Generalized Dirichlet forms and associated Markov processes [1]

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Abstract - We prove that for a certain class of bilinear forms satisfying some regularity conditions which include quasi-regular Dirichlet forms (cf. [3]) and time dependent Dirichlet forms (cf. [5]) as particular cases there exists an associated strong Markov process having nice sample path properties. These forms, called generalized Dirichlet forms, are the sum of a coercive part and a perturbation (e.g. the time derivative in the time dependent case), so that in general neither the sector condition is fulfilled by the sum nor is the associated L^2-semigroup analytic. A wide variety of new examples can be treated in this extended framework of Dirichlet forms including fractional diffusion operators and transformations of time dependent Dirichlet forms by α-excessive functions h (h-transformations).

Formes de Dirichlet généralisées et processus de Markov associés

Résumé - Nous construisons des processus standard spéciaux associés à certaines formes bilinéaires qui satisfont à quelques conditions de régularité. La classe de ces formes que nous appelons formes de Dirichlet généralisées, contient les formes de Dirichlet quasi régulières (cf. [3]) aussi bien que les formes de Dirichlet dépendant du temps (cf. [5]). Une forme de Dirichlet généralisée est la somme d'une forme coercive et d'une perturbation (par exemple la dérivation par rapport au temps dans le cas des formes de Dirichlet dépendant du temps). Donc en général les formes ne sont pas coercives, et les semigroupes associés ne sont pas non plus analytiques. Nous pouvons traiter de nombreux exemples nouveaux dont des opérateurs de diffusions fractionnaires et la transformée d'une forme de Dirichlet dépendant du temps par une fonction α-excessive.

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Version française abrégée - Soit E une espace topologique de Hausdorff, dont la tribu borélienne $\mathcal{B}$ est engendrée par l'ensemble $C(E)$ des fonctions continues. Soit m une mesure positive σ-finie, telle que l'espace de Hilbert (réel) $\mathcal{H} = L^2(E, \mathcal{B}, m)$ soit séparable. Soit $(\mathcal{A}, \mathcal{V})$ une forme coercive sur $\mathcal{H}$. Alors l'espace $\mathcal{V}$ muni du produit scalaire $\tilde{\mathcal{A}}_1(u,v) := 1/2(\mathcal{A}(u,v)+\mathcal{A}(v,u))+(u,v)_{\mathcal{H}}$ est aussi un espace de Hilbert réel et séparable. Nous identifions $\mathcal{H}$ à son dual. Alors $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ densément et continûment.

Soit Λ un opérateur linéaire de domaine $D(\Lambda, \mathcal{V}') \subset \mathcal{V}'$ satisfaisant aux conditions suivantes:

D1 (i) $(\Lambda, D(\Lambda, \mathcal{V}'))$ est le générateur d'un semigroupe $(U_t)_{t\geq 0}$ fortement continu d'opérateurs bornés sur $\mathcal{V}'$.

(ii) On peut restreindre $(U_t)_{t\geq 0}$ à un semigroupe fortement continu de contractions sur $\mathcal{H}$ et à un semigroupe fortement continu d'opérateurs bornés sur $\mathcal{V}$.

Nous désignons par $(\Lambda, D(\Lambda, \mathcal{V}))$ le générateur du semigroupe sur $\mathcal{V}$. L'opérateur dual $(\hat{\Lambda}, D(\hat{\Lambda}, \mathcal{V}'))$ de $(\Lambda, D(\Lambda, \mathcal{V}))$ satisfait aussi aux conditions D1.

Désignons respectivement par $\mathcal{F}$ et $\hat{\mathcal{F}}$ les espaces de Hilbert $D(\Lambda, \mathcal{V}') \cap \mathcal{V}$ muni de la norme $\| u \|_{\mathcal{F}}^2 := \| u \|_{\mathcal{V}}^2 + \| \Lambda u \|_{\mathcal{V}'}^2$ et $D(\hat{\Lambda}, \mathcal{V}') \cap \mathcal{V}$ muni de la norme $\| u \|_{\hat{\mathcal{F}}}^2 := \| u \|_{\mathcal{V}}^2 + \| \hat{\Lambda} u \|_{\mathcal{V}'}^2$ (où $\| \cdot \|_{\mathcal{V}}$ et $\| \cdot \|_{\mathcal{V}'}$ sont les normes dans $\mathcal{V}$ et $\mathcal{V}'$).

Soit $\mathcal{E}$ la forme

$$\mathcal{E}(u,v) := \begin{cases} \mathcal{A}(u,v) - \langle \Lambda u, v \rangle, & u \in \mathcal{F}, v \in \mathcal{V} \\ \mathcal{A}(u,v) - \left\langle \hat{\Lambda} v, u \right\rangle, & u \in \mathcal{V}, v \in \hat{\mathcal{F}}, \end{cases}$$

et soit $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha(u,v)_{\mathcal{H}}$ pour $\alpha > 0$.

Pour $\alpha > 0$ il existe des bijections linéaires et continues $W_\alpha : \mathcal{V}' \to \mathcal{F}$ et $\hat{W}_\alpha : \mathcal{V}' \to \hat{\mathcal{F}}$, telles que $\mathcal{E}_\alpha(W_\alpha f, u) = \langle f, u \rangle = \mathcal{E}_\alpha(u, \hat{W}_\alpha f)$, $\forall f \in \mathcal{V}', u \in \mathcal{V}$ (cf. [2], p. 248, théorème 1.1). Les restrictions $(G_\alpha)_{\alpha>0}$ et $(\hat{G}_\alpha)_{\alpha>0}$ de $(W_\alpha)_{\alpha>0}$ et $(\hat{W}_\alpha)_{\alpha>0}$ à $\mathcal{H}$ sont des résolvantes fortement continues de contractions sur $\mathcal{H}$. On a $\lim_{\alpha\to\infty} \alpha G_\alpha f = f$ et $\lim_{\alpha\to\infty} \alpha \hat{G}_\alpha f = f$ aussi bien dans $\mathcal{V}$ pour $f \in \mathcal{V}$ que dans $\mathcal{F}$ (ou $\hat{\mathcal{F}}$) pour $f \in \mathcal{F}$ (ou $\hat{\mathcal{F}}$).

Nous supposons les conditions suivantes:

D2 $u \in \mathcal{F} \Rightarrow u^+ \wedge 1 \in \mathcal{V}$ et $\mathcal{E}(u, u - u^+ \wedge 1) \geq 0$.

D3 Il existe un sous-espace vectoriel dense $\mathcal{Y}$ de $\mathcal{F}$ tel que $u \wedge \alpha \in \mathcal{Y}$ pour $u \in \mathcal{Y}$ et $\alpha \geq 0$.

Définition La forme $\mathcal{E}$ est appelée **forme semi-Dirichlet généralisée**, si elle satisfait à D1, D2 et D3.

Une fonction $u \in \mathcal{H}$ est appelée **1-excessive** si $\beta G_{\beta+1} u \leq u$, $\forall \beta \geq 0$. Nous désignons par $\mathcal{P}$ l'ensemble des fonctions 1-excessives dans $\mathcal{V}$. Pour $h \in \mathcal{H}$ soit $\mathcal{L}_h = \{u \in \mathcal{H} | u \geq h\}$.

Proposition Si $\mathcal{L}_h \cap \mathcal{F} \neq \emptyset$, alors il existe $e_h \in \mathcal{P}$, telle que

(i) $e_h = \inf\{u | u \in \mathcal{L}_h \cap \mathcal{P}\}$.

(ii) $\mathcal{E}_1(e_h, v) \geq \mathcal{A}_1(e_h, e_h)$, $\forall v \in \mathcal{L}_h \cap \hat{\mathcal{F}}$.

e_h est appelée la fonction **1-réduite** de h. Pour $U \subset E$ ouvert et $h \in \mathcal{H}$ tel que $\mathcal{L}_h \cap \mathcal{F} \neq \emptyset$ soit $h_U := e_{(h 1_U)}$.

Définition Une suite croissante $(F_k)_{k \geq 1}$ d'ensembles fermés est appelée **réseau de** $\mathcal{E}$, si pour chaque $u \in \mathcal{F} \cap \mathcal{P}$, $u_{F_k^c} \to 0$ dans $\mathcal{H}$ et faiblement dans $\mathcal{V}$.

Nous utilisons les notions $\mathcal{E}$**-exceptionnel** et $\mathcal{E}$**-quasi continu** relativement à cette définition.

Définition Une forme semi-Dirichlet généralisée est appelée **quasi régulière**, si elle satisfait aux conditions suivantes:

(i) Il existe un réseau $(F_k)_{k \geq 1}$ d'ensembles compacts.

(ii) Il existe un sous-ensemble dense de $\mathcal{F}$ composé de fonctions $\mathcal{E}$-quasi continues.

(iii) Il existe une suite $\{\tilde{u}_n | n \in \mathbb{N}\}$ de fonctions dans $\mathcal{F}$, $\mathcal{E}$-quasi continues, telle que $\{\tilde{u}_n | n \in \mathbb{N}\}$ sépare les points du complémentaire d'un ensemble $\mathcal{E}$-exceptionnel.

Maintenant, nous pouvons formuler le résultat principal de ce travail:

Théorème Soit $\mathcal{E}$ une forme semi-Dirichlet généralisée quasi régulière. Alors il existe un processus standard spécial et m-tendu $\mathbf{M} = (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (Y_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ associé à $\mathcal{E}$ au sens suivant: pour $\alpha > 0$ et $f \in \mathcal{B}_b(E) \cap \mathcal{H}$,

$$E_{\cdot}[\int_0^\infty e^{-\alpha t} f(Y_t)\, dt]$$

est une m version $\mathcal{E}$-quasi continue de $G_\alpha f$.

Pour les démonstrations détaillées des résultats nous faisons référence à [10].

The purpose of this work is to provide an extended framework for Dirichlet forms which includes both elliptic cases (i.e., those in [3]) and parabolic ones (i.e., those time-dependent ones in [5]). It is thus a continuation both of [3] and [5] as well as of the classical fundamental work on parabolic potential theory in [6],[7] and [8] in an L^2-framework.

We first introduce our framework. Let $\mathcal{H}$ be a separable real Hilbert space with inner product $(\cdot,\cdot)_\mathcal{H}$, $(\mathcal{A},\mathcal{V})$ a real-valued coercive closed form on $\mathcal{H}$. Then $\mathcal{V}$ equipped with inner product $\tilde{\mathcal{A}}_1(u,v) := 1/2(\mathcal{A}(u,v) + \mathcal{A}(v,u)) + (u,v)_\mathcal{H}$ is again a separable real Hilbert space. Let $\|\cdot\|_\mathcal{V}$ be the corresponding norm. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$, we have that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ densely and continuously.

Let Λ with domain $D(\Lambda, \mathcal{V}')$ be a linear operator on $\mathcal{V}'$ satisfying the following conditions:

D1 (i) $(\Lambda, D(\Lambda,\mathcal{V}'))$ generates a strongly continuous semigroup $(U_t)_{t\geq 0}$ of bounded linear operators on $\mathcal{V}'$.

(ii) $(U_t)_{t\geq 0}$ can be restricted both to a strongly continuous **contraction** semigroup on $\mathcal{H}$ and to a strongly continuous semigroup of bounded linear operators on $\mathcal{V}$.

Denote by $(\Lambda, D(\Lambda,\mathcal{H}))$ and $(\Lambda, D(\Lambda,\mathcal{V}))$ the generators corresponding to the restricted semigroups.

The dual operator $(\hat{\Lambda}, D(\hat{\Lambda}, \mathcal{V}'))$ of $(\Lambda, D(\Lambda, \mathcal{V}))$ also satisfies condition D1. We can now define the Hilbert spaces $\mathcal{F} := D(\Lambda, \mathcal{V}') \cap \mathcal{V}$ with norm $\| u \|_{\mathcal{F}}^2 := \| u \|_{\mathcal{V}}^2 + \| \Lambda u \|_{\mathcal{V}'}^2$ and $\hat{\mathcal{F}} := D(\hat{\Lambda}, \mathcal{V}') \cap \mathcal{V}$ with norm $\| u \|_{\hat{\mathcal{F}}}^2 := \| u \|_{\mathcal{V}}^2 + \| \hat{\Lambda} u \|_{\mathcal{V}'}^2$.

Let the form $\mathcal{E}$ be given by

$$(1) \qquad \mathcal{E}(u,v) := \begin{cases} \mathcal{A}(u,v) - \langle \Lambda u, v \rangle & \text{for } u \in \mathcal{F},\ v \in \mathcal{V} \\ \mathcal{A}(u,v) - \left\langle \hat{\Lambda} v, u \right\rangle & \text{for } u \in \mathcal{V},\ v \in \hat{\mathcal{F}}, \end{cases}$$

and $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha (u,v)_{\mathcal{H}}$ for $\alpha > 0$.

Here $\langle \cdot, \cdot \rangle$ denotes the dualization between $\mathcal{V}$ and $\mathcal{V}'$. Note that $\mathcal{E}$ is well-defined. We can also define the co-form $\hat{\mathcal{E}}$ of $\mathcal{E}$ given by

$$\hat{\mathcal{E}}(u,v) := \begin{cases} \mathcal{A}(v,u) - \left\langle \hat{\Lambda} u, v \right\rangle & \text{for } u \in \hat{\mathcal{F}},\ v \in \mathcal{V} \\ \mathcal{A}(v,u) - \langle \Lambda v, u \rangle & \text{for } u \in \mathcal{V},\ v \in \mathcal{F}. \end{cases}$$

It follows from [2], p. 248, Théorème 1.1, that for all $\alpha > 0$ there exist continuous, linear bijections $W_\alpha : \mathcal{V}' \longrightarrow \mathcal{F}$ and $\hat{W}_\alpha : \mathcal{V}' \longrightarrow \hat{\mathcal{F}}$ such that $\mathcal{E}_\alpha(W_\alpha f, u) = \langle f, u \rangle = \mathcal{E}_\alpha(u, \hat{W}_\alpha f)$, $\forall f \in \mathcal{V}'$, $u \in \mathcal{V}$. Restricting W_α and $\hat{W}_\alpha$ to $\mathcal{H}$ we get strongly continuous contraction resolvents $(G_\alpha)_{\alpha > 0}$ and $(\hat{G}_\alpha)_{\alpha>0}$ on $\mathcal{H}$ satisfying $\lim_{\alpha \to \infty} \alpha G_\alpha f = f$ (resp. $\lim_{\alpha \to \infty} \alpha \hat{G}_\alpha f = f$) both in $\mathcal{V}$ for $f \in \mathcal{V}$ and in $\mathcal{F}$ for $f \in \mathcal{F}$ (resp. in $\hat{\mathcal{F}}$ for $f \in \hat{\mathcal{F}}$).

Next we want to generalize the Dirichlet property to this new setting. Let $(E, \mathcal{B}, m)$ be a σ-finite measure space such that $\mathcal{H} := L^2(E, \mathcal{B}, m)$ is separable. We assume the following condition:

D2 $\quad u \in \mathcal{F} \Rightarrow u^+ \wedge 1 \in \mathcal{V}$ and $\mathcal{E}(u, u - u^+ \wedge 1) \geq 0$.

For the terminology in the following proposition we refer to [3], [4].

Proposition 1 (i) D2 holds if and only if $(G_\alpha)_{\alpha>0}$ is sub-Markovian.

(ii) If $(\mathcal{A}, \mathcal{V})$ is a semi-Dirichlet form and $(\Lambda, D(\Lambda, \mathcal{H}))$ a Dirichlet operator, then D2 is satisfied.

Since $\mathcal{F}$ is not inf-stable in general (i.e. $u, v \in \mathcal{F} \not\Rightarrow u \wedge v \in \mathcal{F}$) we impose the following additional regularity condition on the domain $\mathcal{F}$:

D3 There exists a dense linear subspace $\mathcal{Y} \subset \mathcal{F}$ such that $u \wedge \alpha \in \mathcal{Y}$ for $u \in \mathcal{Y}$ and $\alpha \geq 0$.

Definition 2 The form $\mathcal{E}$ with domain $\mathcal{F}\times\mathcal{V}\cup\mathcal{V}\times\hat{\mathcal{F}}$ is called a **generalized semi-Dirichlet form** if D1, D2 and D3 hold and a **generalized Dirichlet form** if both $\mathcal{E}$ and its co-form $\hat{\mathcal{E}}$ are generalized semi-Dirichlet forms.

Examples 3 (i) Let $(\mathcal{A}, \mathcal{V})$ be a semi-Dirichlet form and $\Lambda = 0$. Clearly $\mathcal{F} = \mathcal{V} = \hat{\mathcal{F}}$ and $\mathcal{E} = \mathcal{A}$ is a generalized semi-Dirichlet form.
(ii) Let $\mathcal{E}$ be a time dependent Dirichlet form (cf. [5]), i.e. $(\mathcal{E}^{(t)})_{t\in\mathbb{R}}$ a family of coercive closed forms on $H := L^2(X,\mu)$ with common domain V and sector constant K independent of t - where V is a Hilbert space with corresponding norm $\|\cdot\|_V$ - satisfying the following conditions:

(a) $\mathcal{E}^{(t)}(u,v)$ is a Borel measurable function of $t \in \mathbb{R}$, $\forall\, u, v \in V$.

(b) There exists a positive constant c such that

$$c \parallel u \parallel_V^2 \geq \mathcal{E}_1^{(t)}(u,u) \geq c^{-1} \parallel u \parallel_V^2,\ u \in V,\ t \in \mathbb{R}.$$

(c) Every form $(\mathcal{E}^{(t)}, V)$ is a Dirichlet form.

For $u, v \in \mathcal{V} := L^2(\mathbb{R}, V)$ define $\mathcal{A}(u,v) := \int_{\mathbb{R}} \mathcal{E}^{(t)}(u_t, v_t)\lambda(dt)$, where λ denotes the one-dimensional Lebesgue measure. Then $(\mathcal{A}, \mathcal{V})$ is a Dirichlet form on $\mathcal{H} := L^2(\mathbb{R}, H) \cong L^2(\mathbb{R}\times X, \lambda\otimes\mu)$. As Λ we can take the time derivative d/dt on $L^2(\mathbb{R}, V')$ with domain $H^{1,2}(\mathbb{R}, V')$ since it clearly satisfies condition D1. D2 holds because of Proposition 1, and D3 with $\mathcal{Y} := H^{1,2}(\mathbb{R}, V)$ because of the inf-stability of $H^{1,2}(\mathbb{R}, V)$.
(iii) In the situation of Example (ii) with $X = \mathbb{R}^d$ and $\mu = \lambda^d$ denote the one-sided stable semigroup of order $\alpha \in (0,1)$ by $(\eta_t^\alpha)_{t\geq 0}$ and the semigroup subordinated to $(U_t)_{t\geq 0}$ via $(\eta_t^\alpha)_{t\geq 0}$ by $(U_t^\alpha)_{t\geq 0}$. Then as Λ we can also take the generator $\big((d/dt)^\alpha, D((d/dt)^\alpha, \mathcal{V}')\big)$ which is the time derivative of power α and satisfies D1. D2 holds because of Proposition 1. Since

$$(2)\quad D((\frac{d}{dt})^\alpha, \mathcal{H}) = \{f \in L^2(\mathbb{R}^{d+1}, \lambda^{d+1}) | \int |\hat{f}|^2(x)|x_0|^{2\alpha}\lambda^{d+1}(dx) < \infty\},$$

where $x = (x_0, x_1, \ldots, x_d)$ and $\hat{f}$ denotes the Fourier transform of f, and the right side of (2) is a Dirichlet space, for $\mathcal{Y} := D((d/dt)^\alpha, \mathcal{H}) \cap \mathcal{V}$ condition D3 is satisfied. In the special case where $V = H^{1,2}(\mathbb{R}^d)$ and $\mathcal{E}^{(t)}(u,v) = \int \nabla u \cdot \nabla v \, d\lambda^d$, $\forall t$, the generator $(L^\alpha, D(L^\alpha))$ of the corresponding contraction resolvent is an extension of the fractional diffusion operator $(\partial/\partial t)^\alpha u + \Delta u$, $u \in C_0^\infty(\mathbb{R}^{d+1})$ (cf. [9]).
(iv) Consider Example (ii) again. Let $h \in D(d/dt, \mathcal{H}), h > 0$, be an α-excessive function for some $\alpha > 0$ (i.e. $\beta G_{\beta+\alpha} h \leq h$ for all $\beta \geq 0$). Then we can define the h-transform $\mathcal{E}^h$ of $\mathcal{E}$ on $\mathcal{H}^h := L^2(\mathbb{R} \times X, h^2 \lambda \otimes \mu)$ by

$$D(\mathcal{A}^h) := \left\{ u \in \mathcal{H}^h | uh \in \mathcal{V} \right\}, \quad \mathcal{A}^h(u,v) := \mathcal{A}_\alpha(uh, vh)$$

and $D(\frac{d^h}{dt}, \mathcal{H}) := \{u \in \mathcal{H}^h | uh \in D(\frac{d}{dt}, \mathcal{H})\}$, $\frac{d^h}{dt} u := \frac{1}{h} \frac{d}{dt}(uh)$.
$(d^h/dt, D(d^h/dt, \mathcal{H}^h))$ is closable on $D(\mathcal{A}^h)'$ and its closure $(d^h/dt, D(d^h/dt, D(\mathcal{A}^h)'))$ satisfies condition D1. The strongly continuous contraction resolvent $(G^h_\beta)_{\beta > 0}$ corresponding to $\mathcal{E}^h$ is sub-Markovian, hence D2 holds. Since $h \in D(d/dt, \mathcal{H})$, for $\mathcal{Y} := D(d^h/dt, \mathcal{H}^h) \cap D(\mathcal{A}^h)$ condition D3 is satisfied.

We want to state some results concerning the analytic potential theory of generalized semi-Dirichlet forms. Let $\mathcal{E}$ with domain $\mathcal{F} \times \mathcal{V} \cup \mathcal{V} \times \hat{\mathcal{F}}$ be a generalized semi-Dirichlet form (cf. (1)).

Let $\alpha > 0$. An element $u \in \mathcal{H}$ is called **α-excessive** if $\beta G_{\beta+\alpha} u \leq u$ for $\beta \geq 0$. Denote by $\mathcal{P}$ the set of all 1-excessive elements in $\mathcal{V}$. As in the theory of Dirichlet forms we can prove for $u \in \mathcal{V}$ and $\alpha > 0$: u is α-excessive if and only if $\mathcal{E}_\alpha(u,v) \geq 0$ for all $v \in \hat{\mathcal{F}}$ with $v \geq 0$.

In order to introduce the notion of 1-reduced functions let $\mathcal{L}_h = \{v \in \mathcal{H} | v \geq h\}$ for $h \in \mathcal{H}$. Using [1], p. 232, Théorème 7.1, we can show (by an approximation procedure):

Proposition 4 If $\mathcal{L}_h \cap \mathcal{F} \neq \emptyset$ then there exists $e_h \in \mathcal{V}$ such that:

(i) $e_h = \inf\{u | u \in \mathcal{L}_h \cap \mathcal{P}\}$.

(ii) $\mathcal{E}_1(e_h, v) \geq \mathcal{A}_1(e_h, e_h)$, $\forall v \in \mathcal{L}_h \cap \hat{\mathcal{F}}$.

e_h is called the **1-reduced** function of h.

From now on we assume that E is a Hausdorff topological space and $\mathcal{B}$ its Borel σ-algebra. For simplicity we assume that $\mathcal{B}$ is generated by the set $C(E)$ of all continuous functions on E. For an open subset $U \subset E$ and a function $g \in \mathcal{H}$ let $g_U := e_{(g1_U)}$ if the 1-reduced function $e_{(g1_U)}$ exists.

Definition 5 (i) An increasing sequence of closed subsets $(F_k)_{k\geq 1}$ is called an **$\mathcal{E}$-nest**, if for every function $u \in \mathcal{P} \cap \mathcal{F}$ it follows that $u_{F_k^c} \to 0$ in $\mathcal{H}$ and weakly in $\mathcal{V}$.

(ii) A subset $N \subset E$ is called **$\mathcal{E}$-exceptional** if $N \subset \cap_{k\geq 1} E \setminus F_k$ for some $\mathcal{E}$-nest $(F_k)_{k\geq 1}$.

(iii) A function f defined outside some $\mathcal{E}$-exceptional set is called **$\mathcal{E}$-quasi continuous** if there is an $\mathcal{E}$-nest $(F_k)_{k\geq 1}$ such that $f_{|F_k}$ is continuous for each $k \geq 1$.

Remark 6 (i) If $(\mathcal{A}, \mathcal{V})$ in (1) is a Dirichlet form the last definition implies that every $\mathcal{E}$-nest is an $\mathcal{A}$-nest in the sense of [3], p.75, Definition 2.1.
(ii) We can show that there exists a Choquet capacity defined on the subsets of E whose zero-sets are exactly the $\mathcal{E}$-exceptional sets.

We can now formulate sufficient regularity conditions on our generalized semi-Dirichlet form in order to construct an associated m-tight special standard process (cf. [3], Chapter IV, Section 1]).

Definition 7 The generalized semi-Dirichlet form $\mathcal{E}$ is called **quasi-regular** if:

(i) There exists an $\mathcal{E}$-nest $(F_k)_{k\geq 1}$ of compact sets.

(ii) There exists a dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-q.c. m-versions.

(iii) There exist a countable subset $\{u_n \mid n \in \mathbb{N}\} \subset \mathcal{F}$ with $\mathcal{E}$-q.c. m-versions $\tilde{u}_n$ and an $\mathcal{E}$-exceptional set N, such that $\{\tilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

Examples 8 (i) Every quasi-regular semi-Dirichlet form is a quasi-regular generalized semi-Dirichlet form.
(ii) It can be shown that the generalized semi-Dirichlet forms in Examples 3 (ii) and (iii) are quasi-regular if the form $(\mathcal{E}^{(t)}, V)$ is quasi-regular for one (hence all) $t \in \mathbb{R}$. In particular, every time dependent Dirichlet form in the sense of Oshima (cf. [5]) is quasi-regular.

We can now state the main theorem:

Theorem 9 Let $\mathcal{E}$ be a quasi-regular generalized semi-Dirichlet form. Then there exists an m-tight special standard process $\mathbf{M} = (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (Y_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ with state space E which is associated to the form $\mathcal{E}$ in the following sense: for all $\alpha > 0$ and $f \in \mathcal{B}_b(E) \cap \mathcal{H}$

$$E_\cdot[\int_0^\infty e^{-\alpha t} f(Y_t)\, dt\,]$$

is an $\mathcal{E}$-q.c. m-version of $G_\alpha f$.

Remark 10 Suppose that also the co-form $\hat{\mathcal{E}}$ of $\mathcal{E}$ is a quasi-regular generalized semi-Dirichlet form. We can show that the $\mathcal{E}$-exceptional sets coincide with the $\hat{\mathcal{E}}$-exceptional sets. Thus there exists a m-tight special standard process $\hat{\mathbf{M}} = (\hat{\Omega}, \hat{\mathcal{M}}, (\hat{\mathcal{M}}_t)_{t\geq 0}, (\hat{Y}_t)_{t\geq 0}, (\hat{P}_z)_{z\in E_\Delta})$ such that $\hat{E}_\cdot[\int_0^\infty e^{-\alpha t} f(\hat{Y}_t)\, dt\,]$ is an $\mathcal{E}$-q.c. m-version of $\hat{G}_\alpha f$ for all $\alpha > 0$ and $f \in \mathcal{B}_b(E) \cap \mathcal{H}$. $\hat{\mathbf{M}}$ is then in duality to $\mathbf{M}$ w.r.t. m.

Work is in progress to weaken condition D1 on the perturbation Λ. All essential functional analytic results are still true if $(\Lambda, D(\Lambda, \mathcal{V}'))$ is a closed linear operator on $\mathcal{V}'$ such that $(\Lambda, D(\Lambda, \mathcal{H}))$ - where $D(\Lambda, \mathcal{H}) = \{u \in \mathcal{H} \cap D(\Lambda, \mathcal{V}') | \Lambda u \in \mathcal{H}\}$ - generates a strongly continuous contraction semigroup on $\mathcal{H}$ and for $t \geq 0$ the restriction of U_t to $\mathcal{V}$ is a bounded linear operator on $\mathcal{V}$ such that $(U_t)_{t\geq 0}$ is strongly continuous on $\mathcal{V}$. This framework then includes finite dimensional Dawson-Watanabe processes (cf. [12]) as well as Schrödinger operators with very singular time dependent potentials as new examples (cf. [11]).

Details and proofs of all the results announced in this Note and further examples which can only be covered by our extended framework are contained in [10].

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References

[1] Lions, J. L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Paris: Dunod, Gauthier - Villars, 1969.

[2] Lions, J. L., Magenes, E.: Problèmes aux limites non homogènes et applications, Volume 1. Paris: Dunod, 1968.

[3] Ma, Z. M., Röckner, M.: Introduction to the Theory of (Non-Symmetric) Dirichlet Forms. Berlin - Heidelberg - New York: Springer, 1992.

[4] Ma, Z. M., Overbeck, L., Röckner, M.: Markov Processes associated with semi-Dirichlet Forms. Preprint, 1993.

[5] Oshima, Y.: On a construction of Markov Processes associated with time dependent Dirichlet Spaces. Forum Mathematicum 4, 395 - 415, 1992.

[6] Pierre, M.: Problèmes d'evolution avec constraintes et potentiels paraboliques. Comm. P. D. E. 4, 1149 - 1197, 1979.

[7] Pierre, M.: Representant precis d'un potentiel parabolique. Séminaire Theorie du Potentiel, Université Paris VI. Lecture Notes in Mathematics 814. Berlin - Heidelberg - New York: Springer, 1980.

[8] Pierre, M.: Parabolic capacity and Sobolev spaces. SIAM Journal Math. Anal. 14, 522-533, 1983.

[9] Schneider, W.R., Wyss, W.: Fractional diffusions and wave equations. J. Math. Phys. 30, 134 - 144, 1989.

[10] Stannat, W.: Dirichlet forms and Markov processes: a generalized framework including both elliptic and parabolic cases (in preparation).

[11] Stannat, W.: Doctor degree thesis, University of Bielefeld (in preparation).

[12] Watanabe, S.: On two dimensional Markov Processes with branching property. Trans. Am. Math. Soc. 136, 447 - 466, 1969.