On Joint Eigenvalues of Commuting Matrices

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September 28, 1994

Abstract

A spectral radius formula for commuting tuples of operators has been proved in recent years. We obtain an analog for all the joint eigenvalues of a commuting tuple of matrices. For a single matrix this reduces to an old result of Yamamoto.

1 Introduction, formulation of the result

Let $T = (T_1, \ldots, T_s)$ be an s-tuple of complex $d \times d$-matrices. The *joint spectrum* $\sigma_{pt}(T)$ is the set of all points $\lambda = (\lambda_1, \ldots, \lambda_s) \in C^s$ (called *joint eigenvalues*) for which there exists a nonzero vector $x \in C^d$ (called *joint eigenvector*) satisfying

$$T_j x = \lambda_j x \text{ for } j = 1, \ldots, s. \tag{1}$$

If the $T_i's$ are commuting then $\sigma_{pt}(T) \neq \emptyset$. The joint spectrum can be read off the diagonal of the common triangular form: There exists a unitary $d \times d$ - matrix U such that

$$U^H T_j U = \begin{pmatrix} \lambda_1^{(j)} & \ldots & \ldots & \ldots \\ 0 & \lambda_2^{(j)} & \ldots & \ldots \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_d^{(j)} \end{pmatrix} \text{ for } j = 1, \ldots, s. \tag{2}$$

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Then

$$\sigma_{pt}(T) = \{\lambda_i = (\lambda_i^{(1)}, \dots, \lambda_i^{(s)}) : i = 1, \dots, d\}.$$

We order the joint eigenvalues according to their norms

$$||\lambda_1|| \geq \dots \geq ||\lambda_d||. \tag{3}$$

Here $||.||$ denotes the Euclidean norm in C^r and will later on also denote the associated operator norm for matrices. We omit the reference to the dimensions.

The s-tuple T can be identified with a linear operator mapping C^d into C^{sd}. If $S = (S_1, \dots, S_m)$ is another m-tuple of $d \times d$-matrices, we define as TS the $sm-$tuple of matrices, whose entries are $T_i S_j, i = 1, \dots, s, j = 1, \dots m$, ordered lexicographically. Continuing in this way we define T^m, consisting of s^m entries, each of which is a product of m of the $T_i's$. Identifying again T^m with an operator mapping C^d into $C^{s^m d}$, T^m has d singular values

$$s_1(T^m) \geq s_2(T^m) \geq \dots \geq s_d(T^m). \tag{4}$$

In this note we will prove

Theorem 1 *For any s-tuple $T = (T_1, \dots, T_s)$ of commuting $d \times d$-matrices*

$$lim_{m\to\infty}(s_j(T^m))^{\frac{1}{m}} = ||\lambda_j|| \qquad j = 1, \dots, d. \tag{5}$$

For $j = 1$ this has been proved in [2] , hence we know

$$||\lambda_1|| = lim_{m\to\infty}(s_1(T^m))^{\frac{1}{m}}. \tag{6}$$

We also remark that (6) has been proved in [1] for l_p-norms and in [5] for infinite-dimensional Hilbert spaces. If $s = 1$ then T^m is the usual m-th power of $T = T_1$, and the joint spectrum is the usual spectrum. For this case (5) has been proved by Yamamoto [6], who showed that for a $d \times d-$matrix T with eigenvalues λ_i ordered according to their moduli

$$lim_{m\to\infty}(s_j(T^m))^{\frac{1}{m}} = |\lambda_j| \qquad j = 1, \dots, d. \tag{7}$$

We will prove Theorem 1 in the following section.

2 Proof of the Theorem

It is convenient to introduce a Kronecker-type matrix product "$\tilde{\otimes}$" in the following way:

Let A and B be two (r,s) and (t,u) block matrices

$$A = (A_{ij})_{i=1,\ldots,r,\ j=1,\ldots,s} \quad B = (B_{ij})_{i=1,\ldots,t,\ j=1,\ldots,u}$$

where the A_{ij} and B_{ij} are $d \times d$ matrices. Define

$$A_{ij} B = (A_{ij} B_{kl})_{k=1,\ldots,t,\ l=1,\ldots,u}$$

and the $rt \times su$ - block matrix

$$A \tilde{\otimes} B = \begin{pmatrix} A_{11}B & \ldots & A_{1s}B \\ \vdots & & \vdots \\ A_{r1}B & \ldots & A_{rs}B \end{pmatrix} \tag{8}$$

of dimension $rtd \times sud$. This product is associative. For $d = 1$ this is the usual Kronecker product, which we will denote by "$\otimes$", following the customary notation (see e.g. [4]). Except for $d = 1$ however $A\tilde{\otimes}B$ is different from $A \otimes B$ which is an $rtd^2 \times sud^2$ matrix. So the product depends on d. However in order to avoid an overload of indices and as we keep d fixed throughout, we refrained from stressing this fact in the notation.

The main relation for $\otimes$ carries over to $\tilde{\otimes}$, namely

$$(A\tilde{\otimes}B)(C\tilde{\otimes}D) = AC\tilde{\otimes}BD \tag{9}$$

if all the blocks in B commute with those in C, and the dimensions are fitting. For this it suffices that AC and BD can be formed. We observe that T^m, as defined in the first section, has the representation

$$T^m = T\tilde{\otimes}\ldots\tilde{\otimes}T$$

as the m-fold product of T with itself.

First we show that we can transform T to a simpler form without changing the magnitudes involved in (5). Then we prove the Theorem for this simple form using (6) and (7).

Let S be a nonsingular $d \times d$ - matrix,

$$\tilde{T}_i = ST_iS^{-1} \qquad i = 1, \ldots, s,$$

and

$$\tilde{T} = (\tilde{T}_1, \dots, \tilde{T}_s).$$

Obviously the $\tilde{T}_i' s$ commute too, and $\sigma_{pt}(\tilde{T}) = \sigma_{pt}(T)$. We show

$$s_i(\tilde{T}^m) \leq ||S|| \, ||S^{-1}|| s_i(T^m) \qquad i = 1, \dots, d, \tag{10}$$

which implies that the lefthand side of (5) is not changed if we replace T^m by $\tilde{T}^m$.

T^m consists of s^m blocks of $d \times d$ matrices $C_i, i = 1, \dots, s^m$, each of which is a product of m of the $T_i' s$. Hence the corresponding block $\tilde{C}_i$ of $\tilde{T}^m$ satisfies $\tilde{C}_i = SC_iS^{-1}$. Thus

$$\begin{aligned} (\tilde{T}^m)^H \tilde{T}^m &= \sum_{i=1}^{s^m} \tilde{C}_i^H \tilde{C}_i & (11) \\ &= (S^{-1})^H (\sum_{i=1}^{s^m} C_i^H S^H S C_i) S^{-1} & (12) \\ &\leq ||S||^2 (S^{-1})^H (T^m)^H T^m S^{-1} & (13) \end{aligned}$$

Here "$\leq$" is the Loewner partial ordering. Let $z \in C^d$ and $x = Sz$. The last inequality implies

$$\frac{x^H (\tilde{T}^m)^H \tilde{T}^m x}{x^H x} \leq ||S||^2 ||S^{-1}||^2 \frac{z^H (T^m)^H T^m z}{z^H z}. \tag{14}$$

Using the Courant-Fischer representation of the eigenvalues $\mu_1 \geq \ldots \geq \mu_d$ of a hermitean $d \times d$ matrix B (e.g. [4])

$$\mu_i = min_{dimV = d+1-i} \; max_{x \in V, x \neq 0} \frac{x^H B x}{x^H x}$$

for $B = (\tilde{T}^m)^H \tilde{T}^m$ and then for $B = (T^m)^H T^m$ and taking (14) into account, (10) follows.

Another transformation of T which doesn't change the numbers $||\lambda_i||$ is the following:

Given a unitary $s \times s$-matrix $U = (u_{ij})$, let $W = U \otimes I_d$, where I_d is the unit matrix of dimension d, and

$$\hat{T} = WT, \tag{15}$$

i.e.

$$\hat{T}_i = \sum_{j=1}^{s} u_{ij} T_j \quad i = 1, \dots, s$$

Then it is obvious that the joint spectrum of $\hat{T}$ is given by the vectors $\hat{\lambda}_i = U\lambda_i, \quad i = 1, \ldots, d$, where $\lambda_i \in \sigma_{pt}(T)$. Hence $||\hat{\lambda}_i|| = ||\lambda_i||, \quad i = 1, \ldots, d$. Also by using (9) we get

$$\begin{aligned} \hat{T}^m &= (WT)\tilde{\otimes}\ldots\tilde{\otimes}(WT) && (16) \\ &= (W\tilde{\otimes}\ldots\tilde{\otimes}W)(T\tilde{\otimes}\ldots\tilde{\otimes}T) && (17) \\ &=: W^{(m)}T^m. && (18) \end{aligned}$$

Again by (9) we see that $W^{(m)}$ defined in the last eaquation is a unitary mapping of $C^{s^m d}$ into itself, hence

$$s_i(\hat{T}^m) = s_i(T^m), \qquad i = 1, \ldots, d.$$

Having now assembled our tools, we invoke a result in ([3], Vol.I, p. 224), by which there exists a nonsingular $d \times d-$ matrix S and positive integers $s_1, \ldots, s_t$ with $\sum_{i=1}^t s_i = d$, such that

$$\tilde{T}_i = ST_iS^{-1} = diag(\tilde{T}_i^1, \ldots, \tilde{T}_i^t) \quad i = 1, \ldots, s,$$

where

$$\tilde{T}_i^\nu = \begin{pmatrix} \tilde{\lambda}_i^\nu & \ldots & \ldots \\ 0 & \ddots & \ldots \\ 0 & 0 & \tilde{\lambda}_i^\nu \end{pmatrix} \text{ for } i = 1, \ldots, s \quad \nu = 1, \ldots, t \tag{19}$$

is an $s_\nu \times s_\nu -$ matrix, upper triangular with constant diagonal. Observe that also $(\tilde{T}^m)^H \tilde{T}^m$ is block diagonal with $s_\nu \times s_\nu$ blocks. This shows that we have to prove (5) only for $T_i's$ of the form (19). Clearly then $||\lambda_1|| = \ldots = ||\lambda_d||$. Also by applying a suitable transformation of the form (15), we can assume that $T_2, \ldots, T_d$ have zero diagonals, while the diagonal of T_1 is $||\lambda_1||$.

Now from

$$(T^m)^H T^m \geq (T_1^m)^H T_1^m$$

we get

$$(s_1(T^m))^{\frac{1}{m}} \geq (s_i(T^m))^{\frac{1}{m}} \geq (s_d(T_1^m))^{\frac{1}{m}} \quad i = 1, \ldots, d.$$

But the leftmost term converges to $||\lambda_1||$ by (6), while the rightmost term converges to $min|\lambda_i(T_1)| = ||\lambda_1||$ by (7). Hence (5) holds for $i = 1, \ldots, d$.

This finishes the proof.

References

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