X)} + \|P_n\beta^\mu_H - \beta^\mu_H\|_{L^2(\mu,X)} \le$$

$$\|B_n - \beta^\mu_H\|_{L^2(\mu,X)} + \|P_n\beta^\mu_H - \beta^\mu_H\|_{L^2(\mu,X)} \longrightarrow 0 \quad \text{as} \quad n \to \infty\ .$$

On the other hand, $P_n B_n(x) = \beta_n(P_n x)$, where the maps $\beta_n :\ X_n \to X_n$ can be taken as versions of vector logarithmic derivatives of the projections μ_n of μ on X_n. Indeed, this is an immediate consequence of [14, Lemma 4.2] .

Now we can apply (i) to obtain assertion (ii). □

Remark 2.9 The statement of Theorem 2.8(ii) can also be found in [47]. But there, only the trivial part of the proof is given, namely the reduction to the finite dimensional result. The essential part, i.e., the finite dimensional case (cf. Theorem 2.8(i)) was left out.

3 Regularity of solutions of elliptic equations for measures

Throughout this section we assume that H is a separable Hilbert space, which is continuously and densely embedded into a locally convex space X (e.g. a Banach space). Let $B : X \to X$ be a Borel measurable map. We shall study elliptic equations

$$L^*\mu = 0$$

for measures on X. Here L is defined for $f \in \mathcal{F}C_b^\infty$ by

$$Lf(x) = \Delta_H f(x) + {}_{X^*}\langle f'(x), B(x)\rangle_X, \ \ x \in X \ , \tag{3.1}$$

where

$$\begin{aligned} \Delta_H f(x) \ &:= \ Trace_H f''(x) \\ &= \ \sum_{i,j=1}^m \frac{\partial^2 g}{\partial y_i \partial y_j}(\ell_1(x), \ldots, \ell_m(x))\,(j_H(\ell_i), j_H(\ell_j))_H \ , \end{aligned}$$

if $f = g(\ell_1, \ldots, \ell_m)$, $m \in \mathrm{N}$, $\ell_i \in X^*$, $g \in C_b^\infty(\mathrm{R}^m)$.

For the rest of this section we fix a probability measure μ on X such that

$${}_{X^*}\langle \ell, B\rangle_X \in L^2(\mu) \quad \forall\, \ell \in X^*. \tag{3.2}$$

Recall that by our definition (cf. (0.4)) the equation $L^*\mu = 0$ means the following

$$\int_X Lf(x)\,\mu(dx) = 0 \quad \forall\, f \in \mathcal{F}C_b^\infty \ . \tag{3.3}$$

Note that $Lf \in L^1(\mu)$ by (3.2).

Remark 3.0 Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on $L^2(\mu)$ with generator $\mathcal{L}$. μ is called *invariant* for $(T_t)_{t\geq 0}$ if for all bounded Borel functions f and all $t \geq 0$,

$$\int_X T_t f(x)\,\mu(dx) = \int f(x)\,\mu(dx). \tag{3.3'}$$

In fact, it suffices to have (3.3)' only for $f \in \mathcal{F}C_b^\infty$. (3.3)' is equivalent to (3.3) with $\mathcal{L}$ replacing L holding for all f in a subset of $D(\mathcal{L})$ dense w.r.t. the graph norm of $\mathcal{L}$. Hence if merely $\mathcal{F}C_b^\infty \subset D(\mathcal{L})$ and $\mathcal{L} = L$ on $\mathcal{F}C_b^\infty$, (3.3) holding for all $f \in \mathcal{F}C_b^\infty$ is in general weaker than (3.3)', since $\mathcal{F}C_b^\infty$ is in general not dense w.r.t. the graph norm of $\mathcal{L}$.

In the case when $H = X := \mathrm{R}^n$, equipped with the Euclidean inner product and B is smooth, a classical result says that any solution of (3.3) is an absolutely continuous measure which has a smooth density with respect to Lebesgue measure. An extension of this result to the case of a locally bounded (or locally Lebesgue integrable) map B gives the existence of a density in some Sobolev space. However, in our more general setting the classical theory does not apply since B is not supposed to be locally dx–integrable. Nevertheless, some regularity holds even in this general setting.

Theorem 3.1 *Assume that $H = X := \mathrm{R}^n$ with the Euclidean inner product $(\ ,\ )$ and that*

$$\int Lu(x)\,\mu(dx) = 0 \quad \textit{for all} \quad u \in C_b^\infty(R^n)\ .$$

Then:

(i) $\mu = p\,dx$, *where* $p = \varphi^2$ *with* $\varphi \in H^{1,2}(\mathrm{R}^n; dx)$. *In particular,* μ *is differentiable along all elements in* R^n.

(ii)
$$\frac{1}{4}\int \left(\frac{|\nabla p|}{p}\right)^2 p\,dx = \int |\nabla\varphi|^2 dx \le \frac{1}{4}\int |B(x)|^2 \mu(dx) \tag{3.4}$$
where $|\ |$ *denotes Euclidean norm.*

(iii) $\int \left(\nabla u, B - \frac{\nabla p}{p}\right) p\,dx = 0\ \forall u \in C_b^\infty(\mathrm{R}^n)$ *(i.e.,* $\Gamma(B) = \frac{\nabla p}{p}$*). In particular, if* B *is a generalized* H*-gradient, then* $2\nabla\varphi/\varphi = B$ μ*–a.e.*

(iv) Assertion (i) and (ii) remain true if $(\ ,\ )$ *on* $H(=\mathrm{R}^n)$ *is replaced by* $(A\cdot,\cdot)$ *where* A *is a symmetric strictly positive definite* $n\times n$*-matrix and* $\frac{1}{4}$ *in (ii) is replaced by* $\frac{1}{4}\|A^{-1}\|^2$.

Proof Let g be the standard Gaussian density on R^n. Set
$$f_\varepsilon(x) := g_\varepsilon * \mu(x) = \int g_\varepsilon(x-y)\,\mu(dy),$$
where $g_\varepsilon(x) = \varepsilon^{-n} g(x/\varepsilon)$. Then each f_ε is strictly positive, infinitely differentiable, and tends to zero at infinity together with all its derivatives by Lebesgue's dominated convergence theorem. Furthermore, $\int f_\varepsilon dx = 1\ \forall \varepsilon > 0$ and $f_\varepsilon dx \underset{\varepsilon\to 0}{\longrightarrow} \mu$ vaguely. Assertion (i) is proved by the following claim: there exists a subsequence $(\varepsilon_n)_{n\in\mathrm{N}}$ with $\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$ and $p \in L^1(\mathrm{R}^n; dx)$ such that $f_{\varepsilon_n} \underset{n\to\infty}{\longrightarrow} p$ in $L^1_{\mathrm{loc}}(\mathrm{R}^n; dx)$ and $\varphi := \sqrt{p} \in H^{1,2}(\mathrm{R}^n; dx)$. This claim, however, follows from the estimate
$$\int |\nabla\sqrt{f_\varepsilon}|^2 dx \le \frac{1}{4}\int |B|^2 d\mu\ . \tag{3.5}$$
Indeed, if (3.5) holds, then for $\varphi_n := \sqrt{f_{\frac{1}{n}}}$
$$\int \left(\varphi_n^2 + |\nabla\varphi_n|^2\right)\,dx \le 1 + \frac{1}{4}\int |B|^2 d\mu\ .$$
Since $H^{1,2}(D; dx) \subset L^2(D; dx)$ compactly for every open ball $D \subset \mathrm{R}^n$, there exists $\varphi \in H^{1,2}(\mathrm{R}^n; dx)$ and a subsequence $(n_k)_{k\in\mathrm{N}}$ such that $\varphi_{n_k} \underset{k\to\infty}{\longrightarrow} \varphi$ weakly in $H^{1,2}(\mathrm{R}^n; dx)$ and strongly in $L^2(D; dx)$. Hence take $p := \varphi^2$. So, it remains to prove (3.5). For any $u \in C_0^\infty(\mathrm{R}^n)$, integrating by parts and using Fubini's theorem we obtain that
$$\begin{aligned}
&\int u\left(\Delta f_\varepsilon - \sum_{i=1}^n \partial_i\left[g_\varepsilon * (B_i\mu)\right]\right) dx = \int \Delta u\, f_\varepsilon dx + \int \sum_{i=1}^n (\partial_i u)\left[g_\varepsilon * (B_i\mu)\right]\,dx \\
&= \iint \Delta u(x) g_\varepsilon(x-y)\mu(dy)\,dx + \iint \sum_{i=1}^n (\partial_i u)(x) g_\varepsilon(x-y) B_i(y)\mu(dy)\,dx \\
&= \int \Delta u * g_\varepsilon(y)\,\mu(dy) + \int \sum_{i=1}^n \partial_i u * g_\varepsilon(y)\,B_i(y)\,\mu(dy) \\
&= \int \Delta u * g_\varepsilon(y)\,\mu(dy) + \int \sum_{i=1}^n \partial_i(u * g_\varepsilon)(y)\,B_i(y)\,\mu(dy) \\
&= \int L(u * g_\varepsilon)(y)\,\mu(dy) = 0,
\end{aligned}$$

since $u * g_\varepsilon \in C_b^\infty$. Therefore,

$$\Delta f_\varepsilon - \sum_{i=1}^n \partial_i [g_\varepsilon * (B_i \mu)] = 0 \, dx\text{-a.e.}, \tag{3.6}$$

and for any positive c

$$\log(f_\varepsilon + c)\Delta f_\varepsilon = \log(f_\varepsilon + c) \sum_{i=1}^n \partial_i [g_\varepsilon * (B_i\mu)] \ . \tag{3.7}$$

We shall need the following estimate. Let $\psi \in L^2(\mu)$. Then

$$\int \frac{(g_\varepsilon * (\psi\mu))^2(x)}{f_\varepsilon(x)} \, dx \leq \int \psi^2(x) \, \mu(dx). \tag{3.8}$$

Indeed, we may assume that $\psi \geq 0$. Then we have that for every $x \in \mathrm{R}^n$

$$\begin{aligned} [g_\varepsilon * (\psi\mu)(x)]^2 &= \Big(\int g_\varepsilon(x-y) \, \psi(y) \, \mu(dy) \Big)^2 \\ &\leq \int \psi^2(y) g_\varepsilon(x-y) \, \mu(dy) \int g_\varepsilon(x-y) \, \mu(dy) = \int \psi^2(y) g_\varepsilon(x-y) \, \mu(dy) f_\varepsilon(x). \end{aligned}$$

Hence

$$\begin{aligned} \int (g_\varepsilon * (\psi\mu))^2 f_\varepsilon^{-1} \, dx &\leq \int \int \psi^2(y) \, g_\varepsilon(x-y) \, \mu(dy) \, dx \\ &= \int \psi^2(y) \Big(\int g_\varepsilon(x-y) \, dx \Big) \, \mu(dy) = \int \psi^2(y) \, \mu(dy), \end{aligned}$$

as $\int g_\varepsilon(x-y) \, dx = \int g_\varepsilon(z) \, dz = 1$, and (3.8) is shown.

Define $h := \log(f_\varepsilon + c)$, $F_i := \partial_i f_\varepsilon - (g_\varepsilon * (B_i\mu))$, $1 \leq i \leq n$, and $F := (F_1, \ldots, F_n)$. Then $h \cdot \operatorname{div}(F)$, $(\nabla h, F), h|F| \in L^1(\mathrm{R}^n; dx)$ which easily follows by (3.6) resp. Fubini's theorem, since

$$\partial_i f_\varepsilon(x) = -\varepsilon^{-2} \int (x_i - y_i) g_\varepsilon(x-y) \, \mu(dy), \quad 1 \leq i \leq n \ ,$$

$\mu(\mathrm{R}^n) = 1$, and $|B| \in L^2(\mu)$.

Hence we can integrate by parts in (3.7) to obtain

$$\begin{aligned} 0 &= \int \log(f_\varepsilon + c)\Delta f_\varepsilon \, dx - \int \log(f_\varepsilon + c) \sum_{i=1}^n \partial_i \, [g_\varepsilon * (B_i\mu)] \, dx \\ &= -\int \frac{|\nabla f_\varepsilon|^2}{f_\varepsilon + c} \, dx + \int \sum_{i=1}^n \frac{\partial_i f_\varepsilon}{f_\varepsilon + c} [g_\varepsilon * (B_i\mu)] \, dx. \end{aligned}$$

Hence

$$\begin{aligned} \int \frac{|\nabla f_\varepsilon|^2}{f_\varepsilon + c} \, dx &= \left| \int \sum_{i=1}^n \frac{\partial_i f_\varepsilon}{f_\varepsilon + c} [g_\varepsilon * (B_i\mu)] \, dx \right| \\ &\leq \Big(\int \frac{|\nabla f_\varepsilon|^2}{f_\varepsilon + c} \, dx \Big)^{1/2} \Big(\int \sum_{i=1}^n \frac{[g_\varepsilon * (B_i\mu)]^2}{f_\varepsilon + c} \, dx \Big)^{1/2}. \end{aligned}$$

Taking into account (3.8) we get

$$\begin{aligned}\int \frac{|\nabla f_\varepsilon|^2}{f_\varepsilon + c}\,dx &\leq \int \sum_{i=1}^n \frac{[g_\varepsilon * (B_i\mu)]^2}{f_\varepsilon + c}\,dx \leq \int \sum_{i=1}^n \frac{[g_\varepsilon * (B_i\mu)]^2}{f_\varepsilon}\,dx \\ &\leq \int \sum_{i=1}^n B_i^2(x)\mu(dx) = \int |B(x)|^2\,\mu(dx)\ .\end{aligned}$$

Since $|\nabla\sqrt{f_\varepsilon}|^2 = \frac{1}{4}|\nabla f_\varepsilon|^2 f_\varepsilon^{-1}$, by Fatou's lemma this implies (3.5).

To prove assertion (iii) one notes that by Theorem 2.8(i) $\nabla p/p$ is a generalized H-gradient. So, it suffices to check that for any $u \in C_b^\infty(\mathbb{R}^n)$ one has

$$\int \left(\nabla u, B - \frac{\nabla p}{p}\right) p\,dx = 0\ .$$

But integrating by parts we see that the left hand side is equal to $\int Lu\ d\mu$ which is zero by assumption.

Assertion (iv) easily follows from (i) and (ii) by a change of coordinates. □

Remark 3.2 Under the additional assumption that there exists a diffusion process with drift b having μ as invariant measure the assertion of Theorem 3.1 follows from [36]. We do not assume the existence of a process here and even if there existed one generated by L, by Remark 3.0 it is not clear wether μ is an invariant measure of this process. It is worth mentioning that a proof similar to that above works also for operators of type

$$Lf(x) = \sum_{i,j=1}^n a^{ij}(x)\,\partial_{x_i x_j} f(x) + \sum_{i=1}^n b_i(x)\,\partial_{x_i} f(x),$$

where the matrix-valued function $A = (a^{ij})$ is uniformly nondegenerate, bounded and Lipschitzian. This more general case is studied in [21].

Proposition 3.3 *Assume that μ satisfies (3.3). Then for any continuous linear surjection $P:\ X \to \mathbb{R}^n$ the induced measure μ_P on $\mathbb{R}^n$ is differentiable along all elements in $\mathbb{R}^n$ and possesses a density $p := \varphi^2$ with $\varphi \in H^{1,2}(\mathbb{R}^n; dx)$.*

Proof Since $j_H(X^*)$ is dense in X, $P(j_H(X^*)) = \mathbb{R}^n$. Hence we can find $\ell_1, \ldots, \ell_n \in X^*$ such that $e_1 := j_H(\ell_1), \ldots, e_n := j_H(\ell_n)$ are orthonormal in H and $a_1 := Pe_1, \ldots, a_n := Pe_n$ span $\mathbb{R}^n$ and $e_1, \ldots, e_n$ span $(\ker P)^\perp$. Since then

$$P(j_H(\ell)) = \sum_{i=1}^n {}_{X^*}\langle \ell_i, j_H(\ell)\rangle_X\, P(j_H(\ell_i)) \quad \forall\, \ell \in X^*\ ,$$

by continuity and the definition of a_i it follows that

$$Px = \sum_{i=1}^n \ell_i(x) a_i \quad \forall x \in X\ .$$

Defining a new inner product $(\cdot,\cdot)_{\mathrm{R}^n} := (P^{-1}\cdot, P^{-1}\cdot)_H$ on R^n with corresponding norm $|\ |_{\mathrm{R}^n}$ we hence have that $a_1,\dots,a_n$ are orthonormal w.r.t. $(\cdot,\cdot)_{\mathrm{R}^n}$ and

$$|Px|^2_{\mathrm{R}^n} = \sum_{i=1}^n \ell_i(x)^2 \ . \tag{3.9}$$

We complement $e_1,\dots,e_n$ to an orthonormal basis $\{e_i\}$ of H. By (3.2) and (3.9) we have that $|PB|_{\mathrm{R}^n} \in L^2(\mu)$. Hence there exists the conditional $\mu-$expectation B_P of the map PB with respect to the σ-field σ_P, generated by P. Then $B_P(x) = b(Px)$ for some Borel map $b : \mathrm{R}^n \to \mathrm{R}^n$, $b = \sum_{i=1}^n b^i a_i$. Clearly, $|b|_{\mathrm{R}^n} \in L^2(\mu_P)$ since $|B_P|_{\mathrm{R}^n} \in L^2(\mu)$. The measure μ_P satisfies the equation $L_P^*\mu_P = 0$, where

$$L_P g := \Delta_n g + (\nabla g, b)_{\mathrm{R}^n}, \qquad g \in C_b^\infty(\mathrm{R}^n) \ , \tag{3.10}$$

and

$$\Delta_n g := \sum_{i=1}^n \partial^2_{a_i} g, \quad (\nabla g, b)_{\mathrm{R}^n} := \sum_{i=1}^n b^i \partial_{a_i} g.$$

Indeed if $f(x) := g(Px)$, $x \in X$, one has that

$$\partial_{e_i} f(x) = \partial_{a_i} g(Px), \quad \partial^2_{e_i} f(x) = \partial^2_{a_i} g(Px),$$

and

$${}_{X^*}\langle f'(x), B(x)\rangle_X = (\nabla g(Px), PB(x))_{\mathrm{R}^n}.$$

In addition, $\partial_{e_k} f(x) = 0$ for any $k > n$, since $Pe_k = 0$. Therefore,

$$\begin{aligned}
&\int L_P\, g(x)\, \mu_P(dx) = \int L_P\, g(Px)\, \mu(dx) \\
&= \int \sum_{i=1}^n \partial^2_{a_i} g(Px)\, \mu(dx) + \int (\nabla g(Px), b(Px))_{\mathrm{R}^n}\, \mu(dx) \\
&= \int \sum_{i=1}^n \partial^2_{a_i} g(Px)\, \mu(dx) + \int (\nabla g(Px), PB(x))_{\mathrm{R}^n}\, \mu(dx) \\
&= \int \sum_{i=1}^n \partial^2_{e_i} f(x)\, \mu(dx) + \int {}_{X^*}\langle f'(x), B(x)\rangle_X\, \mu(dx) = \int Lf(x)\, \mu(dx) = 0.
\end{aligned}$$

Applying Theorem 3.1 we get the assertion. □

Remark 3.4 (i) Differentiability of the finite dimensional projections of a measure μ does not imply in general differentiability of μ or differentiability of the conditional measures. A counter-example can be found in [22, e.g. Example 4.1]. In how far differentiability of the conditional measures is enough for differentiability of μ has been clarified in [8, Theorem 2.5]. The results in [30], [37], [44] can be interpreted as differentiability of the conditional measures on some finite-dimensional subspaces.

(ii) Projections of an invariant measure μ of the Ornstein-Uhlenbeck process under a smooth (in Malliavin's sense) nonlinear map to a finite-dimensional manifold have been studied in [1]. It would be interesting to investigate this problem in a more general case, for example, for solutions of our equation $L^*\mu = 0$.

Theorem 3.5 *Let γ be a centred Gaussian measure on X with reproducing kernel Hilbert space H. Assume μ satisfies equation (3.3), where $B(x) = -x + v(x)$, $v : X \to H$, $|v|_H \in L^2(\mu)$. (Recall B is still assumed to satisfy (3.2)) Then:*

(i) μ is absolutely continuous with respect to γ and its Radon-Nikodym derivative p admits the representation $p = \varphi^2$, where φ is in the Sobolev space $H^{1,2}(X;\gamma)$. (cf. e.g. [60].)

(ii) β_H^μ exists and $\beta_H^\mu(x) = -x + u(x)$, where $u : X \to H$ is the orthogonal projection of v on the closure of the set $\{\nabla_H f | \ f \in \mathcal{F}C_b^\infty\}$ in the Hilbert space $L^2(\mu, H)$. Furthermore, $u = 2\nabla_H\varphi/\varphi$ (see e.g. [60] for the definition of $\nabla_H\varphi$).

(iii) μ is differentiable along all vectors $h \in H$.

Proof (i): The method of the proof of Proposition 3.3 applies here with some technical alterations. However, in order to keep this proof self-contained we repeat some of the steps.We assume first that X is a Souslin space, i.e., the continuous image of a polish space (e.g. a separable Banach space). We may assume that $u = v$, since μ satisfies equation (3.3) with $B_1(x) = -x + u(x)$. This follows from the equality

$$\begin{aligned}\int & {}_{X^*}\langle f'(x), u(x) - v(x)\rangle_X \, \mu(dx) \\ &= \int (\nabla_H f(x), u(x) - v(x))_H \mu(dx) = 0 \quad \forall f \in \mathcal{F}C_b^\infty.\end{aligned}$$

It is known (see e.g. [12, Lemma 5.6]) that there is a countable set $\{y_n : \ n \in \mathrm{N}\} \subset X^*$ generating $\mathcal{B}(X)$. Applying the Gram–Schmidt orthonormalization procedure we get an orthonormal basis $\{e_n\}$ in H, such that $e_n = j_H(\ell_n)$, where $\ell_n \in$ span $\{y_i : i \in \mathrm{N}\}$.

For any μ-square-integrable map $F : \ X \to H$ the sequence $F_n = E_\mu[F|\sigma_n]$ of the conditional expectations of F with respect to the σ-fields σ_n, generated by $\ell_1, ..., \ell_n$, converges to F in $L^2(\mu, H)$ (see e.g. [67, Chapter II, Theorem 4.1]). Let $H_n :=$ span $(e_1, ..., e_n)$ and $P_n x = \ell_1(x)e_1 + ... + \ell_n(x)e_n$. The space H_n is regarded with the inner product from H. Note that $\ell_i(e_j) = (e_i, e_j)_H$ for any i, j. Hence $P_n|_H$ is the orthogonal projection on H_n with respect to the inner product of H and $\|P_n h - h\|_H \to 0$ as $n \to \infty$ $\forall h \in H$. Therefore, if F and F_n are as above then $(P_n F_n)_{n\in\mathrm{N}}$ converges to F in $L^2(\mu, H)$. Indeed,

$$\begin{aligned}\int & |F(x) - P_n F_n(x)|_H^2 \, \mu(dx) \\ \le & \int |F(x) - P_n F(x)|_H^2 \, \mu(dx) + \int |P_n F(x) - P_n F_n(x)|_H^2 \, \mu(dx) \\ \le & \int |F(x) - P_n F(x)|_H^2 \, \mu(dx) + \int |F(x) - F_n(x)|_H^2 \, \mu(dx) \underset{n\to\infty}{\longrightarrow} 0 \ ,\end{aligned}$$

since the first term in the right-hand side tends to 0 by Lebesgue's dominated convergence theorem.

Set

$$v_n := E_\mu[P_n v|\sigma_n] \ \ (= P_n E_\mu[v|\sigma_n])$$

and

$$b_n := E_\mu[P_n B|\sigma_n] \ .$$

As it has been just explained,

$$v_n \underset{n\to\infty}{\longrightarrow} v \quad \text{in} \quad L^2(\mu, H).$$

Note that $b_n(x) = -P_n x + v_n(x)$. Let $\mu_n := \mu \circ P_n^{-1}$. There exist $\tilde{b}_n : H_n \to H_n$ measurable such that $b_n = \tilde{b}_n \circ P_n$ μ-a.e. As we have seen in the proof of Proposition 3.3 the measure μ_n on H_n satisfies the equation

$$L_n^* \mu_n = 0 \ ,$$

where

$$L_n u = \sum_{i=1}^{n} \partial_{e_i}^2 u + (\nabla_H u, \tilde{b}_n)_{H_n}, \quad \forall u \in C_b^\infty(H_n).$$

Therefore, by Proposition 3.3 μ_n has a density f_n with respect to the standard Gaussian measure γ_n on H_n. Let q_n be the standard Gaussian density on H_n (the space H_n is equipped with the Lebesgue measure dx associated with the inner product in H_n). By Theorem 3.1 $p_n := f_n q_n \in H^{1,1}(H_n; dx)$ and $\beta_{H_n}^{\mu_n} = \nabla_H p_n / p_n$ with $|\beta_{H_n}^{\mu_n}|_{H_n} \in L^2(\mu_n)$. Hence

$$\beta_{H_n}^{\mu_n}(z) = -z + \frac{\nabla_H f_n(z)}{f_n(z)} \quad \text{for } \mu_n\text{-a.e. } z \in H_n. \tag{3.11}$$

On the other hand, by virtue of Theorem 3.1(iii)

$$\tilde{b}_n(z) = \beta_{H_n}^{\mu_n}(z) + d_n(z) \quad \text{for } \mu_n\text{-a.e. } z \in H_n, \tag{3.12}$$

where $d_n : \ H_n \to H_n$ such that

$$\int_{H_n} (\nabla_H f(z), d_n(z))_{H_n} \mu_n(dz) = 0 \ \ \forall f \in C_b^\infty(H_n). \tag{3.13}$$

From (3.11) and (3.12) we obtain that

$$\begin{aligned} v_n(x) = b_n(x) + P_n x &= \beta_{H_n}^{\mu_n}(P_n x) + d_n(P_n x) + P_n x \\ &= \frac{\nabla_H f_n(P_n x)}{f_n(P_n x)} + d_n(P_n x) \quad \text{for } \mu\text{-a.e. } x \in X. \end{aligned}$$

By (3.13) it follows that

$$\int_X \left(\frac{\nabla_H f_n(P_n x)}{f_n(P_n x)}, d_n(P_n x) \right)_H \mu(dx) = 0. \tag{3.14}$$

Indeed, by (3.11) this can be seen as follows. According to Theorem 2.8 there exist functions $q_i \in C_0^\infty(H_n)$, $i \in \mathrm{N}$ such that

$$\nabla_H q_i \underset{i\to\infty}{\longrightarrow} \beta_{H_n}^{\mu_n} \quad \text{in } L^2(\mu_n, H_n).$$

The map $S : \ z \mapsto -z$ on H_n coincides with $\nabla_H Q$, where $Q(z) = -\frac{1}{2}(z,z)_{H_n}$. Hence it is easy to check that the map S belongs to the closure of $\{\nabla_H f | \ f \in C_b^\infty(H_n)\}$ in $L^2(\mu_n, H_n)$ as well, using the fact that $S \in L^2(\mu_n, H_n)$. (The latter follows because both b_n and v_n

are μ-square-integrable). Since $v_n \underset{n\to\infty}{\longrightarrow} v$ in $L^2(\mu, H)$, (3.14) implies that $d_n \circ P_n \underset{n\to\infty}{\longrightarrow} d$ and $\frac{\nabla_H f_n \circ P_n}{f_n \circ P_n} \underset{n\to\infty}{\longrightarrow} G$ in $L^2(\mu, H)$. By (3.13) it is easy to see that d is orthogonal to $\{\nabla_H f : f \in \mathcal{F}C_b^\infty\}$, hence by the arguments above also to G. But since we assumed $v = u$, d is orthogonal to v, hence $d = 0$.

Therefore,

$$\frac{\nabla_H f_n \circ P_n}{f_n \circ P_n} \underset{n\to\infty}{\longrightarrow} v \text{ in } L^2(\mu, H). \tag{3.15}$$

We shall use Gross's well-known logarithmic Sobolev inequality (see [41]), i.e.

$$\int_X F(x)^2 \log |F(x)| \,\gamma(dx) \le 2 \int_X \|\nabla_H F(x)\|_H^2 \,\gamma(dx) + \|F\|_{L^2(\gamma)}^2 \ln \|F\|_{L^2(\gamma)} \quad \forall F \in H^{1,2}(X;\gamma) \ .$$

Since $\sqrt{f_n} \in H^{1,2}(H_n;\gamma_n)$ by Theorem 3.1, we have that

$$\varphi_n := \sqrt{f_n} \circ P_n \in H^{1,2}(H;\gamma)$$

and we can apply the logarithmic Sobolev inequality to φ_n. Moreover,

$$\|\varphi_n\|_{L^2(\gamma)}^2 = \int_X f_n(P_n x)\,\gamma(dx) = \int_{H_n} f_n(x)\,\gamma_n(dx) = \int_{H_n} d\mu_n = 1,$$

and

$$\begin{aligned} 4\int_X |\nabla_H \varphi_n(x)|_H^2 \,\gamma(dx) &= \int_{H_n} |\frac{\nabla_H f_n(x)}{\sqrt{f_n(x)}}|_{H_n}^2 \,\gamma_n(dx) \\ &= \int_{H_n} |\frac{\nabla_H f_n(x)}{f_n(x)}|_{H_n}^2 f_n(x)\,\gamma_n(dx) = \int_{H_n} |\frac{\nabla_H f_n(x)}{f_n(x)}|_{H_n}^2 \,\mu_n(dx) \\ &= \int_X |\frac{\nabla_H f_n(P_n x)}{f_n(P_n x)}|_H^2 \mu(dx). \end{aligned} \tag{3.16}$$

(where the use of the chain rule is justified by replacing f_n by $f_n + \varepsilon$ and letting $\varepsilon \to 0$). By virtue of (3.15) the norms of the mappings $\nabla_H f_n / f_n$ in $L^2(\mu, H)$ are bounded by some constant C. Therefore,

$$\sup_n \int_X \varphi_n(x)^2 \log |\varphi_n(x)| \gamma(dx) \le C^2.$$

This estimate provides the uniform integrability of $f_n \circ P_n = \varphi_n^2$ on (X, γ). Furthermore, $(f_n \circ P_n)_{n\in\mathrm{N}}$ is a $(\sigma_n)_{n\in\mathrm{N}}$-martingale (under γ), thus it converges to some function $p \in L^1(\gamma)$. Let $\lambda := p \cdot \gamma$. Then for $\lambda_n := \varphi_n^2 \lambda,\ n \in \mathrm{N}$, and for any $\ell = c_1\ell_1 + ... + c_m\ell_m$ one has that

$$\int_X \exp(i\ell(x))\,\lambda_n(dx) \underset{n\to\infty}{\longrightarrow} \int_X \exp(i\ell(x))\,\lambda(dx).$$

On the other hand, for k such that $n_k \geq m$ we have that

$$\int_X \exp(i\ell(x))\lambda_n(dx) = \int_X \exp(i\ell(P_n x)) f_n(P_n x)\,\gamma\,(dx) = \int_{H_n} \exp(i\ell(x)) f_n(x)\gamma_n(dx)$$

$$= \int_{H_n} \exp(i\ell(x))\mu_n(dx) = \int_X \exp(i\ell(P_n x))\mu(dx) = \int_X \exp(i\ell(x))\mu(dx).$$

Therefore, $\mu = \lambda$. Furthermore, since $\varphi_n \to \sqrt{p} =: \varphi$ in γ-measure and since by (3.16)

$$\|\nabla_H \varphi_n\|^2_{L^2(\gamma,H)} \leq \frac{C^2}{4} \ ,$$

by a standard argument we obtain that $\varphi \in H^{1,2}(X;\gamma)$ and assertion (i) is proved.

(ii) and (iii): Let $h \in H$. Since $\beta_h^\gamma = Jh$ (cf. Sect. 1) we have that for sufficiently small positive c

$$\int \exp\left(-c(\beta_h^\gamma)^2\right)\, d\gamma < \infty \ .$$

Hence applying the logarithmic Sobolev inequality to φ, we obtain that $\varphi\beta_h^\gamma \in L^2(\gamma)$. It follows e.g. by [60, Proposition 2.1] that β_h^μ exists, $\beta_h^\mu \in L^2(\mu)$, and

$$\beta_h^\mu = \beta_h^\gamma + (h, 2\varphi^{-1}\nabla_H\varphi)_H \ , \tag{3.17}$$

which proves (iii). Since $\beta_h^\gamma = -\ell$ if $h = j_H(\ell)$ for some $\ell \in X^*$ (cf. Example 2.1(i)), (3.17) also implies that β_H^μ exists with

$$\beta_H^\mu(x) = -x + 2\varphi^{-1}\,\nabla_H\varphi(x) \quad \text{for } \mu - a.e.\ x \in X \ .$$

It remains to be shown that $2\varphi^{-1}\nabla_H\varphi = u$ which is now equivalent to

$$\beta^\mu_{j_H(\ell)} = -\ell + {}_{X^*}\langle \ell, u\rangle_X \quad \forall \ell \in X^* \ .$$

So, let $\ell \in X^*$, and let H_n, $n \in$ N, be as in the proof of (i). We may assume that $h := j_H(\ell) \in H_m$ for some $m \in$ N (otherwise we increase $\{y_i :\ i \in \text{N}\}$ accordingly). We adopt the notation of the proof of (i). Since by assumption

$$\ell = {}_{X^*}\langle \ell, u\rangle_X - {}_{X^*}\langle \ell, B\rangle_X = (h,u)_H - {}_{X^*}\langle \ell, B\rangle_X \in L^2(\mu) \ ,$$

and $\ell = \ell \circ P_n$ for $n \geq m$, we have that $\ell \in L^2(\lambda_n)$ hence $\varphi_n^2\beta_h^\gamma$ and thus (e.g. by [60, Proposition 2.1])

$$\beta_h^{\lambda_n} = 2\varphi_n^{-1}(\nabla_H\varphi_n, h)_H - \ell \ .$$

Hence

$$\begin{aligned}
-i\,{}_{X^*}\langle \ell, h\rangle_X\,\tilde{\mu}(\ell) &= \lim_{n\to\infty} -i\,{}_{X^*}\langle \ell, h\rangle_X\,\tilde{\lambda}_n(\ell) \\
&= \lim_{n\to\infty} \int \exp(i\ell)\,\beta_h^{\lambda_n}\,d\lambda_n \\
&= \lim_{n\to\infty} \int \exp(i\ell)\,[(f_n\circ P_n)^{-1}(\nabla_H f_n \circ P_n, h)_H - \ell]\,d\lambda_n \\
&= \lim_{n\to\infty} \int \exp(i\ell)\,[(f_n\circ P_n)^{-1}(\nabla_H f_n \circ P_n, h)_H - \ell]\,d\mu \\
&= \int \exp(i\ell)[(u,h)_H - \ell]\,d\mu \ ,
\end{aligned}$$

where we used (3.15) in the last step. Consequently,

$$\beta^{\mu}_{j_H(\ell)} = -\ell + {}_{X^*}\langle \ell, u\rangle_X \ .$$

Now let us explain the necessary changes in the proof for a general locally convex space X. First of all we can find $y_n \in X^*$, $n \in \mathrm{N}$, such that any γ-measureable function φ is γ-equivalent to a $\sigma(\{y_n : n \in \mathrm{N}\})$-measurable function. (This follows from the existence of metrizable compacts K_n, $n \in \mathrm{N}$, with $\gamma(K_n) \underset{n\to\infty}{\longrightarrow} 1$, see [66]). Increasing $\{y_n : n \in \mathrm{N}\}$, we may assume that there is an orthonormal basis $\{e_n\}$ in H such that $e_n = j_H(\ell_n)$, $\ell_n \in \mathrm{span}\{y_i\}$, $n \in \mathrm{N}$. By the same argument as above, we see that on the σ-field $\sigma(\{y_n : n \in \mathrm{N}\})$ we have that $\mu = f \cdot \gamma$. Now let $\ell \in X^*$ be arbitrary. Then again $\mu = f^\ell \cdot \gamma$ on $\sigma(\{y_n : n \in \mathrm{N}\} \cup \{\ell\})$. However, by our choice of $\{y_n : n \in \mathrm{N}\}$ f^ℓ is γ-equivalent to some $\sigma(\{y_n : n \in \mathrm{N}\})$-measurable function g. Therefore $f = f^\ell\ \gamma - a.e.$. In particular $\tilde{\mu}(\ell) = \widetilde{f\gamma}(\ell)$ which implies that $\mu = f\gamma$ because both are Radon measures. □

Remark 3.6 (i) Theorem 3.5 (i) gives a positive answer to a problem raised by Shigekawa [65] and Vintschger [68]. Note that in those papers additional restrictions were imposed on v. In [65] under the assumption that $\sup |v|_H < \infty$ Shigekawa established the (existence and) uniqueness for invariant probabilities which are absolutely continuous with respect to γ. Hence by our result the latter restriction can be dropped. Some other conditions, among which the inclusion $|v|_H \in L^2(\gamma)$, were used in [68]. However, even for measures on the line uniqueness is not guaranteed if one only assumes that $|v|_H \in L^2(\mu)$ (see Example 6.1 below). However, by our methods we expect to be able to show that if in the situation of Theorem 3.5 two probability measures μ and ν on X solve $L^*\mu = L^*\nu = 0$, satisfy (3.2), and $|v|_H \in L^2(\gamma) \cap L^2(\mu) \cap L^2(\nu)$, then $\mu = \nu$. This will be treated in a forthcoming paper.

(ii) A result of a similar type was obtained in [29] in a different situation. In our case the method of [29] works if v is bounded and Lipschitzian. (In this case the same follows from a combination of [65] and [51].) We refer to [20] for details.

Now we shall systematically discuss some useful consequences of the existence of β^{μ}_{H}.

Proposition 3.7 *Suppose that β^{μ}_{H} exists. Then:*

(i) $L^*\mu = 0$ if and only if $\int {}_{X^*}\langle u', B - \beta^{\mu}_{H}\rangle_X \, d\mu = 0 \ \forall\, u \in \mathcal{F}C^\infty_b$.

(ii) If $L^*\mu = 0$, then $(L, \mathcal{F}C^\infty_b)$ is negative definite on $L^2(\mu)$.

(iii) Suppose $B, \beta^{\mu}_{H} \in L^2(\mu, X)$ and $H = X$. Then $L^*\mu = 0$ if and only if $\Gamma(B) = \beta^{\mu}_{H}$. In this case $(L, \mathcal{F}C^\infty_b)$ is closable on $L^2(\mu)$.

Remark 3.8 The conditions in Proposition 3.8(iii) (as well as in Theorem 3.10(iii)and Remark 3.11 below) are basically only fulfilled if $X = \mathrm{R}^n$.

For the proof of Proposition 3.7 we first state the following simple

Lemma 3.9 *Suppose β_H^μ exists and let $u, v \in \mathcal{F}C_b^\infty$. Then:*

(i) $L(uv) = uLv + vLu + 2(\nabla_H u, \nabla_H v)_H$.

(ii) Define $L^{H,\mu} f := \Delta_H f + {}_{X^*}\langle f', \beta_H^\mu \rangle_X$, $f \in \mathcal{F}C_b^\infty$. *Then*

$$-\int L^{H,\mu} u \ v \, d\mu = \int (\nabla_H u, \nabla_H v)_H \, d\mu \ .$$

(iii) $-\int Lu\, v \ d\mu = \int (\nabla_H u, \nabla_H v)_H \, d\mu - \int \ {}_{X^*}\langle u', B - \beta_H^\mu \rangle_X \, v \, d\mu$.

(iv) If $(L, \mathcal{F}C_b^\infty)$ is symmetric on $L^2(\mu)$, then $L^*\mu = 0$.

Proof Writing $u = \phi(\ell_1, \ldots, \ell_m), v = \psi(\ell_1, \ldots, \ell_m)$ with $\ell_i \in X^*$, $1 \le i \le m$, $\phi, \psi \in C_b^\infty(\mathrm{R}^m)$, and $j_H(\ell_1), \ldots, j_H(\ell_m)$ orthonormal in H one derives (i), (ii) by an elementary calculation. (iii) is an immediate consequence of (ii), and (iv) follows since $1 \in \mathcal{F}C_b^\infty$ and $L1 = 0$. □

Proof of Proposition 3.7 (i): Take $v \equiv 1$ in 3.9 (iii). Then the assertion follows.

(ii): If $u \in \mathcal{F}C_b^\infty$, then $uu' = \frac{1}{2}(u^2)'$ and $u^2 \in \mathcal{F}C_b^\infty$. Hence by (i) and 3.9 (iii)

$$-\int Lu\, u \, d\mu = \int |\nabla_H u|_H^2 \, d\mu \ge 0 \ , \tag{3.18}$$

i.e., L is negative definite on $L^2(\mu)$.

(iii): The equivalence is obvious by (i) and Theorem 2.8. To prove the last part of the assertion let $u_n \in \mathcal{F}C_b^\infty, n \in \mathrm{N}$, such that $u_n \xrightarrow[n\to\infty]{} 0$ and $Lu_n \xrightarrow[n\to\infty]{} w$ in $L^2(\mu)$. (3.18) now implies that $|\nabla_H u_n|_H \xrightarrow[n\to\infty]{} 0$ in $L^2(\mu)$. Since $H = E, \beta_H^\mu \in L^2(\mu, X)$, and $\mathcal{F}C_b^\infty$ is dense in $L^2(\mu)$, it hence follows by 3.9 (iii) that $Lu_n \xrightarrow[n\to\infty]{} 0$ weakly in $L^2(\mu)$. Consequently, $w = 0$ and the proof is complete. □

As an easy consequence we obtain the following partial extension of A. Kolmogorov's classical result in [49]. For the corresponding result in probabilistic terms we refer to Theorem 4.1 below.

Theorem 3.10 *(i) $(L, \mathcal{F}C_b^\infty)$ is symmetric on $L^2(\mu)$ (i.e., $\int Lu\, v\, d\mu = \int u\, Lv \, d\mu \ \forall u, v \in \mathcal{F}C_b^\infty$) if and only if β_H^μ exists and $B = \beta_H^\mu$. In particular, if $(L, \mathcal{F}C_b^\infty)$ is symmetric on both $L^2(\mu)$ and $L^2(\nu)$ for some other probability measure ν on X, then β_H^μ and β_H^ν exist and have a joint Borel version, namely B.*

(ii) Let $(L, \mathcal{F}C_b^\infty)$ be symmetric on $L^2(\mu)$ (hence β_H^μ exists). Then μ is the only such measure if and only if μ is the only measure having β_H^μ as vector logarithmic derivative.

(iii) Suppose $B \in L^2(\mu, X)$ and $X = H$. Then $(L, \mathcal{F}C_b^\infty)$ is symmetric on $L^2(\mu)$ if and only if β_H^μ exists in $L^2(\mu, X), L^*\mu = 0$, and $B = \Gamma(B)$.

Proof (i): Assume that $(L, \mathcal{F}C_b^\infty)$ is symmetric on $L^2(\mu)$. Then $L^*\mu = 0$ by 3.9 (iv). Hence $\forall u, v \in \mathcal{F}C_b^\infty$ by 3.9 (i)

$$0 = \int L(uv)d\mu = 2\int u\, Lv\, d\mu + 2\int (\nabla_H u, \nabla_H v)_H\, d\mu\ ,$$

i.e.,

$$-\int u\, Lv\, d\mu = \int (\nabla_H u, \nabla_H v)_H\, d\mu\ . \tag{3.19}$$

Let $g_n \in C_b^\infty(\mathrm{R}), n \in \mathrm{N}$, such that $g_n(t) = t$ on $[-n, n]$ and $\sup\{|g_n'(t)| + |g_n''(t)| : n \in \mathrm{N}, t \in \mathrm{R}\} < \infty$. Let $\ell \in X^*$. Applying (3.19) to $v := g_n(\ell)$ we can take $n \to \infty$ according to Lebesgue's dominated convergence theorem, and since

$$L(g_n(\ell)) = g_n''(\ell)|j_H(\ell)|_H^2 + g_n'(\ell)\ {}_{X^*}\!< \ell, B >_X\ ,\ \nabla_H(g_n(\ell)) = g_n'(\ell) j_H(\ell)\ ,$$

we obtain that

$$-\int u\ {}_{X^*}\!\langle \ell, B\rangle_X\, d\mu = \int \frac{\partial u}{\partial j_H(\ell)}\, d\mu\ .$$

Hence β_H^μ exists and $\beta_H^\mu = B$. The converse follows from 3.9 (ii) and thus (i) is proved.

(ii): The assertion follows immediately from (i).

(iii): Suppose $L^*\mu = 0$ and $B = \Gamma(B)$. Then $(L, \mathcal{F}C_b^\infty)$ is symmetric on $L^2(\mu)$ by 3.7(iii) and 3.9(ii). The converse follows from assertion (i) by 3.9(iv) and 3.7(iii). □

Remark 3.11 (i) Assume that β_H^μ exists in $L^2(X, \mu)$ and that $X = H$. Consider the set $\mathcal{M}$ of all operators $L = L^B$ of type (3.1) with $B \in L^2(X, \mu)$ and $L^*\mu = 0$. Then by 3.9 (iii) and 3.10 (iii) the (according to 3.10(i)) unique operator $L \in \mathcal{M}$ which is symmetric on $L^2(\mu)$ is exactly the one whose drift has minimal norm in $L^2(\mu, X)$.

(ii) Theorem 3.10(ii) is a result with many applications in statistical mechanics, since e.g. in lattice models with finite range interactions the Gibbs states μ are exactly the ones for which an operator $(L, \mathcal{F}C_b^\infty)$ of the above type (uniquely determined by the corresponding local specification) is symmetric on $L^2(\mu)$. Hence it is an important question whether a measure μ is uniquely determined by its logarithmic derivative, since it implies uniqueness of Gibbs states in these models (cf. [7]). In Section 6 we shall come back to this problem and prove a characterization for such measures in the general case.

To conclude this section we prove the following useful characterization of $L^*\mu = 0$ resp. "$(L, \mathcal{F}C_b^\infty)$ symmetric on $L^2(\mu)$" in terms of the Fourier transform $\tilde{\mu}$ of μ defined by

$$\tilde{\mu}(\ell) := \int e^{i\ell(x)}\mu(dx),\ \ \ell \in X^*\ .$$

Proposition 3.12 *(i) $L^*\mu = 0$ if and only if*

$$-|j_H(\ell)|_H^2\ \ \tilde{\mu}(\ell) + i\int\ {}_{X^*}\!\langle \ell, B\rangle_X\, e^{i\ell} d\mu = 0 \quad \forall\, \ell \in X^*.$$

(ii) $(L, \mathcal{F}C_b^\infty)$ is symmetric on $L^2(\mu)$ if and only if

$$-\,(j_H(k), j_H(\ell))_H\ \tilde{\mu}(\ell) + i\int\ {}_{X^*}\!\langle k, B\rangle_X\, e^{i\ell} d\mu = 0 \quad \forall\, k, \ell \in X^*\ . \tag{3.20}$$

Proof (i): Let $u := e^{i\ell}, \ell \in X^*$. Then, extending L to complex valued functions we obtain

$$Lu = \left(-|j_H(\ell)|_H^2 + i \ {}_{X^*}\langle \ell, B\rangle_X\right) e^{i\ell} \ .$$

Hence if $L^*\mu = 0$ (by separation into real and imaginary parts) we obtain that

$$-|j_H(\ell)|_H^2 \ \tilde{\mu}(\ell) + i \int \ {}_{X^*}\langle \ell, B\rangle_X \ e^{i\ell} d\mu = 0 \quad \forall \ell \in X^* \ .$$

Conversely, if this holds we obtain for u which are finite linear combinations of $e^{i\ell}, \ell \in X^*$, that

$$\int Lu \, d\mu = 0 \ .$$

Hence if $u = g(\ell_1, \ldots, \ell_m)$ with $m \in \mathrm{N}$, $\ell_1, \ldots, \ell_m \in X^*$, and $g \in C_b^\infty(\mathrm{R}^m)$, writing g as the Fourier transform of a function in the Schwartz test function space, one easily sees that then also $\int Lu \, d\mu = 0$. Obviously, another limit argument allows to drop the restriction that g has compact support and (i) is proved.

(ii): As in the proof of (i) one sees that $(L, \mathcal{F}C_b^\infty)$ is symmetric on $L^2(\mu)$ if and only if $\forall \ \ell, k \in X^*$

$$\begin{aligned} &-|j_H(k)|_H^2 \ \tilde{\mu}(\ell + k) + i \int \ {}_{X^*}\langle k, B\rangle_X \ e^{i(\ell+k)} \, d\mu \\ &= -|j_H(\ell)|_H^2 \ \tilde{\mu}(\ell + k) + i \int \ {}_{X^*}\langle \ell, B\rangle_X \ e^{i(\ell+k)} \, d\mu \ . \end{aligned}$$

Renaming $\ell + k$ by k this is equivalent with

$$\begin{aligned} &-|j_H(\ell)|_H^2 \ \tilde{\mu}(\ell) + i \int \ {}_{X^*}\langle \ell, B\rangle_X \ e^{i\ell} d\mu \\ &= 2\left(-(j_H(k), j_H(\ell))_H + i \int \ {}_{X^*}\langle k, B\rangle_X \ e^{i\ell} \, d\mu\right) \quad \forall \ \ell, k \in X^* \ . \end{aligned}$$

Since the left hand side of this equality does not depend on k, this is equivalent with the right hand side being equal to zero for all $\ell, k \in X^*$ and (ii) is proved. □

Remark 3.13 (i) In the Proposition 3.12 it suffices to assume that ${}_{X^*}\langle \ell, B\rangle_X \in L^1(\mu)$ $\forall \ \ell \in X^*$ instead of (3.2)

(ii) Note that if $|B|_X \in L^1(\mu)$, 3.12 (ii) says that $(L, \mathcal{F}C_b^\infty)$ is symmetric on $L^2(\mu)$ if and only if the function $F : X^* \to X$ defined by

$$F(\ell) = -j_H(\ell) \ \tilde{\mu}(\ell) + i \int_X B e^{i\ell} \, d\mu$$

vanishes on X^*, while 3.12 (i) says that $L^*\mu = 0$ if and only if ${}_{X^*}\langle \ell, F(\ell)\rangle_X = 0 \ \forall \ \ell \in X^*$.

Corollary 3.14 *Assume that μ satisfies (3.3) and that for every continuous linear map $P : X \to \mathrm{R}^2$, $E_\mu[PB|\sigma(P)] = \nabla_H(f_P \circ P)$ μ-a.e. for some $f_P \in D(\mathcal{E}_\varphi^+)$ (cf. the proof of 2.8) where $\varphi \in H^{1,2}(\mathrm{R}^2; dx)$ such that $\mu_P := \mu \circ P^{-1} = \varphi^2 dx$. ($\varphi$ exists by Proposition 3.3.) Then $(L, \mathcal{F}C_b^\infty)$ is symmetric on $L^2(\mu)$, in particular $\beta_H^\mu (= B)$ exists.*

Proof We shall verify (3.20). So let $k, \ell \in X^*$. We may assume that k, ℓ are independent since otherwise we just apply 3.12 (i). For the same reason we can then assume that for $e_1 := j_H(\ell)$, $e_2 := j_H(k)$ we have $(e_i, e_j) = \delta_{ij}$, $i, j \in \{1, 2\}$. Define $P : X \to$ span $\{e_1, e_2\}$ by

$$Px := \ell(x)e_1 + k(x)e_2, x \in X .$$

Then $\ell \circ P = \ell$, $k \circ P = k$, and hence

$$\int {}_{X^*}\langle k, B\rangle_X e^{i\ell} d\mu = \int E_\mu[k(P(B))|\sigma(P)] e^{i\ell\circ P} d\mu$$

$$= \iint\limits_{\mathrm{R}^2} \frac{\partial f_P}{\partial x_2} e^{ix_1} \varphi^2(x_1, x_2)\, dx_1 dx_2 \tag{3.21}$$

where we identified R^2 with span (e_1, e_2). By the proof of Proposition 3.3 we know that $L_P^* \mu_P = 0$ where L_P is as in (3.10) with $b = \nabla f_P$. But hence by Theorem 3.1 (iii), $\nabla f_P = 2\varphi^{-1}\nabla\varphi$, and thus the right hand side of (3.20) is equal to

$$\iint \frac{\partial \varphi^2}{\partial x_2}(x_1, x_2)\, dx_2 e^{ix_1}\, dx_1$$

which in turn is equal to zero, since $\varphi \in H^{1,2}(\mathrm{R}^d; dx)$. $\square$

4 Logarithmic derivatives as drifts to symmetrizable diffusions

A diffusion $(\xi_t)_{t\geq 0}$ with the transition semigroup $(T_t)_{t\geq 0}$ is called *symmetrizable* if there exists a measure μ such that for all bounded Borel functions f, g

$$\int T_t f(x)\, g(x)\, \mu(dx) = \int f(x)\, T_t g(x)\, \mu(dx).$$

In this case $(\xi_t)_{t\geq 0}$ is also called $\mu-$*symmetric*. It is clear that μ is an invariant measure for the diffusion $(\xi_t)_{t\geq 0}$ with the additional property that the generator L of $(T_t)_{t\geq 0}$ on $L^2(\mu)$ is symmetric on $L^2(\mu)$.

In the celebrated paper [49] A. N. Kolmogorov studied the following problem. Let $(\xi_t)_{t\geq 0}$ be a diffusion on a Riemannian manifold X with generator $L = \Delta/2 + B$, where B is a smooth vector field on X. When is the process $\eta_t = \xi_{T-t}$ governed by the same equation, that is has the same generator L? The answer found by Kolmogorov says: if and only if B is a gradient (see also [63], [45]). In [49] Kolmogorov discussed only solutions of the Fokker - Planck - Kolmogorov equations as the densities of the transition probabilities, because stochastic integration was not developed then. Moreover, mainly the case of compact X was considered, when an invariant probability always exists. In this case the property studied by Kolmogorov is equivalent to the symmetrizability of $(\xi_t)_{t\geq 0}$. Therefore, Kolmogorov's result is a criterion for the symmetrizability of a diffusion on a compact manifold. In the noncompact case one needs additional conditions to ensure the existence of invariant probabilities. The best known conditions of this type were found by Hasminskii [42]. These conditions involve certain Lyapunov functions. For example, if $X = \mathbb{R}^n$ it suffices that B be continuous and outside of some ball $(B(x), x) \leq c < 0$. Symmetric diffusions with singular drifts in finite-dimensional spaces were investigated e.g. in [6], [38] (see also the references therein).

The difference between the existence of an invariant measure and the symmetrizability of a diffusion $(\xi_t)_{t\geq 0}$ with generator $L = \Delta/2 + B$ can be read of Proposition 3.7 and Theorem 3.10. We only mention that in the case the diffusion $(\xi_t)_{t\geq 0}$ is not symmetric, and the reversed process $(\eta_t)_{t\geq 0}$ is a diffusion with (a different) drift B^*, the drifts B and B^* are related by the duality equation $B + B^* = 2\nabla p/p$ (see [35]).

In recent years there has been a lot of progress in the study of analogous questions both in the finite- and infinite dimensional setting (see e.g. [31], [35] [36], [43], [55]).

In particular, the following result (whose first assertion was already proved in [12]) shows that in infinite dimensions vector logarithmic derivatives are exactly the drifts to symmetric diffusions. It can be considered as the probabilistic counterpart of Theorem 3.10 (i).

Theorem 4.1 *Let X be a Banach space, $H \subset X$ a densely and continuously embedded separable Hilbert space such that there is a Gaussian measure γ on X with $H = H(\gamma)$, and μ a measure on X such that $X^* \subset L^2(\mu)$. Assume that β_H^μ exists and $|\beta_H^\mu|_X \in L^2(\mu)$. Then one can find an X-valued Wiener process $(w_t)_{t\geq 0}$, with covariance given by H, such that the stochastic differential equation*

$$d\xi_t = \sqrt{2}dw_t + B(\xi_t)dt,\ \xi_0 = x,$$

where $B = \beta_H^\mu$, admits a weak solution $(\xi_t)_{t\geq 0}$ for μ-a.e. (or even quasi–every, cf. [12]) $x \in X$, which is a μ-symmetric diffusion with invariant measure μ. Conversely, if there is a μ-symmetric diffusion $(\xi_t)_{t\geq 0}$, satisfying the equation above, for μ-a.e. $x \in X$ and $\|B\|_X \in L^2(\mu)$, then μ is differentiable along all directions in $j_H(X^*)$, β_H^μ exists, and $\beta_H^\mu = B$ (μ-a.e.).

Proof The first implication is just [12, Theorem 6.10]. Conversely, it is easy to see (and well–known) that the generator L of the transition semigroup $(T_t)_{t\geq 0}$ of $(\xi_t)_{t\geq 0}$ on $L^2(\mu)$ has $\mathcal{F}C_b^\infty$ in its domain and is of type (3.1). Hence the dual implication follows from Theorem 3.10 (i). □

It is worth mentioning that Theorem 4.1 gives a certain characterization of mappings $\beta : X \to X$ which are vector logarithmic derivatives of measures on X. The question of finding necessary and/or sufficient conditions for a map to be a vector logarithmic derivative was raised in [15]. Some sufficient conditions of a rather special character can be found in Theorem 5.2 (cf. also Remark 5.5(ii)) below and also in [48].

5 Existence of solutions of elliptic equations for measures

In this section let X be a separable (real) Hilbert space with inner product $(\cdot,\cdot)_X$ and norm $|\cdot|_X$. Let $H \subset X$ be a separable Hilbert space densely embedded into X by a

Hilbert-Schmidt map. Then there exists a nonnegative definite injective self-adjoint Hilbert-Schmidt operator T on X such that $H = T(X)$ and $|\cdot|_H = |T^{-1}\cdot|_X$. Let $\{h_n\}$ be a corresponding eigenbvasis with respective eigenvalues t_n, $n \in \mathrm{N}$. The vectors $e_n := t_n h_n$ form an orthonormal basis in H and clearly $\sum_{n=1}^{\infty} t_n^2 < \infty$. Furthermore, we fix $B : X \to X$ $\mathcal{B}(X)/\mathcal{B}(X)$-measurable and define as before

$$Lf := \Delta_H f + {}_{X^*}\langle f', B\rangle_X, \quad f \in \mathcal{F}C_b^\infty.$$

Clearly, for all $x \in X$, $f \in \mathcal{F}C_b^\infty$,

$$\begin{aligned} Lf(x) &= \sum_{n=1}^{\infty} \left[t_n^2 f''(x)(h_n, h_n) + {}_{X^*}\langle f'(x), h_n\rangle_X (B(x), h_n)_X\right] \\ &= \sum_{n=1}^{\infty} \left[t_n^2\, \partial_{h_n}^2 f(x) + \partial_{h_n} f(x)(B(x), h_n)_X\right]. \end{aligned}$$

For $n \in \mathrm{N}$ let P_n denote the orthogonal projection onto $X_n := \mathrm{span}\{h_1, \ldots, h_n\}$ in X.

The following lemma is a modification of a standard moment estimate for diffusions (see e.g. [71]). Here, however, we do not assume the existence of a process, but only deal with operators. Some additional difficulties in the proof arise from the fact that we are dealing with weak solutions.

Lemma 5.1 *Suppose μ is a probability measure on X such that ${}_{X^*}\langle \ell, B\rangle_X \in L^1(\mu)$ $\forall\, \ell \in X^*$ and $L^*\mu = 0$ (in the sense of (3.3)). Assume, furthermore that $x \mapsto (B(x), x)_X$ is bounded on balls and that*

$$\lim_{|x|_X \to \infty} (B(x), x)_X = -\infty \tag{5.1}$$

Let $k \in \mathrm{N}$ and $V_k(x) := (x, x)_X^k$, $x \in X$. Then

$$\int V_k \; d\mu \leq M_k, \tag{5.2}$$

where

$$\begin{aligned} M_k &:= \sup_{|x|_X \leq R_k} |LV_{k+1}(x)| + R_k^{2k} \\ &\leq R_k^{2k}[2(k+1)\, Trace(T^2) + 4k(k+1)\, \|T^2\| \\ &\qquad + 2(k+1) \sup_{|x|_X \leq R_k} (B(x), x)_X + 1] \end{aligned} \tag{5.3}$$

and

$$R_k := \inf\left\{R > 0 : \sup_{|x|_X \geq R} (B(x), x)_X \leq -[Trace(T^2) + 2k\|T^2\| + (2(k+1))^{-1}]\right\}$$

Proof: Note first that Δ_H and hence L acts on V_k since $H \subset X$ is Hilbert-Schmidt. We have that for all $x \in X$

$$\begin{aligned} LV_k(x) &= 2k(x, x)_X^{k-1}\, Trace(T^2) + 4k(k-1)(x, x)_X^{k-2}(T^2 x, x)_X \\ &\quad + 2k(x, x)_X^{k-1}(B(x), x)_X. \end{aligned}$$

By definition, if $|x|_X \geq R_{k-1}$,

$$(B(x), x)_X \leq -(Trace(T^2) + 2(k-1)\|T^2\| + (2k)^{-1}),$$

hence $LV_k(x) \leq -(x, x)_X^{k-1}$. Consequently,

$$LV_k \leq M_{k-1} - V_{k-1} \quad \text{on } X \tag{5.4}$$

Formal integration of (5.4) gives the desired estimate. To justify this, fix $\epsilon < 0$ and $m \in \mathrm{N}$, and consider

$$W_{\epsilon,m} := \frac{V_k \circ P_m}{1 + \epsilon V_k \circ P_m} = \epsilon^{-1}\left(1 - \frac{1}{1 + \epsilon V_k \circ P_m}\right) .$$

Then $\forall n \in \mathrm{N}$

$$\partial_{h_n} W\epsilon, m = \frac{\partial_{h_n}(V_k \circ P_m)}{(1 + \epsilon V_k \circ P_m)^2}$$

and

$$\begin{aligned}
\partial_{h_n}^2 W_{\epsilon,m} &= \frac{\partial_{h_n}^2(V_k \circ P_m)(1 + \epsilon V_k \circ P_m)^2 - 2\epsilon(\partial_{h_n}(V_k \circ P_m))^2(1 + \epsilon V_k \circ P_m)}{(1 + \epsilon V_k \circ P_m)^2} \\
&\leq \frac{\partial_{h_n}^2(V_k \circ P_m)}{(1 + \epsilon V_k \circ P_m)^2}.
\end{aligned}$$

Since $L(V_k \circ P_m) = (LV_k) \circ P_m$, by (5.4) we conclude that

$$LW_{\epsilon,m} \leq \frac{(LV_k) \circ Pm}{(1 + \epsilon V_k \circ P_m)^2} \leq \frac{M_{k-1} - V_{k-1} \circ P_m}{(1 + \epsilon V_k \circ P_m)^2} \quad \text{on } X. \tag{5.5}$$

But $W_{\epsilon,m} \in \mathcal{F}C_b^\infty$ (in particular $LW_{\epsilon,m} \in L^1(\mu)$) hence by (5.5)

$$\int \frac{V_{k-1} \circ P_m}{(1 + \epsilon V_k \circ P_m)^2} \, d\mu \leq M_{k-1}.$$

First letting $\epsilon \downarrow 0$ and then $m \nearrow \infty$ we obtain the assertion. □

The main result of this section is the following

Theorem 5.2 *Assume that*

$$\lim_{|x|_X \to \infty} (B(x), x)_X = -\infty,$$

that every ${}_{X^*}\langle \ell, B\rangle_X$, $\ell \in X^*$, is weakly continuous on X, and that there exist $C_1, C_2, d \in]0, \infty[$ such that

$$|B(x)|_X \leq C_1 + C_2 |x|_X^d \quad \forall\, x \in X \ . \tag{5.6}$$

Then there exists a probability measure μ on X satisfying (3.2) such that $L^*\mu = 0$. In particular, Proposition 3.3 and Corollary 3.14 apply.

We first prove the finite dimensional case where we can even drop the continuity assumption, i.e., we have

Theorem 5.3 *Assume that $X = H = \mathbb{R}^n$ (but possibly with different inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_H$ on X resp. H). Assume that*

$$\lim_{|x|\to\infty} (B(x), x) = -\infty$$

and that there exist $C_1, C_2, d \in]0,\infty[$ such that

$$|B(x)| \le C_1 + C_2|x|^d \quad \forall x \in \mathbb{R}^n.$$

Then there exists a probability measure μ on $\mathbb{R}^n$ satisfying (3.2) such that $L^*\mu = 0$. In particular, Theorem 3.1 applies.

Proof We may asume that $C_1, C_2, d \ge 1$. We may also assume that $(\cdot,\cdot)_X$ is the Euclidean inner product on $\mathbb{R}^n$. For each $j \in \mathbb{N}$ let $B_j : \mathbb{R}^n \to \mathbb{R}^n$ denote the map which coincides with B on the closed ball K_j of radius j and $B_j(x) = -x$ if x is not in this ball. According to [72, Theorem 2.3 and the following Remark] there exists a probability measure μ_j on $\mathbb{R}^n$ equivalent to the standard normal distribution γ on $\mathbb{R}^n$ with $\frac{d\mu_j}{d\gamma} \in H^{1,2}(\mathbb{R}^n;\gamma)$, satisfying $(\Delta_H + B_j\cdot\nabla)^*\mu_j = 0$. In particular, $|B_j| \in L^1(\mu_j)$ (since $|B_j| \in L^2(\gamma)$) and by Lemma 5.1 we conclude that $|B_j| \in L^2(\mu_j)$. Hence we can apply Theorem 3.1(iv) to obtain that $\mu_j = \varphi_j^2\, dx$ for some $\varphi_j \in H^{1,2}(\mathbb{R}^n; dx)$ and

$$\int |\nabla\varphi_j|_H^2\, dx \le const. \int |B_j|^2\, d\mu_j \le const. \int (C_1 + C_2(x,x)^d)^2\, \mu_j(dx) \le C$$

where by Lemma 5.1 C is a constant independent of j. Hence the same arguments as in the proof of Theorem 3.1 imply the convergence of a subsequence $(\varphi_{j_k})_{k\in\mathbb{N}}$ in $L^2_{\mathrm{loc}}(dx)$ to some $\varphi \in H^{1,2}(\mathbb{R}^n; dx)$. But since by Lemma 5.1 and (5.3) μ_j, $j \in \mathbb{N}$, have uniformly bounded moments, the Chebychev-Markov inequality and Prohorov's theorem imply that $(\mu_j)_{j\in\mathbb{N}}$ has a weakly convergent subsequence. Hence $\int \varphi^2\, dx = 1$ and clearly $\mu := \varphi^2\, dx$ has finite moments, in particular $|B| \in L^2(\mu)$. Furthermore, $\beta_H^\mu = 2\varphi^{-1}\nabla\varphi \in L^2(\mu, \mathbb{R}^n)$ and, since B is locally bounded, we have that

$$0 = \lim_{j\to\infty} \int (\Delta_H f + {}_{X^*}\langle f', B_j\rangle_X)\, \varphi_j^2\, dx = \int (\Delta_H f + {}_{X^*}\langle f', B\rangle_X)\, d\mu \quad \forall f \in C_0^\infty(\mathbb{R}^n)\ . \tag{5.7}$$

Since $|B| \in L^2(\mu)$, (5.7) and a simple approximation argument imply that

$$0 = \int (\Delta_H f + {}_{X^*}\langle f', B\rangle_X)\, d\mu \qquad \forall f \in C_b^\infty(\mathbb{R}^n),$$

i.e., μ satisfies (3.3). □

Remark 5.4 If we replace the polynomial growth condition (5.6) in Theorem 5.2 by the assumption: $\forall \ell \in X^*$ there exist $C_1(\ell), C_2(\ell), d(\ell)$ such that

$$|{}_{X^*}\langle \ell, B(x)\rangle_X| \le C_1(\ell) + C_2(\ell)|x|_X^{d(\ell}} \qquad \forall x \in X,$$

we would get (5.12) in the proof of Theorem 5.2 below only $\forall\, \ell \in K := R_X^{-1}(\bigcup_{n\in\mathbb{N}} X_n)$ and not for all $\ell \in X^*$. Then by the same arguments as in the proof of Proposition 3.12(i) one can show that then

$$\int Lu\, d\mu = 0$$

for all $u \in \mathcal{F}C_b^\infty(K) := \{g(\ell_1,\dots,\ell_m)|\ m \in \mathbb{N},\ g \in C_b^\infty(\mathbb{R}^m),\ \ell_1,\dots,\ell^m \in K\}$ $(\subsetneq \mathcal{F}C_b^\infty)$ which is a little less than (3.3).

Proof of Theorem 5.2. For $n \in \mathbb{N}$ define maps $B_n : X_n \to X_n$ by

$$B_n(x) := P_n(B(x))\,, \quad x \in X_n.$$

Since $(B_n(x), x)_{X_n} = (B(x), x)_X\ \forall x \in X_n$, by Theorem 5.3 there exists a probability measure μ_n on X_n such that $|B_n|_X \in L^2(\mu_n)$ and $L_n^*\mu_n = 0$ where

$$L_n = \Delta_n + B_n \cdot \nabla$$

with

$$\Delta_n = \sum_{i=1}^n t_i^2\, \partial_{h_i}^2 \ \text{ and }\ B_n \cdot \nabla = \sum_{i=1}^n (h_i, B)_X\, \partial_{h_i}.$$

We consider μ_n as a measure on X by extending it on $X \setminus X_n$ by zero. We claim that μ_n, $n \in \mathbb{N}$, have uniformly bounded moments of any fixed order. Indeed, if for $k \in \mathbb{N}$ we denote the restriction of V_k to X_n by $V_{k,n}$ and if R_k is defined as in Lemma 5.1, then

$$\begin{aligned}\int V_k\, d\mu_n &= \int_{X_n} V_{k,n}\, d\mu_n \\ &\le \sup\{|L_n V_{k+1,n}(x)| : x \in X_n,\ |x|_{X_n} \le R_k\} + R_k^{2k}.\end{aligned}$$

But $L_n V_{k+1,n} = LV_{k+1}$ on X_n. Hence (5.3) and the assumption on B yield our claim. Since balls are weakly compact, the claim and the Chebychev-Markov inequality imply that μ_n, $n \in \mathbb{N}$, are uniformly tight on X w.r.t. the weak topology. Applying Prohorov's theorem (cf.[64, p.379] for the version needed here) we conclude that $\{\mu_n : n \in \mathbb{N}\}$ is relatively compact w.r.t. to the topology τ_w on the space $M_1(X)$ of probability measures on X which is generated by the bounded weakly continuous functions on X. Recall that $\mathcal{B}(X)$ coincides with the σ-algebra generated by the weakly open sets in X. Hence there exists $\mu \in M_1(X)$ and a subsequence $(\mu_{n_i})_{i\in\mathbb{N}}$ of $(\mu_n)_{n\in\mathbb{N}}$ such that

$$\tau_w - \lim_{i\to\infty} \mu_{n_i} = \mu.$$

(We note that since closed balls in X are compact and metrizable, it is easy to see that really a convergent subsequence and not just a convergent subnet exists.) We now claim that μ has finite moments of every order. Indeed, fix $k \in \mathbb{N}$ and let $m, M \in \mathbb{N}$. Then $x \to |P_m x|_X^{2k} \wedge M$ is a bounded weakly continuous function on X. Hence

$$\begin{aligned}\int |P_m x|_X^{2k} \wedge M\, d\mu &= \lim_{i\to\infty} \int |P_m x|_X^{2k} \wedge M d\mu_{n_i} \\ &\le sup_n \textstyle\int V_k\, d\mu_n =: C_k,\end{aligned} \tag{5.8}$$

and $C_k < \infty$ as shown above. Letting $m, M \to \infty$ we prove the claim. Fix $\ell \in X^*$. By assumption we can now conclude that ${}_{X^*}\langle \ell, B\rangle_X \in L^2(\mu)$. Furthermore, if $n \in \mathbb{N}$ and $\ell \in R_X^{-1}(X_n)$, then $j_{H_n}(\ell|_{X_n}) = j_H(\ell)$, (cf. Example 1.1(iii)), where $H_n = X_n$, but equipped with the inner product $(\cdot,\cdot)_{H_n}$ induced by $(\cdot,\cdot)_H$, and ${}_{X_n^*}\langle \ell|_{X_n}, B_n\rangle_{X_n} = {}_{X_n^*}\langle \ell, B\rangle_{X_n}$. Hence by Proposition 3.12(i)

$$\begin{aligned} 0 &= -|j_{H_n}(\ell|_{X_n})|^2_{H_n} + i\int {}_{X_n^*}\langle \ell|_{X_n}, B_n\rangle_{X_n} e^{i\ell}\, d\mu_n \\ &= -|j_H(\ell)|^2_H + i\int {}_{X^*}\langle \ell, B\rangle_X e^{i\ell}\, d\mu_n. \end{aligned} \tag{5.9}$$

We claim that $\forall\, \ell \in X^*$

$$\lim_{i\to\infty} \int {}_{X^*}\langle \ell, B\rangle_X e^{i\ell}\, d\mu_{n_i} = \int {}_{X^*}\langle \ell, B\rangle_X\, e^{i\ell}\, d\mu. \tag{5.10}$$

Indeed, if $f := {}_{X^*}\langle \ell, B\rangle_X e^{i\ell}$, $\quad, K_r :=$ closed ball of radius r, and $c_r := sup_{K_r}|f|$, then $\forall\, n \in \mathbb{N},\ \ r > 0$

$$\begin{aligned} &\left|\int f\, d\mu_n - \int f\, d\mu\right| \\ &\le \left|\int_{K_r} f\, d\mu_n - \int_{K_r} f\, d\mu\right| + \int_{X\setminus K_r} |f|\, d\mu_n + \int_{X\setminus K_r} |f|\, d\mu \\ &\le \left|\int f\wedge c_r\, d\mu_n - \int f\wedge c_r\, d\mu\right| + 2\Big(\int_{X\setminus K_r} |f|\, d\mu_n + \int_{X\setminus K_r} |f|\, d\mu\Big). \end{aligned} \tag{5.11}$$

But by assumption for some $k \in \mathbb{N}$

$$\int_{X\setminus K_r} |f|\, d\mu_n \le \mu_n(X\setminus K_r)^{1/2} C_k$$

and

$$\int_{X\setminus K_r} |f|\, d\mu \le \mu(X\setminus K_r)^{1/2} C_k,$$

where C_k is as in (5.8). Again by (5.8) for $\epsilon > 0$ there exists $r > 0$ such that

$$\sup_n \mu_n(X\setminus K_r) + \mu(X\setminus K_r) < \epsilon^2/C_k^2.$$

Since $f \wedge c_r$ is a bounded weakly continuous function, (5.11) now implies (5.10). Hence we can take the limit over the subsequence $(n_i)_{i\in\mathbb{N}}$ in (5.9) to conclude that $\forall \ell \in R_X^{-1}(\bigcup_{n\in\mathbb{N}} X_n)$

$$0 = -|j_H(\ell)|^2_H + i\int {}_{X^*}\langle \ell, B\rangle_X e^{i\ell}\, d\mu. \tag{5.12}$$

Let $\ell \in X^*$. There exist $\ell_n \in R_X^{-1}(\bigcup_{m\in\mathbb{N}} X_m)$, $n \in \mathbb{N}$, such that $\ell_n \underset{n\to\infty}{\longrightarrow} \ell$ in X^*. As ${}_{X^*}\langle \ell, B\rangle_X \in L^2(\mu)$, we can take the limit for $n \to \infty$ in (5.12) which hence also holds for ℓ and the proof is complete because of Proposition 3.12(i). $\square$

Remark 5.5 (i) Theorem 5.2 is an extension of [47] in two directions. First, in [47] the coercitivity condition figurates in a stronger form (with some compact operator), and, second, B has to be the uniform limit of its restrictions to finite dimensional subspaces (which implies weak continuity). If there exists a compactly embedded Hilbert space $E \subset X$ such that $H \subset E$, the natural embedding $H \subset E$ is a Hilbert-Schmidt map, $B(E) \subset E$, and

$$\lim_{|x|_E \to \infty} (B(x), x)_E = -\infty,$$

then instead of the weak continuity of the functions ${}_{X^*}\langle \ell, B\rangle_X$, $\ell \in X^*$, it suffices to assume their continuity in the X-norm. Indeed, in this case $\{\mu_n : n \in \mathrm{N}\}$ is relatively weakly compact on X with the norm topology, since by similar arguments as in the proof of Theorem 5.2 this sequence has uniformly bounded moments on the space E. This covers the existence result in [48].

(ii) We point out that Theorem 5.2 in connection with Proposition 3.3 and Corollary 3.14 provides a sufficient condition for the given map B to be the vector logarithmic derivative of a measure.

6 Uniqueness problem

Let X be a locally convex space over R. Let H be a separable real Hilbert space such that $H \subset X$ continuously and densely and let $j_H : X^* \to H$ be as in Section 1. Fix a linear subspace K of X^*, dense w.r.t the strong topology, and a probability measure μ on $\mathcal{B}(X)$ such that all partial logarithmic derivatives $\beta^\mu_{j_H(\ell)}$ for $\ell \in K$ exist. For each $\ell \in K$ we choose and fix a Borel version of $\beta^\mu_{j_H(\ell)}$, again denoted by $\beta^\mu_{j_H(\ell)}$. The reader is warned that various quantities below depend on this choice. Define $\mathcal{G}_K^{\beta^\mu}$ to be the set of all probability measures ν on $\mathcal{B}(X)$ such that for every $\ell \in K$ the partial logarithmic derivative $\beta^\nu_{j_H(\ell)}$ of ν along $j_H(\ell)$ exists and $\beta^\nu_{j_H(\ell)} = \beta^\mu_{j_H(\ell)}$ $\nu - a.e.$. As already mentioned in Remark 3.11(ii) the question whether

$$\sharp \mathcal{G}_K^{\beta^\mu} = 1 \tag{6.1}$$

is of interest, in particular w.r.t. uniqueness of Gibbs states. In this section we shall also study the question whether

$$\sharp \mathcal{G}_{K,ac}^{\beta^\mu} = 1 \ , \tag{6.2}$$

where

$$\mathcal{G}_{K,ac}^{\beta^\mu} := \left\{ \nu \in \mathcal{G}_K^{\beta^\mu} : \nu \ll \mu \right\}. \tag{6.3}$$

We note that both $\mathcal{G}_K^{\beta^\mu}$ and $\mathcal{G}_{K,ac}^{\beta^\mu}$ are convex sets.

a) Some results on uniqueness / non-uniqueness

In general (6.2) and so (6.1) do not hold, even if $\dim X = 1$, as the following example shows.

Example 6.1 Let $K = H = X = \mathrm{R}$ with the euclidean inner product and $\varphi \in C_b^\infty(\mathrm{R}) \cap H^{1,2}(\mathrm{R}; dx)$ is such that $\varphi(0) = 0$, $\varphi > 0$ on $\mathrm{R} \setminus \{0\}$, and $\int \varphi^2 \, dx = 1$. Let $\psi := \frac{1}{2}\varphi$ on $[0, \infty[$ and $\psi := c\varphi$ on $]-\infty, 0]$ where $c \in]0, \infty[$ is such that $\int \psi^2 \, dx = 1$. Let $\mu := \varphi^2 \, dx$, $\nu := \psi^2 \, dx$. Then $\mu \neq \nu$ and $\beta_H^\mu = 2\frac{\nabla \varphi}{\varphi} = \beta_H^\nu$ (cf. Remark 1.3), hence $\sharp \mathcal{G}^{\beta^\mu} > 1$.

More concretely, we can take $\varphi(x) := x\sqrt{g(x)}$, $x \in \mathrm{R}$, where g is the standard Gauss density on R. Since in this case $\beta_H^\mu(x) = -x + \frac{2}{x}$, hence , in particular, by Theorem 3.10(ii) the uniqueness result in [65] mentioned in Remark 3.6(i) does not hold if one drops the condition $\sup \|v\|_H < \infty$.

On the other hand we have the following positive result in the finite dimensional case.

Theorem 6.2 *Let $K = H = X = \mathrm{R}^n$ equipped with the Euclidean inner product. Let $\varphi \in H^{1,2}(\mathrm{R}^n; dx)$ and $\mu := \varphi^2\, dx$. If $|\nabla\varphi|/\varphi \in L^1_{\mathrm{loc}}(\mathrm{R}^n; dx)$, then $\mathcal{G}_K^{\beta^\mu} = \{\mu\}$.*

Remark 6.3 Since $\nabla\varphi = 0 \quad dx - a.e.$ on $\{\varphi = 0\}$, in Example 6.1 and Theorem 6.2 below we set

$$\frac{\nabla\varphi}{\varphi} \equiv 0 \quad \text{ on } \quad \{\varphi = 0\}.$$

For the proof of Theorem 6.2 we need

Lemma 6.4 *Let $\varphi \in H^{1,1}_{\mathrm{loc}}(\mathrm{R}^n; dx)$ (i.e., φ, $|\nabla\varphi| \in L^1_{\mathrm{loc}}(\mathrm{R}^n; dx)$) such that $\varphi \geq 0\, dx - a.e.$ and $|\nabla\varphi|/\varphi \in L^1_{\mathrm{loc}}(\mathrm{R}^n; dx)$. Then $\log\varphi \in H^{1,1}_{\mathrm{loc}}(\mathrm{R}^n; dx)$ and $\nabla\log\varphi = \nabla\varphi/\varphi$.*

Proof Let $\delta \in]0,1]$. Then $\log(\varphi + \delta) \in H^{1,1}_{\mathrm{loc}}(\mathrm{R}^n; dx)$ and

$$\begin{aligned} |\nabla \log(\varphi+\delta)| &= |\nabla\varphi|/(\varphi+\delta) \\ &\leq |\nabla\varphi|/\varphi \quad dx - a.e. \quad \text{ on } \{\varphi > 0\} \end{aligned} \tag{6.4}$$

But on $\{\varphi = 0\}$, $\log(\varphi+\delta) = \log\delta$ hence $\nabla\log(\varphi+\delta) = 0$ on $\{\varphi = 0\}$ and (6.4) holds $dx - a.e.$ on R^n. Hence by Poincare's inequality we have for every open ball K in R^n that, if $|K| = \int_K dx$,

$$\begin{aligned} &\int_K \left| |\log(\varphi+\delta)| - \frac{1}{|K|}\left|\int_K \log(\varphi+\delta)\, dx\right| \right| dx \\ &\qquad \leq \int_K |\nabla\log(\varphi+\delta)|\, dx \\ &\qquad \leq \int_K |\nabla\varphi|/\varphi\, dx \quad (< \infty)\ . \end{aligned} \tag{6.5}$$

Now suppose that

$$\int_K |\log\varphi|\, dx = \infty\ . \tag{6.6}$$

Then

$$\begin{aligned} \left|\int_K \log(\varphi+\delta)\, dx\right| &\geq \left|\int_{K\cap\{\varphi+\delta<1\}} \log(\varphi+\delta)\, dx\right| - \int_{K\cap\{\varphi+\delta\geq 1\}} \log(\varphi+\delta)\, dx \\ &\geq \int_{K\cap\{\varphi+\delta<1\}} -\log(\varphi+\delta)\, dx - \int_K \varphi\, dx. \end{aligned}$$

Since (6.6) implies that

$$-\int_{K\cap\{\varphi<1\}} \log\varphi\, dx = +\infty,$$

it follows by the monotone convergence theorem that

$$\lim_{n\to\infty}\left|\int_K \log\left(\varphi+\frac{1}{n}\right)\,dx\right| = \infty.$$

Thus by Fatou's lemma

$$\liminf_{n\to\infty}\int_{K\cap\{\varphi>0\}}\left|\left|\log\left(\varphi+\frac{1}{n}\right)\right| - \frac{1}{|K|}\left|\int_K \log\left(\varphi+\frac{1}{n}\right)\right|\right|\,dx \geq \infty,$$

since $dx(\{\varphi>0\})>0$ because $\int\varphi^2\,dx = 1$. This contradicts (6.5), hence (6.6) is false and therefore $\log\varphi\in L^1_{\mathrm{loc}}(\mathrm{R}^n;dx)$. Note that in $L^1_{\mathrm{loc}}(\mathrm{R}^n;dx)$

$$\log\left(\varphi+\frac{1}{n}\right) \quad \underset{n\to\infty}{\longrightarrow} \quad \log\varphi$$

and

$$\nabla\log\left(\varphi+\frac{1}{n}\right) = \nabla\varphi/\left(\varphi+\frac{1}{n}\right) \quad \underset{n\to\infty}{\longrightarrow} \quad \nabla\varphi/\varphi\ .$$

Hence $\log\varphi\in H^{1,1}_{\mathrm{loc}}(\mathrm{R}^n;dx)$ and $\nabla\log\varphi = \nabla\varphi/\varphi$. $\square$

Proof of Theorem 6.2 Since $|\varphi|\in H^{1,2}_{\mathrm{loc}}(\mathrm{R}^n;dx)$, we may assume that $\varphi\geq 0\ dx-a.e.$. Let $\nu_0\in\mathcal{G}^{\beta\mu}$. Then by Remark 1.3 there exists $\psi_0\in H^{1,2}(\mathrm{R}^n;dx)$ such that $\nu_0=\psi_0^2\,dx$. Set $\psi := \sqrt{\frac{1}{2}(\varphi^2+\psi_0^2)}$. Then $\nu := \psi^2\,dx\in\mathcal{G}^{\beta\mu}$. Since by Lemma 6.4 $\log\varphi\in H^{1,1}_{\mathrm{loc}}(\mathrm{R}^n;dx)$, in particular $\varphi>0\ dx-a.e.$, hence $\psi\geq\sqrt{\frac{1}{2}}\varphi>0$, and so $\frac{\nabla\psi}{\psi}=\frac{\nabla\varphi}{\varphi}\ dx-a.e.$. Applying Lemma 6.4 again we conclude that $\log\psi\in H^{1,1}_{\mathrm{loc}}(\mathrm{R}^n;dx)$ and that $\nabla\log\varphi=\nabla\log\psi\ dx-a.e.$.

Consequently, $\log\varphi-\log\psi$ is constant $dx-a.e.$. But both φ^2 and φ^2 are probability densities. Thus $\varphi=\psi\,dx-a.e.$, hence $\varphi=\psi_0\,dx-a.e.$. $\square$

We can also prove and disprove (6.1) in some infinite dimensional (Gaussian) cases.

Theorem 6.5 *Consider the situation described in Example 2.1(i) and let A,γ^A be as defined there. Assume that $K := j_H^{-1}(D(A))$ separates the points of X and that $Aj_H(K)\subset j_H(X^*)$. Define*

$$M := \bigcap_{\ell\in K}\{x\in X|\,{}_{X^*}\langle j_H^{-1}Aj_H(\ell),x\rangle_X = 0\}.$$

(i) Let σ be a probability measure on X such that $\sigma(M)=1$. Then $\gamma^A*\sigma\in\mathcal{G}_K^{\gamma^A}$.

(ii) $\mathcal{G}_K^{\gamma^A}=\{\gamma^A\}$ if and only if $M=\{0\}$.

Proof (i): Let $\ell\in K$. Then for all $f\in\mathcal{F}C_b^\infty$

$$\begin{aligned}\int\partial_{j_H(\ell)}f\,d(\gamma^A*\sigma) &= \iint\partial_{j_H(\ell)}f(x+y)\,\gamma^A(dx)\,\sigma(dy)\\ &= -\iint {}_{X^*}\langle(j_H^{-1}Aj_H)(\ell),x\rangle_X\,f(x+y)\,\gamma^A(dx)\,\sigma(dy)\\ &= -\int(j_H^{-1}Aj_H)(\ell)\,f\,d(\gamma^A*\sigma)\end{aligned}$$

where the last step follows since $\sigma(M)=1$. Since by Example 2.1(i), $\beta^{\gamma^A}_{j_H(\ell)}=(j_H^{-1}Aj_H)(\ell)$, the assertion follows.

(ii): Suppose $\mathcal{G}_K^{\gamma^A}=\{\gamma^A\}$ and let $x\in M$. Then by (i) $\gamma^A*\delta_x\in\mathcal{G}_K^{\gamma^A}$, hence $x=\{0\}$.

Conversely, suppose that $M=\{0\}$ and let $\nu\in\mathcal{G}_K^{\gamma^A}$. Fix $\ell\in K$ and set $\ell_A:=(j_H^{-1}Aj_H)(\ell)(\in X^*)$. Then for all $s\in\mathrm{R}$

$$\frac{d}{ds}\tilde{\nu}(s\ell_A)=i\int \ell_A e^{is\ell_A}\,d\nu \tag{6.7}$$

Since $-\ell_A=\beta^{\gamma^A}_{j_H(\ell)}=\beta^{\nu}_{j_H(\ell}\ (\in L^2(\nu))$ by Example 2.1(i) and the assumption, the right hand side of (6.7) is equal to

$$i\int \partial_{j_H(\ell)}(e^{is\ell_A})\,d\nu=-((Aj_H)(\ell),j_H(\ell))_H\, s\,\tilde{\nu}(s\ell_A)\ .$$

Consequently , since $\tilde{\nu}(0)=1$, we have that for all $s\in\mathrm{R}$

$$\begin{aligned}\tilde{\nu}(s\ell_A)&=e^{-\frac12 s^2(Aj_H(\ell),j_H(\ell))_H^2}\\&=\tilde{\gamma}^A(s\ell_A).\end{aligned}$$

Thus, $\tilde{\nu}(\ell)=\tilde{\gamma}^A(\ell)$ for all ℓ in a linear subset of X^* that separates the points of X. Since both are Radon, it follows that $\nu=\gamma^A$. □

Remark 6.6 (i) Unfortunately, in Theorem 6.5 we were not able to prove that

$$\mathcal{G}_K^{\gamma^A}=\{\gamma^A*\sigma\mid \sigma \text{ probability measure on } X \text{ such that } \sigma(M)=1\}. \tag{6.8}$$

However, this is known to be true if X is a separable Hilbert space and the semigroup generated by A on H extends to a strongly continuous semigroup on X. In this case it can be shown that $\mathcal{G}_K^{\beta^{\gamma^A}}$ coincides with the set $\mathcal{M}$ of probability measures μ satisfying $(L^{H,\gamma^A})^*\mu=0$ and that $\mathcal{M}$ coincides with the set $\mathcal{M}_1$ of invariant measures for the semigroup generated by L^{H,γ^A} which is given by a Mehler type formula (cf. [24]). But $\mathcal{M}_1$ is known to be just the right hand side of (6.8) (cf. [28, Theorem 11.7] and [24, Theorem 4.4]).

(ii) The converse of Theorem 6.5 was first proved in [62] in the case where $A=$ identity and (X,H,γ) is the classical Wiener space.

Note that though A in Theorem 6.5 is strictly positive, it can happen that $M\neq\{0\}$ as the following example shows.

Example 6.7 Let $X:=\mathcal{D}'(\mathrm{R}^d)$ (i.e., the space of *Schwartz distributions*), $X^*=K:=C_0^\infty(\mathrm{R}^d)$, $H:=L^2(\mathrm{R}^d;dx)$ and $A:=-\Delta+1$, $D(A):=H^{2,2}(\mathrm{R}^d;dx)$. Then γ^A is just the *free Euclidean field of quantum field theory* which has been studied extensively (cf. [56]). Furthermore, M is the set of $(-\Delta+1)$-harmonic functions in $\mathcal{D}'(\mathrm{R}^d)$ which is certainly non-empty. (Note, however, that if $\mathcal{S}'(\mathrm{R}^d)$ denotes the space of *tempered Schwartz distributions*, then $\gamma^A(\mathcal{S}'(\mathrm{R}^d))=1$, but $M\cap\mathcal{S}'(\mathrm{R}^d)=\{0\}$). It has been shown in [58] that the set on the right hand side of (6.8) is exactly the set of all Gibbs measures for the corresponding "free field specification" (cf. [58] for details).

By taking σ to be a Dirac measure it follows by Theorem 6.5 (and Example 6.7) that $\mathcal{G}_K^{\gamma^A}$ in general contains other, but non-centred, Gaussian measures. It is surprising that even if X is a separable Hilbert space there exist two *centred* Gaussian measures on X having the same vector logarithmic derivative which is even a continuous linear operator on X. (Note that this is entirely in contrast to the finite dimensional case). That this is possible, is based on the earlier observation that the range of R in Example 2.5 might not be dense (as is the case in the following theorem).

Theorem 6.8 *There exist separable real Hilbert spaces H, X such that $H \subset X$ is Hilbert-Schmidt, and two different centred Gaussian measures μ_1, μ_2 on X such that there exists a bounded linear operator A on X which is a joint Borel version for $\beta_H^{\mu_1}$ and $\beta_H^{\mu_2}$.*

Proof We shall use Example 2.5.

Let $X := L^2(\,]0,1[\,; dx)$, $H := H_0^{4,2}(\,]0,1[\,; dx)$ (i.e., the Sobolev space of functions with absolutely continuous third derivatives such that

$$f^{(4)} \in L^2(\,]0,1[\,; dx),\ f^{(j)}(0) = f^{(j)}(1) = 0\ ,\ 1 \le j \le 3\ .$$

Define operators $(A_i, D(A_i))$, $i = 1, 2$, on X by

$$A_1 := -\Delta\ ,\quad D(A_1) := \{u \in H^{2,2}(\,]0,1[\,; dx) | u(0) = u(1) = 0\},$$

$$A_2 := -\Delta\ ,\quad D(A_2) := \{u \in H^{2,2}(\,]0,1[\,; dx) | u(1) - u'(1) = 0 = u(0) + u'(0)\}.$$

Then, as is well-known, $(A_i, D(A_i))$, $i = 1, 2$, are injective non-negative definite self-adjoint operators on X such that $T_i := A_i^{-1}$, $i = 1, 2$, injective non-negative definite self-adjoint Hilbert-Schmidt operators on X. Since$H \subset X$ is Hilbert-Schmidt, there exists an injective non-negative definite self-adjoint Hilbert-Schmidt operator Q on X such that $Q(X) = H$ and $|\cdot|_H = |Q^{-1}\cdot|_X$ (cf. Example 2.1(i)). Clearly,

$$Q(X) = H \subset T_1^2(X) \cap T_2^2(X)$$

and

$$T_1^{-2}h = T_2^{-2}h \quad \forall\ h \in H\ .$$

Since $Q^2(X) \subset Q(X)$ and $T_i^2(X) \subset T_i(X)$, $i = 1, 2$, the assertion now follows by Example 2.5 by taking μ_i to be the centered Gaussian measure on X with covariance operator T_i^2, $i = 1, 2$, and $A := R^*$ with $R := T_1^{-2}\,Q^2 = T_2^{-2}\,Q^2$. $\square$

Now we are ready to give an example of a linear equation (0.1) with continuous linear operator B such that several Gaussian measures are solutions, which is at the same time an example of a nondegenerate Gaussian diffusion with nonunique invariant probability. Such an example is not possible in the finite dimensional case, but as noticed first by J. Zabczyk is possible in infinite dimensions. In [69], [70] there is an example of a diffusion $d\xi_t = A\xi_t dt + b\,d\omega_t$ with a fixed vector b and unbounded linear A such that the transitions probabilities are positive on nonvoid open sets, but an invariant probability for $(\xi_t)_{t\ge0}$ is not unique. Later modifications were suggested (see [54], [51]) even with bounded A. Below we describe an easy example of a diffusion governed by the equation $d\xi_t = A\xi_t\,dt + \sqrt{2}\,\omega_t$ and possessing different invariant probabilities, which are nondegenerate Gaussian measures. In addition, $(\xi_t)_{t\ge0}$ is symmetric with respect to these Gaussian measures.

Corollary 6.9 *Let μ_1, μ_2 be the centred Gaussian measures and A the bounded linear operator on X as in Theorem 6.8. Then the unique mild solution $(\xi_t)_{t\geq 0}$ of the stochastic differential equation*

$$d\xi_t = d\omega_t + A\xi_t\, dt, \quad \xi_0 = x \ (\in X)$$

has μ_1 and μ_2 as invariant measures and $(\xi_t)_{t\geq 0}$ is symmetric w.r.t. both μ_1 and μ_2.

Proof First of all, note that since the operator A is bounded, the equation above has a unique solution which is given by the formula (see [28, Theorem 5.4])

$$\begin{aligned} \xi_t &= e^{tA}x + \omega_t + \int_0^t Ae^{(t-s)A}\omega_s\, ds \\ &= e^{tA}x + \int_0^t e^{(t-s)A} d\omega_s\ , \quad t \geq 0\ . \end{aligned}$$

Hence, for $t > 0$ by [28, Theorem 5.2] the distribution P_t of the random vector ξ_t is a Gaussian measure with mean $e^{tA}x$ and covariance $(Q_t v, v)_X = \int_0^t (e^{(t-s)A}Qe^{(t-s)A^*}v, v)_X\, ds$, where Q is the covariance of the law of ω_1. Since the integrand above is continuous and Q is injective, it follows that Q_t is injective as well, that is, P_t has full support. It remains to note that since in this case the transition semigroup sends the linear span of the functions $\exp(il)$, $l \in X^*$, into itself, any measure satisfying our elliptic equation is in fact an invariant measure of the process $(\xi_t)_{t\geq 0}$. So both μ_1 and μ_2 are invariant measures and $(\xi_t)_{t\geq 0}$ is then symmetric w.r.t. both by Theorem 3.10(i). $\square$

b) Characterization of uniqueness

Now we want to state and prove the main result of this section, i.e., the characterization of all probability measures μ as above for which $\mathcal{G}_{K,ac}^{\beta^\mu} = \{\mu\}$ resp. $\mathcal{G}_K^{\beta^\mu} = \{\mu\}$. In order to do so we have to recall the definition of the (maximal) Dirichlet form $(\mathcal{E}^\mu, D(\mathcal{E}^\mu))$ associated with μ (introduced in [11], see also [8]) where μ is as specified at the beginning of this section. For simplicity from now on we assume X to be a Souslin space. We emphasize, however, that everything below is true for general locally convex spaces. The same holds for previous papers on classical Dirichlet forms by S. Albeverio and one of the authors. The reason is that since μ is Radon, its images under projections are Radon, $\mathcal{F}C_b^\infty$ is dense in $L^2(\mu)$, and regular conditional probabilities w.r.t. subspaces of finite codimension exist. These are the only properties used below which, in particular, Souslin spaces have. Now we recall (cf. [34], [52])

Definition 6.10 *(i) A closed non-negative definite symmetric bilinear form $(\mathcal{E}, D(\mathcal{E}))$ with domain $D(\mathcal{E})$ dense in $L^2(\mu)$ is called a* Dirichlet form *if $u^\sharp := u^+ \wedge 1 \in D(\mathcal{E})$ and $\mathcal{E}(u^\sharp, u^\sharp) \leq \mathcal{E}(u,u)$ for all $u \in D(\mathcal{E})$.*

(ii) A Dirichlet form is called irreducible *if $\mathcal{E}(u,u) = 0$ implies $u = constant\ \mu - a.e.$.*

Fix $k \in X$ such that β_k^μ exists and $\beta_k^\mu \in L^2(\mu)$. We define $D(\mathcal{E}_k^\mu)$ to be the space of all $u \in L^2(\mu)$ such that there exists $f_k \in L^2(\mu)$ so that

$$\int u\frac{\partial v}{\partial k}\, d\mu = -\int f_k\, v\, d\mu - \int uv\beta_k^\mu\, d\mu \qquad \forall v \in \mathcal{F}C_b^\infty\ . \tag{6.9}$$

In this case f_k is uniquely determined by u and we set

$$\frac{\partial^\mu u}{\partial k} := f_k,$$

and for $u, v \in D(\mathcal{E}_k^\mu)$

$$\mathcal{E}_k^\mu(u,v) := \int \frac{\partial^\mu u}{\partial k}\,\frac{\partial^\mu v}{\partial k}\,d\mu.$$

In order to show that $(\mathcal{E}_k^\mu, D(\mathcal{E}_k^\mu))$ is a Dirichlet form one needs the characterization in Proposition 6.11 below. For this we recall that for any closed subspace X_k of X such that $X = k\mathrm{R} \oplus X_k$, if $\pi_k : X \to X_k$ is the canonical projection, there exists a function $\varrho_k :\ X_k \times \mathrm{R} \to [0,\infty[$ such that $\varrho_k(y,s)\,ds$ is a subprobability kernel from $(X_k, \mathcal{B}(X_k))$ to $(\mathrm{R}, \mathcal{B}(\mathrm{R}))$ and for all $u : X \to [0,\infty[$, $\mathcal{B}(X)$-measurable,

$$\int_X u(x)\mu(dx) = \int_{X_k}\int_{\mathrm{R}} u(y+sk)\varrho_k(y,s)\,ds\,\nu_k(dy)\ .$$

Here $x = y+sk$ with $y \in X_k,\ s \in \mathrm{R}$ uniquely determined, and $\nu_k := \mu\circ\pi_k^{-1}$. Furthermore, for $\nu_k - a.e.\, y \in X_k,\ \varrho_k(y,\cdot)$ satisfies Hamza's condition (H). We recall that a $\mathcal{B}(\mathrm{R})$-measurable function $\varrho :\ \mathrm{R} \to [0,\infty[$ satisfies (H) if $\varrho = 0\ ds - a.e.$ on $\mathrm{R} \setminus R(\varrho)$ where

$$R(\varrho) := \left\{ t \in \mathrm{R} \middle| \int_{t-\epsilon}^{t+\epsilon} \varrho(s)^{-1}\,ds < \infty \ \text{ for some } \ \epsilon > 0 \right\}\ ,$$

i.e., $R(\varrho)$ is the largest open set $U \subset \mathrm{R}$ such that $\varrho^{-1} \in L^1_{\mathrm{loc}}(U; ds)$. (Here $\varrho^{-1}(s) := +\infty$ if $\varrho(s) = 0$.) All this follows from the main result (i.e., Theorem 3.2, particularly part (ii)) in [11] (see also [8, Proposition 2.2]).

We also note that correspondingly $L^2(\mu)$ can then be written as a direct integral of L^2-spaces over R,i.e.,

$$L^2(\mu) = \int_{X_k}^{\oplus} L^2(\mathrm{R}; \varrho_k(y,s)\,ds)\,\nu_k(dy) \tag{6.10}$$

in the sense that each $u \in L^2(\mu)$ corresponds to a "field of vectors"$(u_y)_{y\in X_k}$ where $u_y := u(y+\cdot k),\ y \in X_k$ (cf. [11] for details and references).

Proposition 6.11 *Let $u \in L^2(\mu)$. Then $u \in D(\mathcal{E}_k^\mu)$ if and only if for the corresponding element $(u_y)_{y\in X_k}$ in $\int_{X_k}^{\oplus} L^2(\mathrm{R}; \varrho_k(y,s)\,ds)\,\nu_k\,(dy)$ (see (6.10)) for $\nu_k - a.e.\ y \in X_k$ there exists a locally absolutely continuous ds-version $\tilde{u}_y$ of u_y on $R(\varrho_k(y,\cdot))$ and $\left(\frac{d\tilde{u}_y}{ds}\right)_{y\in X_k} \in \int^{\oplus} L^2(\mathrm{R}; \varrho_k(y,s)\,ds)\,\nu_k\,(dy)$. In this case*

$$\frac{\partial^\mu u}{\partial k} = \left(\frac{d\tilde{u}_y}{ds}\right)_{y\in X_k}$$

in the sense of (6.10). In particular, $(\mathcal{E}_k^\mu, D(\mathcal{E}_k^\mu))$ is a Dirichlet form on $L^2(\mu)$.

Proof [60, Remark 1.5(i)] or [14, Lemma 2.2] resp. [11, Theorems 3.2(i) and 3.8] □

Proposition 6.11 will also be crucial in the proof of our main characterization theorem. (See the proof of Lemma 6.14 below).

Corollary 6.12 *Let $D(\mathcal{E}_K^\mu)$ denote the set of all $u \in L^2(\mu)$ such that $u \in D(\mathcal{E}^\mu_{j_H(\ell)})$ for all $\ell \in K$ and there exists $\nabla_H^\mu u : X \to H$, $\mathcal{B}(X)/\mathcal{B}(H)$-measurable, such that $(\nabla_H^\mu u, j_H(\ell))_H = \frac{\partial^\mu u}{\partial j_H(\ell)}$ $\mu - a.e.$ $\forall \ell \in K$ and $\int |\nabla_H^\mu u|_H^2 \, d\mu < \infty$. Let*

$$\mathcal{E}_K^\mu(u,v) = \int (\nabla_H^\mu u, \nabla_H^\mu v)_H \, d\mu \ ; \quad u, v \in D(\mathcal{E}_K^\mu) \ .$$

Then $(\mathcal{E}_K^\mu, D(\mathcal{E}_K^\mu))$ is a Dirichlet form on $L^2(\mu)$.

Proof [11, Theorem 3.10] □

Remark 6.13 By [12, Lemma 5.6] there exists $K_0 \subset K$ such that $j_H(K_0)$ is an orthonormal base of H. Hence if $u \in D(\mathcal{E}_K^\mu)$ such that $\frac{\partial^\mu u}{\partial j_H(\ell)} = 0$ $\mu - a.e.$ for all $\ell \in K$, then $|\nabla_H^\mu u|_H = 0$ $\mu - a.e.$.

The following lemma is the main part of the proof of Theorem 6.15 below.

Lemma 6.14 *Let $\varrho \in L^1(\mu)$, $\varrho \geq 0$, $\nu := \varrho \cdot \mu$, and $\ell \in K$ such that for $k := j_H(\ell)$ we have that β_k^ν exists and that $\beta_k^\nu \in L^2(\nu)$. Assume furthermore that*

$$\varrho^{1/2} \beta_k^\mu \in L^2(\mu). \tag{6.11}$$

Then $\sqrt{\varrho} \in D(\mathcal{E}_k^\mu)$ and $\frac{\partial^\mu \sqrt{\varrho}}{\partial k} = \frac{1}{2}\varrho^{1/2}(\beta_k^\nu - \beta_k^\mu)$.

Proof We know by [8, Theorem 2.5 and Definition 2.1] that we may assume that for every $y \in X_k$, $\varrho_k(y,\cdot) \in L^1(\mathrm{R}; ds)$ with distributional derivative $\frac{\partial}{\partial s}\varrho_k(y,\cdot) \in L^1_{\mathrm{loc}}(\mathrm{R}; ds)$ and (cf. [8, Proposition 2.4]) there exists a ds-version $\tilde{\varrho}_k(y,\cdot)$ of $\varrho_k(y,\cdot)$ which is absolutely continuous on R and

$$R(\varrho_k(y,\cdot)) = \{\tilde{\varrho}_k(y,\cdot) > 0\}. \tag{6.12}$$

Note that the set $\mathcal{L}_0$ of all functions $T : X_k \to \mathrm{R}$ of the form

$$T(y) = f(\ell_1(y), \ldots, \ell_m(y)) \ , \ y \in X_k,$$

where $m \in \mathrm{N}$, $f \in C_b^\infty(\mathrm{R}^m)$, $\ell_1, \ldots, \ell_m \in X_k^*$, is dense in $L^2(X_k; \nu_k)$. Let $\ell \in X^*$ such that $\ell(k) = 1$ and $\ell(y) = 0$ $\forall y \in X_k$. (Then, of course, $\pi_k(x) = x - \ell(x)k$ $\forall x \in X$). Applying (1.1) to $f := (T \circ \pi_k)(g \circ \ell)$, $g \in C_0^\infty(\mathrm{R})$, $T \in \mathcal{L}_0$, we obtain that there exists $\Omega \in \mathcal{B}(X_k)$ with $\nu_k(\Omega) = 1$ and for all $y \in \Omega$, $g \in C_0^\infty(\mathrm{R})$

$$\int \frac{dg}{ds} \varrho(y+sk)\varrho_k(y,s)\, ds = -\int g(s)\varrho(y+sk)\beta_k^\nu(y+sk)\varrho_k(y,s)\, ds. \tag{6.13}$$

and

$$\varrho(y+\cdot k)\varrho_k(y,\cdot),\ \varrho(y+\cdot k)\,\beta_k^\nu(y+\cdot k)\varrho_k(y,\cdot) \in L^1(\mathrm{R}; ds) \tag{6.14}$$

Fix $y \in \Omega$.

Claim $\varrho(y+\cdot k) \in L^1_{\mathrm{loc}}(R(\varrho_k(y,\cdot)); ds)$ and its distributional derivative on $R(\varrho_k(y,\cdot))$ is in $L^1_{\mathrm{loc}}(R(\varrho_k(y,\cdot); ds)$ and is equal to $\varrho(y+\cdot k)[\beta_k^\nu(y+\cdot k) - \beta_k^\mu(y+\cdot k)]$.

If $[a,b] \subset R(\varrho_k(y,\cdot))$, by [8, Proposition 2.4(i)] we know that $\beta_k^\mu(y+\cdot k) \in L^1([a,b];ds)$ and

$$\tilde{\varrho}_k(y,s) = \tilde{\varrho}_k(y,a)\, \exp\left[\int_a^s \beta_k^\mu(y+tk)\,dt\right]\ ,\ \ s\in[a,b]. \tag{6.15}$$

Hence the claim follows by (6.12)-(6.14).

Let us now first assume that $\varrho \geq \epsilon$ for some $\epsilon > 0$. Then by the claim on $R(\varrho_k(y,\cdot))$

$$\frac{\partial}{\partial s}\varrho^{1/2}(y+\cdot k) = \frac{1}{2}\varrho^{1/2}(y+\cdot k)[\beta_k^\nu(y+\cdot k) - \beta_k^\mu(y+\cdot k)]\ .$$

By assumption $\frac{1}{2}\varrho^{1/2}(\beta_k^\nu - \beta_k^\mu) \in L^2(\mu)$. Hence by Proposition 6.11 and (6.10) the assertion follows.

In the general case consider $\varrho_\epsilon := (\varrho+\epsilon)$. Then $(\varrho_\epsilon)^{1/2}\beta_k^\mu \in L^2(\mu)$,

$$\beta_k^{(\varrho_\epsilon\mu)} = (\varrho+\epsilon)^{-1}(\varrho\beta_k^\nu + \epsilon\beta_k^\mu) \in L^2(\varrho_\epsilon\cdot\mu)\ ,$$

and hence

$$\frac{\partial^\mu}{\partial k}\sqrt{\varrho_\epsilon} = \frac{1}{2}\varrho_\epsilon^{1/2}\frac{\varrho}{\varrho+\epsilon}(\beta_k^\nu - \beta_k^\mu)\ .$$

Since $(D(\mathcal{E}_k^\mu), \mathcal{E}_k^\mu)$ is closed, we obtain the assertion by letting $\epsilon \to 0$. □

Now we can state and prove our characterization theorem.

Theorem 6.15 *(i)* $\mathcal{G}_{K,ac}^{\beta^\mu} = \{\mu\}$ *if and only if* $(\mathcal{E}_K^\mu, D(\mathcal{E}_K^\mu))$ *is irreducible.*

(ii) $\mathcal{G}_K^{\beta^\mu} = \{\mu\}$ *if and only if* $(\mathcal{E}_K^\nu, D(\mathcal{E}_K^\nu))$ *is irreducible for all* $\nu \in \mathcal{G}_K^{\beta^\mu}$.

Proof

(i) : Suppose $\mathcal{G}_{K,ac}^{\beta^\mu} = \{\mu\}$ and let $\varrho \in D(\mathcal{E}_K^\mu)$ such that $\mathcal{E}_K^\mu(\varrho,\varrho) = 0$, i.e. $\nabla_H^\mu \varrho = 0$ $\mu - a.e.$.

Case 1. Assume that $\varrho \geq c$ where $c \in \mathrm{R}$. Set $\varrho_0 := (\varrho - c)\cdot(\int(\varrho-c)\,d\mu)^{-1}$. Then $\int \varrho_0\,d\mu = 1$ and $|\nabla_H^\mu \varrho_0|_H = 0$. Define $\nu := \varrho_0\cdot\mu$. Then by (6.9) for all $v \in \mathcal{F}C_b^\infty$, $\ell \in K$,

$$\int \frac{\partial v}{\partial j_H(\ell)}\,d\nu = -\int \frac{\partial^\mu \varrho_0}{\partial j_H(\ell)}\,v\,d\mu - \int \varrho_0 v \beta_{j_H(\ell)}^\mu\,d\mu.$$

Since $\frac{\partial^\mu\varrho_0}{\partial j_H(\ell)} = 0$ $\mu - a.e.$ it follows that $\nu \in \mathcal{G}_{K,ac}^{\beta^\mu}$, hence $\mu = \nu$ by assumption, i.e., $\varrho_0 \equiv 1$, and thus $\varrho = const.$ $\mu - a.e.$.

Case2. ϱ arbitrary.

Since also $|\nabla_H^\mu|\varrho||_H = 0$ (because $(\mathcal{E}_K^\mu, D(\mathcal{E}_K^\mu))$ is a Dirichlet form) we conclude by Case1 that $|\varrho| = const.$ $\mu - a.e.$, in particular $\varrho \geq c$ $\mu - a.e.$ for some $c \in \mathrm{R}$. Applying Case1 again we obtain that $\varrho = const.$ $\mu - a.e.$ and hence $(\mathcal{E}_K^\mu, D(\mathcal{E}_K^\mu))$ is irreducible.

Now assume that $(\mathcal{E}_K^\mu, D(\mathcal{E}_K^\mu))$ is irreducible and let $\nu \in \mathcal{G}_{K,ac}^{\beta^\mu}$. Then $\nu = \varrho\cdot\mu$ for some $\varrho \in L^1(\mu)$, $\varrho \geq 0$, and for all $\ell \in K$, $\beta_{j_H(\ell)}^\mu = \beta_{j_H(\ell)}^\nu \in L^2(\nu)$, which is equivalent with $\varrho^{1/2}\beta_{j_H(\ell)}^\mu \in L^2(\mu)$. Hence by Lemma 6.14 and Remark 6.13 it follows that $\sqrt{\varrho} \in D(\mathcal{E}_K^\mu)$ and $|\nabla_H^\mu\sqrt{\varrho}|_H = 0$. Consequently, by assumption $\sqrt{\varrho} = const.$, thus $\varrho = const.$ $\mu - a.e.$. Hence $\nu = \mu$, i.e., $\mathcal{G}_K^{\beta^\mu} = \{\mu\}$.

(ii) : Assume that $(\mathcal{E}_K^\nu, D(\mathcal{E}_K^\nu))$ is irreducible for all $\nu \in \mathcal{G}_K^{\beta^\mu}$. Let $\nu \in \mathcal{G}_K^{\beta^\mu}$. Define $\mu_0 := \frac{1}{2}(\mu + \nu)$. Then $\mu_0 \in \mathcal{G}_K^{\beta^\mu}$ and $\nu \in \mathcal{G}_{K,ac}^{\beta^{\mu_0}}$ (where for μ_0 and $\ell \in K$ we choose and fix a Borel version of $\beta_{j_H(\ell)}^{\mu_0}$, which coincides with the chosen one of $\beta_{j_H(\ell)}^{\mu}$ everywhere on X). Hence by (i) and the assumption $\nu = \mu_0$, therefore $\nu = \mu$ and thus $\mathcal{G}_K^{\beta^\mu} = \{\mu\}$.

The converse trivially follows by (i). $\square$

Remark 6.16 (i) Since irreducibility is so crucial for our uniqueness problem we recall here some characterizations of it in terms of the semigroup $(T_t)_{t>0}$ and generator $(L, D(L))$ of $(\mathcal{E}_K^\mu, D(\mathcal{E}_K^\mu))$ on $L^2(\mu)$ (cf. [52, Chap.I] for this terminology). The following are equivalent:

1. $(\mathcal{E}_K^\mu, D(\mathcal{E}_K^\mu))$ is irreducible.
2. $(T_t)_{t>0}$ is irreducible, i.e., if $g \in L^2(\mu)$ such that $T_t(gf) = gT_tf$ for all $t > 0$, $f \in L^2(\mu)$, then $g = const.$
3. If $g \in L^2(\mu)$ such that $T_tg = g$ for all $t > 0$ then $g = const.$
4. $\int (T_tg - \int g\, d\mu)^2\, d\mu \underset{t\to\infty}{\longrightarrow} 0$ for all $g \in L^2(\mu)$.
5. If $u \in D(L)$ with $Lu = 0$, then $u = const.$

The corresponding proofs are entirely standard and essentially well-known. More details on this and applications to Gibbs states will be published in [13], [2] and [7].
(ii) We note that Theorem 6.15(i) implies that in the situation of Theorem 6.2 the corresponding Dirichlet form is irreducible.

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