Regularity of invariant measures for a class of perturbed Ornstein–Uhlenbeck operators

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Abstract

In the framework of [5] we prove regularity of invariant measures μ for a class of Ornstein–Uhlenbeck operators perturbed by a drift which is not necessarily bounded or Lipschitz continuous. Regularity here means that μ is absolutely continuous with respect to the Gaussian invariant measure of the unperturbed operator with the square root of the Radon–Nikodym density in the corresponding Sobolev space of order 1.

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1 Introduction and main result

The purpose of this paper is to apply the main result in [3], see also [6], and a modification of a technique from [2] to prove regularity of invariant measures

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in the cases considered in [5] but without assuming that the non–linear part of the drift is bounded or Lipschitz continuous.

Consider the stochastic equation

$$\begin{cases} dX = (AX + F(X))dt + dW, \\ X(0) = x \in H \end{cases} \tag{1}$$

on a real separable Hilbert space H, with inner product $< \cdot, \cdot >$ and norm $|\cdot|$. We assume throughout this paper that A is a non negative definite self–adjoint operator on H such that A^{-1} is trace–class. Let $S(t) := e^{tA},\ t \geq 0$. For $F = 0$ it is known that (1) has a well defined H–valued mild solution given by

$$X(t) = S(t)x + \int_0^t S(t-s)dW(s), \tag{2}$$

which has as unique invariant (probability) measure the Gaussian measure $\mu_0 = \mathcal{N}(0, \frac{1}{2}A^{-1})$ (cf.[4]).

Suppose that the following condition holds

$$F \text{ is bounded and Lipschitz continuous.} \tag{3}$$

Then the integral version of (1), i.e.

$$X(t) = S(t)x + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)dW(s), t \geq 0, \tag{4}$$

has a unique solution $X(t,x),\ t \geq 0$,(cf.[4]). It also follows (cf.[5, Theorem 3.5]) that this solution has a unique invariant measure μ_F, μ_F is absolutely continuous with respect to μ_0 and

$$\frac{d\mu_F}{d\mu_0} \in W^{1,2}(H, \mu_0).$$

Here the Sobolev space $W^{1,2}(H, \mu_0)$ is defined as follows: Let $\mathcal{F}C_b^\infty(H)$ be the set of all functions φ on H of the form

$$\varphi(x) = f(< e_1, x >, ..., < e_n, x >),\ x \in H,$$

where $n \in \mathbb{N},\ f \in C_b^\infty(\mathbb{R}^n)$, and $\{e_j |\ j \in \mathbb{N}\}$ is an eigenbasis of A which we *fix from now on.* For $\varphi \in \mathcal{F}C_b^\infty(H)$ let φ' denote its Fréchet derivative, and for $x \in H$ let $D_x\varphi(x)$ denote the element in H corresponding to $\varphi'(x) \in H^*$. Then $W^{1,2}(H, \mu_0)$ is defined as the completion of $\mathcal{F}C_b^\infty(H)$ with respect to the norm given by

$$||\varphi||^2_{W^{1,2}(H,\mu_0)} = ||\varphi||^2_{L^2(H,\mu_0)} + \int_H |D_x\varphi(x)|^2\mu_0(dx). \tag{5}$$

Since each e_n is in the Cameon–Martin space (=reproducing kernel Hilbert space) of μ_0 it follows that $W^{1,2}(H,\mu_0) \subset L^2(H,\mu_0)$, (cf. [1]). Note that in [5] in the above definition $C_b^\infty(\mathbb{R}^n)$ was replaced by $\mathcal{P}_n,\ n \in \mathbb{N}$, i.e. the space of all polynomials in n variables. But since μ_0 is Gaussian the resulting space $W^{1,2}(H,\mu_0)$ is the same.

For $\varphi \in \mathcal{F}C_b^\infty(H)$ let φ'' denote its second Fréchet derivative, and for $x \in H$ let $D_{xx}\varphi(x)$ denote the corresponding element in $\mathcal{L}(H)$ (the set of all bounded linear operators on H) . Note that $D_{xx}\varphi(x)$ is trace class and that

$$D_x\varphi = \sum_{j=1}^{n} \partial_j f(< e_1, \cdot >, ..., < e_n, \cdot >)e_j, \tag{6}$$

if $\varphi = f(< e_1, \cdot >, ..., < e_n, \cdot >)$ and $\partial_j f(y) = \frac{\partial f}{\partial y_j}(y),\ y \in \mathbb{R}^n$. Hence $D_x\varphi(x) \in D(A)$ for all $x \in H$ and we can define

$$\mathcal{A}_0\varphi(x) = \ \text{Tr}\ [D_{xx}\varphi(x)] + \langle x, AD_x\varphi(x)\rangle,\ x \in H \tag{7}$$

and

$$\mathcal{A}_F\varphi(x) = \mathcal{A}_0\varphi(x) + \langle F, D_x\varphi(x)\rangle,\ x \in H. \tag{8}$$

By [5, Theorems 2.10 and 3.1] it follows that (since (3) holds)

$$\int_H \mathcal{A}_F\varphi(x)\mu_F(dx) = 0 \ \text{ for all } \varphi \in \mathcal{F}C_b^\infty(H), \tag{9}$$

and

$$\int_H < e_j, x >^2 \mu_F(dx) < +\infty \ \text{ for all } j \in \mathbb{N}. \tag{10}$$

Since F is bounded, we also have that

$$\int_H |F(x)|^2\mu_F(dx) < +\infty \ \text{ for all } j \in \mathbb{N}. \tag{11}$$

Note that $\mathcal{A}_F$ is well defined on $\mathcal{F}C_b^\infty(H)$ if $F : H \to H$ is merely Borel measurable. Hence (9)–(11) give rise to a definition of *generalized* invariant measures, in case (3) does not hold and (1) might not even have a solution.

Definition 1.1 *Let $F : H \to H$ be Borel measurable. A probability μ on $\mathcal{B}(H)$(:=Borel $\sigma-$ algebra on H) is called a* generalized invariant measure *if* (9)–(11) *hold with μ replacing μ_F. In this case we write $\mathcal{A}_F^*\mu = 0$.*

We note that if (10),(11) hold, then $\mathcal{A}_F\varphi \in L^2(H,\mu_F)$ and (9) is hence well–defined. In this paper we do not discuss the existence of measures satisfying $\mathcal{A}_F^*\mu = 0$, but we refer to [2, §5] in this respect. Here we are only interested in regularity properties of such μ. Our main result is the following:

Theorem 1.1 *Let $F : H \to H$ be Borel measurable and let μ be a probability measure on $\mathcal{B}(H)$ such that $\mathcal{A}_F^*\mu = 0$. Then $\mu = \rho\mu_0$ with $\sqrt{\rho} \in W^{1,2}(H, \mu_0)$.*

The proof will be given in the next section. It is based on three main ingedients. The first is that by [3]

$$W^{1,2}(H, \mu_0) \subset L^2(H, \mu_0) \text{ is compact.} \tag{12}$$

The second is Theorem 3.1 in [2], i. e. a regularity result in finite dimensions. The third ingredient is a modified version of a technique used in the proof of [2, Theorem 3.5] to "lift" [2, Theorem 3.1] to infinite dimensions. This "lifting" will be presented in detail and in a self–contained way below.
It should be noted that $\mathcal{F}C_b^\infty(H)$ and hence Definition 1.1 depend on the choice of $\{e_j \,| j \in \mathbb{N}\}$. But the conclusions in Theorem 1.1 hold for any such choice.

Finally, we would like to describe the connection of Theorem 1.1 with Theorem 3.5 in [2]. The latter is the situation studied by I. Shigekawa in [7] which (in the terminology used above) is the case where $\mathcal{A}_0$ in (7) has the form

$$\mathcal{A}_0\varphi(x) = \ \text{Tr}\,[QD_{xx}\varphi(x)] + \langle x, D_x\varphi(x)\rangle, \ x \in H$$

(i.e. $A = \ \text{id}_H)$. Here $Q \in \mathcal{L}(H)$ and it is trace class. Furthermore, F in the definition of $\mathcal{A}_F$ (cf. (8)) is then assumed to take values in $Q^{1/2}H$ such that $|Q^{-1/2}F| \in L^2(H, \mu_0)$, where μ_0 is the centered Gaussian measure on H with covariance operator Q (i.e., $(Q^{1/2}H, \langle Q^{-1/2}\cdot, Q^{-1/2}\cdot\rangle)$ is the Cameron–Martin = reproducing kernel Hilbert space of μ_0). In this case the embedding of the corresponding Sobolev space into $L^2(H, \mu_0)$ is not compact, but instead a log–Sobolev inequality holds which is essential in the proof of [2, Theorem 3.5].
For more details and the underlying background literature we refer to [2]. Both above results can be combined into one, and we think that extensions to more general situations are possible. This will be the subject of future research.

2 Proof of Theorem 1.2

Let $\mathcal{G}$ denote the closure of the set $\{D_x\varphi | \varphi \in \mathcal{F}C_b^\infty(H)\}$ in the Hilbert space $L^2(H \to H, \mu)$. Let F_0 denote the orthogonal projection of F onto $\mathcal{G}$ in $L^2(H \to H, \mu)$. Then

$$\int_H \langle F(x) - F_0(x), D_x\varphi(x)\rangle \mu(dx) = 0 \text{ for all } \varphi \in \mathcal{F}C_b^\infty(H),$$

consequently,

$$\mathcal{A}_{F_0}^*\mu = 0,$$

and we may hence assume without loss of generality that

$$F \in \mathcal{G}. \tag{13}$$

Let $-\alpha_n,\ n \in \mathbb{N}$, be the eigenvalues of A corresponding to the eigenbasis $\{e_n\ |n \in \mathbb{N}\}$ fixed in §1.

For $n \in \mathbb{N}$ let P_n be the orthogonal projection from H onto span$\{e_1, ..., e_n\}$ which we identify with $\mathbb{R}^n$, hence $P_n : H \to \mathbb{R}^n$ and the euclidean inner product on $\mathbb{R}^n$ is just $< \cdot, \cdot >$ (i.e. the one induced by $(H, < \cdot, \cdot >)$ on span$\{e_1, ..., e_n\}$). Let $E_\mu[\ \ |P_n]$ denote the conditional expectation of μ given the σ–algebra generated by P_n. For any $G \in L^2(H \to H, \mu)$ e.g. by [8, Ch. II, Theorem 4.1]

$$G_n := E_\mu[G|P_n] \overset{n\to\infty}{\to} G \ \text{ in } \ L^2(H \to H, \mu).$$

Hence

$$\begin{aligned}
&\int_H |G(x) - P_n G_n(x)|^2 \mu(dx)\\
&\le 2\int_H |G(x) - P_n G(x)|^2 \mu(dx) + 2\int_H |P_n G(x) - P_n G_n(x)|^2 \mu(dx)\\
&\le 2\int_H |G(x) - P_n G(x)|^2 \mu(dx) + 2\int_H |G(x) - G_n(x)|^2 \mu(dx) \overset{n\to\infty}{\to} 0.
\end{aligned}$$

Therefore, if

$$F_n^j := E_\mu[< e_j, F > |P_n],\ 1 \le j \le n,$$

and

$$F_n := (F_n^1, ..., F_n^n) \left(\equiv \sum_{j=1}^n F_n^j e_j \right),$$

then we have

$$F_n \overset{n\to\infty}{\to} F \ \text{ in } \ L^2(H \to H, \mu). \tag{14}$$

Fix $n \in \mathbb{N}$ and let $\widetilde{F_n^j} : \mathbb{R}^n \to \mathbb{R}^n$ be Borel–measurable such that $F_n^j = \widetilde{F_n^j} \circ P_n,\ 1 \le j \le n$, and set

$$\widetilde{F_n} := (\widetilde{F_n^1}, ..., \widetilde{F_n^n}).$$

Let $\mu_n := \mu \circ P_n^{-1}$ and

$$\mathcal{G}_n := \ \text{closure of } \{\nabla u |\ u \in C_b^\infty(\mathbb{R}^n)\} \ \text{ in } L^2(\mathbb{R}^n \to \mathbb{R}^n, \mu_n). \tag{15}$$

Set

$$\overline{\alpha_n} := (\alpha_1, ..., \alpha_n).$$

Claim 1 — $\mu_n = p_n dx$(where dx denotes Lebesgue measure on $\mathbb{R}^n$) with $\sqrt{p_n} \in W^{1,2}(\mathbb{R}^n, dx)$ and $\frac{\nabla p_n}{p_n}$ is the orthogonal projection of $(-\overline{\alpha_n}, \cdot) + \widetilde{F}_n$ onto $\mathcal{G}_n$ in $L^2(\mathbb{R}^n \to \mathbb{R}^n, \mu_n)$.

To prove this claim define for $u \in C_b^\infty(\mathbb{R}^n)$

$$L_n u = \sum_{i=1}^n \partial_i^2 u + \left\langle -\overline{\alpha_n} + \widetilde{F}_n, \nabla u \right\rangle.$$

Note that for $1 \leq j \leq n$ by assumption on μ

$$\begin{aligned} \int_{\mathbb{R}^n} (\alpha_j z_j)^2 \mu_n(dz) &= \alpha_j^2 \int_H \langle e_j, P_n x\rangle^2 \mu(dx) \\ &= \alpha_j^2 \int_H \langle e_j, x\rangle^2 \mu(dx) < \infty, \end{aligned} \tag{16}$$

$$\begin{aligned} \int_{\mathbb{R}^n} (\widetilde{F}_n^j(z))^2 \mu_n(dz) &= \int_H \left(E_\mu\left[\langle e_j, F\rangle | P_n\right]\right)^2(x)\mu(dx) \\ &= \int_H \langle e_j, F(x)\rangle^2 \mu(dx) < \infty, \end{aligned} \tag{17}$$

and for all $u \in C_b^\infty(\mathbb{R}^n)$, if $\varphi := u \circ P_n$,

$$\begin{aligned} &\int_{\mathbb{R}^n} L_n u(x)\ \mu_n(dx) = \int_H \Big(\ \mathrm{Tr}\ D_{xx}\varphi(x) \\ &+\sum_{j=1}^n \Big[-\alpha_j \partial_j f(< e_1, x >, \cdots, < e_n, x >) < e_j, x > \\ &+E_\mu\left[\langle e_j, F\rangle | P_n\right](x) \langle e_j, D_x\varphi(x)\rangle \Big]\Big)\mu(dx), \\ &= \int_H L\varphi(x)\ \mu(dx) = 0, \end{aligned} \tag{18}$$

where we used (6) in the second last step. By [2, Theorem 3.1] (16) –(18) imply the assertion of Claim 1.

For the next step we note that if $\mu_{0,n} := \mu_0 \circ P_n^{-1}$, then

$$\mu_{0,n} = q_n \cdot dx \tag{19}$$

with

$$q_n(z) := (2\pi\alpha_1 \cdots \alpha_n)^{-n/2} e^{-\frac{1}{2}\sum_{i=1}^n \alpha_i z_i^2},\ z \in \mathbb{R}^n. \tag{20}$$

Claim 2 — Let $\rho_n := \frac{p_n}{q_n}$. Then

$$\frac{\nabla \rho_n}{\rho_n} = \langle \overline{\alpha_n}, \cdot \rangle + \frac{\nabla p_n}{p_n} \in \mathcal{G}_n.$$

In particular, $\sqrt{\rho_n} \in W^{1,2}(\mathbb{R}^n, \mu_{0,n})$.

The equality in the assertion of the claim is clear. The fact that $\sqrt{\rho_n} \in W^{1,2}(\mathbb{R}^n, \mu_{0,n})$ follows then by an elementary calculation from (16). Since we know that $\frac{\nabla p_n}{p_n} \in \mathcal{G}_n$ by Claim 1, we only need to show that $\langle \overline{\alpha_n}, \cdot \rangle \in \mathcal{G}_n$. But if

$$f(z) := \frac{1}{2} \sum_{i=1}^{n} \alpha_i z_i^2, \; z \in \mathbb{R}^n,$$

then $\nabla f = \langle \overline{\alpha_n}, \cdot \rangle$. A simple truncation argument and (16) imply that $\langle \overline{\alpha_n}, \cdot \rangle \in \mathcal{G}_n$.

Claim 3 — $\frac{\nabla \rho_n}{\rho_n} \circ P_n \overset{n\to\infty}{\to} F$ in $L^2(H \to H, \mu)$.

Let $d_n := \langle -\overline{\alpha_n}, \cdot \rangle + \widetilde{F}_n - \frac{\nabla p_n}{p_n}$. Then by Claim 1 $d_n \perp \mathcal{G}_n$ in $L^2(\mathbb{R}^n \to \mathbb{R}^n, \mu_n)$ and hence

$$\widetilde{F}_n = d_n + \frac{\nabla \rho_n}{\rho_n}$$

(which holds by Claim 2) is an orthogonal decomposition of $\widetilde{F}_n$ in $L^2(\mathbb{R}^n \to \mathbb{R}^n, \mu_n)$. Hence by (14)

$$d_n \circ P_n \overset{n\to\infty}{\to} d, \; \frac{\nabla \rho_n}{\rho_n} \circ P_n \overset{n\to\infty}{\to} G \; \text{ in } \; L^2(H \to H, \mu)$$

for some $d, G \in L^2(H \to H, \mu)$ which are orthogonal. Since $d_n \perp \mathcal{G}_n$ in $L^2(\mathbb{R}^n \to \mathbb{R}^n, \mu_n)$ it easily follows that $d \perp \mathcal{G}$ in $L^2(H \to H, \mu)$. Hence by (13) we have that $d \perp F$. Since $F = d + G$, this implies that $d = 0$, i.e. , $F = G$ and Claim 3 is proved.

By Claim 3 and (12) it is now easy to complete the proof: Since $\sqrt{\rho_n} \in W^{1,2}(\mathbb{R}^n, \mu_{0,n})$ (cf. Claim 2), by approximation it follows that

$$\sqrt{\rho_n} \circ P_n \in W^{1,2}(H, \mu_0) \;\&D_x \left(\sqrt{\rho_n} \circ P_n\right) = (\nabla \sqrt{\rho_n}) \circ P_n, \tag{21}$$

where we denote the closure of $(D_x, \mathcal{F}C_b^\infty(H))$ defined on all $W^{1,2}(H, \mu_0)$ again by D_x. By definition of ρ_n and (19) we have that

$$\rho_n \mu_{0,n} = p_n dx = \mu_n = \mu \circ P_n^{-1}. \tag{22}$$

Consequently by (21)

$$
\begin{aligned}
&4\int_H \left|D_x\left(\sqrt{\rho_n}\circ P_n\right)(x)\right|^2\mu_0(dx)\\
&=4\int_{\mathbb{R}^n}\left|\frac{\nabla\sqrt{\rho_n}}{\sqrt{\rho_n}}(z)\right|^2\mu_n(dz)\\
&=\int_{\mathbb{R}^n}\left|\frac{\nabla\rho_n}{\rho_n}(z)\right|^2\mu_n(dz)\\
&=\int_H\left|\frac{\nabla\rho_n}{\rho_n}\circ P_n(x)\right|^2\mu(dx)
\end{aligned}
\tag{23}
$$

Furthermore, by Claim 2

$$
\int_H\left(\sqrt{\rho_n}\circ P_n\right)^2\mu_0(dx)=\int_{\mathbb{R}^n}p_n dx=\int_{\mathbb{R}^n}\mu_n(dx)=1.
$$

Hence (23) and Claim 3 imply that $\left\{\sqrt{\rho_n}\circ P_n \middle|\, n\in\mathbb{N}\right\}$ is bounded in $W^{1,2}(H,\mu_0)$ and thus by (12)

$$
\sqrt{\rho_{n_l}}\circ P_{n_l}\overset{l\to\infty}{\rightharpoonup}\sqrt{\rho}\ \text{ in }\ L^2(H,\mu_0)
$$

for some $\sqrt{\rho}\in W^{1,2}(H,\mu_0)$ and some subsequence $(n_l)_{l\in\mathbb{N}}$. It remains to show that $\mu=\rho\mu_0$. To this end let

$$
h:=\sum_{i=1}^m c_i e_i,\ m\in\mathbb{N},\ c_1,\ldots,c_m\in\mathbb{R}.
$$

Then, since $\rho_{n_l}\circ P_{n_l}\overset{l\to\infty}{\rightharpoonup}\rho$ in $L^1(H,\mu_0)$,

$$
\begin{aligned}
&\int_H e^{i<h,x>}\rho(x)\mu_0(dx)\\
&=\lim_{l\to\infty}\int_H e^{i<h,x>}(\rho_{n_l}\circ P_{n_l})(x)\mu_0(dx).
\end{aligned}
$$

On the other hand if $n_l\geq m$, we have that $<h,x>=\langle h,P_{n_l}x\rangle$ for all $x\in H$. Hence

$$
\begin{aligned}
&\int_H e^{i<h,x>}(\rho_{n_l}\circ P_{n_l})(x)\mu_0(dx)=\int_{\mathbb{R}^{n_l}}e^{i\sum_{j=1}^m c_j z_j}\rho_{n_l}(z)\mu_{0,n_l}(dz)\\
&=\int_H e^{i<h,x>}\mu(dx)
\end{aligned}
$$

where we used (22) in the last step. Consequently $\mu = \rho\mu_0$. ■

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