Circular Coloring of Weighted Graphs

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Abstract

Suppose that G is a finite simple graph and w is a weight function which assigns to each vertex of G a nonnegative real number. Let C be a circle of length t. A t-circular coloring of (G, w) is a mapping Δ of the vertices of G to arcs of C such that $\Delta(x) \cap \Delta(y) = \emptyset$ if $xy \in E(G)$ and $\Delta(x)$ has length $w(x)$. The circular-chromatic number of (G, w) is the least t for which there is a t-circular coloring of (G, w). This paper discusses basic properties of circular chromatic number of a weighted graph and relations between this parameter and other graph parameters. We are particularly interested in graphs G for which the circular-chromatic number of (G, w) is equal to the fractonal clique weight of (G, w) for arbirtary weight function w. We call such graphs star-superperfect. We prove that odd cycles and the complement of odd cycles are star-superperfect. We then consider circular chromatic number of lexicographic product of graphs, which provides a tool of constructing new star-superperfect graphs from old ones.

1 Introduction

Let $G = (V, E)$ be a graph, and $w : V \to [0, \infty)$ be a weight function. Let C be a circle of length t in the Euclidean space R^2. A *t-circular coloring* of (G, w) is a mapping Δ of the vertices of G to open arc of C such that

1. if $(x, y) \in E$ then $\Delta(x)$ and $\Delta(y)$ are disjoint; and

2. for all vertices $x \in V$ the length of the arc $\Delta(x)$ is at least $w(x)$.

The *circular-chromatic number* $\chi_c(G, w)$ of (G, w) is defined as

$$\chi_c(G, w) = inf\{t : \text{there is a } \ t - \text{circular coloring of } (G, w)\}.$$

The circular coloring and circular chromatic number of weighted graphs can be used to model traffic control systems at traffic intersections. Each vertex of the graph represents a traffic flow, and two vertices are adjacent if the corresponding traffic flows are incompatible. The weight of a vertex may be proportional to the amount of traffic in the corresponding traffic flow. The traffic control system assigns to each traffic flow an interval of time in which the light is green. The pattern of the red-green lights will be repeated periodically. In this setting, the circle C then represents a whole period, and the arc assigned to a vertex represents the time interval in which the light is green for the corresponding traffic. As adjacent vertices are assigned to disjoint arcs, non-compatible traffic flows will not have green lights at the same time. It is then natural that we would like to minimize the total length of the period, that is the total length of the circle, subject to the condition that each vertex x is assigned an arc of pre-assigned length, $w(x)$. Circular graphs were used to model traffic control before, (cf. [5]). However, different objectives are approached there.

The parameter χ_c is closely related to many other graph parameters. First we observe that $\chi_c(G, w)$ can be regarded as a generalization of the star-chromatic number $\chi^*(G)$ to weighted graphs.

The star-chromatic number of a simple graph G is a natural generalization of the chromatic number of graph. It was first introduced by A. Vince in 1988, [8]. For integers $k \geq 2d$, a (k, d)-coloring of a graph G is a coloring Δ of the vertices of G with k colors $\{0, 1, 2, \cdots, k-1\}$ so that for any two adjacent vertices x, y we have $d \leq |\Delta(x) - \Delta(y)| \leq k - d$. The ordinary k-coloring is just $(k, 1)$-coloring of G. The star-chromatic number $\chi^*(G)$ of G is the minimum of the ratios k/d for which there is a (k, d)-coloring of G.

A result in [10] can be restated as that $\chi^*(G) = \chi_c(G, w)$, where w is the weight function which takes constant value 1. In this paper, we discuss some basic properties of the circular chromatic number of a weighted graph.

We shall prove that $\chi_c(G, w)$ can always be attained; and in case all the weights are rationals, so is the circular chromatic number. These results generalize some results in [2, 8, 10] about the star-chromatic number of a graph. We then discuss the relation among the circular chromatic number and two other parameters of weighted graphs: the interval chromatic number and the fractional chromatic number.

The definition of interval chromatic number of a weighted graph is similar to that of the circular chromatic number. The only difference is that instead of coloring vertices of G by arcs of a circle, we color them by sub-intervals of an interval, [5]. Let I be an interval of length t. A t-interval coloring of (G, w) is a mapping Δ of the vertices of G to open sub-intervals of I such that for each vertex x, the length of $\Delta(x)$ is equal to $w(x)$ and adjacent vertices are mapped to disjoint sub-intervals. The interval chromatic number $\chi(G, w)$ of (G, w) is the minimum number t for which there is a t-interval coloring of G.

As noted in [5], $\chi(G, w) = \chi(G)$ in case $w(x) = 1$ for all vertices x.

The fractional chromatic number of a weighted graph (G, w) is obtained by solving a linear programming problem. A *fractional coloring* of (G, w) is an assignment Δ of non-negative weights to independent sets X of G so that for each vertex $x \in V(G)$, we have $\Sigma_{x \in X} \Delta(X) \geq w(x)$. The *fractional chromatic number* $\chi_f(G, w)$ of (G, w) is the minimum total weight (i.e., the sum of the weights of all independent sets) of a fractional coloring of (G, w). Thus the fractional chromatic number of (G, w) can be obtained by solving a linear programming problem.

We note that an alternate definition of the fractional chromatic number of (G, w) can be obtained by slightly modifying the definition of the interval-chromatic number of (G, w). To be precise, we define a t-fractional coloring of (G, w) to be a mapping Δ of $V(G)$ to measurable subsets of I such that $\Delta(x)$ has measure $w(x)$ and adjacent vertices are mapped to disjoint subsets of I. Here I is an interval of length t. Then the fractional chromatic number of (G, w) is the minimum t for which there is a t-fractional coloring of (G, w). The equivalence of these two definitions are easy to prove, and a detailed proof for the unweighted case can be found in [12].

The dual of the linear programming above defines the *fractional clique weight* $\omega_f(G, w)$ of (G, w). The *clique weight* $\omega(G, w)$ is defined as the maximum total weight of a clique of G.

These parameters of a weighted graph (G, w) are related as follows:

$$\omega(G, w) \leq \omega_f(G, w) = \chi_f(G, w) \leq \chi_c(G, w) \leq \chi(G, w).$$

All the three inequalities follows from definitions. (For the second inequality, it is better to use the alternate definition of fractional chromatic number given above). The equality follows from the duality theorem of linear programming.

Interesting problems arose when one considers for which graphs, certain inequalities become equality. For example, those graphs G for which $\chi(G, w) = \omega(G, w)$ for all 0-1 valued wight functions are *perfect graphs*; and those graphs G for which $\chi(G, w) = \omega(G, w)$ for arbitrary weight functions are *superperfect graphs*, [3, 5].

When do the other inequalities become equalities? Are these inequalities related to each other? It is trivial that $\chi(G, w) = \omega(G, w)$ implies all the other inequalities become equalities. However there are also some other implications. By results of [2, 8], $\chi(G) = \lceil \chi^*(G) \rceil$. Thus for a 0-1 valued weight function w, $\chi_c(G, w) = \omega(G, w)$ if and only if $\chi(G, w) = \omega(G, w)$. Suppose G is a graph and X is a clique of G. Let $a_X : V(G) \to \{0, 1\}$ be the characteristic function of X, which can be regarded as a vector in $R^{|V|}$. Let $CLIQ(G)$ be the convex hull of all vectors a_X such that X is a clique of G. Let $QCLIQ(G)$ be the convex set $\{a \in R^{|V|} : \Sigma_{x \in S} a(x) \leq 1$ for all independent sets S of $G\}$. It was known (cf. [7]) that a graph G is perfect if and only if $CLIQ(G) = QCLIQ(G)$. It is easy to see that $CLIQ(G) = QCLIQ(G)$ if and only if $\omega(G, w) = \omega_f(G, w)$ for all weight functions w. Therefore $\chi(G, w) = \omega(G, w)$ for all 0-1 valued weight functions w if and only if $\omega(G, w) = \omega_f(G, w)$ for all weight functions w. Very likely there are other implications. For example, the strong perfect graph conjecture implies that $\chi_c(G, w) = \chi(G, w)$ for all 0-1 valued weight functions if and only if $\chi(G, w) = \omega(G, w)$ for all 0-1 valued weight functions. It is also probably true that $\chi_c(G, w) = \chi(G, w)$ for arbitrary weight functions if and only if $\chi(G, w) = \omega(G, w)$ for arbitrary weight functions.

This paper concentrates on inequalities concerning $\chi_c(G, w)$. We say a graph G is star-perfect if $\chi_c(G, w) = \omega_f(G, w)$ for all 0-1 valued weight functions; and say G is star-superperfect if $\chi_c(G, w) = \omega_f(G, w)$ for arbitrary weight functions. We shall prove that cycles and the complement of odd cycles are star-superperfect. Thus there are star-superperfect graphs which are not perfect. By considering the circular chromatic number of products of weighted graphs, we can construct new star-superperfect graphs from old ones.

Suppose (G, w) is a weighted graph, and X is a subset of $V(G)$. We denote by $w(X)$ the total weight of X, which is equal to the sum of the weights of the vertices in X.

2 Circular chromatic number

The circular chromatic number $\chi_c(G, w)$ of a weighted graph is defined as the infimum of $\{t :$ there is a t-circular coloring of $(G, w)\}$. First we show that the infimum is attained.

Theorem 1 *Let (G, w) be a weighted graph with vertex set $\{x_1, x_2, \cdots, x_n\}$. If $\chi_c(G, w) = t$, then there is a t-circular coloring of (G, w).*

Proof. Suppose $\chi_c(G, w) = t$. By definition there is a decreasing sequence $t_j \to t$ (as j goes to ∞) such that for each j, there is a t_j-circular coloring Δ_j of (G, w). Each Δ_j is a mapping of $V(G)$ to open arcs of C_j, where C_j is a circle of length t_j, such that $\Delta(x)$ has length $w(x)$ for each vertex x and $\Delta(x) \cap \Delta(y) = \emptyset$ if $(x, y) \in E(G)$. By evenly "shrinking" the circles C_j to a circle C of length 1, we obtain mappings Δ'_j which map $V(G)$ to arcs of C such that $\Delta'_j(x)$ has length $w(x)/t_j$ and $\Delta'_j(x) \cap \Delta'_j(y) = \emptyset$ if $(x, y) \in E(G)$. Consider the sequence $\{\Delta'_j(x_1), j = 1, 2, \cdots\}$ of arcs of C. There exits a convergent subsequence, say $\{\Delta'_{1j} : j = 1, 2, \cdots\}$. Let $\Delta'(x_1)$ be the limit arc, whose length is the limit of $\lim_{j\to\infty} w(x_1)/t_{1j} = w(x_1)/t$. In case the limit arc is a closed, or half closed interval, we simply delete the end point from it to make it open. Then we consider the sequence $\{\Delta'_{1j}(x_2) : j = 1, 2, \cdots\}$ and choose a convergent subsequence, say $\{\Delta'_{2j} : j = 1, 2, \cdots\}$, and let $\Delta'(x_2)$

be the limit (open) arc, which has length $w(x_2)/t$. Similarly, we can define $\Delta'(x_3), \Delta'(x_4), \cdots, \Delta'(x_n)$. Then $\Delta'(x) = w(x)/t$ and it is a routine to verify that $\Delta'(x) \cap \Delta'(y) = \emptyset$ when $(x, y) \in E(G)$. By "stretching" the circle C to a circle of length t, we then obtain a t-circular coloring of (G, w). Then Δ' induces a t-circular coloring of (G, w). ∎

Theorem 2 *Let (G, w) be a weighted graph with vertex set $\{x_1, x_2, \cdots, x_n\}$. If $w(x_i)$ are rationals, then $\chi_c(G, w)$ is also a rational. Moreover if $w(x_i) = a_i/b$ have a common denominator then $\chi_c(G, w) = a/c$ for some $a \leq \Sigma_{i=1}^{n} a_i$.*

Proof. Suppose that $w(x_i) = a_i/b$ are rationals with a common denominator b, and that $\chi_c(G, w) = t$. Let Δ be a t-circular coloring of (G, w). Thus Δ maps each vertex x of G to an arc $\Delta(x)$ of a length t circle C. For each arc $\Delta(x)$, we label its two end points "left" and "right" respectively so that the direction goes from left to right along the arc is the clockwise direction.

Claim: There is a sequence $\Delta(x_{\alpha_1}), \Delta(x_{\alpha_2}), \cdots, \Delta(x_{\alpha_k})$ of distinct arcs such that the right end point of $\Delta(x_{\alpha_i})$ is the left end point of $\Delta(x_{\alpha_{i+1}})$, and the right end point of $\Delta(x_{\alpha_k})$ is the left end point of $\Delta(x_{\alpha_1})$.

Proof of the claim: Construct a digraph D on vertices $\{x_1, x_2, \cdots, x_n\}$ by putting a directed edge from x_i to x_j if and only if the right end point of $\Delta(x_i)$ is the left end point of $\Delta(x_j)$.

If the claim is not true, then D has no directed cycles. We define the level $\ell(x_i)$ of a vertex x_i to be the length of the longest directed path in D which ends at x (we use the convention that the length of a directed path is the number of vertices of the path). Let $\delta = \min\{d(p, p') : p, p'$ are distinct end points of the arcs $\Delta(x_i), i = 1, 2, \cdots, n\}$. For each vertex x_i, we move the arc $\Delta(x_i)$ to the left a distance $1/2^{\ell(x_i)} \cdot \delta$, and thus obtain a new mapping Δ' of $V(G)$ to the arcs of C.

We now show that Δ' is also a t-circular coloring of (G, w). Certainly the length of the arc $\Delta'(x)$ is equal to $w(x)$, since it has the same length as $\Delta(x)$. Suppose (x_i, x_j) is an edge of G, then $\Delta(x_i)$ and $\Delta(x_j)$ are disjoint. If $\Delta(x_i)$ and $\Delta(x_j)$ have no common end points, then by the choice of δ, it is obvious that $\Delta'(x_i)$ and $\Delta'(x_j)$ are still disjoint. If $\Delta(x_i)$ and $\Delta(x_j)$ do have a common end point, then either the right end point $\Delta(x_i)$ is the left end point of $\Delta(x_j)$ or the right end point of $\Delta(x_j)$ is the left end point $\Delta(x_i)$. Without loss of generality we assume that the right end point $\Delta(x_i)$ is the left end point of $\Delta(x_j)$. Then $\ell(x_i) > \ell(x_j)$. Thus $\Delta(x_i)$ is moved further to the left than $\Delta(x_j)$. Therefore $\Delta'(x_i)$ and $\Delta'(x_j)$ are still disjoint. So Δ' is indeed a t-circular coloring of G.

Moreover, the above argument also shows that the right end point of $\Delta'(x)$ is not the left end point of any other arcs $\Delta'(y)$. Let $\delta' = \min\{d(p, p') : p, p'$ are distinct end points of the arcs $\Delta'(x_i), i = 1, 2, \cdots, n\}$. Let A be any arc of C of length δ' which contains no end points. We cut off A from C and joining the cuts to obtain a new circle C' of length $t - \delta'$. For any arc $\Delta'(x_i)$, if $\Delta'(x)$ contains A, then we move the right end point of $\Delta'(x)$ to the right a distance δ', while fixing the left end point. Since the right end point of $\Delta'(x)$ is not the left end point of any other arcs $\Delta'(y)$, this will not creat any new intersections among the arc $\Delta'(x_i)$ by the choice of δ'. Thus we obtain a $t - \delta'$-circular coloring of G, contrary to the assumption that $\chi_c(G, w) = t$. The claim is proved.

Let $\Delta(x_{\alpha_1}), \Delta(x_{\alpha_2}), \cdots, \Delta(x_{\alpha_k})$ be distinct arcs such that the right end point of $\Delta(x_{\alpha_i})$ is the left end point of $\Delta(x_{\alpha_{i+1}})$, and the right end point of $\Delta(x_{\alpha_k})$ is the left end point of $\Delta(x_{\alpha_1})$. Then the union of the k arcs $\Delta(x_{\alpha_1}), \Delta(x_{\alpha_2}), \cdots, \Delta(x_{\alpha_k})$ wind around the circle C exactly s times for some integer s. Thus the total length of the k arcs is st, and therefore $st = \Sigma_{i=1}^{k} w(x_{\alpha_i}) = \Sigma_{i+1}^{k} a_{\alpha_i}/b$. Hence $t = \Sigma_{i+1}^{k} a_{\alpha_i}/bs = p/q$, which is a rational and $p \leq \Sigma_{i+1}^{n} a_i$. ∎

As the star-chromatic number $\chi^*(G)$ of a graph G equals the circular chromatic number $\chi_c(G, w)$ where $w(x) = 1$ for each vertex x, we have the following

Corollary 1 ([2, 8]) *Let G be a graph on n vertices. Then $\chi^*(G)$ is a rational p/q and $p \leq n$.*

We noted in the previous section that $\chi_c(G, w) \leq \chi(G, w)$. The next theorem shows that $\chi_c(G, w)$ can not be much smaller than $\chi(G, w)$.

Theorem 3 *Let (G, w) be a weighted graph with vertices $\{x_1, x_2, \cdots, x_n\}$ such that $w(x_1) \geq w(x_2) \geq \cdots \geq w(x_n)$. If $\chi_c(G, w) = t$, then $\chi(G, w) < t - w(x_k)$, where i satisfies $\Sigma_{i=1}^{k-1} w(x_i) < t$.*

Proof. Let Δ be a t-circular coloring of (G, w). Since $\Sigma_{i=1}^{k-1} w(x_i) < t$, there is a point p on the circle C which is not contained in any of the arcs $\Delta(x_i), i = 1, 2, \cdots, x_{k-1}$. Cut the circle at p, we obtain an interval I of length t. We then view Δ as a mapping which sends vertices of G to sub-intervals of I. However for those vertices x for which $\Delta(x)$ contains p, the corresponding arcs are cutted into two pieces. Let r be the maximum length of all such pieces. By the choice of p, we know that $r < w(x_k)$. We extend I by adding at one of its end an interval of length r. Then we can move all those pieces at the other end to this added part of I, and thus obtain a $t + r$-interval coloring of (G, w). This completes the proof. ∎

3 Star-superperfect graphs

We call a graph G *star-perfect* if $\chi_c(G, w) = \omega_f(G, w)$ for every 0-1 valued weight functions, and call G *star-superperfect* if $\chi_c(G, w) = \omega_f(G, w)$ for arbitrary weight function w. As we noted before, if the maximum fractional clique weight $\omega_f(G, w)$ is replaced by the maximum clique weight $\omega(G, w)$, and the circular chromatic number $\chi_c(G, w)$ is replaced by the interval chromatic number $\chi(G, w)$, then we would have defined the class of perfect graphs and the class of superperfect graphs respectively.

First we note that if G is perfect, than G is star-perfect; and if G is superfect than G is star-superperfect. Theorems 4 and 5 below show that the converse is not true.

Theorem 4 *For every integer $k \geq 1$, the odd cycle C_{2k+1} is star-superperfect.*

Proof. Suppose the vertices of C_{2k+1} are $\{0, 1, \cdots, 2k\}$ and the edges are $\{(0, 1), (1, 2), \cdots, (2k - 1, 2k), (2k, 1)\}$. Let $w : \{0, 1, \cdots, 2k\} \to [0, \infty)$ be an arbitrary weight function. Let $\omega = \omega(C_{2k+1}, w) = \max\{w(i) + w(i + 1) : i = 0, 1, \cdots, 2k\}$ where $w(2k + 1) = w(0)$, and let $\tau = \tau(C_{2k+1}, w) = \frac{1}{k}\Sigma_{i=0}^{2k} w(i)$, and $t = \max\{\omega, \tau\}$. Then obviously $\omega_f = \omega_f(C_{2k+1}, w) \geq t$.

In the following we prove that (C_{2k+1}, w) is t-circular colorable. Note that this would imply that $\omega_f = \chi_c = t$, and hence C_{2k+1} is star-superperfect.

We first consider the case that $\tau \geq \omega$. Let C be a circle of length $t = \tau = 1/k \cdot \Sigma_{i=0}^{2k} w(i)$. Define a t-circular coloring Δ of C_{2k+1} as follows: Let $\Delta(0)$ be any arc of C of length $w(0)$. Suppose $\Delta(i)$ has been defined then let $\Delta(i+1)$ be the arc of length $w(i+1)$ whose left end point is the right end point of $\Delta(i)$. As $\Sigma_{i=0}^{2k} w(i) = kt$, the union of all the arcs $\Delta(i)$ winds around the circle C exactly k times. Thus the left end point of $\Delta(2k)$ is the right end point of $\Delta(0)$. Consider an edge $(i, i+1)$ of C_{2k+1} (if $i = 2k$, then $i+1 = 0$). The left end point of $\Delta(i)$ is the right end point of $\Delta(i+1)$. Since $w(i) + w(i+1) \leq t$, $\Delta(i)$ and $\Delta(i+1)$ are disjoint. Therefore Δ is a t-circular coloring of (C_{2k+1}, w). So the claim is true in this case.

Now suppose that $\tau < \omega$. Let $a = kt - \Sigma_{i=0}^{2k} w(i) = k(\omega - \tau) > 0$, and let $b = \Sigma_{i=0}^{2k}(\omega - (w(i) + w(i+1)))$. Then $b = (2k+1)\omega - 2\Sigma_{i=0}^{2k} w(i) = \omega + 2a > 2a$. We choose $2k+1$ real numbers $\delta_0, \delta_1, \delta_{2k}$ such that $0 \leq \delta_i \leq \omega - (w(i) + w(i+1))$ and $\Sigma_{i=0}^{2k} = a$. Let C be a circle of length $t = \omega$, and define a mapping Δ from the vertices of C_{2k+1} to arcs of C as follows: Let $\Delta(0)$ be an arbitrary arc of C of length $w(0)$. Suppose $\Delta(i)$ has been defined, we then let $\Delta(i+1)$ be the arc of C of length $w(i+1)$ whose right end point has distance δ_i from the left end point of $\Delta(i)$. Since $w(i) + w(i+1) + \delta_i \leq t$ we know that $\Delta(i)$ and $\Delta(i+1)$ are disjoint, for $i = 0, 1, \cdots, 2k-1$. Since $\Sigma_{i=0}^{2k} w(i) + \delta_i = kt$, and $\delta_{2k} + w(0) + w(2k) \leq t$ we also have $\Delta(0)$ and $\Delta(2k)$ are disjoint. Therefore Δ is a t-circular coloring of (C_{2k+1}, w). ∎

Theorem 5 *For any integer k, the complement of the odd cycle C_{2k+1} is star-superperfect*

Suppose C_{2k+1} is an odd cycle with vertices $\{0, 1, \cdots, 2k\}$ and edges $\{(0,1), (1,2), \cdots, (2k-1, 2k), (2k, 1)\}$. Let $G = \overline{C_{2k+1}}$ be the complement. Thus $(i, j) \in E(G)$ if and only if $2 \leq |i - j| \leq 2k - 1$. Let w be a weight function of G. Let $\omega(G, w)$ be the maximum clique weight of G and let $\tau(G, w) = \frac{1}{2} w(V(G))$.

Before proving Theorem 5, we first prove the following lemma.

Lemma 1 *Let G be the complement of C_{2k+1}, and w be a weight function, and let $\tau(G, w), \omega(G, w)$ be defined as above. If $\tau(G, w) < \omega(G, w)$, then there is a vertex, say i, of G for which any clique X of G containing i has weight $w(X) < \omega$.*

Proof. Let $t = \omega(G, w)$. Suppose to the contrary that every vertex of G is contained in a clique of weight t. Let $X = \{i_1, i_2, \cdots, i_s\}$ be a clique of weight t. Assume that $i_1 < i_2 < \cdots < i_s$. We define a *segment* of X to be a subset of circular consecutive vertices of X, i.e., subsets of the form $\{i_u, i_{u+1}, \cdot, i_{u+r},$ or of the form $\{i_u, i_{u+1}, \cdots, i_s, i_1, \cdots, i_r\}$. (We assume that the additions in the indices are carried out modular s, then the second form is the same as the first one.) For any other clique Y of weight t, let $\phi(X, Y)$ be the size of a largest segment of X contained in $X - Y$. Let X, X' be two cliques of weight t for which $\phi(X, X') \geq \phi(Y, Y')$ for any cliques Y, Y' of weight t.

First we note that $X - X' \neq \emptyset$, for otherwise $X \cap X' = \emptyset$ and $w(V(G)) = \Sigma_{i=0}^{2k} w(i) \geq \Sigma_{x \in X} w(x) + \Sigma_{x' \in X'} w(x') = 2t$, contrary to our assumption that $\tau(G, w) < \omega(G, w)$.

Assume that the largest segment of X in $X - X'$ is $U = \{i_u, i_{u+1}, \cdots, i_{u+r}\}$. Then $i_{u-1}, i_{u+r+1} \in X \cap X'$. (Additions and substractions in the indices are carried out modular s). Let A be the set of those vertices of X' contained in the set $\{i_{u-1} + 1, i_{u-1} + 2, \cdots, i_{u+r+1} - 1\}$, and let $B = X' - A$. (Additions and substractions in the vertices are carried out modular $2k + 1$.)

Let $V = X - U$. If $w(A) < w(U)$ then $w(B) > w(V)$ as $w(A) + w(B) = w(U) + w(V) = t$. It is easy to see that $B \cup U$ is a clique of G. However $w(B \cup U) > w(V \cup U) = w(X) = t$, contrary to our assumption that $t = \omega(G, w)$. Similarly if $w(A) > w(U)$ then we obtain a clique $A \cup V$ of G which has weight greater than t. Therefore we have $w(A) = w(U)$ and $w(B) = w(V)$. So without loss of generality, we can assume that $B = V$, i.e., $X = U \cup V$, $X' = A \cup V$.

We now consider the vertex $i_{u-1} - 1$. By our assumption, there is a clique, say Z, of G of weight t which contains $i_{u-1} - 1$. Then certainly $i_{u-1} \notin Z$, as $i_{u-1} - 1$ is not adjacent to i_{u-1}.

Note that $Z \cap U \neq \emptyset$. For otherwise $A \cup \{i_{u-1}\} \subset X - Z$ is contained in a segment of X. This implies that $\phi(X, Z) > \phi(X, X')$, contrary to our choice of X, X'.

Let $j \in Z \cap (A \cup U)$ be a vertex in the intersection of Z and $A \cup U$ such that $\{i_{u-1} + 1, i_{u-1} + 2, \cdots, j - 1\} \cap (Z \cap (A \cup U)) = \emptyset$. In other words, j is the first vertex of Z in $A \cup U$ (if traversed from i_{u-1} towards i_{u+r}).

We first consider the case that $j \in U$. Let $W \subset Z - X$ be the maximal segment of $Z - X$ containing $i_{u-1} - 1$ and let $W' \subset X - Z$ be the maximal segment of $X - Z$ containing i_{u-1}. Similar to the argument in the fourth previous paragraph, we can show that $Z' = W \cup (X - W')$ is a clique of G of weight t. It is routine to verify that $\phi(Z', X') > \phi(X, X')$, contrary to our choice of X and X'.

The case that $j \in A$ is similar. We can combine a segment of Z with a segment of X' to form a new clique, say Z'' for which $\phi(X, Z'') > \phi(X, X')$, and thus obtain a contradiction. This completes the proof of the lemma. ∎

Proof of Theorem 5 Let $t = \max\{\omega(G, w), \tau(G, w)\}$. Obviously $\omega_f(G, w) \geq \omega(G, w)$. Since each independent set of G has cardinality ≤ 2, the assignment $x_i = 1/2, i = 0, 1, \cdots, 2k$ satisfies the constraints $\Sigma_{x \in X} x_i \leq 1$ for all independent sets X of G. Thus $\omega_f(G, w) \geq \frac{1}{2} w(V(G)) = \tau(G, w)$. Therefore $\omega_f(G, w) \geq t$. We claim that G is t-circular colorable. As in the case of odd cycles, this implies that $\omega_f(G, w) = \chi_c(G, w) = t$ and that G is star-superperfect, as w is an arbitrary weight function.

We consider two cases.

Case 1: Supppose first that $\omega(G, w) \leq \tau(G, w)$. Then $t = \tau(G, w)$. Let C be a circle of length t. We define a mapping Δ from the vertices of G to arcs of C as follows: Let $\Delta(0)$ be an arbitrary arc of C of length $w(0)$. Suppose $\Delta(i)$ has been defined, then let $\Delta(i+2)$ be the arc of length $w(i+2)$ of C whose right end point is the left end point of $\Delta(i)$. Here the addition of the indices are carried out modular $2k + 1$. Since 2 and $2k + 1$ are coprime, $\Delta(i)$ is defined for all $i = 0, 1, \cdots, 2k$. We observe that since $\Sigma_{i=0}^{2k} w(i) = 2t$, the unions of all the arcs $\Delta(i)$ winds around the circle C exactly twice. Thus the right end point of $w(0)$ is the left end point of $\Delta(2k - 1)$. Now we show that Δ is a t-circular coloring of G. By definition we have $\Delta(i)$ is an arc of length $w(i)$. Thus we only need to show that if (i, j) is an edge of G then $\Delta(i)$ and $\Delta(j)$ are disjoint. Suppose that there are two adjacent vertices i, j for which $\Delta(i)$ and $\Delta(j)$ intersect each other. Without loss of generality, we asume that $i < j$. If $j - i$ is even, then $i, i + 2, i + 4, \cdots, j$ forms a clique of G. However, the right end point of $\Delta(i+2)$ is the left end point of $\Delta(i)$, the right end point of $\Delta(i+4)$ is the left end point of $\Delta(i+2), \cdots$, and the right end point of $\Delta(j)$ is the left end point of $\Delta(j - 2)$. Since $\Delta(j)$ intersects $\Delta(i)$, we see that the union of the arcs $\Delta(i), \Delta(i + 2), \Delta(i + 4), \cdots, \Delta(j)$

covers the whole circle C more than once. Therefore the sum of the lengths of these arcs is greater than t. On the other hand, the sum of the lengths of these arcs is just the total weight of the clique $\{i, i+2, \cdots, j\}$, contrary to our assumption that $\omega(G, w) \leq t$. If $j - i$ is odd, then $j, j+2, j+4, \cdots, i$ is a clique of G, and a similar argument applies to this case. Thus Δ is a t-circular coloring of (G, w) and hence $\chi_c(G, w) = t = \omega_f(G, w)$.

Case 2: Suppose now that $\tau(G, w) < \omega(G, w)$. By Lemma 1, there is a vertex i of G such that every clique X of G containing i has weight $w(X) < t$. Let $\delta = \min\{t - W(X) : X$ is a clique of G containing $i\}$. Let $\epsilon = \min\{\delta, t - \tau(G, w)\}$. We define a new weight function w' on G as follows: $w'(j) = w(j)$ if $j \neq i$, and $w'(i) = w(i) + \epsilon$. Since $\epsilon > 0$, we have $\tau(G, w') > \tau(G, w)$. By the choice of δ and ϵ, we have $\omega(G, w') = \omega(G, w) = t$. If $\tau(G, w')$ is still less than $\omega(G, w') = t$, we may apply Lemma 1 again. In the end, we will obtain a weight function w^* of G such that $w^*(i) \geq w(i)$ for each vertex of G, and $\omega(G, w^*) = \tau(G, w^*) = t$. Thus by Case 1, we know that (G, w^*) is t-circular colorable. Since $w^*(i) \geq w(i)$ for all vertices i of G, any t-circular coloring of (G, w') induces a t-circular coloring of (G, w). This completes the proof of Theorem 5.

4 Lexicographic product

In this section, we discuss the circular chromatic number of lexicographic product of graphs. This product will allow us to construct new star-superperfect graphs from old ones.

Suppose (G, w) and (G', w') are weighted graphs. Then the lexicographic product $(G[G'], ww')$ is the weighted graph with vertex set $\{(x, y) : x \in V(G), y \in V(G')\}$ and (x, y) is adjacent to (u, v) if either x is adjacent to u, or $x = u$ and y is adjacent to v. The weight of (x, y) is $w(x)w'(y)$. The graph $G[G']$ can be regarded as obtained from G by replacing each vertex of G by a copy of G'. In this section, we consider how the circular chromatic number of $(G[G'], ww')$ is related to the circular chromatic number of (G, w) and that of (G', w').

Theorem 6 *Suppose (G, w) is a weighted graph for which the non-zero* *weight vertices induces a graph without isolated vertices. Then for any weighted graph (G', w'), we have $\chi_f(G, w)\chi(G', w') \leq \chi_c(G[G'], ww') \leq \chi_c(G, w)\chi(G', w')$.*

Proof. First we prove that $\chi_f(G, w)\chi(G', w') \leq \chi_c(G[G'], ww')$.

Suppose $\chi_c(G, w)\chi(G', w') = t$ and let Δ be a t-circular coloring of $(G[G'], ww')$. Thus Δ sends each vertex of $G[G']$ to an arc of a length t circle C. For each point p of C, let S_p be the set of vertices $x \in V(G)$ such that $p \in \Delta(x, y)$ for some $y \in V(G')$. Obviously S_p is an independent set of G. We say an interval A of C is an *elementary interval* if A is a maximal interval with the property that $S_p = S_{p'}$ for any two points $p, p' \in A$. If A is an elementary interval and $p \in A$, then we associate the set S_P to the elementary interval A. Note that an independent set of G could be asssociated to more than one elementary interval, or none elemenatary interval, and distinct elementary intervals are disjoint. We now define a mapping c from all the independent sets of G to $[0, \infty)$ by setting $c(S) = \Sigma\ell(A)$ where $\ell(A)$ is the length of the interval A and the sum is over all the elementary intervals associated to the independent set S. (In case S is associated to no elementary intervals, $c(S) = 0$).

For a vertex $x \in V(G)$, we consider the sum $\Sigma_{x\in S}c(S) \geq w(x)\chi(G', w')$. It is easy to see that this sum is equal to the sum of the lengths of all the arcs $\Delta(x, y), y \in V(G')$. Since there is a vertex of nonzero weight which is adjacent to x, the union of this arcs $\cup_{y\in V(G')}\delta(x, y)$ does not cover the whole circle C. Thus we may consider the union of these arcs as an interval I, and regard Δ as a mapping which sends each vertex $y \in V(G')$ to the subinterval $\Delta(x, y)$, which has length $w(x)w'(y)$. Therefore the total length of the interval is at least $w(x)\chi(G', w')$. Thus $\Sigma_{x\in S}c(S) \geq w(x)\chi(G', w')$. By definition, we then have $\Sigma c(S) \geq \chi_f(G, w)\chi(G', w')$, where the sum is over all the independent sets of G. On the other hand, the union of all the elementary intervals is the circle C, thus $\Sigma c(S) = t$. Therefore $t \geq \chi_f(G, w)\chi(G', w')$.

We now prove that $\chi_c(G[G'], ww') \leq \chi_c(G, w)\chi(G', w')$. Suppose $\chi_c(G, w) = r$, and let Δ be an r-circular coloring of (G, w), i.e., Δ is a mapping which sends each vertex x of G to an arc of length $w(x)$ of a circle C, which has length r. We now stretch the circle C to a circle C' of length $r \cdot \chi(G', w')$. Then the arc $\Delta(x)$ of C is stretched to an arc $\Delta'(x)$

of C' of length $w(x)\chi(G',w')$. We then define a mapping Δ_x which sends each vertex y of G' to an subinterval of $\Delta'(x)$ such that $\Delta_x(y)$ has length $w(x)w'(y)$, and adjacent vertices of G' are send to disjoint subintervals. Such a mapping exists by the definition of the interval chromatic number. Now the mapping Δ^* maps each vertex (x,y) of $G[G']$ to the arc $\Delta_x(y)$ of C' is a $\chi_c(G,w)\chi(G',w')$-circular coloring of $(G[G'],ww')$. ∎

Corollary 2 *If $\chi_f(G,w)=\chi_c(G,w)$ and the subgraph of G induced by non-zero weighted vertices contains no isolated vertices, then for any weighted graph (G',w') we have $\chi_c(G[G'],ww')=\chi_c(G,w)\cdot\chi(G',w')$.*

We note the condition that the graph induced by non-zero weighted vertices contains no isolated vertices is necessary. For example, if G contains a single vertex x, then it is easy to see that $\chi_c(G[G'],ww') = \chi_c(G,w)\chi_c(G',w') = w(x)\chi(G',w')$. In case there are isolated vertices in the graph induced by non-zero weighted vertices, the resulting product weighted graph $(G[G'],ww')$ contains components corresponding to these isolated vertices. Suppose that $X=\{x_1,x_2,\cdots,x_k\}$ is the set of isolated vertices in the graph induced by non-zero weighted vertices of (G,w), and that $w(x_1)\geq w(x_i)$ for all $i=1,2,\cdots,k$. Let $G''=G-X$. By using Theorem 6, and observe that those components corresponding of $(G[G'],ww')$ corresponding to isolated vertices have circular chromatic number equals to $w(x_1)\chi_c(G',w')$, we see that $\chi_c(G[G'],ww')=\max\{\chi_c(G'',w)\chi(G',w'),w(x_1)\chi_c(G',w')\}$. In particular, if $\chi(G',w')=\chi_c(G',w')$ then the condition that "the subgraph induced by non-zero weighted vertices contains no isolated vertices" can be dropped from both Theorem 6 and Corollary 2.

If G is an odd cycle or the complement of an odd cycle, then G is star-superperfect by Theorems 4 and 5. Thus $\chi_c(G,w)=\omega_f(G,w)$ for arbitrary weight function w. If G' is any superperfect graph, then for any weight function w' we have $\chi(G',w')=\omega(G',w')$, which implies that $\chi_c(G',w')=\chi(G',w')$. Thus by the above Corollary, we have

Corollary 3 *For odd cycles C_{2k+1}, we have $\chi_c(C_{2k+1}[G'],ww') = \chi_c(C_{2k+1},w)\cdot\chi(G',w')$ and $\chi_c(\overline{C_{2k+1}}[G'],ww')=\chi_c(\overline{C_{2k+1}},w)\cdot\chi(G',w')$ for arbirtay weight function w and arbitrary weighted graph (G',w').*

The special case of Corollary 3 in which all the weight functions are constant functions was obtained in [4]. This special case generalizes a result of Stahl [9], which asserts that $\chi(C_{2k+1}[G']) = 2\chi(G') + \lceil \frac{\chi(G')}{k} \rceil$. (This result can be derived from Corollary 3 by noting that $\chi_c(C_{2k+1}) = 2 + \frac{1}{k}$ and by using Theorem 3.)

Lemma 2 *Let* (G, w) *and* (G', w') *be weighted graphs. Then* $\chi_f(G[G'], ww') = \chi_f(G, w)\chi_f(G', w')$.

The special case of this lemma for unweighted graphs was proved in [4]. We omit the proof of this lemma, as it is easy and similar to the proof for unweighted graphs.

Theorem 7 *If* G *is a star-superperfect graph, and* G' *is a superperfect graph, then* $G[G']$ *is a star-superperfect graph.*

Proof. By Corollary 2 and the comments below that corollary, for any weight functions w, w' of G and G' respectively, we have $\chi_c(G[G'], ww') = \chi_c(G, w)\chi(G', w')$. By Lemma 2, $\chi_f(G[G'], ww') = \chi_f(G, w)\chi_f(G', w')$. Since the fractional chromatic number of a weighted graph is the same as its fractional clique weight, we have $\omega_f(G[G'], ww') = \omega_f(G, w)\omega_f(G', w')$. Because G is star-superperfect, and G' is superperfect, we have $\chi_c(G, w) = \omega(G, w)$ and $\chi(G', w') = \omega(G', w')$. Therefore $\chi_c(G[G'], ww') = \omega_f(G[G'], ww')$. ∎

We close this section with some open problems:

Is it true that $\chi_c(G, w) = \chi(G, w)$ for arbitrary weight function if and only if G is superperfect?

Is it true that $\chi_c(G, w) = \chi(G, w)$ for every 0-1 valued weight function if and only if G is perfect?

For integers $k \geq 2d \geq 1$, let G_k^d be the graph with vertices $\{0, 1, \cdots, k-1\}$ and edges $\{(i, j) : d \leq |i - j| \leq k - d\}$. Are the graphs G_k^d star-superperfect?

It is known that to determine the circular chromatic number of a weighted graph is NP-complete. However suppose a graph G is star-superperfect, and w is a weight function. Can one determine the circular chromatic number of (G, w) in polynomial time?

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