Indecomposable representations of generalized Weyl algebras

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Abstract

For a class of generalized Weyl algebras which includes the Weyl algebras A_n the tame criteria is given for the problem of describing indecomposable weight and generalized weight modules with supports from a fixed orbit. In tame cases all the indecomposable modules are described.

Introduction

Let an algebra $A = \otimes_1^n \mathcal{A}_i$ be the tensor product over an algebraically closed field K of generalized Weyl algebras $\mathcal{A}_i = \mathcal{D}_i(\sigma_i, a_i)$ of degree 1 (see subsec. 1.1 and 1.4) with basic polynomial rings $\mathcal{D}_i = K[H_i]$ in one variable, where σ_i is any automorphism of $\mathcal{D}_i$ and a_i is any non-zero element of $\mathcal{D}_i$. Then A is a generalized Weyl algebra of degree n. If $a_i = H_i$ and $\sigma_i(H_i) = H_i - 1$ for $i = 1, \ldots, n$ the algebra A is isomorphic to the Weyl algebra A_n of degree n (subsec. 1.4). Remark that considered generalized Weyl algebras of degree 1 include the quantum Weyl algebra $A_1(q)$, the quantum plane, the algebra of functions $A(S_q^2)$ on the quantum 2-dimensional sphere and others (see subsec. 1.3).

In the present paper we use the methods of [G], [GP] (see also [Kh], [D2], [DGO]) for study indecomposable weight and generalized weight A-modules. More precisely, we give the criteria when the problem of describing such modules (with support from some fixed orbit) is tame, Theorem 1 and 2, and in the tame cases all such indecomposable modules are classified.

Before the formulation of main results (Theorem 1 and 2) let us give some definitions. On the set of maximal ideals of $\mathcal{D}_i$ resp. $\otimes_{i=1}^n \mathcal{D}_i$ which can be indentified with K resp. K^n acts the group $\mathbf{Z}$ resp. $\mathbf{Z}^n$ by the automorhism σ_i resp. automorphisms $\{\sigma_i\}$. In the first case an orbit $\mathcal{O}_i$ is caled *degenerate* resp. *infinite* if it contains a root of the polynomial a_i resp. infinitely many elements. In the second case an orbit $\mathcal{O}$ is the direct product $\prod_{i=1}^n \mathcal{O}_i$ of orbits $\mathcal{O}_i$. A-module M is *generalized weight* resp. *weight* provided $M = \oplus_{\lambda=(\lambda_1,\ldots,\lambda_n)\in K^n} M^\lambda$ resp. $M = \oplus M_\lambda$ where the finite dimensional vector space M^λ resp. M_λ consists of elements $m \in M$ such that $(H_i - \lambda_i)^k m = 0$ resp. $(H_i - \lambda_i)m = 0$ for some $k = k(m)$. The set of all λ such that $M^\lambda \neq 0$ is called the *support* of M. It is easy to prove that support of each (generalized) weight modules belongs to some orbit

$\mathcal{O}$, say. Denote by $\mathcal{W}(\mathcal{O})$ resp. $\mathcal{GW}(\mathcal{O})$ the category of weight resp. generalized weight A-modules with supports from $\mathcal{O}$. One of the aims of the present paper is to prove the following Theorems.

Theorem 1. *1. The problem of describing $\mathcal{W}(\mathcal{O})$ is tame if and only if the number δ of non-degenerate infinite orbits is more or equal to $n-2$, moreover for $\delta = n-2$ all the orbits are infinite and each degenerate orbit contain exactly one root of the polynomial a_i.*

2. If all orbits are infinite and there is no more than one degenerate orbit, then $\mathcal{W}(\mathcal{O})$ contains only finitely many non-isomorphic indecomposable modules.

Theorem 2. *The problem of describing of $\mathcal{GW}(\mathcal{O})$ is tame if and only if $n = 1$, the orbit $\mathcal{O}$ is infinite and contains no more than 2 roots of the polynomial a_1, and if it contains exactly two roots, then both of them are simple, and if one, then its multiplicity is less or equal to 2.*

In the tame cases of Theorem 1 and 2 all the indecomposable modules from the categories $\mathcal{W}(\mathcal{O})$ and $\mathcal{GW}(\mathcal{O})$ are described. Remark, that the category $\mathcal{W}(\mathcal{O})$ is described in [DGO] in the case $n = 1$ and for any commutative ring D.

The plan of the paper is as follows. In sec. 1 all definitions and examples of generalized Weyl algebras are given. Theorems 1 and 2 are proved in sec. 2. In sec. 3 indecomposable weight and generalized weight modules are described in the tame cases. In sec. 4 the results of sec. 2, 3 are used for the classifications of indecomposable weight and generalized weight modules (in tame cases) over the Weyl algebra $A_n(K)$, the quantum plane, the quantum Weyl algebra $A_1(q)$, the factoralgebra $U(\lambda)$ of the universal enveloping algebra $Usl(2)$ of the Lie algebra $sl(2)$, the algebra of functions $A(S_q^2)$ on the quantum 2-dimensional sphere.

1 Generalized Weyl algebras, examples

1.1 Definition of the generalized Weyl algebras ([B1],[B2]).

Let D be a ring, $\sigma = (\sigma_1, ..., \sigma_n)$ a set of commuting automorphisms of D, ($\sigma_i\sigma_j = \sigma_j\sigma_i$, for all i, j), $a = (a_1, ..., a_n)$ be a set of (non-zero) elements of the centre $Z(D)$ of D, such that $\sigma_i(a_j) = a_j$ for all $i \neq j$

The *generalized Weyl algebra* $A = D(\sigma, a)$ (briefly GWA) of degree n with a *basic* ring D is a ring generated by D and the $2n$ indeterminates $X_1^+, ..., X_n^+, X_1^-, ..., X_n^-$ subject to the relations:

$$X_i^- X_i^+ = a_i, \qquad X_i^+ X_i^- = \sigma_i(a_i),$$

$$X_i^{\pm}\alpha = \sigma_i^{\pm 1}(\alpha)X_i^{\pm}, \forall\ \alpha \in D,$$

$$[X_i^-, X_j^-] = [X_i^+, X_j^+] = [X_i^+, X_j^-] = 0,\ \forall\ i \neq j,$$

where $[x,y] = xy - yx$. We say that a and σ are the sets of *defining* elements and automorphisms of A respectively.

Suppose that $A = D(\sigma, a), \ldots, A' = D'(\sigma', a')$ are generalized Weyl algebras of degree $n, \ldots, m$ respectively, then their tensor product (with respect to a basic field or ring)

$$A \otimes \ldots \otimes A' = D \otimes \ldots \otimes D'(\sigma \cup \ldots \cup \sigma', a \cup \ldots \cup a'),$$

is generalized Weyl algebra of degree $n + \ldots + m$ with basic ring $D \otimes \ldots \otimes D'$. An opposite algebra A^o to a generalized Weyl algebra A is also the generalized Weyl algebra:

$$A^0 = D^0(\sigma^{-1}, \sigma(a)), \text{ where } \sigma^{-1} = (\sigma_i^{-1}), \sigma(a) = (\sigma_i(a_i)).$$

So there is left-right symmetry of modules, ideals etc.

1.2 Grading.

Let $A(\sigma, a)$ be a generalized Weyl algebra of degree n. For any vector $k = (k_1, ..., k_n) \in \mathbf{Z}^n$ we put $v_k = v_{k_1}(1)...v_{k_n}(n)$ where for any $1 \leq i \leq n$ and $m \geq 0$ $v_{\pm m}(i) = (X_i^{\pm})^m$, $v_0(i) = 1$. In the case $n = 1$ we write v_m for $v_m(1)$. It follows from the definition of a GWA that

$$A = \oplus_{k \in \mathbf{Z}^n} A_k$$

is a $\mathbf{Z}^n$-graded algebra ($A_k A_e \subset A_{k+e}, \forall k, e \in \mathbf{Z}^n$), where $A_k = Dv_k = v_k D$. So A is a free left and right D-module.

Using the grading and defining relations of A we see that

$$v_{\mathbf{n}} v_{\mathbf{m}} = (\mathbf{n}, \mathbf{m}) v_{\mathbf{n}+\mathbf{m}} = v_{\mathbf{n}+\mathbf{m}} < \mathbf{n}, \mathbf{m} > \text{ where } \mathbf{n}, \mathbf{m} \in \mathbf{Z}^n,$$

for some $(\mathbf{n}, \mathbf{m}), < \mathbf{n}, \mathbf{m} > \in D$. Moreover

$$(\mathbf{n}, \mathbf{m}) = \prod_{i=1}^n (n_i, m_i), \ < \mathbf{n}, \mathbf{m} > = \prod_{i=1}^n < n_i, m_i >,$$

where (n_i, m_i) and $< n_i, m_i >$ are defined in the next subsection.

The existense of the grading reduces the problem of describing indecomposable (generalized) weight A-modules to the same but over (as usually) the finite dimensional categories (algebras).

Proposition 1.2.1 *([B]).Let A be a generalized Weyl algebra with basic ring D. Then*

1. *if D is left (right) Noetherian, then A is left (right) Noetherian;*
2. *if D has no zero divisors and $a_i \neq 0$, $\forall\ i = 1, ..., n$, then A has no zero divisors.* ∎

1.3 Generalized Weyl algebras of degree 1, examples.

Let $\mathcal{A} = \mathcal{D}(\sigma, a)$ be a generalized Weyl algebra of degree 1, $a \in Z(\mathcal{D})$, $\sigma \in \mathrm{Aut}(\mathcal{D})$.

That is the ring $\mathcal{A}$ which is generated by $\mathcal{D}$ and the 2 indeterminates $X = X_1^+$ and $Y = X_1^-$ subject to the relations:

$$X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y,\ \forall \alpha \in \mathcal{D},\ YX = a \text{ and } XY = \sigma(a)$$

Recall that the algebra $\mathcal{A} = \oplus_{n\in\mathbf{Z}} \mathcal{A}_n$ is $\mathbf{Z}$-graded, where $\mathcal{A}_n = \mathcal{D}v_n$, and we put

$$v_n = X^n\ (n > 0),\ v_n = Y^{-n}\ (n < 0),\ v_0 = 1.$$

It follows from the defining relations that

$$v_n v_m = (n, m)v_{n+m} = v_{n+m} < n, m >$$

for some $(n, m) = \sigma^{-n-m}(< n, m >) \in D$. If $n > 0$ and $m > 0$ then

$$n \geq m:\ (n, -m) = \sigma^n(a)...\sigma^{n-m+1}(a),\ (-n, m) = \sigma^{-n+1}(a)...\sigma^{-n+m}(a),$$

$$n \leq m:\ \ (n, -m) = \sigma^n(a)...\sigma(a),\ \ (-n, m) = \sigma^{-n+1}(a)...a,$$

in other cases $(n, m) = 1$.

Let $A_{(i)} = D_i(\sigma_i, a_i)\,(i = 1, ..., n)$ be a GWA of degree 1 over some fixed field K and every automorphism σ_i is a K-automorphism, then their tensor product

$$\otimes_1^n A_{(i)} = (\otimes_1^n D_i)((\sigma_i), (a_i))$$

is the GWA of degree n over K.

Examples of generalized Weyl algebras of degree 1. Let K be an algebraically closed field. From now on an algebra means a K-algebra.

Example 1. $A = K[H](\sigma, a), \sigma$ is an arbitrary automorphism of the polynomial ring $K[H]$ in one variable, i.e. $\sigma(H) = \lambda H + \mu$, $\lambda \neq 0, \mu \in K$, $a \in K[H]$.

Next algebras are special examples of generalized Weyl algebras from Example 1.

Example 2. $A = K[H](\sigma, a)$, $\sigma(H) = H - 1$, $a \in K[H]$. In [B] the properties of these algebras (the global homological dimension, the Krull dimension, two-sided ideals etc.) and the category of simple modules (the classification of simple modules, the finite dimension of the vector spaces Ext_i^A and Tor_i^A for simple modules etc.) are studied. In [Hod] the analog of the Bernstein-Beilinson Theorem [BB] is established, the Grothendieck group $K_0(A)$ and the global dimension is calculated.

The first *Weyl* algebra $A_1 =< X, \partial \,|\, \partial X - X\partial = 1 >$ is isomorphic to the above generalized Weyl algebra where $a = H$:

$$A_1 \to K[H](\sigma, a = H),\ X \to X^+,\ \partial \to X^-,\ \partial X \to H.$$

Let $F_m = A_1^G$ be the fixed ring where G is the cyclic group of order m, acting by multiplication $\partial \to \omega\partial$, $X \to \omega^{-1}X$, where ω is a primitive m'th root of unity. Then

$$F_m = K < \partial^m, \partial X, X^m > \simeq K[H](\sigma, a = m^m H(H-1/m)\dots(H-(m-1)/m),$$

$$\partial^m \leftrightarrow Y, X^m \leftrightarrow X, \partial X/m \leftrightarrow H,$$

is the GWA of degree 1, where $\sigma(H) = H-1$, char $K = 0$, [Hod].

Example 3. The *quantum Weyl* algebra $A_1(q) =< x, \partial | \partial x - qx\partial = 1 >$ of degree 1 over K $(q \neq 0 \in K)$ is a generalized Weyl algebra of degree 1:

$$A_1(q) \simeq K[H](\sigma, a = H)$$

with basic polynomial ring $K[H]$ and defining automorphism $\sigma:\ \sigma(H) = q^{-1}(H-1)$.

Let $U = Usl(2)$ be the universal enveloping algebra of the Lie algebra

$$sl(2) =< X, Y, Z | [H, X] = X, [H, Y] = -Y, [X, Y] = 2H >$$

over a field K, $C = H(H+1) + YX$ be *Casimir* element. Evidently,

$$U \simeq K[H, C](\sigma, a = C - H(H+1)),\ X \to X,\ Y \to Y,\ H \to H,\ H(H+1) + YX \to C,$$

where $\sigma: H \to H-1, C \to C$.

Example 4. For any $\lambda \in K$ the factor algebra $U(\lambda) := U/U(C-\lambda)$ is the GWA from Example 2:

$$U(\lambda) \simeq K[H](\sigma, a = \lambda - H(H+1)).$$

Example 5. $A = K[H](\sigma, a), \sigma(H) = \lambda H, \lambda \neq 0 \in K$. Let $\Lambda = K < X, Y | XY = \lambda YX >$ (*the quantum plane*), then

$$\Lambda \simeq K[H](\sigma, a = H),\ X \to X,\ Y \to Y,\ H \to H.$$

Example 6. Let $A(S^2_\lambda) = K < X, Y, H \,|\, XH = \lambda HX,\ YH = \lambda^{-1}HY,\ \lambda^{1/2}YX = -(c-H)(d+H),\ \lambda^{-1/2}XY = -(c-\lambda H)(d+\lambda H) >$ be the algebra of functions on the quantum 2-dimensional sphere, then

$$A(S^2_\lambda) \simeq K[H](\sigma, a = -\lambda^{-1/2}(c-H)(d+H)),\ X \to X,\ Y \to Y,\ H \to H,$$

where $\sigma(H) = \lambda H$.

1.4 Main object.

Let $\mathcal{A}_i = \mathcal{D}_i(\sigma_i, a_i)$, $i = 1, \dots, n$, be a generalized Weyl algebra from Example 1 (see subsection 1.3), i.e. $\mathcal{D}_i = K[H_i]$ is a polynomial ring in 1 variable, $\sigma_i(H_i) = \lambda_i H_i + \mu_i$, $\lambda_i \neq 0, \mu_i \in K$. From now on let the algebra

$$A = \bigotimes_{i=1}^n \mathcal{A}_i$$

be the tensor product over K of the generalized Weyl algebras of degree 1. Then A is the generalized Weyl algebra of degree n

$$A = (\bigotimes_{i=1}^{n} \mathcal{D}_i)(\sigma = (\sigma_1, \dots, \sigma_n), a = (a_1, \dots, a_n))$$

with basic ring $D = \bigotimes_{i=1}^{n} \mathcal{D}_i = K[H_1, \dots, H_n]$, the polynomial ring in n variables, and with defining automorphisms $\{\sigma_i : \sigma_i(H_i) = \lambda_i H_i + \mu_i, \sigma_i(H_j) = H_j$, for all $j \neq i\}$. The aim of the present paper is to study indecomposable generalized weight A-modules.

The Weyl algebra A_n. Define the n-th *Weyl algebra*, $A_n = A_n(K)$, over a field (a ring) K to be the associative K-algebra with identity generated by the $2n$ indeterminates $X_1, ..., X_n, \partial_1, ..., \partial_n$, subject to the relations:

$$[X_i, X_j] = [\partial_i, \partial_j] = [\partial_i, X_j] = 0 \text{ for } i \neq j, \ [\partial_i, X_i] = 1 \text{ for all } i.$$

The Weyl algebra A_n is the generalized Weyl algebra $A = D(\sigma; a)$ of degree n with a basic polynomial ring $D = K[H_1, ..., H_n]$ in n variables with coefficients in K. The sets of defining elements and automorphisms of A are $\{a_i = H_i | 1 \leq i \leq n\}$ and $\{\sigma_i | \sigma_i(H_j) = H_j - \delta_{ij})$, respectively, where $\delta_{ij} = 1$, if $i = j$; and 0, otherwise. Moreover, the map

$$A_n \to A, \ X_i \to X_i^+, \ \partial_i \to X_i^-, \ \partial_i X_i \to H_i, \ i = 1, \dots, n,$$

is the isomorphism of algebras.

The *quantum Weyl algebra* of degree n

$$A_n(q) = A_1(q) \otimes \dots \otimes A_1(q), \ (n \text{ times}), \ q \neq 0 \in K$$

is another example (see Example 3).

1.5 The equivalence relation on the set of maximal ideal of the polynomial ring $D = K[H_1, \dots, H_n]$.

Let $\mathcal{D} = K[H]$ be a polynomial ring in 1 variable H and σ its K-automorphism, i.e. $\sigma(H) = \lambda H + \mu$ for some $\lambda \neq 0$, $\mu \in K$. On the set $\mathcal{M}$ of maximal ideals of $\mathcal{D}$ the group G, generated by σ, acts in an obvious way. Since K is algebraically closed we can indentify $\mathcal{M}$ with K by the map $(H - \gamma) \leftrightarrow \gamma$, $\mathcal{M} \equiv K$. Under this identification σ acts on K as the map $\tau(\gamma) = \lambda^{-1}(\gamma - \mu)$. Each orbit $\mathcal{O}(\lambda)$ equals to $\mathcal{O}(\lambda) = \{\sigma^i(\lambda), i \in \mathbf{Z}\}$ for some scalar λ of K. An orbit is *cyclic* of the length n (respectively, *linear*) if it contains a finite (respectively, infinite) number $n = |\mathcal{O}|$ of elements. The set of cyclic (linear) orbits denote by *Cyc* (*Lin*).

Fix some non-zero polynomial $a \in K[H]$, futher it is a defining element of a generalized Weyl algebra. An orbit $\mathcal{O}$ is *degenerate*, if it contains a root of a. Denote by *Cycd* and *Lind* (*Cycn* and *Linn*) respectively the set of all (non-)degenerate cyclic and linear orbits. The degenerate orbit which contains more than one root are called *strong degenerate* (otherwise *weak degenerate*) and we denote by *Cycds* and *Linds* the set of all such cyclic and linear orbits respectively (otherwise by *Cycdw* and *Lindw*).

Each linear orbit $\mathcal{O}(\lambda)$, $\lambda \in \mathcal{O}$, by the map $\mathcal{O}(\lambda) \to \mathbf{Z}, \tau^i(\lambda) \to i$, can be identified with the set of integers $\mathbf{Z}$. Therefore, for $\mathcal{O}(\lambda) \in Lin$ we can use, without mentioning it explicitly, all the definitions and notations which are employed for $\mathbf{Z}$ (such that the order, the segment, the semiaxis, etc.). For example, $\tau^i(\lambda) \le \tau^j(\lambda)$ iff $i \le j$; $(-\infty, \lambda] = \{\tau^i(\lambda), i \le 0\}$. For a cyclic orbit $\mathcal{O}$ of the length n we use only the definition of the segment $(\mu, \lambda] = \{\tau^i(\mu), 0 < i \le j \le n\}$ where j is the minimal positive integer such that $\tau^j(\mu) = \lambda$. The roots $\lambda_1 < \ldots < \lambda_s$ (respectively, λ_1, $\lambda_2 = \sigma^{i(2)}(\lambda_1), \ldots, \lambda_s = \sigma^{i(s)}(\lambda_1)$, $0 < i(2) < \ldots < i(s) < n$) of the degenerate linear (respectively, cyclic) orbit $\mathcal{O}$ divide it into $s+1$ (respectively, s) parts,

$$\Gamma_1 = (-\infty, \lambda_1],\ \Gamma_2 = (\lambda_1, \lambda_2], \ldots, \Gamma_{s+1} = (\lambda_s, \infty),$$

$$(\text{respectively, } \Gamma_1 = (\lambda_s, \lambda_1], \Gamma_2 = (\lambda_1, \lambda_2], \ldots, \Gamma_s = (\lambda_{s-1}, \lambda_s]).$$

We say that scalars (i.e. maximal ideals) λ and μ are *equivalent* ($\lambda \sim \mu$) if they belong either to a non-degenerate orbit or to some Γ_i. An equivalence class Γ containing a root of a is caled *marked* (otherwise *non-marked*). Let $\hat{\mathcal{O}}$ be the set of equivalence classes from the orbit $\mathcal{O}$. Denote by $\mathcal{O}(\gamma)$ and $\Gamma(\gamma)$ respectively the orbit and equivalence class of $\gamma \in K$.

Now for $\mathcal{D}_i, \sigma_i, a_i$ from subsection 1.4 we have the action of the group $G_i =< \sigma_i > \simeq < \tau_i >$ on the set of maximal ideal $\mathcal{M}_i \equiv K$ by the map $\tau_i(\gamma) = \lambda_i^{-1}(\gamma - \mu_i)$ and the equivalence relation $\sim_i$. The set $\mathcal{M}$ of maximal ideal of $D = \bigotimes_{i=1}^n \mathcal{D}_i = K[H_1, \ldots, H_n]$ can be identified with $\prod_{i=1}^n \mathcal{M}_i \equiv K^n$, so define the equivalence relation $\sim$ and action of $G = \prod_{i=1}^n G_i$ on $\mathcal{M} \equiv K^n$ respectively as the direct product of equivalence relations ($\sim = \prod_{i=1}^n \sim_i$) and actions. Then an equivalence class $\Gamma(\alpha)$ (respectively, an orbit $\mathcal{O}(\alpha)$) of $\alpha = (\alpha_i) \in K^n$ is the direct product $\prod_{i=1}^n \Gamma(\alpha_i)$ ($\prod_{i=1}^n \mathcal{O}(\alpha_i)$) of equivalence classes (respectively, orbits).

1.6 Weight and generelazed weight A-modules.

Let A be the generalized Weyl algebra from subsection 1.4. Denote by $\mathcal{H}$ and $\mathcal{H}^*$ respectively the n-dimensional vector space with basis $H_1, \ldots, H_n$ and its dual. Element χ from $\mathcal{H}^*$ is called a *weight*.

A-module M is called *generalized weight* (respectively, *weight*) if

$$M = \oplus_{\chi \in \mathcal{H}^*} M^\chi,\ M^\chi = \left\{ m \in M : (h - \chi(h))^k m = 0 \text{ for all } h \in \mathcal{H}, k = k(m) \right\},$$

$$\dim M^\chi < \infty \text{ for all } \chi \in \mathcal{H}^*,$$

$$(\text{respectively, } M = \oplus_{\chi \in \mathcal{H}^*} M_\chi,\ M_\chi = \{m \in M : (h - \chi(h))m = 0 \text{ for all } h \in \mathcal{H}\},$$

$$\dim M_\chi < \infty \text{ for all } \chi \in \mathcal{H}^*.$$

$\mathrm{Supp}(M) = \{\chi \in \mathcal{H}^* : M^\chi \neq 0\}$ is the *support* of M. The category of generalized weight A-modules is a full subcategory of the category of A-modules. Moreover, if $0 \to N \to M \to L \to 0$ is an exact sequence of generalized weight modules, then $\mathrm{Supp}\, M = \mathrm{Supp}\, N + \mathrm{Supp}\, L$ and $\dim M^\chi = \dim N^\chi + \dim L^\chi$ for each $\chi \in \mathcal{H}^*$. Using the defining relations of A we see that

$$X_i^\pm M^\chi \subseteq M^{\tau_i{}^{\pm 1}(\chi)}, \ \ (X_i^\pm M_\chi \subseteq M_{\tau_i{}^{\pm 1}(\chi)}) \text{ for all } \chi \in \mathcal{H}^*. \tag{1.6.1}$$

Thus each generalized weight (respectively, weight) module M decomposes into the direct sum of A-submodules

$$M = \oplus M^{\mathcal{O}}, \ M^{\mathcal{O}} := \bigoplus_{\chi \in \mathcal{O}} M^\chi \ , (M = \oplus M_{\mathcal{O}}, \ M_{\mathcal{O}} := \bigoplus_{\chi \in \mathcal{O}} M_\chi \ ) \tag{1.6.2}$$

where $\mathcal{O}$ runs over all orbits of $\mathcal{M}$. If M is indecomposable, then its support belong to some orbit $\mathcal{O}$.

Lemma 1.6.1 *If M is a generalized weight (resp., weight) A-module, then* $\dim M^\chi = \dim M^\gamma$ *(resp.,* $\dim M_\chi = \dim M_\gamma$ *) for every equivalent weights $\chi \sim \gamma$ from K^n.*

Proof. It is enough to prove for $\gamma = \tau_i(\chi)$. Since the vector space M^χ is finite dimensional from the relations $X_i^- X_i^+ = a_i$ and $X_i^+ X_i^- = \sigma_i(a_i)$ we conclude that the linear maps $X_i^+ : M^\chi \to M^{\tau_i(\chi)}$ and $X_i^- : M^{\tau_i(\chi)} \to M^\chi$ are bijections. ∎

Corollary 1.6.2 *A support of generalized weight module is a union of equivalence classes.* ∎

Denote by $\mathcal{W}(\mathcal{O})$ and $\mathcal{GW}(\mathcal{O})$ respectively the category of weight and generalized weight modules with supports from an orbit $\mathcal{O}$. Let $\mathrm{Ind}\, \mathcal{W}(\mathcal{O})$ and $\mathrm{Ind}\, \mathcal{GW}(\mathcal{O})$ be the set of isoclasses of indecomposable modules from $\mathcal{W}(\mathcal{O})$ and $\mathcal{GW}(\mathcal{O})$ respectively. As usually we identify the isoclass of modules with one of its representative.

2 The proof of Theorem 1 and 2

From now on all the considered categories $\mathcal{C}$ will be *K-linear*, i.e. for every two objects $i, j \in \mathcal{C}$ the set of morphisms $\mathcal{C}(i,j)$ is a vector space over the field K and the composition map $\mathcal{C}(j,k) \times \mathcal{C}(i,j) \to \mathcal{C}(i,k)$ is bilinear for all objects i, j, k. We say that a $\mathcal{C}$-module $V = \{V_i\}$ is a *local finite dimensional* if $\dim V_i < \infty$ for each object $i \in \mathcal{C}$. Given categories $\mathcal{C}, \mathcal{D}$, the category $\mathcal{E}$ is the tensor product $\mathcal{E} = \mathcal{C} \otimes \mathcal{D}$ of categories $\mathcal{C}$ and $\mathcal{D}$ if $Ob\,\mathcal{E} = Ob\mathcal{C} \times OB\,\mathcal{D}$, $\mathcal{E}((i,k),(j,l)) = \mathcal{C}(i,j) \otimes_K \mathcal{D}(k,l)$ and the composition of morphisms is defined in the following way $(a \otimes b)(c \otimes d) = ac \otimes bd$.

2.1 Category $\mathcal{C}_{\mathcal{O}}$.

For each orbit $\mathcal{O} \in K^n$ define the category $\mathcal{C}_{\mathcal{O}}$ as K-linear category with set of objects $\mathcal{O}$, generated over K by the set of morphisms $\{ X_{\alpha,i}, Y_{\alpha,i}, H_{\alpha,i} \,|\, \alpha \in \mathcal{O},\, i = 1, \dots, n \}$ where

$$X_{\alpha,i} : \alpha \to \tau_i(\alpha),\ Y_{\alpha,i} : \tau_i(\alpha) \to \alpha,\ H_{\alpha,i} : \alpha \to \alpha,$$

subject to the relations

$$Y_{\alpha,i}\, X_{\alpha,i} = a_i(H_{\alpha,i}),\ X_{\alpha,i}\, Y_{\alpha,i} = \sigma_i(a_i)(H_{\tau_i(\alpha),i}),\ X_{\alpha,i}\, H_{\alpha,i} = \sigma_i(H_{\tau_i(\alpha),i})\, X_{\alpha,i},$$

$$Y_{\alpha,i}\, H_{\tau_i(\alpha),i} = {\sigma_i}^{-1}(H_{\alpha,i})\, Y_{\alpha,i},\ U_{\alpha,i}\, V_{\beta,j} - V_{\gamma,j}\, U_{\delta,i} = 0 \text{ for all } i \neq j$$

and all possible $U, V \in \{H, X, Y\}$, $\alpha, \beta, \gamma, \delta \in \mathcal{O}$, for which the last equality makes sense (where $\sigma_i(H_*) = \lambda_i H_* + \mu_i$). It is clear that

$$\mathcal{C}_{\mathcal{O}} \simeq \otimes_{i=1}^{n} \mathcal{C}_{\mathcal{O}_i}$$

where $\mathcal{C}_{\mathcal{O}_i}$ is the above defined categories in the case $n = 1$, $\sigma = \sigma_i$. Remark that *each generalized weight A-module with support from $\mathcal{O}$ is $\mathcal{C}_{\mathcal{O}}$-module and vice versa.*

Given a generalized weight module M with support from $\mathcal{O}$, each element

$$h_{\alpha,i} := H_{\alpha,i} - \alpha_i,\ \alpha = (\alpha_i) \in \mathcal{O},\ i = 1, \dots, n,$$

is a nilpotent endomorphism of the finite dimensional vector space M^{α}. Choosing new generators $\{X_{\alpha,i},\, Y_{\alpha,i},\, h_{\alpha,i}\}$ for the category $\mathcal{C}_{\mathcal{O}}$ the defining relations can be rewritten as

$$Y_{\alpha,i}\, X_{\alpha,i} = a_i(h_{\alpha,i} + \alpha_i),\ X_{\alpha,i}\, Y_{\alpha,i} = a_i(\lambda_i h_{\tau_i(\alpha),i} + \alpha_i),\ X_{\alpha,i}\, h_{\alpha,i} = \lambda_i h_{\tau_i(\alpha),i}\, X_{\alpha,i},$$

$$h_{\alpha,i}\, Y_{\alpha,i} = \lambda_i\, Y_{\alpha,i}\, h_{\tau_i(\alpha),i},\ U_{\alpha,i}\, V_{\beta,j} - V_{\gamma,j}\, U_{\delta,i} = 0 \text{ for all } i \neq j$$

and all possible $U, V \in \{h, X, Y\}$, $\alpha, \beta, \gamma, \delta \in \mathcal{O}$, for which the last equality makes sence.

Denote by $\mathcal{C}_{\mathcal{O}}(nil)$ (respectively, $\mathcal{C}_{\mathcal{O}}(0)$) the category of local finite dimensional representations of $\mathcal{C}_{\mathcal{O}}$ with nilponent (respectively, zero) action of all $h_{\alpha,i}$-th. Then the following equivalences of categories is evident,

$$\mathcal{GW}(\mathcal{O}) \simeq \mathcal{C}_{\mathcal{O}}(nil),\ \mathcal{W}(\mathcal{O}) \simeq \mathcal{C}_{\mathcal{O}}(0). \tag{2.1.1}$$

Till the end of subsec. 2.1 we suppose that $A = \mathcal{D}(\sigma, a)$ is a generalized Weyl algebra of degree 1 and for the sake of simplicity of formulas we do not write the second index 1 in expresions like $X_{\alpha,1}$.

Let $\mathcal{O} \in Lin$, $\alpha, \beta = \tau^i(\alpha) \in \mathcal{O}$, $i \in \mathbf{Z}$. Denote by $l_{\beta\alpha}$ the "minimal" way from α to β, i.e. $l_{\beta\alpha} = X_{\tau^{i-1}(\alpha)} \dots X_{\tau(\alpha)} X_{\alpha}$ if $i > 0$, $l_{\alpha\alpha} = 1_{\alpha}$, $l_{\beta\alpha} = Y_{\tau^{-i}(\alpha)} \dots Y_{\tau^{-1}(\alpha)}$ if $i < 0$.

Lemma 2.1.1 *Let $\mathcal{O}, \alpha$ and β be as above, then*

1. $\mathcal{C}_{\mathcal{O}}(\alpha,\alpha) = K[H_\alpha]$;

2. $\mathcal{C}_{\mathcal{O}}(\alpha,\beta) = \mathcal{C}_{\mathcal{O}}(\beta,\beta)\, l_{\beta\alpha} = l_{\beta\alpha}\mathcal{C}_{\mathcal{O}}(\alpha,\alpha)$.

Proof. It is an immediate consequence of the above defining relations. ∎

Let $\mathcal{O} \in Cyc$, $|\mathcal{O}| = m$, $\alpha \neq \beta \in \mathcal{O}$, then $\beta = \tau^i(\alpha)$ for some $0 < i \leq m-1$. Let

$$c_{\alpha,+} = X_{\tau^{m-1}(\alpha)} \dots X_{\tau(\alpha)} X_\alpha,\ \ c_{\alpha,-} = Y_{\tau^{-m}(\alpha)} \dots Y_{\tau^{-1}(\alpha)}.$$

Lemma 2.1.2 *Let $\mathcal{O}, \alpha$ and β be as above, then $\mathcal{C}(\alpha,\alpha) = K < H_\alpha,\, c_{\alpha,+},\, c_{\alpha,-} >\simeq K[H_\alpha](\sigma^m, (-m,m)(H_\alpha))$ is the generalized Weyl algebra of degree 1 where $(-m,m)(H) = \sigma^{-m+1}(a)\dots a \in K[H]$.* ∎

Denote by $l_{\beta\alpha,+}$ and $l_{\beta\alpha,-}$ the two "minimal" ways from α to β, in $\mathcal{C}_{\mathcal{O}}$ i.e.

$$l_{\beta\alpha,+} = X_{\tau^{i-1}(\alpha)} \dots X_{\tau(\alpha)} X_\alpha,\ \ l_{\beta\alpha,-} = Y_{\tau^{-(n-i)}(\alpha)} \dots Y_{\tau^{-1}(\alpha)}.$$

Lemma 2.1.3 *1.* $c_{\beta,\pm}\, l_{\beta\alpha,\pm} = l_{\beta\alpha,\pm}\, c_{\alpha,\pm}$;

2. $c_{\beta,-}\, l_{\beta\alpha,+} = l_{\beta\alpha,-}\,(-i,i)(H_\alpha)$;

3. $c_{\beta,+}\, l_{\beta\alpha,-} = l_{\beta\alpha,+}\,(m-i,-m+i)(H_\alpha)$;

4. $l_{\beta\alpha,+}\, c_{\alpha,-} = (i,-i)(H_\beta)\, l_{\beta\alpha,-}$;

5. $l_{\beta\alpha,-}\, c_{\alpha,+} = (-n+i, n-i)(H_\beta)\, l_{\beta\alpha,+}$.

Proof. Prove for example the second equality, the others can be shown similarly:

$$c_{\beta,-}\, l_{\beta\alpha,+} = Y_{\tau^{-m}(\beta)} \dots Y_{\tau^{-1}(\beta)} X_{\tau^{i-1}(\alpha)} \dots X_{\tau(\alpha)} X_\alpha =$$
$$Y_{\tau^{-m}(\beta)} \dots Y_{\tau^{-2}(\beta)} a(H_{\tau^{i-1}(\alpha)}) X_{\tau^{i-2}(\alpha)} \dots$$
$$X_{\tau(\alpha)} X_\alpha =$$
$$Y_{\tau^{-m}(\beta)} \dots Y_{\tau^{-2}(\beta)} X_{\tau^{i-2}(\alpha)} \dots X_{\tau(\alpha)} X_\alpha \sigma^{-i+1}(a)(H_\alpha) = \dots = l_{\beta\alpha,-}\,(-i,i)(H_\alpha).\ ∎$$

Lemma 2.1.4 $\mathcal{C}_{\mathcal{O}}(\alpha,\beta) = \mathcal{C}_{\mathcal{O}}(\beta,\beta)\, l_{\beta\alpha,+} + \mathcal{C}_{\mathcal{O}}(\beta,\beta)\, l_{\beta\alpha,-}$.

Proof is evident. ∎

Define (also for an arbitary n)

$$\mathcal{N}_{\alpha,\beta} = \mathcal{N}_{\mathcal{O},\alpha,\beta} := \mathcal{C}_{\mathcal{O}}(\beta,\alpha)\mathcal{C}_{\mathcal{O}}(\alpha,\beta).$$

By Lemmas 2.1.1–2.1.4 we have Lemma 2.1.5.

Lemma 2.1.5 *1. Let $\mathcal{O} \in Lin$, $\alpha \neq \beta = \tau^i(\alpha) \in \mathcal{O}, i \in \mathbf{Z}$. Then $\mathcal{N}_{\alpha,\beta} = \mathcal{C}_{\mathcal{O}}(\alpha,\alpha)(-i,i)(H_\alpha)$.*

2. Let $\mathcal{O} \in Cyc$, $\alpha \neq \beta = \tau^i(\alpha) \in \mathcal{O}, i > 0$. Then $\mathcal{N}_{\alpha,\beta} = \mathcal{C}_{\mathcal{O}}(\alpha,\alpha)(c_{\alpha,+},\ c_{\alpha,-},\ (-i,i)(H_\alpha),\ (m-i,-m+i)(H_\alpha))$, i.e. the left $\mathcal{C}_{\mathcal{O}}(\alpha,\alpha)$-module $\mathcal{N}_{\alpha,\beta}$ is generated by the morphisms $c_{\alpha,+},\ c_{\alpha,-},\ (-i,i)(H_\alpha)$ and $(m-i,-m+i)(H_\alpha)$. ∎

Since for an arbitrary n the category $\mathcal{C}_{\mathcal{O}}$ is the tensor product of the categories $\mathcal{C}_{\mathcal{O}_i}$, it is clear that

$$\mathcal{N}_{\mathcal{O},\alpha,\beta} = \otimes_{i=1}^n \mathcal{N}_{\mathcal{O}_i,\alpha_i,\beta_i}. \tag{2.1.2}$$

2.2 Categories $\mathcal{C}_{\mathcal{O},nil}$ and $\mathcal{C}_{\mathcal{O},1}$.

For each natural number $m \in \mathbf{N}$ define the category $\mathcal{C}_{\mathcal{O},m}$:

$$\mathcal{C}_{\mathcal{O},m} := \mathcal{C}_{\mathcal{O}}/({h_{\alpha,i}}^m,\ \text{for all } \alpha \in \mathcal{O},\ i = 1,\ldots,n).$$

Then

$$\mathcal{C}_{\mathcal{O},m} = \otimes_{i=1}^n \mathcal{C}_{\mathcal{O}_i,m}. \tag{2.2.1}$$

If we define $\mathcal{N}_{\alpha,\beta,m} \equiv \mathcal{N}_{\mathcal{O},\alpha,\beta,m} := \mathcal{C}_{\mathcal{O},m}(\beta,\alpha)\mathcal{C}_{\mathcal{O},m}(\alpha,\beta)$, then

$$\mathcal{N}_{\alpha,\beta,m} = \otimes_{i=1}^n \mathcal{N}_{\alpha_i,\beta_i,m}. \tag{2.2.2}$$

Let

$$\mathcal{C}_{\mathcal{O},nil} := \lim_{\leftarrow} \mathcal{C}_{\mathcal{O},m}$$

be the inverse limit of the sequence of functors $\ldots \to \mathcal{C}_{\mathcal{O},m} \to \ldots \to \mathcal{C}_{\mathcal{O},1}$, and $\mathcal{C}_{\mathcal{O},nil}$-mod (respectively, $\mathcal{C}_{\mathcal{O},1}$-mod) be the category of local finite dimensional $\mathcal{C}_{\mathcal{O},nil}$-modules (resprectively, $\mathcal{C}_{\mathcal{O},1}$-modules).

Lemma 2.2.1 *$\mathcal{C}_{\mathcal{O}}(nil) \simeq \mathcal{C}_{\mathcal{O},nil}$, and $\mathcal{C}_{\mathcal{O}}(0) \simeq \mathcal{C}_{\mathcal{O},1}$.* ∎

2.3 Categories $\mathcal{L}_{\mathcal{O},nil}$ and $\mathcal{L}_{\mathcal{O},1}$.

Suppose that $\Gamma = (\Gamma_1,\ldots,\Gamma_n)$ is an equivalence class from an orbit $\mathcal{O}$. Then each Γ_i contains no more than one root of the defining element a_i of the generalized Weyl algebra $\mathcal{A}_i$. If such a root exists denote by $<\Gamma, i>$ its multiplicity. For each Γ and $i \in \{1,\ldots,n\}$ let $\Gamma[i]$ equals to Γ, if $\tau_i(\Gamma) \subseteq \Gamma$, and otherwise to the uniquely determined equivalence class Γ' such that $\Gamma' \neq \Gamma$ and $\Gamma' \cap \tau_i(\Gamma) \neq \oslash$. Let $\Gamma[i,0] := \Gamma$ and by induction $\Gamma[i,j] := \Gamma[i,j-1][i]$ for $j \geq 1$. It is clear that $\Gamma[i] = \Gamma[i,1]$.

For each orbit $\mathcal{O} = \mathcal{O}_1 \times \ldots \times \mathcal{O}_n$ consider the category $\mathcal{L}_{\mathcal{O}}$ with objects $\Gamma \in \hat{\mathcal{O}}$ and generating set of morphisms:

$$h_{\Gamma,i} : \Gamma \to \Gamma, \text{ for each } \Gamma \in \hat{\mathcal{O}},$$

$$x_{\Gamma,i} : \Gamma \to \Gamma[i],\ y_{\Gamma,i} : \Gamma[i] \to \Gamma, \text{ for each marked } \Gamma_i \in \hat{\mathcal{O}},$$

$$g_{\Gamma,i},\ {g_{\Gamma,i}}^{-1} : \Gamma \to \Gamma, \text{ for each } \Gamma \in \hat{\mathcal{O}} \text{ such that } \mathcal{O}_i \in Cycn,$$

where $i \in 1, \ldots, n$, subject to the relations:

$$y_{\Gamma,i}\, x_{\Gamma,i} = {h_{\Gamma,i}}^{<\Gamma,i>},\ x_{\Gamma,i}\, y_{\Gamma,i} = \lambda_i^{|\Gamma[i]_i|<\Gamma,i>}\, {h_{\Gamma[i],i}}^{<\Gamma,i>},\ x_{\Gamma,i}\, h_{\Gamma,i} = \lambda_i^{|\Gamma[i]_i|}\, h_{\Gamma[i],i}\, x_{\Gamma,i},$$

$$h_{\Gamma,i}\, y_{\Gamma,i} = \lambda_i^{|\Gamma[i]_i|}\, y_{\Gamma,i}\, h_{\Gamma[i],i},\ u_{\Gamma,i}\, v_{L,j} - v_{M,j}\, u_{N,i} = 0, \text{ for } i \neq j$$

and all possible $u, v \in \{x, y, h, g,\}$ and $\Gamma, L, M, N \in \hat{\mathcal{O}}$, for which the above equality makes sense; where $|\Gamma[i]_i|$ is a number of the set $\Gamma[i]_i$ (the ith component of $\Gamma[i]$) when it is finite and $|\Gamma[i]_i| = 1$, otherwise.

Let

$$\mathcal{L}_{\mathcal{O},m} := \mathcal{L}_{\mathcal{O}}/({h_{\Gamma,i}}^m,\ \forall \Gamma \in \hat{\mathcal{O}},\ i = 1, \ldots, n),\ m \in \mathbf{N},\ \mathcal{L}_{\mathcal{O},nil} := \varprojlim \mathcal{L}_{\mathcal{O},m}.$$

It is evident that

$$\mathcal{L}_{\mathcal{O}} \simeq \otimes_{i=1}^n \mathcal{L}_{\mathcal{O}_i} \text{ and } \mathcal{L}_{\mathcal{O},m} \simeq \otimes_{i=1}^n \mathcal{L}_{\mathcal{O}_i,m}. \tag{2.3.1}$$

Recall that a full subcategory $\mathcal{C}$ of $\mathcal{B}$ is called a *skeleton* of $\mathcal{B}$ provided objects of $\mathcal{C}$ pairwise non-isomorphic and any object of $\mathcal{C}$ is isomorphic to some object of $\mathcal{B}$. It is clear that if $\mathcal{C}$ is a skeleton of $\mathcal{B}$, then the categories of $\mathcal{C}$-modules and $\mathcal{B}$-modules are equivalent.

Proposition 2.3.1 *The categories $\mathcal{L}_{\mathcal{O},nil}$ and $\mathcal{L}_{\mathcal{O},1}$ are the sceletons of the categories $\mathcal{C}_{\mathcal{O},nil}$ and $\mathcal{C}_{\mathcal{O},1}$ respectively.*

Proof. It is enough to prove that the category $\mathcal{L}_{\mathcal{O},m}$ is the skeleton of the category $\mathcal{C}_{\mathcal{O},m}$ for each $m \in \mathbf{N}$.

Let us prove that objects α and β of the category $\mathcal{C}_{\mathcal{O},m}$ are isomorphic if and only if they are equivalent.

"$\Leftarrow$" Suppose that α and β are equivalent and neighbouring, i.e. $\beta = \tau_i(\alpha)$ (or $\alpha = \tau_i(\beta)$) for some i. Then α_i is not a root of the polynomial a_i, and since ${h_{\alpha,i}}^m = 0$ the morphism $a_i(h_{\alpha,i} + \alpha_i) : \alpha \to \alpha$ is an isomorphism in $\mathcal{C}_{\mathcal{O},m}$. Using relations $Y_{\alpha,i}\ X_{\alpha,i} = a_i(h_{\alpha,i} + \alpha_i)$ and $X_{\alpha,i}\ Y_{\alpha,i} = a_i(\lambda_i\, h_{\beta,i} + \alpha_i)$ we see that morphisms $X_{\alpha,i} : \alpha \to \beta$ and $Y_{\alpha,i} : \beta \to \alpha$ are isomorphisms.

If $\alpha \sim \beta$, then by the definition of the equivalence we can choose a chain $\alpha = \gamma_1, \ldots, \gamma_k = \beta$ such that $\gamma_j,\ \gamma_{j+1}$ are equivalent and neihgbouring for all $j = 1, \ldots, k-1$, so $\alpha \simeq \beta$.

"$\Rightarrow$" Suppose that α and β are not equivalent and $\beta = \tau_1^{j_1} \dots \tau_n^{j_n}(\alpha)$, $(j_1, \dots, j_n) \in \mathbf{Z}^n$, then $\alpha_i \not\sim \beta_i$ for some $i \in \{1, \dots, n\}$. Without less of generality we can assume that $j_i > 0$.

Case 1, $\mathcal{O}_i \in Lin$. Since the polynomial $(-j_i, j_i)(H) \in K[H]$ has a root ${\tau_i}^k(\alpha)$ for some $k \in \{0, \dots, j_i - 1\}$ the morphism $(-j_i, j_i)(H_{\alpha,i}) : \alpha \to \alpha$ is not an isomorphism in $\mathcal{C}_{\mathcal{O},m}$, so by Lemma 2.1.5 and (2.2.2) $\alpha \not\simeq \beta$.

Case 2, $\mathcal{O}_i \in Cyc$, $|\mathcal{O}_i| = s$. By definition of $\sim_i$ ${\tau_i}^k(\alpha_i)$ and ${\tau_i}^l(\alpha_i)$ are roots of the polynomial a_i for some $0 \leq k < j_i$ and $j_i \leq l < s$, so morphisms $(-j_i, j_i)(H_{\alpha,i})$, $(s - j_i, -s + j_i)(H_{\alpha,i})$ are not isomorphisms. By Lemma 2.1.2 $\mathcal{C}_{\mathcal{O}_i,m}(\alpha_i, \alpha_i)$ is the generalized Weyl algebra $\mathcal{D}_{\alpha_i,m}({\sigma_i}^s, (-s, s)(H_{\alpha,i}))$ with basic ring $\mathcal{D}_{\alpha_i,m} := K[H_{\alpha,i}]/((H_{\alpha,i} - \alpha_i)^m)$, then using Lemma 2.1.5

$$\mathcal{N}_{\alpha_i,\beta_i,m} \cap \mathcal{D}_{\alpha_i,m} = \mathcal{D}_{\alpha_i,m}((-j_i, j_i)(H_{\alpha,i}),\ (s-j_i, -s+j_i)(H_{\alpha,i}),\ (s, -s)(H_{\alpha,i}),\ (-s, s)(H_{\alpha,i}))$$

$$\subseteq \mathcal{D}_{\alpha_i,m} h_{\alpha,i},$$

hence, by (2.2.2) $\alpha \not\simeq \beta$.

So there is 1-1 correspondence between objects of the skeleton of the category $\mathcal{C}_{\mathcal{O},m}$ and the equivalence classes of the orbit $\mathcal{O}$.

Suppose that $n = 1$ (as usually in the case $n = 1$ the lower index 1 is ommitted), now it is easy to verify that the functor

$$P : \mathcal{L}_{\mathcal{O},m} \to \mathcal{C}_{\mathcal{O},m}$$

is the equivalence of the categories where

$$P : Ob\, \mathcal{L}_{\mathcal{O},m} \to Ob\, \mathcal{C}_{\mathcal{O},m},\ \Gamma \to P(\Gamma),$$

and $P(\Gamma) = \lambda$, if $\Gamma = (-\infty, \lambda]$, $(\mu, \lambda]$ or $(\tau^{-1}(\lambda), \infty)$; in the remaining cases λ is an arbitrary fixed element from Γ.

$$P : Mor\, \mathcal{L}_{\mathcal{O},m} \to Mor\, \mathcal{C}_{\mathcal{O},m},\ u \to P(u),$$

where $P(h_\Gamma) = h_{P(\Gamma)}$ for each equivalence class Γ from $\mathcal{O}$. If $\mathcal{O} = \Gamma \in Cycn$, then $P(g_\Gamma) = c_{P(\Gamma),+}$.

If $\mathcal{O} \in Lind \cup Cycd$ and the equivalence class Γ from $\mathcal{O}$ contains a root $\alpha = P(\Gamma)$ of the polynomial a, i.e. $\Gamma = (-\infty, \alpha]$ or $\Gamma = (\gamma, \alpha]$, then $\beta = P(\Gamma[1]) = \tau^i(\alpha)$ for some $i > 0$. Put

$$P(y_\Gamma) = l_{\alpha\beta},\ P(x_\Gamma) = l_{\beta\alpha} b_{\alpha,\beta}^{-1}(H_\alpha),$$

where $l_{\alpha\beta} l_{\beta\alpha} = {h_\alpha}^{<\Gamma,1>} b_{\alpha\beta}(H_\alpha)$ and $b_{\alpha\beta}(H) := \sigma^{-i+1}(a) \dots a/(H - \alpha)^{<\Gamma,1>}$.

For an arbitrary n the tensor product of functors

$$P := \otimes_{i=1}^n P_i : \mathcal{L}_{\mathcal{O},m} = \otimes_{i=1}^n \mathcal{L}_{\mathcal{O}_i,m_i} \to \mathcal{C}_{\mathcal{O},m} = \otimes_{i=1}^n \mathcal{C}_{\mathcal{O}_i,m_i}$$

is the equivalence of the categories where the functor $P_i : \mathcal{L}_{\mathcal{O}_i,m_i} \to \mathcal{C}_{\mathcal{O}_i,m_i}$ is defined above (for $n = 1$). $\blacksquare$

2.4 Wild categories.

A category $\mathcal{C}$ is *wild* if there is a finitely generated $\mathcal{C} - K < x, y >$-bimodule M which is free as a right $K < x, y >$- module and such that the functor $M \otimes_{K<x,y>} -$ from the category of finite-dimensional $K < x, y >$-modules to $\mathcal{C}$-mod preserves indecomposability and isomorhism classes [D1, CB2].

We list below the wild categories which are used in the proofs of Theorems 1 and 2.

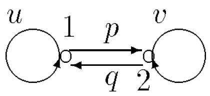
A : $K < u, v | uv = vu, u^s = 0, v^t = 0 >$

B : $K < u, u^{-1}, v, v^{-1} | uv = vu >$

C : $K < u, u^{-1}, v | uv = vu, v^t = 0 >$

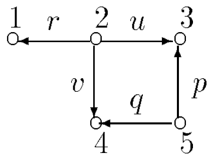
D : $K < u, v, w | st = 0, s, t \in \{u, v, w\} >$

$\mathbf{E(m)}, m \geq 3$:

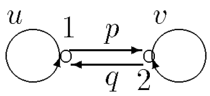


$qp = u^m, pq = v^m, pu = vp,$
$pu = vp, u^s = 0, v^t = 0$

F :

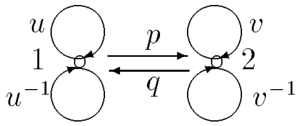


G :



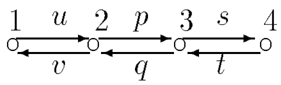
$pu = vp, qv = uq, pq = 0,$
$qp = 0, u^s = 0, v^t = 0$

H :



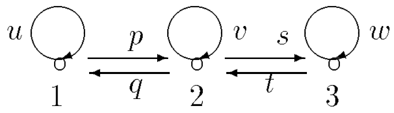
$pu = vp, qv = uq,$
$pq = 0, qp = 0$

I :

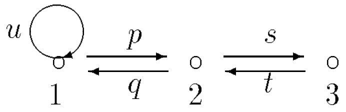


$uv = qp = \gamma, \gamma^l = 0,$
$pq = ts = \delta, \delta^m = 0$

J :



$pu = vp, qv = uq, sv = ws,$
$vt = tw, u^2 = qp, w^2 = st,$
$v^2 = pq = ts, v^l = 0$

K : 

$pu = tsp, qts = uq, u^2 = qp,$
$pq = (ts)^2 = \gamma, \gamma^l = 0$

The wildness of the categories **A** and **D** follows from [Ri], **F** from [Na], $\mathbf{E}(m)$, $m \geq 3$ from [Ge], **G** from [Be] (see also [HoM]).

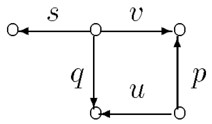
Using the wildness of the category **A** it is easy to show the wildness of **B** and **C**. In fact, for each finite-dimensional **A**-module M we can take into correspondence the **B**- and **C**-module N and L respectively, where

$$M(1) = N(1) = L(1),\ N(u) = L(u) = 1 + M(u),\ N(v) = 1 + M(v),\ L(v) = M(v).$$

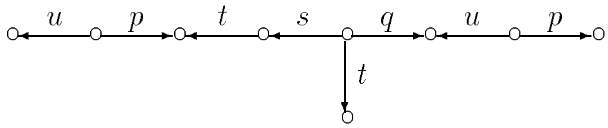
Using the same sort of arguments the category of finite-dimensional **G**-modules can be included into **H**-mod. Namely, let M be a finite-dimensional **G**-module, define **H**-module as follows:

$$N(i) = M(i), i = 1, 2, N(p) = M(p), N(q) = M(q), N(u) = 1 + M(u), N(v) = 1 + M(v).$$

The category **J** is wild since it "contains" the well-known wild subcategory [Na]:



Consider in the category of finite-dimensional **K**-modules the wild full subcategory consisting of the modules defining by the following diagram [Na]:



So **K** is wild.

It is not difficult to verify that $\mathbf{I} - K < x, y >$-bimodule M defined below is free as a right $K < x, y >$- module and such that the functor $M \otimes_{K<x,y>} -$ from the category of finite-dimensional $K < x, y >$-modules to **I**-mod preserves indecomposability and isomorhism classes. Let

$$M(1) = S^3,\ M(i) = S^4,\ i = 2, 3, 4,\ \text{where } S = K < x, y >,$$

and $\{m_{ik}\}$ are natural free generators of $M(i)$ for each $i = 1, \dots, 4$, then define the non-zero actions of the morphisms $u, \dots, t$ as follows:

$$um_{11} = m_{21},\ um_{12} = m_{23},\ um_{13} = m_{24},\ vm_{21} = m_{12},\ vm_{2-2} = m_{13},\ pm_{21} = m_{31},$$

$$pm_{22} = m_{32},\ qm_{31} = m_{23},\ qm_{32} = m_{24},\ tm_{41} = m_{32} + m_{33},\ tm_{42} = m_{31} + m\_34,$$

$$sm_{31} = m_{43},\ sm_{32} = xm_{43} + ym_{44},\ sm_{33} = m_{43},\ sm_{34} = m_{44-}.$$

2.5 Tame categories.

Remind that K is an algebraically closed field. A category $\mathcal{C}$ is called *tame* if for any integer $d > 0$ exist finitely many $\mathcal{C} - K[x]$-bimodules M_i, $i = 1, \dots, n_d$, such that the following two conditions satisfied [D1]:

1. for each $i = 1, \dots, n_d$ the right $K[x]$-module M_i is finitely generated and free;
2. each finitely generated indecomposable $\mathcal{C}$-module which has the dimension d is isomorphic to $M_i \otimes_{K[x]} S$ for some simple $K[x]$-module S.

We list below the tame categories which are used in the proofs of Theorems 1 and 2, their tameness is proved in Section 3.

$\mathcal{A} : K$

$\mathcal{B} : K < a | a^l = 0 >$

$\mathcal{C} : K < a, a^{-1} >$

$\mathcal{D}$:
$$\begin{tikzcd} \overset{1}{\circ} \arrow[r, shift left, "a"] & \underset{2}{\circ} \arrow[l, shift left, "b"] \end{tikzcd} \qquad (ab)^l = 0$$

$\mathcal{E}$:
$$\begin{tikzcd} \overset{1}{\circ} \arrow[loop left, "a_{11}"] \arrow[r, shift left, "a_{21}"] & \underset{2}{\circ} \arrow[l, shift left, "a_{12}"] \arrow[loop right, "a_{22}"] \end{tikzcd} \qquad \begin{array}{l} a_{11}^2 = a_{12}a_{21}, a_{22}^2 = a_{21}a_{12}, \\ a_{21}a_{11} = a_{22}a_{21}, a_{12}a_{22} = a_{11}a_{12}, a_{11}^l = 0 \end{array}$$

$\mathcal{F}$:
$$\begin{tikzcd} 1\ \circ \arrow[r, shift left, "a_2"] \arrow[d, shift right, "b_1"'] & \circ\ 2 \arrow[l, shift left, "b_2"] \arrow[d, shift right, "a_3"'] \\ 0\ \circ \arrow[u, shift right, "a_1"'] \arrow[r, shift left, "b_4"] & \circ\ 3 \arrow[l, shift left, "a_4"] \arrow[u, shift right, "b_3"'] \end{tikzcd} \qquad \begin{array}{l} a_ib_i = b_ia_i = 0, i = 1, ..., 4, \\ a_ia_j = b_lb_m \text{ for any } i, j, l, m \in \{1, 2, 3, 4\}, \\ \text{if it makes sense} \end{array}$$

$\mathcal{G}:$
$$\begin{tikzcd} \underset{1}{\circ} \arrow[r, shift left, "d"] & \underset{0}{\circ} \arrow[l, shift left, "c"] \arrow[r, shift left, "a"] & \underset{2}{\circ} \arrow[l, shift left, "b"] \end{tikzcd} \qquad ba = dc, (ab)^l = 0$$

$\mathcal{H}^m, m \geq 1:$
$$\begin{tikzcd} \underset{0}{\circ} \arrow[r, shift left, "a_1"] & \underset{1}{\circ} \arrow[l, shift left, "b_1"] \;\ldots\; \underset{m-1}{\circ} \arrow[r, shift left, "a_m"] & \underset{m}{\circ} \arrow[l, shift left, "b_m"] \end{tikzcd} \qquad \begin{array}{l} a_ib_i = b_ia_i = 0, \\ i = 1, ..., m \end{array}$$

$\mathcal{I}^m, m \geq 1:$
$$\begin{tikzcd} 1\ \circ \arrow[r, shift left, "a_1"] \arrow[d, shift right, "b_m"'] & \circ\ 2 \arrow[l, shift left, "b_1"] \arrow[d, shift right, "a_2"'] \\ m\ \circ \arrow[u, shift right, "a_m"'] \;\cdots & \circ\ 3 \arrow[u, shift right, "b_2"'] \end{tikzcd} \qquad a_ib_i = b_ia_i = 0, i = 1, ..., m$$

2.6 Proof of Theorem 1.

By Lemmas 2.1, 2.2 and Proposition 2.3 the tameness of the problem of describing the category $\mathcal{W}(\mathcal{O})$ of weight modules with supports from an orbit $\mathcal{O}$ is equivalent to the tameness of the category $\mathcal{L}_{\mathcal{O},1}$.

The proof of Theorem 1 can be devided into two part. In the first it is proved that the category $\mathcal{L}_{\mathcal{O},1}$ is wild in the case when the conditions of Theorem 1 are not satisfied. In the second part the tameness is proved in the case when the condition of Theorem 1 are satisfied. In both cases all the possible situations are considered and every time the problem is reduced to the same for above categories from subsections 2.4 and 2.5 respectively. Let $\mathcal{O} = \mathcal{O}_1 \times \ldots \times \mathcal{O}_n$ be an orbit.

Wild cases.

W1. Suppose that *three degenerated orbits exist*, say $\mathcal{O}_i, \mathcal{O}_j, \mathcal{O}_k$. Then $\mathbf{F}$ is isomorphic to the factorcategory of $\mathcal{L}_{\mathcal{O},1}$ for this it is enough to put

$$\mathcal{L}_{\mathcal{O},1} \to \mathbf{F},\ x_{\Gamma,i} \to u,\ x_{\Gamma,j} \to v,\ x_{\Gamma,k} \to r,\ y_{\Gamma[i],j} \to p,\ y_{\Gamma[j],i} \to q,$$

where $\Gamma = (\Gamma_1, \ldots, \Gamma_n) \in \hat{\mathcal{O}}$ and the equivalence classes $\Gamma_i, \Gamma_j, \Gamma_k$ contain the roots of the polynomoials a_i, a_j, a_k respectively, and the remaining generators map to zero or to 1, i.e. $x_* \to 0, y_* \to 0,\ g_*^{\pm 1} \to 1$.

W2. Suppose that in $\mathcal{O}$ there are *two cyclic orbits* $\mathcal{O}_i, \mathcal{O}_j \in Cyc$. Fix some equivalence class Γ from $\hat{\mathcal{O}}$ and let P_Γ be the full subcategory which consists of the arrows with starts and ends in Γ. Then P_Γ is generated by the arrows

$$x_k := x_{\Gamma[k,d(k)-1],k} \ldots x_{\Gamma[k,1],k}\, x_{\Gamma[k,0],k},\ y_k := y_{\Gamma[k,0],k} y_{\Gamma[k,1],k} \ldots y_{\Gamma[k,d(k)-1],k},$$

when the orbit $\mathcal{O}_k \in Cycd$ is cyclic degenerate, $d(k) = |\hat{\mathcal{O}}_k|$, and generated by g_k, g_k^{-1} when the orbit $\mathcal{O}_k \in Cycn$ is cyclic non-degenerate. Then exist factorcategories of P_Γ which are isomorphic to the categories **A**, **B**, and **C** respectively in the cases:

$$(a)\quad \mathcal{O}_i, \mathcal{O}_j \in Cycd,\ (b)\quad \mathcal{O}_i, \mathcal{O}_j \in Cycn,\ (c)\quad \mathcal{O}_i \in Cycn,\ \mathcal{O}_j \in Cycd.$$

For this purpose map the generators of P_Γ to as follows

$$(a)\ \ x_i \to u,\ x_j \to v;\ (b)\ \ g_i \to u,\ g_j \to v;\ (c)\ \ g_i \to u,\ x_j \to v,$$

$$x_* \to 0,\ y_* \to 0,\ g_*^{\pm 1} \to 1,\ \text{(remining generators)}.$$

W3. Suppose that $\mathcal{O}_i \in Cyc$ is a *unique cyclic orbit* in $\mathcal{O}$ and $\mathcal{O}_j \in Lind$. Let Γ be an equivalence class from $\hat{\mathcal{O}}$ such that Γ_j contains a root of the polynomial a_i. Then the full subcategory P_Γ of $\mathcal{L}_{\mathcal{O},1}$ which contains only two objects Γ and $\Gamma[j]$ is generated by the following elements:

(a) $\mathcal{O}_i \in Cycd$,

$$x_i := x_{\Gamma[i,d(i)-1],i} \ldots x_{\Gamma[i,1],i}\, x_{\Gamma[i,0],i}, \quad y_i := y_{\Gamma[i,0],i} y_{\Gamma[i,1],i} \ldots y_{\Gamma[i,d(i)-1],i},$$

$$x_i' := x_{\Gamma[j][i,d(i)-1],i} \ldots x_{\Gamma[j][i,1],i}\, x_{\Gamma[j][i,0],i},\ \ y_i' := y_{\Gamma[j][i,0],i} y_{\Gamma[j][i,1],i} \ldots y_{\Gamma[j][i,d(i)-1],i},\ x_{\Gamma,j}\ y_{\Gamma,j},$$

where $d(i)$ is a number of the roots of the polynomial a_i belonging to the orbit $\mathcal{O}_i$.

$(b)\mathcal{O}_i \in Cycn$,

$$g_{\Gamma,i},\ g_{\Gamma,i}^{-1},\ g_{\Gamma[j],i},\ g_{\Gamma[j],i}^{-1},\ x_{\Gamma,j},\ y_{\Gamma,j}.$$

In both cases there exists the factorcategory of P_Γ which is isomorphic to the category **H** and **G** respectively for this it enough to map

$$(a)\quad x_i \to u,\ x_i' \to v,\ x_{\Gamma,j} \to p,\ y_{\Gamma,j} \to q,$$

$$(b)\quad g_{\Gamma,i} \to u,\ g_{\Gamma[j],i} \to v,\ x_{\Gamma,j} \to p,\ y_{\Gamma,j} \to q,$$

and remaining arrows $x_* \to 0,\ y_* \to 0,\ g_*^{\pm 1} \to 1$.

W4. Suppose that $\mathcal{O}_i \in Lind, \mathcal{O}_j \in Linds$. Then **F** is isomorphic to the factorcategory of $\mathcal{L}_{\mathcal{O},1}$ for this it is enough to put

$$x_{\Gamma[j],j} \to u,\ y_{\Gamma[j],i} \to v,\ y_{\Gamma,j} \to r,\ y_{\Gamma[j,1],i} \to p,\ y_{\Gamma[j],i} \to q,$$

and $x_* \to 0,\ y_* \to 0,\ g_* \to 1$, where Γ_j and $\Gamma[j]_j$ contain the roots of the polynomial a_j; and Γ_i the root of a_i.

Tame cases.

T1. Suppose that $\delta = n-2, \mathcal{O}_i, \mathcal{O}_j \in Lindw$, then the map

$$\mathcal{L}_{\mathcal{O},1} \to \mathcal{F},\ y_{\Gamma,j} \to a_1,\ x_{\Gamma,j} \to b_1,\ x_{\Gamma,i} \to a_2,$$

$$y_{\Gamma,i} \to b_2,\ x_{\Gamma[i],j} \to a_3,\ y_{\Gamma[i],j} \to b_3,\ y_{\Gamma[-j],i} \to a_4,\ x_{\Gamma[-j],i} \to b_4,$$

is the equivalence of the categories.

T2. Suppose that $\delta = n-1$, and the orbit $\mathcal{O}_i \in Lind$ contains m roots $\lambda_1 < \ldots < \lambda_m$ of the polynomial a_i which belong to the equivalence classes $\Gamma_1, \ldots, \Gamma_m$ respectively. Then the map

$$\mathcal{L}_{\mathcal{O},1} \to \mathcal{H}^m,\ \ x_{\Gamma_i} \to a_i,\ y_{\Gamma_i} \to b_i,\ \ i = 1, \ldots, m,$$

is the equivalence of the categories.

T3. Suppose that $\delta = n-1$, and the orbit $\mathcal{O}_i \in Cycd$ contains m roots $\lambda_1 < \ldots < \lambda_m$ of the polynomial a_i which belong to the equivalence classes $\Gamma_1, \ldots, \Gamma_m$ respectively. Then the map

$$\mathcal{L}_{\mathcal{O},1} \to \mathcal{I}^m,\ \ x_{\Gamma_i} \to a_i,\ y_{\Gamma_i} \to b_i,\ \ i = 1, \ldots, m,$$

is the equivalence of the categories.

T4. Suppose that $\delta = n-1$, and $\mathcal{O}_i \in Cycn$, then the categories $\mathcal{L}_{\mathcal{O},1}$ and $\mathcal{C}$ are equivalent ($g_{\Gamma,i} \to a$).

T5. In the last case $\delta = n$ it is evident that the categories $\mathcal{L}_{\mathcal{O},1}$ and $\mathcal{A}$ are equivalent.

■

2.7 Proof of Theorem 2.

By Lemmas 2.1, 2.2 and Proposition 2.3 the tameness of the problem of describing the category $\mathcal{GW}(\mathcal{O})$ of generalized weight A-modules with supports from an orbit $\mathcal{O}$ is equivalent to the tameness of the category $\mathcal{L}_{\mathcal{O},nil}$. The structure of the proof of Theorem 2 is the same as Theorem 1.

Wild cases .

W 1. Suppose that $n > 1$. For any object Γ of the category $\mathcal{L}_{\mathcal{O},nil}$ define the category P_Γ as above. Then $\mathbf{A}$ is isomorphic to the factorcategory of P_Γ, to see this it is enough to map

$$h_{\Gamma,1} \to u,\ h_{\Gamma,2} \to v,\ h_* \to 0,\ y_* \to 0,\ g_*^{\pm 1} \to 1.$$

So from now on $n = 1$ and for simplicity of formulas the index 1 is dropped

W 2. Suppose that the orbit $\mathcal{O} \in Lind$ contains a root of the multiplicity $m \geq 3$ which belongs to same equivalence class Γ. Then the full subcategory $\mathcal{C}$ of $\mathcal{L}_{\mathcal{O},nil}$ which contains only two objects Γ and $\Gamma[1]$ is isomorphic to $\mathbf{E}(m)$ ($h_\Gamma \to u,\ \lambda^{|\Gamma[1]|} h_{\Gamma[1]} \to v,\ x_\Gamma \to p,\ y_\Gamma \to q$).

W 3. Suppose that the orbit $\mathcal{O} \in Lind$ contains two neighbour roots $\lambda < \mu$ of the polynomial a each of them has the multiplicity 2. Denote by Γ the equivalence class of λ. Then the full subcategory $\mathcal{C}$ of $\mathcal{L}_{\mathcal{O},nil}$ with objects $\Gamma, \Gamma[1]$ and $\Gamma[1,2]$ is isomorphic to the category $\mathbf{J}$ ($h_\Gamma \to u$, $\lambda^{|\Gamma[1]|}h_{\Gamma[1]} \to v$, $x_\Gamma \to p$, $y_\Gamma \to q$, $\lambda^{2|\Gamma[1]|}x_{\Gamma[1]} \to s$, $y_{\Gamma[1]} \to t$, $\lambda^{|\Gamma[1,2]|+|\Gamma[1]|}h_{\Gamma[1,2]} \to w$.)

W4. Suppose that the orbit $\mathcal{O} \in Lind$ contains two neighbour roots λ and μ of the polynomial a of the multiplicity 2 and 1 respectively. Let $\lambda < \mu$, (the case $\lambda > \mu$ can be considered in the same way) and $\mathcal{C}$, Γ be as above.Then

$$\mathcal{C} \to \mathbf{K}, \quad h_\Gamma \to u,\ x_\Gamma \to p,\ y_\Gamma \to q, \lambda^{|\Gamma[1]|}x_{\Gamma[1]} \to s,\ y_{\Gamma[1]} \to t,$$

is the equivalence of the categories.

W 5. Suppose that the orbit $\mathcal{O} \in Lind$ contains three neighbour roots of the polynomial a each of them has the multiplicity 1 and belongs to the equivalence classes $\Gamma, \Gamma[1]$ and $\Gamma[1,2]$ respectively. Then the full sabcategory $\mathcal{C}$ of $\mathcal{L}_{\mathcal{O},nil}$ with objects $\Gamma, \Gamma[1]$ and $\Gamma[1,2]$ is isomorhic to the category $\mathbf{I}$ ($x_\Gamma \to u$, $y_\Gamma \to v, -\lambda^{|\Gamma[1]|}x_{\Gamma[1]} \to p$, $y_{\Gamma[1]} \to q$, $\lambda^{|\Gamma[1,2]|}x_{\Gamma[1,2]} \to s$, $y_{\Gamma[1,2]} \to t$).

W 6. Suppose that $\mathcal{O} \in Cycd$ and Γ is an arbitrary equivalence class from $\mathcal{O}$. Let P_Γ be the full subcategory of $\mathcal{L}_{\mathcal{O},nil}$ which consists of the arrows with starts and ends in Γ and I be the two-sided ideal of P_Γ generated by the pathes of the length more or equal 2. Then the factorcategory P_Γ/I is isomorphic to $\mathbf{D}$ ($h_\Gamma \to u$, $\prod_{i=0}^{m} x_{\Gamma[1,i]} \to v$, $\prod_{j=m}^{0} y_{\Gamma[1,j]} \to w$, where $m = |Ob\,\mathcal{L}_{\mathcal{O},nil}|$).

W 7. If $\mathcal{O} \in Cycn$ then the category $\mathcal{L}_{\mathcal{O},nil}$ is equivalent to $\mathbf{C}$ ($g_\Gamma \to u,\ h_\Gamma \to v$).

Tame cases.

T 1. Let the orbit $\mathcal{O} \in Lind$ contains exactly two simple roots $\lambda_1 < \lambda_2$ from the equivalence classes Γ and $\Gamma[1]$ respectively. Then the map

$$\mathcal{L}_{\mathcal{O},nil} \to \mathcal{G},\ \ -\lambda^{|\Gamma[1]|}x_\Gamma \to d,\ y_\Gamma \to c,\ x_{\Gamma[1]} \to a,\ y_{\Gamma[1]} \to b,$$

is the equivalence of the categories.

T 2. If the orbit $\mathcal{O}$ contains exactly one root of the multiplicity 2 and Γ its equivalence class, then the categories $\mathcal{L}_{\mathcal{O},nil}$ and $\mathcal{E}$ are equivalent ($x_\Gamma \to a_{21}$, $y_\Gamma \to a_{12}$, $h_\Gamma \to a_{11}$, $\lambda^{|\Gamma[1]|}h_{\Gamma[1]} \to a_{22}$).

T 3. If the orbit $\mathcal{O}$ contains exactly one simple root and Γ its equivalence class, then the categories $\mathcal{L}_{\mathcal{O},nil}$ and $\mathcal{D}$ are equivalent ($x_\Gamma \to a$, $y_\Gamma \to b$).

T 4. In the last case, $\mathcal{O} \in Linn$, the categories $\mathcal{L}_{\mathcal{O},nil}$ and $\mathcal{B}$ are equivalent ($h_\Gamma \to a$). ∎

3 Indecomposable (generalized) weight modules

For a given category $\mathcal{C}$ denote by $Ind(\mathcal{C})$ the set of isoclasses of indecomposable finite dimensional $\mathcal{C}$-modules.

3.1 Ind (A)-Ind (D).

Remind the classification of the indecomposable modules over the categories $\mathcal{A} - \mathcal{D}$. The set $Ind(\mathcal{A})$ consists of the 1-dimensional vector space K.

$$Ind(\mathcal{B}) = \{\mathcal{B}_n := K[a]/(a^n) | n \in \mathbf{N}\};$$

$$Ind(\mathcal{C}) = \{\mathcal{C}_{f,n} := K[a]/(f^n) | n \in \mathbf{N}, f \in Irr\, K[a] \backslash \{a\}\},$$

where $Irr\, K[a]$ is the set of irreducible polynomials with the highest coefficients 1.

$$Ind(\mathcal{D}) = \{\mathcal{D}_{n,i} \,|\, n \in \mathbf{N},\ i = 1, 2\},$$

where the module $\mathcal{D}_{n,i}$ has the K-basis $e_1, \ldots, e_n$, moreover the vector spaces $\mathcal{D}_{n,1}(1)$, $\mathcal{D}_{n,2}(2)$ and $\mathcal{D}_{n,1}(2)$, $\mathcal{D}_{n,2}(1)$ contain vectors e_jth with respectively odd and even indices; $ue_j = e_{j+1}$ and $ve_j = e_{j+1}$ (where $e_{n+1} = 0$).

3.2 The category $\mathcal{I}^m$, $m \geq 1$.

The problem of finding the set of indecomposable representations for $\mathcal{I}^m$ can be reduced to the same one but for $\mathcal{I}^1$ which is solved in [GP]. We give the description of $Ind(\mathcal{I}^m)$ using the results of [DGO].

Consider a free monoid W generated by two letters a and b. For each word $w = w_1 \ldots w_n$ of the length n and $j \in \mathbf{Z}_m := \mathbf{Z}/m\mathbf{Z}$ define the $\mathcal{I}^m$-module $\mathcal{I}_{j,w} = \mathcal{I}^m_{j,w}$ as follows (module of the first kind in terms of [GP]).

Consider vectors $e_0, \ldots, e_n$, then a K-basis of the vector space $\mathcal{I}_{j,w}(l)$ is the set of e_kth such that $j + k \equiv l (mod\, m)$. Define the action of a_l and b_{l-1} by the rules:

$$a_l e_k = \begin{cases} e_{k+1} & \text{, if } k \neq n \text{ and } w_{k+1} = a; \\ 0 & \text{, otherwise.} \end{cases}$$

$$b_{l-1} e_k = \begin{cases} e_{k-1} & \text{, if } w_k = b; \\ 0 & \text{, otherwise.} \end{cases}$$

If $w = \oslash$ (the empty word), then $\mathcal{I}_{j,\oslash}$ is the simple $\mathcal{I}^m$-module corresponding to j. For $m = 1$ we put $\mathcal{I}^1_w := \mathcal{I}^1_{0,w}$.

A word w is called an *m-word* if its lenght n is a multiple of m, and *non-periodic* if it is not a power of another m-word.

For each non-indecomposable polynomial $f = \alpha_1 + \ldots + \alpha_d x^{d-1} + x^d \neq x^d$ and each non-periodic m-word w define $\mathcal{I}^m$-module $\mathcal{I}_{w,f} = \mathcal{I}^m_{w,f}$ (of the second kind, [GP]) as follows.

Consider elements e_{ks}, $k = 0, \ldots, n-1, s = 1, \ldots, d$. Take for a basis of the vector space $\mathcal{I}_{w,f}(l)$ vectors e_{ks} such that $k \equiv l (mod\, m)$. Then a_l and b_{l-1} act on this basis by the rules:

$$a_l e_{ks} = \begin{cases} e_{k+1,s} & \text{, if } k \neq n-1 \text{ and } w_{k+1} = a; \\ e_{0,s+1} & \text{, if } k = n-1, w_n = a \text{ and } s \neq d; \\ -\sum_r \alpha_r e_{0,r} & \text{, if } k = n-1, w_n = a \text{ and } s = d; \\ 0 & \text{, otherwise.} \end{cases}$$

$$b_{l-1} e_{ks} = \begin{cases} e_{k-1,s} & \text{, if } k \neq 0 \text{ and } w_k = b; \\ e_{n-1,s+1} & \text{, if } k = 0, w_n = b \text{ and } s \neq d; \\ -\sum_r \alpha_r e_{n-1,r} & \text{, if } k = 0, w_n = b \text{ and } s = d; \\ 0 & \text{, otherwise.} \end{cases}$$

Two m-word $w = w_{11} \dots w_{km}$ and $w' = w_{sm+1} \dots w_{km} w_{11} \dots w_{sm}$ are *equivalent*, $1 \leq s \leq k$. Denote by Ω_m the set of equivalence classes of all non-periodic m-words, and by $Ind\, K[x]$ the set of indecomposable polynomials with highest coefficient 1 except of $\{x^d, d \geq 1\}$.

Proposition 3.2.1 $Ind\mathcal{I}^m = \{\mathcal{I}^m_{j,w}, \mathcal{I}^m_{z,f} \mid z \in \Omega_m,\ w \in W,\ f \in Ind\, K[x],\ j = 0, \dots, m-1\}$.∎

3.3 $Ind\,\mathcal{F}$.

Since $\mathcal{F}$ is the factorcategory of $\mathcal{I}^4$, Proposition 3.3.1 follows from Proposition 3.2.1.

Proposition 3.3.1 $Ind\mathcal{F} = \{\mathcal{F}_{1,x-1} = \mathcal{I}^4_{aabb,x-1}, \mathcal{F}_{2,x-1} = \mathcal{I}^4_{bbaa,x-1}, \mathcal{F}_{1,f} = \mathcal{I}^4_{abab,f}, \mathcal{F}_{2,f} = \mathcal{I}^4_{baba,f}, \mathcal{F}_{j,w} = \mathcal{I}^4_{j,w} \mid f \in Ind\, K[x],\ j = 0, \dots, m-1,\ w = b^\gamma (ab)^k a^\delta,\ k \geq 0,\ \gamma, \delta \in \{0, 1\}\}$. ∎

3.4 Category $\mathcal{H}^m, m \geq 1$.

Using the functor $\mathcal{H}^m \to \mathcal{I}^{m+2}$, $i \to i$, $a_i \to a_i$, $b_i \to b_i$, by Proposition 3.2.1 we have the following result.

Proposition 3.4.1 $Ind\mathcal{H}^m = \{\mathcal{H}^m_{j,w} := \mathcal{I}^{m+2}_{j,w} \mid |w| \leq m-j, w \in W, j = 0, \dots, m-1\}$. ∎

3.5 Category $\mathcal{E}$.

Suppose that the field K is not finite and char $K \neq 2$, (if K is finite the categories $\mathcal{L}_\mathcal{O}$ and $\mathcal{E}$ are not equivalent).

Suppose that W is the free monoid from subsection 3.2 then define the action of the group $G = < e, g \mid g^2 = e >$ on W as follows: $g(a) = b, g(\oslash) = \oslash$. The word $w = w_1 \dots w_n$ is called *symmetric* (otherwise, *asymmetric*) if either $w = \oslash$ or $w \neq \oslash$ and $w_i = g(w_{n-i+1})$ for each $i = 1, \dots, n$. Denote by SW and AW respectively the set of all symmetric and asymmetric words. The equivalence class of non-periodic 1-word is called *symmetric* if it contains a symmetric 1-word (otherwise, *asymmetric*). Let $S\Omega_1$ and $A\Omega_1$ be the set of all symmetric and asymmetric equivalence classes respectively. An indecomposable polynomial $f = \alpha_1 + \ldots + \alpha_d x^{d-1} + x^d \neq x^d$ is *symmetric* (otherwise, *asymmetric*) if $\alpha_{d-i+1} = \alpha_1\alpha_{1+i}$ for each $i = 0, \ldots, d$ where we put $\alpha_{d+1} = 1$. The set of all symmetric and asymmetric polynomials denote by $Ind^0\, K[x]$ and $Ind^1\, K[x]$ respectively.

Now we introduce some series of $\mathcal{E}$-modules which, as show Proposition 3.5.1, are the complete list of indecomposable $\mathcal{E}$-modules.

Let $\mathcal{I}^1$-module M be either $\mathcal{I}^1_w$, $w \in AW$, or $\mathcal{I}^1_{w,f}$ where $w \in A\Omega_1$ or $f \in Ind^1\, K[x]$. Define the $\mathcal{E}$-module N as follows. The vector spaces $N(i)$, $i = 1, 2$ and $M(1)$ are isomorphic, denote by γ_i the bijection from $N(i)$ to $M(1)$ which is an isomorphism of the vector spaces. Then the action of arrows a_{ij} of the category $\mathcal{E}$ on the vector space $N = N(1) \oplus N(2)$ is define by the rule

$$a_{ij}n := \gamma_i^{-1}((a + (-1)^{i+j}b)\gamma_j(n)),\ n \in N.$$

Denote by $\mathcal{E}_w$ and $\mathcal{E}_{w,f}$ the $\mathcal{E}$-module N with respect to two possibilities.

For each $w = w_1 \ldots w_n \in SW$ consider the $\mathcal{I}^1$-module $\mathcal{I}^1_w$ and the K-linear map $\varphi : \mathcal{I}^1_w \to \mathcal{I}^1_w$, $e_k \to e_{n-k}$, $k = 0, \ldots, n$ (see subsection 3.2).

Then the $\mathcal{E}$-module $\mathcal{E}_{w,1}$ is defined as

$$\mathcal{E}(1) = Im\,(E + \varphi),\ \mathcal{E}(2) = Im\,(E - \varphi),\ a_{ij} = (E + (-1)^{i+1}\varphi)a(E + (-1)^{j+1}\varphi)/4,$$

where E–is the unit map of $\mathcal{I}^1_w(1)$.

Define $\mathcal{E}$-module $\mathcal{E}_{w,2}$:

$$\mathcal{E}_{w,2}(1) = \mathcal{E}_{w,1}(2),\ \mathcal{E}_{w,2}(2) = \mathcal{E}_{w,1}(1),\ \mathcal{E}_{w,2}(a_{ij}) = \mathcal{E}_{w,1}(a_{s(i)s(j)}),$$

where $s(1) = 2$ and $s(2) = 1$.

For each $w = w_1 \ldots w_n \in S\Omega_1$ and $f \in Ind^0\, K[x]$ consider the $\mathcal{I}^1$-module $\mathcal{I}^1_{w,f}$ from subsection 3.2 and K-linear map ψ of $\mathcal{I}^1_{w,f}(1)$:

$$\psi(e_{ks}) = \begin{cases} e_{0,d-s} & \text{, if } k = 0,\ s < d; \\ -\alpha_1^{-1}\sum_{r=1}^{d}\alpha_{r+1}e_{0,r} & \text{, if } k = 0,\ s = d\ ; \\ e_{n-k,d-s+1} & \text{, if } k \neq 0, \end{cases}$$

where $\alpha_{d+1} = 1$.

Then the $\mathcal{E}$-modules $\mathcal{E}_{w,f,1}$ and $\mathcal{E}_{w,f,2}$ are defined as follows:

$$\mathcal{E}_{w,f,1}(1) = Im\,(E+\psi),\ \mathcal{E}_{w,f,1}(2) = Im\,(E-\psi),\ a_{ij} = (E+(-1)^{i+1}\psi)a(E+(-1)^{j+1}\psi)/4,$$

where E-is the unit map of $\mathcal{I}^1_{w,f}(1)$ and

$$\mathcal{E}_{w,f,2}(1) = \mathcal{E}_{w,f,1}(2),\ \mathcal{E}_{w,f,2}(2) = \mathcal{E}_{w,f,1}(1),\ \mathcal{E}_{w,f,2}(a_{ij}) = \mathcal{E}_{w,f,1}(a_{s(i)s(j)}),$$

where $s(1) = 2$ and $s(2) = 1$.

Proposition 3.5.1 *Ind* $\mathcal{E} = \{\mathcal{E}_w,\ \mathcal{E}_{u,1},\ \mathcal{E}_{u,2},\ \mathcal{E}_{v,f},\ \mathcal{E}_{z,g,1},\ \mathcal{E}_{z,g,2} \mid w \in AW,\ u \in SW,\ z \in S\Omega_1,\ g \in Ind^0 K[x],\ (v,f) \in \Omega_1 \times Ind\, K[x],\ (v,f) \notin S\Omega_1 \times Ind^0 K[x]\}$.

Proof. Let $G =< e, g \,|\, g^2 = e >$ be a cyclic group of order 2 and $\mathcal{I}G$ be the skew group category, i.e Ob $\mathcal{I}^1 G$ = Ob $\mathcal{I}^1$, Mor $\mathcal{I}^1 G = \{c[g] \,|\, c \in \text{Mor } \mathcal{I}, g \in G\}$ and multiplication is given by $c[f]d[h] = cf(d)[fh]$ where c, d are morphisms from $\mathcal{I}^1$, $f, h \in G$, and $g(a) = b$, $g(b) = a$. Denote by ε_1 and ε_2 the units corresponding respectively to the objects 1 and 2 of the category $\mathcal{I}^1$ (or $\mathcal{I}^1 G$ or $\mathcal{E}$).

Since the characteristic of the field K is not equal 2, the category $\mathcal{I}^1 G$ has two pairwise ortogonal idempotents $E_1 := (\varepsilon_1[e] + \varepsilon_1[g])/2$ and $E_2 := (\varepsilon_1[e] - \varepsilon_1[g])/2$. Then the map $\varepsilon_1 \to E_1$, $\varepsilon_2 \to E_2$ defines the equivalence of the categories $\mathcal{E}$ and $\mathcal{I}^1 G$. Remark that group G acts in an obvious way on the categories $\mathcal{I}^1$-mod. Consider the induced functor

$$ind : \mathcal{I}^1 - \text{mod} \to \mathcal{I}^1 G - \text{mod},\ M \to \mathcal{I}^1 G \otimes_{\mathcal{I}^1} M.$$

Then the following Lemma is an immediate corollary from [DOF] (see also [Kh]).

Lemma 3.5.2 *1. Each indecomposable $\mathcal{I}^1 G$-module is isomoprhic to a direct summand of $ind\, M$ for some indecomposable $\mathcal{I}^1$-module M.*

2. If an $\mathcal{I}^1$-module M is indecomposable and non-isomorphic to the twisted by the element $g \in G$ $\mathcal{I}^1$-module $g(M)$, then $ind\, M$ is indecomposable.

3. If an $\mathcal{I}^1$-module M is indecomposable and $g(M) \simeq M$, then $ind\, M \simeq N \oplus g(N)$ for some indecomposable $\mathcal{I}^1 G$-module N.

Suppose that $\mathcal{I}^1$-modules M and $g(M)$ are isomorphic. Then induced module $ind\, M$ is the direct sum of $\mathcal{I}^1$-submodules $M_1 := \varepsilon_1[e] \otimes M$ and $M_2 := \varepsilon_1[g] \otimes M$ which are isomorhic to the $\mathcal{I}^1$-modules M and $g(M)$ respectively.

Suppose that exists an $\mathcal{I}^1$-isomorphism $\Phi : M_1 \to M_2$ such that

$$\varepsilon_1[g]\Phi(\varepsilon_1[e] \otimes m) = \Phi^{-1}(\varepsilon_1[g] \otimes m),\ \text{for all } m \in M_1. \tag{3.5.1}$$

Let E_i be the unit map from M_i to M_i, $i = 1, 2$, prove that the projection maps

$$P_1 = \begin{pmatrix} E_1 & \Phi^{-1} \\ \Phi & E_2 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} E_1 & -\Phi^{-1} \\ -\Phi & E_2 \end{pmatrix}$$

are the $\mathcal{I}^1 G$-homomorphisms of the module $ind\, M$. Since both maps Φ and Φ^{-1} are the $\mathcal{I}^1$-homomorphisms it is enough to verify that the equality $P_i(\varepsilon_1[g]l) = \varepsilon_1[g]P_i(l)$ holds for all $l \in ind\, M$ and $i = 1, 2$. Check it for P_1 (for P_2 it can be done similarly).

Let $l = \varepsilon_1[e] \otimes m_1 + \varepsilon_1[g] \otimes m_2$, then $\varepsilon_1[g]l = \varepsilon_1[e] \otimes m_2 + \varepsilon_1[g] \otimes m_1$ and

$$P_1(\varepsilon_1[g]l) = (1/2)(\varepsilon_1[e] \otimes m_2 + \Phi^{-1}(\varepsilon_1[g] \otimes m_1) + \Phi(\varepsilon_1[e] \otimes m_2) + \varepsilon_1[g] \otimes m_1).$$

On the other hand,

$$\varepsilon_1[g]P_1(l) = (1/2)\varepsilon_1[g](\varepsilon_1[e]\otimes m_1 + \Phi^{-1}(\varepsilon_1[g]\otimes m_2) + \Phi(\varepsilon_1[e]\otimes m_1) + \varepsilon_1[g]\otimes m_2) =$$

$$(1/2)(\varepsilon_1[g]\otimes m_1 + \varepsilon_1[g]\Phi^{-1}(\varepsilon_1[g]\otimes m_2) + \varepsilon_1[g]\Phi(\varepsilon_1[e]\otimes m_1) + \varepsilon_1[e]\otimes m_2).$$

Now using (3.5.1) we see that right-hand sides of the above equalities coinside, i.e. P_1 is $\mathcal{I}^1G$-homomorphism. So $1_{ind\,M} = P_1 + P_2$, $P_1P_2 = P_2P_1 = 0$, hence, $\mathcal{I}^1G$-module $ind\,M = Im\,P_1 \oplus Im\,P_2$, the direct sum of $\mathcal{I}^1G$-submodules.

It is easy to see that $\mathcal{I}$-modules $\mathcal{I}^1_w$ and $\gamma(\mathcal{I}^1_w)$ (respectively, $\mathcal{I}^1_{w,f}$ and $\gamma(\mathcal{I}^1_{w,f})$) are isomorphic if and only if $w \in SW$ (respectively, $w \in S\Omega_1, f \in Ind^0\,K[x]$.) Remind that categories $\mathcal{E}$ and $\mathcal{I}^1G$ are isomorphic. To verify that $\mathcal{E}$-modules $\mathcal{E}_w$ and $ind\mathcal{I}^1_w$ (respectively, $\mathcal{E}_{w,f}$ and $ind\mathcal{I}^1_{w,f}$) are isomorphic for $w \in AW$ (respectively, for $(w,f) \notin S\Omega_1 \times Ind^0\,K[x]$), it is enough to put

$$e_{ik} = (\varepsilon_1[e] + (-1)^{i+1}\varepsilon_1[g]) \otimes e_k,\ (\text{respectively}, e_{iks} = (\varepsilon_1[e] + (-1)^{i+1}\varepsilon_1[g]) \otimes e_{ks}),$$

where $e_{ik} = \gamma_i^{-1}(e_k)$ (respectively, $e_{iks} = \gamma_i^{-1}(e_{ks})$).

Let $M = \mathcal{I}^1_w$, $w \in SW$ (respectively, $M = \mathcal{I}^1_{w,f}, w \in S\Omega_1$, $f \in Ind^0\,K[x]$), then $\mathcal{I}^1$-isomorphism Φ (see (3.5.1))can be defined as follows:

$$\Phi(\varepsilon_1[e]\otimes m) = \varepsilon_1[g]\otimes\varphi(m),\ (\text{respectively},\ \Phi(\varepsilon_1[e]\otimes m) = \varepsilon_1[g]\otimes\psi(m)).$$

Denote by N the image $Im\,P_1$, the $\mathcal{I}^1G$-submodule of $ind\,M$. Then choosing the basis of N as follows

$$e'_k := P_1(\varepsilon_1[e]\otimes e_k),\ (\text{respectively}, e'_{ks} := P_1(\varepsilon_1[e]\otimes e_{ks})),$$

we can indentify $\varepsilon_1[e] \equiv E$, $a[e] \equiv a$, $\varepsilon_1[g] \equiv \varphi$ (respectively, $\varepsilon_1[g] \equiv \psi$), it means that $N \simeq \mathcal{E}_{w,1}$ (respectively, $N \simeq \mathcal{E}_{w,f,1}$). The proof of Proposition 3.5.1 is complete. ▪

3.6 The category $\mathcal{G}$.

The problem of classification of indecomposable finite dimensional $\mathcal{G}$-modules (the Gelfand problem [G]) was solved in [NR] (see also [Kh], [Bo], [CB 1]). We give the description of $ind\,\mathcal{G}$ in the case when the basic field K has the characteristic char $K \neq 2$. For the category $\mathcal{L}$:

$$\underset{1}{\circ} \underset{y}{\overset{p}{\rightleftarrows}} \underset{0}{\circ} \underset{s}{\overset{x}{\rightleftarrows}} \underset{2}{\circ} \qquad\qquad xp = ys = 0,$$

sx and py are nilpotent, using [WW] we give the classification of indecomposable finite dimensional $\mathcal{L}$-modules. For this purpose consider the monoid W generated by the letters x, y, p and s elements of which are called words. Denote by St the set which consists of three (empty) words $\oslash_0, \oslash_1, \oslash_2$ and of words from W which do not contain as subwords the following words: $xx, yy, ss, pp, px, ps, ys, sy, xy, xp$. Element of St is called a *chain*. A chain $w = w_1 \dots w_n \in St$ is called a *cycle* if $w_1 = x$ and $w_n = y$. A cycle is *non-periodic* if it is not equals to v^m for some cycle v and $n > 1$. Two non-periodic cycles w and w' are *equivalent* if $w = uv$ and $w' = vu$ for some chains u and v. Denote by B_a the set of equivalence classes of non-periodic cycles.

For each chain $w = w_1 \dots w_n \in St$ define the $\mathcal{L}$-module $\mathcal{L}_w$: symbols $e_0, \dots e_n$ are a K-basis of $\mathcal{L}_w$, moreover, if w_i equals respectively to x, y, s and p then $e_{i-1} \in \mathcal{L}_w(\alpha)$ and $e_i \in \mathcal{L}_w(\beta)$ where (α, β) equals respectively to $(0,2), (1,0), (2,0)$ and $(0,1)$. The action of arrows define by the rules:

$$xe_i = \begin{cases} e_{i+1} & \text{, if } i \neq n \text{ and } w_{i+1} = x; \\ 0 & \text{, otherwise;} \end{cases}$$

$$se_i = \begin{cases} e_{i+1} & \text{, if } i \neq n \text{ and } w_{i+1} = s; \\ 0 & \text{, otherwise;} \end{cases}$$

$$ye_i = \begin{cases} e_{i-1} & \text{, if } i \neq 0 \text{ and } w_i = y; \\ 0 & \text{, otherwise;} \end{cases}$$

$$pe_i = \begin{cases} e_{i-1} & \text{, if } i \neq 0 \text{ and } w_i = p; \\ 0 & \text{, otherwise.} \end{cases}$$

In particular, $xe_n = 0$, $se_n = 0$, $ye_0 = 0$ and $pe_0 = 0$. Each empty word, say $\oslash_i$, corresponds simple $\mathcal{L}_w$-module $\mathcal{L}_{\oslash_i}$ (which corresponds to the object i).

For each non-periodic cycle $w = w_1 \dots w_n$ and $f \in Ind\, K[x]$ define $\mathcal{L}$-module $\mathcal{L}_{w,f}$: symbols e_{ks}, $k = 0, \dots, n-1$, $s = 1, \dots, d$, are a K-basis of $\mathcal{L}_{w,f}$, moreover, if w_i equals respectively to x, y, s and p then $e_{i-1,s} \in \mathcal{L}_{w,f}(\alpha)$ and $e_{i,s} \in \mathcal{L}_{w,f}(\beta)$ where (α, β) equals respectively to $(0,2), (1,0), (2,0)$ and $(0,1)$. The action of arrows define by the rules:

$$xe_{ks} = \begin{cases} e_{k+1,s} & \text{, if } k \neq 0 \text{ and } w_{k+1} = x; \\ e_{1,s+1} & \text{, if } k = 0 \text{ and } s \neq d; \\ -\sum_r \alpha_r e_{1,r} & \text{, if } k = 0 \text{ and } s = d; \\ 0 & \text{, otherwise.} \end{cases}$$

$$se_{ks} = \begin{cases} e_{k+1,s} & \text{, if } w_{k+1} = s; \\ 0 & \text{, otherwise;} \end{cases}$$

$$ye_{ks} = \begin{cases} e_{k-1,s} & \text{, if } k \neq 0 \text{ and } w_k = y; \\ e_{n-1,s} & \text{, if } k = 0; \\ 0 & \text{, otherwise;} \end{cases}$$

$$pe_{ks} = \begin{cases} e_{k-1,s} & \text{, if } w_k = p; \\ 0 & \text{, otherwise.} \end{cases}$$

Proposition 3.6.1 $Ind\,\mathcal{L} = \{\mathcal{L}_w,\ \mathcal{L}_{v,f}\,|\, w \in St,\ v \in B_a,\ f \in Ind\,K[x]\,\}$. ∎

The group $G =< e, g\,|\,g^2 = e >$ acts on the set St : $g(x) = y$, $g(p) = q$, $g(\oslash_1) = \oslash_2$, and $g(\oslash_0) = \oslash_0$. A non-empty word $w = w_1 \dots w_n$ is called *symmetric* if $w_i = g(w_{n-i+1})$ for all $i = 1, \dots, n$. By definition $\oslash_0$ is symmetric and $\oslash_i$, $i = 1, 2$, are not. Denote by SSt and ASt the set of symmetric and non-symmetric chains respectively. An eqivalence class of non-periodic cycle is called *symmetric* if it contains a symmetric word (otherwise, assymetric). Then the set B_a of non-perodic cycles can be decomposed into $B_a = SB_a \cup AB_a$ where SB_a and AB_a are respectively the set of symmetric and non-symmetric equivalence classes.

Now using Proposition 3.6.1 we introduce some series of $\mathcal{G}$-modules which as it is proved in Proposition 3.6.2 are the complete list of $\mathcal{G}$-modules.

Let $\mathcal{L}$-module M be equals to $\mathcal{L}_w$, $w \in ASt$ (respectively, to $\mathcal{L}_{w,f}$ where either $w \in AB_a$ or $f \in Ind^1\,K[x]$). Then define the $\mathcal{G}$-modules $\mathcal{G}_w$ and $\mathcal{G}_{w,f}$. In both cases denote by $N(i)$ the vector space which corresponds to i'th point ($i = 0, 1, 2$) and fix some isomorphisms of the vector spaces $\gamma_0 : N(0) \to M(1) \oplus M(2)$, $\gamma_i : N(i) \to M(0)$, $i = 1, 2$. Then the action of the arrows from $\mathcal{G}$ we define by the rules ($n \in N$) :

$$an = \gamma_2^{-1}(p+s)\gamma_0(n)/2,\ cn = \gamma_1^{-1}(p-s)\gamma_0(n)/2,$$

$$bn = \gamma_0^{-1}(y+x)\gamma_2(n)/2,\ dn = \gamma_0^{-1}(y-x)\gamma_1(n)/2,$$

For each $w = w_1 \dots w_n \in SSt$ consider the $\mathcal{L}$-module $\mathcal{L}_w$ with K-basis $e_0, \dots, e_n$ as above and define the linear map $\varphi : \mathcal{L}_w \to \mathcal{L}_w$, $e_k \to e_{n-k}$, $k = 0, \dots, n$. Then define the $\mathcal{G}$-modules $\mathcal{G}_{w,1}$ and $\mathcal{G}_{w,2}$ as follows:

$$\mathcal{G}_{w,1}(0) = \mathcal{L}_w(1),\ \mathcal{G}_{w,1}(1) = (E-\varphi)(\mathcal{L}_w(0)),\ \mathcal{G}_{w,1}(2) = (E+\varphi)(\mathcal{L}_w(0)),$$

$$a = (E+\varphi)p/2,\ c = (E-\varphi)p/2,\ d = y(E-\varphi)/2,\ b = y(E+\varphi)/2,$$

and

$$\mathcal{G}_{w,2}(1) = \mathcal{G}_{w,1}(2),\ \mathcal{G}_{w,2}(2) = \mathcal{G}_{w,1}(1),\ \mathcal{G}_{w,2}(0) = \mathcal{G}_{w,1}(0),$$

$$\mathcal{G}_{w,2}(a) = \mathcal{G}_{w,1}(c),\ \mathcal{G}_{w,2}(c) = \mathcal{G}_{w,1}(a),\ \mathcal{G}_{w,2}(b) = \mathcal{G}_{w,1}(d),\ \mathcal{G}_{w,2}(d) = \mathcal{G}_{w,1}(b).$$

For each $w = w_1 \dots w_n \in SB_a$ and $f \in Ind^0\,K[x]$ consider the $\mathcal{L}$-module $\mathcal{L}_{w,f}$ with basis $\{e_{ks}\}$ as above and K-linear map $\psi : \mathcal{L}_{w,f} \to \mathcal{L}_{w,f}$:

$$\psi(e_{ks}) = \begin{cases} e_{0,d-s} & \text{, if } k = 0,\ s < d; \\ -\alpha_1^{-1}\sum_{r=1}^{d}\alpha_{r+1}e_{0,r} & \text{, if } k = 0,\ s = d\ ; \\ e_{n-k,d-s+1} & \text{, if } k \neq 0, \end{cases}$$

where $\alpha_{d+1} = 1$.

Suppose that w and f as above the last two series $\mathcal{G}_{w,f,1}$ and $\mathcal{G}_{w,f,2}$ of $\mathcal{G}$-modules can be defined as follows:

$$\mathcal{G}_{w,f,1}(0) = \mathcal{L}_{w,f}(1),\ \mathcal{G}_{w,f,1}(1) = (E-\psi)\mathcal{L}_{w,f}(0),\ \mathcal{G}_{w,f,1}(2) = (E+\psi)\mathcal{L}_{w,f}(0),$$

$$a=(E+\psi)p/2,\ c=(E-\psi)p/2,\ d=y(E-\psi)/2,\ d=y(E+\psi)/2,$$

and

$$\mathcal{G}_{w,f,2}(1)=\mathcal{G}_{w,f,1}(2),\ \mathcal{G}_{w,f,2}(2)=\mathcal{G}_{w,f,1}(1),\ \mathcal{G}_{w,f,2}(0)=\mathcal{G}_{w,f,1}(0),$$
$$\mathcal{G}_{w,f,2}(a)=\mathcal{G}_{w,f,1}(c),\ \mathcal{G}_{w,f,2}(c)=\mathcal{G}_{w,f,1}(a),\ \mathcal{G}_{w,f,2}(b)=\mathcal{G}_{w,f,1}(d),\ \mathcal{G}_{w,f,2}(d)=\mathcal{G}_{w,f,1}(b).$$

Proposition 3.6.2 *$Ind\,\mathcal{G} = \{\mathcal{G}_w,\ \mathcal{G}_{v,1},\ \mathcal{G}_{v,2},\ \mathcal{G}_{u,f},\ \mathcal{G}_{z,g,1},\ \mathcal{G}_{z,g,2} \mid w \in ASt,\ v \in SSt,\ z \in SB_a,\ g \in Ind^0\,K[x],\ (u,v) \in B_a \times Ind\,K[x],\ (u,f) \notin SB_a \times Ind^0\,K[x]\,\}$.*

Proof . The kategory $\mathcal{G}$ is isomorphic to the skeleton of the skew group category $\mathcal{LG}$, where $G =< e, g \mid g^2 = e >$, $g(\varepsilon_1) = \varepsilon_2$, $g(\varepsilon_0) = \varepsilon_0$, $g(x) = y$, and $g(s) = p$. Now the proof can be done in the same way as the proof of Proposition 3.5.1. ∎

3.7 Weight and generalized weight A-modules.

We use the notations of the previous section. Using the equivalence P of the categories $\mathcal{L}_{\mathcal{O},m}$ and $\mathcal{C}_{\mathcal{O},m}$ (or $\mathcal{L}_{\mathcal{O},nil}$ and $\mathcal{C}_{\mathcal{O},nil}$) (see 2.1–2.3) for each $\mathcal{L}_{\mathcal{O},m}$-module M we define the corresponding $\mathcal{C}_{\mathcal{O},m}$-module and A-module which are denoted by the same way $M_{\mathcal{O}}$ (for example, if $M = \mathcal{I}^1_{w,f}$, then $M_{\mathcal{O}} = \mathcal{I}^1_{\mathcal{O},w,f}$).

1. A $\mathcal{C}_{\mathcal{O},m}$-module $M_{\mathcal{O}}$. For each $\alpha \in \Gamma$ we put $M_{\mathcal{O}}(\alpha) = M(\Gamma)$ and the action of generators we define in the following way:

$M_{\mathcal{O}}(H_{\alpha,i}) := \lambda_1^{s_1}...\lambda_n^{s_n} M(h_{\Gamma,i}) + \alpha_i E, i = 1, ..., n$, where $\alpha \in \Gamma$ and $P(\Gamma) = \tau_1^{s_1}...\tau_n^{s_n}(\alpha)$;

$$M_{\mathcal{O}}(X_{\alpha,i}) := \begin{cases} M(x_{\Gamma,i})M_{\mathcal{O}}(b_{\alpha_i\beta_i}(H_{\alpha,i})), & \text{if } \alpha_i \text{ is a root of } a_i;\\ M(g_{\Gamma,i}), & \text{if } \mathcal{O}_i \in Cycn \text{ and } \alpha_i = P(\Gamma)_i,\\ E, & \text{otherwise,}\end{cases}$$

where the polynomials $b_{\alpha_i\beta_i}$ are defined in subsec. 2.3;

$$M_{\mathcal{O}}(Y_{\alpha,i}) := \begin{cases} M_{\mathcal{O}}(a(H_{\alpha,i}))M^{-1}(g_{\Gamma,i}), & \text{if } \mathcal{O}_i \in Cycn \\ & \text{and } \alpha_i = P(\Gamma)_i,\\ M(y_{\Gamma,i})M_{\mathcal{O}}^{-1}(a_i(H_{\tau_i^{t_i-1}(\alpha),i}))...M_{\mathcal{O}}^{-1}(a_i(H_{\tau_i(\alpha),i}), & \text{if } \alpha_i \text{ is a root of } a_i,\\ M_{\mathcal{O}}(a_i(H_{\alpha,i})), & \text{otherwise,}\end{cases}$$

where $P(\Gamma[i]) = \tau_i^{t_i}(P(\Gamma))$.

2. An A-module $M_{\mathcal{O}}$. As a vector space $M_{\mathcal{O}} = \oplus_{\alpha\in\mathcal{O}} M(\alpha)$. Define the action of generators on the spaces $M(\alpha)$ by the rules:

$$X_i m := M_{\mathcal{O}}(X_{\alpha,i})m,\ Y_i m := M_{\mathcal{O}}(Y_{\tau_i^{-1}(\alpha),i})m,\ H_i m := M_{\mathcal{O}}(H_{\alpha,i})m,$$

where $m \in M(\alpha)$.

4 Examples

In this section we apply the results of sections 2 and 3 for describing indecomposable weight and generalized weight modules (in the tame cases) for some well-known algebras: the Weyl algebra A_n, the quantum plane Λ, the quantum Weyl algebra $A_1(q)$, the algebra of functions $A(S_q^2)$ on the quantum 2-dimensional sphere, the factoralgebra $U(\lambda)$ of the universal enveloping algebra $Usl(2)$ of the Lie algebra $sl(2)$. We use all the notations and definitions from sec. 3.

We say that $\mathcal{W}(\mathcal{O})$ (or $\mathcal{GW}(\mathcal{O})$) is tame if the problem of describing of these modules is tame.

4.1 The first Weyl algebra A_1.

1. Weight modules. By Theorem 1 $\mathcal{W}(\mathcal{O})$ is tame for each orbit $\mathcal{O}=\mathcal{O}_1$.

If $\mathcal{O}\neq\mathcal{O}(0)$ and char $K=0$, then $\mathcal{W}(\mathcal{O})\simeq\mathcal{A}$-mod and Ind $\mathcal{W}(\mathcal{O})=\{\mathcal{A}_\mathcal{O}\}$.

If $\mathcal{O}\neq\mathcal{O}(0)$ and char $K>0$, then $\mathcal{W}(\mathcal{O})\simeq\mathcal{C}$-mod, Ind $\mathcal{W}(\mathcal{O})=\{\mathcal{C}_{\mathcal{O},f,n}\,|\,n\in\mathbf{N}, f\in Irr\,K[x]\backslash(x)\}$.

If $\mathcal{O}=\mathcal{O}(0)$ and char $K=0$, then $\mathcal{W}(\mathcal{O})\simeq\mathcal{H}^1$-mod and Ind $\mathcal{W}(\mathcal{O})=\{\mathcal{H}^1_{\mathcal{O},j,w}\}$.

If $\mathcal{O}=\mathcal{O}(0)$ and char $K>0$, then $\mathcal{W}(\mathcal{O})\simeq\mathcal{I}^1$-mod and Ind $\mathcal{W}(\mathcal{O})=\{\mathcal{I}^1{}_{\mathcal{O},w},\mathcal{I}^1_{\mathcal{O},z,f}\}$.

2. Generalized weight modules. By Theorem 2 $\mathcal{GW}(\mathcal{O})$ is tame if and only if char $K=0$. So, if $\mathcal{O}\neq\mathcal{O}(0)$ and char $K=0$, then $\mathcal{GW}(\mathcal{O})\simeq\mathcal{B}$-mod and Ind $\mathcal{GW}(\mathcal{O})=\{\mathcal{B}_{\mathcal{O},n}\,|\,n\in\mathbf{N}\}$.

If $\mathcal{O}=\mathcal{O}(0)$ and char $K=0$, then $\mathcal{GW}(\mathcal{O})\simeq\mathcal{D}$-mod and Ind $\mathcal{GW}(\mathcal{O})=\{\mathcal{D}_{\mathcal{O},n.i}\,|\,n\in\mathbf{N}, i=1,2\}$.

4.2 The Weyl algebra A_2.

Weight modules, char $K=0$. By Theorem 1 $\mathcal{W}(\mathcal{O})$ is tame for each orbit $\mathcal{O}$.

If an orbit $\mathcal{O}$ is non-degenerate, i.e. $0\notin\mathcal{O}_1$ and $0\notin\mathcal{O}_2$, then $\mathcal{W}(\mathcal{O})\simeq\mathcal{A}$-mod, Ind $\mathcal{W}(\mathcal{O})=\{\mathcal{A}_\mathcal{O}\}$.

If an orbit $\mathcal{O}$ is degenerate but $(0,0)\notin\mathcal{O}$, i.e. one of the orbits $\mathcal{O}_i$, $i=1,2$, is degenerate and another is not, then $\mathcal{W}(\mathcal{O})\simeq\mathcal{H}^1$-mod, Ind $\mathcal{W}(\mathcal{O})=\{\mathcal{H}^1_{\mathcal{O},j,w}\}$.

If $(0,0)\in\mathcal{O}$, then $\mathcal{W}(\mathcal{O})\simeq\mathcal{F}$-mod,
Ind $\mathcal{W}(\mathcal{O})=\{\mathcal{F}_{\mathcal{O},1,x-1},\ \mathcal{F}_{\mathcal{O},2,x-1},\ \mathcal{F}_{\mathcal{O},1,f},\ \mathcal{F}_{\mathcal{O},2,f},\ \mathcal{F}_{\mathcal{O},j,w}\}$.

char $K>0$. By Theorem 1 $\mathcal{W}(\mathcal{O})$ is wild for each orbit $\mathcal{O}$.

Generalized weight modules. By Theorem 2 $\mathcal{GW}(\mathcal{O})$ is wild for each orbit $\mathcal{O}$.

4.3 The Weyl algebras $A_n, n>2$.

Weight modules, char $K=0$. If all orbits $\mathcal{O}_i$, $i=1,\ldots,n$, are non-degenerate, then $\mathcal{W}(\mathcal{O})\simeq\mathcal{A}$-mod, Ind $\mathcal{W}(\mathcal{O})=\{\mathcal{A}_\mathcal{O}\}$.

If the number of non-degenerate orbits $\mathcal{O}_i$ is $n-1$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{H}^1$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{H}^1_{\mathcal{O},j,w}\}$.

If the number of non-degenerate orbits $\mathcal{O}_i$ is $n-2$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{F}$-mod, Ind $\mathcal{W}(\mathcal{O})$ $= \{\mathcal{F}_{\mathcal{O},1,x-1}, \mathcal{F}_{\mathcal{O},2,x-1}, \mathcal{F}_{\mathcal{O},1,f}, \mathcal{F}_{\mathcal{O},2,f}, \mathcal{F}_{\mathcal{O},j,w}\}$.

If the number of non-degenerate orbits $\mathcal{O}_i$ is less than $n-2$, then $\mathcal{W}(\mathcal{O})$ is wild by Theorem 1.

char $K > 0$. By Theorem 1 $\mathcal{W}(\mathcal{O})$ is wild for each orbit.

4.4 The algebra $U(\lambda)$.

We suppose that char $K = 0$.

Weight modules. By Theorem 1 $\mathcal{W}(\mathcal{O})$ is tame for each orbit $\mathcal{O}$.

$\lambda = \mathbf{-1/4}$. In this case the polynomial $a(H)$ has a unique root $1/2$ of the multiplisity 2. If $\mathcal{O} \neq \mathcal{O}(1/2)$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{A}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{A}_{\mathcal{O}}\}$. If $\mathcal{O} = \mathcal{O}(1/2)$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{H}^1$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{H}^1_{\mathcal{O},j,w}\}$.

$\lambda = (\mathbf{n^2-1})/\mathbf{4}, \mathbf{n} \in \mathbf{N}$. In this case the polynomial $a(H)$ has two simple roots $(n-1)/2$ and $-(n+1)/2$ from the same orbit $\mathcal{O}(1/2)$. If $\mathcal{O} \neq \mathcal{O}(1/2)$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{A}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{A}_{\mathcal{O}}\}$. If $\mathcal{O} = \mathcal{O}(1/2)$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{H}^2$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{H}^2_{\mathcal{O},j,w}\}$.

$\lambda \in \mathbf{K}\backslash\{(\mathbf{n^2-1})/\mathbf{4}, \mathbf{n} = \mathbf{0}, \mathbf{1}, \ldots\}$. In this case each orbit contains no more than one root of $a(H)$ which is simple. If $\mathcal{O}$ is non-degenerate, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{A}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{A}_{\mathcal{O}}\}$. If $\mathcal{O}$ is degenerate, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{H}^1$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{H}_{\mathcal{O},j,w}\}$.

Generalized weight modules. By Theorem 2 $\mathcal{GW}(\mathcal{O})$ is tame for each orbit.

$\lambda = \mathbf{-1/4}$. If $\mathcal{O} \neq \mathcal{O}(1/2)$, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{B}$-mod, Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{B}_{\mathcal{O},n} \,|\, n \in \mathbf{N}\}$.

If $\mathcal{O} = \mathcal{O}(1/2)$, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{E}$-mod, Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{E}_w, \mathcal{E}_{u,1}\,\mathcal{E}_{u,2}, \mathcal{E}_{v,f}, \mathcal{E}_{z,g,1}, \mathcal{E}_{z,g,2} \,|\, w \in AW, u \in SW, z \in S\Omega_1, g \in Ind^0\,K[x], (v,f) \in \Omega_1 \times Ind\,K[x], (v,f) \notin S\Omega_1 \times Ind^0\,K[x]\}$.

$\lambda = (\mathbf{n^2-1})/\mathbf{4}, \mathbf{n} \in \mathbf{N}$. If $\mathcal{O} \neq \mathcal{O}(1/2)$, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{B}$-mod, Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{B}_{\mathcal{O},n} \,|\, n \in \mathbf{N}\}$.

If $\mathcal{O} = \mathcal{O}(1/2)$, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{G}$-mod, Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{G}_w, \mathcal{G}_{u,1}\,\mathcal{G}_{u,2}, \mathcal{G}_{v,f}, \mathcal{G}_{z,g,1}, \mathcal{G}_{z,g,2} \,|\, w \in ASt, u \in SSt, z \in SB_a, g \in Ind^0\,K[x], (v,f) \in B_a \times Ind\,K[x], (v,f) \notin SB_a \times Ind^0\,K[x]\}$.

$\lambda \in \mathbf{K}\backslash\{(\mathbf{n^2-1})/\mathbf{4}, \mathbf{n} = \mathbf{0}, \mathbf{1}, \ldots\}$. If $\mathcal{O}$ is non-degenerate, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{B}$-mod, Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{B}_{\mathcal{O},n} \,|\, n \in \mathbf{N}\}$. If $\mathcal{O}$ is degenerate, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{D}$-mod, Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{D}_{\mathcal{O},n,i} \,|\, n \in \mathbf{N}, i = 1, 2\}$.

4.5 The quantum plane Λ.

λ is not a root of 1. Weight modules. If $\mathcal{O} \neq \{0\}$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{A}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{A}_{\mathcal{O}}\}$. If $\mathcal{O} = \{0\}$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{I}^1$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{I}^1_{\mathcal{O},w}, \mathcal{I}^1_{\mathcal{O},z,f}\}$.

Generalized weight modules. If $\mathcal{O} \neq \{0\}$, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{B}$-mod, Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{B}_{\mathcal{O},n} \,|\, n \in \mathbf{N}\}$. If $\mathcal{O} = \{0\}$, then by Theorem 2 $\mathcal{GW}(\mathcal{O})$ is wild.

λ **is a root of 1.** By Theorem 2 $\mathcal{GW}(\mathcal{O})$ is wild for each orbit $\mathcal{O}$.

Weight modules. If $\mathcal{O} \neq \{0\}$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{C}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{C}_{\mathcal{O},f,n}\}$. If $\mathcal{O} = \{0\}$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{I}^1$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{I}^1_{\mathcal{O},w}, \mathcal{I}^1_{\mathcal{O},z,f}\}$.

4.6 The quantum Weyl algebra $A_1(q)$.

q **is not a root of 1. Weight modules.** If $\mathcal{O} \neq \mathcal{O}(0)$ and $\mathcal{O} \in Lin$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{A}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{A}_{\mathcal{O}}\}$.

If $\mathcal{O} \neq \mathcal{O}(0)$ and $\mathcal{O} \in Cyc$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{C}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{C}_{\mathcal{O},f,n}\}$.

If $\mathcal{O} = \mathcal{O}(0)$ and $\mathcal{O} \in Lin$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{H}^1$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{H}^1_{\mathcal{O},j,w}\}$.

If $\mathcal{O} = \mathcal{O}(0)$ and $\mathcal{O} \in Cyc$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{I}^1$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{I}^1_{\mathcal{O},w}, \mathcal{I}^1_{\mathcal{O},z,f}\}$.

Generalized weight modules. By Theorem 2 $\mathcal{GW}(\mathcal{O})$ is wild for each cyclic orbit $\mathcal{O}$.

Suppose that $\mathcal{O} \in Lin$. If $\mathcal{O} \neq \{0\}$, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{B}$-mod, Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{B}_{\mathcal{O},n} \mid n \in \mathbf{N}\}$. If $\mathcal{O} = \{0\}$, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{D}$-mod, Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{D}_{\mathcal{O},n,i} \mid n \in \mathbf{N}, i = 1,2\}$.

q **is an** m**'th root of 1,** $m > 1$. In this case each orbit is cyclic, so by Theorem 2 resp. Theorem 1 $\mathcal{GW}(\mathcal{O})$ is wild resp. $\mathcal{W}(\mathcal{O})$ is tame for each $\mathcal{O}$.

If $\mathcal{O} \neq \mathcal{O}(0)$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{C}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{C}_{\mathcal{O},f,n}\}$. If $\mathcal{O} = \mathcal{O}(0)$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{I}^1$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{I}^1_{\mathcal{O},w}, \mathcal{I}^1_{\mathcal{O},z,f}\}$.

4.7 The algebra $A(S^2_q)$ of functions on the quantum 2-dimensional sphere.

By Theorem 1 resp. 2 $\mathcal{W}(\mathcal{O})$ is tame for each orbit $\mathcal{O}$, resp. $\mathcal{GW}(\mathcal{O})$ is tame if and only if $\mathcal{O} \in Lin$.

$\mathbf{c = -d \neq 0}$. The polynomial $a(H)$ has a unique root c.

Weight modules. If $\mathcal{O} \neq \mathcal{O}(c)$, $\mathcal{O} \neq \mathcal{O}(0)$ and q is not a root of 1, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{A}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{A}_{\mathcal{O}}\}$.

If $\mathcal{O} = \mathcal{O}(c)$, and q is not a root of 1, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{H}^1$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{H}^1_{\mathcal{O},j,w}\}$.

If $\mathcal{O} = \mathcal{O}(c)$, and q is a root of 1, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{I}^1$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{I}^1_{\mathcal{O},w}, \mathcal{I}^1_{\mathcal{O},z,f}\}$.

If either $\mathcal{O} \neq \mathcal{O}(c)$, q is a root of 1, or $\mathcal{O}(0)$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{C}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{C}_{\mathcal{O},f,n}\}$.

Generalized weight modules. If $\mathcal{O} \neq \mathcal{O}(c)$, $\mathcal{O} \neq \mathcal{O}(0)$ and q is not a root of 1, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{B}$-mod, Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{B}_{\mathcal{O},n} \mid n \in \mathbf{N}\}$.

If $\mathcal{O} = \mathcal{O}(c)$, q is not a root of 1 and char $K > 2$, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{E}$-mod, Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{E}_w, \mathcal{E}_{u,1}\, \mathcal{E}_{u,2}, \mathcal{E}_{v,f}, \mathcal{E}_{z,g,1}, \mathcal{E}_{z,g,2} \mid w \in AW, u \in SW, z \in S\Omega_1, g \in Ind^0 K[x], (v,f) \in \Omega_1 \times Ind\, K[x], (v,f) \notin S\Omega_1 \times Ind^0\ K[x]\}$ (when char $K = 2$ see [CB 1]). In the remaining cases $\mathcal{GW}(\mathcal{O})$ is wild.

c=d=0. The polynomial $a(H)$ has one multiple root 0 which belongs to the orbit $\mathcal{O} = \{0\}$.

Weight modules. If $\mathcal{O} \neq \{0\}$, and q is not a root of 1, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{A}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{A}_{\mathcal{O}}\}$.

If $\mathcal{O} \neq \{0\}$, and q is a root of 1, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{C}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{C}_{\mathcal{O},f,n}\}$.

If $\mathcal{O} = \{0\}$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{I}^1$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{I}^1_{\mathcal{O},w}, \mathcal{I}^1_{\mathcal{O},z,f}\}$.

Generalized weight modules. If $\mathcal{O} \neq \{0\}$, and q is not a root of 1, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{B}$-mod, Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{B}_{\mathcal{O},n} \,|\, n \in \mathbf{N}\}$. In the remaining cases $\mathcal{GW}(\mathcal{O})$ is wild.

$\mathbf{d = -q^s c \neq 0, s \neq 0 \in Z}$. The polynomial $a(H)$ has two distinct roots belonging to the same orbit $\mathcal{O}(c)$.

Weight modules. If $\mathcal{O} \neq \mathcal{O}(c)$, $\mathcal{O} \neq \mathcal{O}(0)$ and q is not a root of 1, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{A}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{A}_{\mathcal{O}}\}$.

If $\mathcal{O} = \mathcal{O}(c)$, and q is not a root of 1, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{H}^2$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{H}^2_{\mathcal{O},j,w}\}$.

If $\mathcal{O} = \mathcal{O}(c)$, and q is a root of 1, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{I}^2$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{I}^2_{\mathcal{O},j,w}, \mathcal{I}^2_{\mathcal{O},z,f}\}$.

If either $\mathcal{O} \neq \mathcal{O}(c)$, q is a root of 1, or $\mathcal{O} = \mathcal{O}(0)$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{C}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{C}_{\mathcal{O},f,n}\}$.

Generalized weight modules. If $\mathcal{O} \neq \mathcal{O}(c)$, $\mathcal{O} \neq \mathcal{O}(0)$ and q is not a root of 1, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{B}$-mod, Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{B}_{\mathcal{O},n} \,|\, n \in \mathbf{N}\}$.

If $\mathcal{O} = \mathcal{O}(c)$, and q is not a root of 1, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{G}$-mod, and for char $K > 2$ Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{G}_w, \mathcal{G}_{u,1}\mathcal{G}_{u,2}, \mathcal{G}_{v,f}, \mathcal{G}_{z,g,1}, \mathcal{G}_{z,g,2} \,|\; w \in ASt,\, u \in SSt,\, z \in SB_a,\, g \in Ind^0 K[x],\ (v,f) \in B_a \times Ind\, K[x],\, (v,f) \notin SB_a \times Ind^0\ K[x]\,\}$. In the remaining cases $\mathcal{GW}(\mathcal{O})$ is wild.

$\mathbf{d \neq -q^s c, s \in Z}$. Each orbit contains no more than one (simple) root of $a(H)$.

Weight modules. If $\mathcal{O} \neq \mathcal{O}(c)$, $\mathcal{O} \neq \mathcal{O}(-d)$, $\mathcal{O} \neq \mathcal{O}(0)$ and q is not a root of 1, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{A}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{A}_{\mathcal{O}}\}$.

If $\mathcal{O} \in \mathcal{O}(c) \cup \mathcal{O}(-d)$ and q is not a root of 1, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{H}^1$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{H}^1_{\mathcal{O},j,w}\}$.

If $\mathcal{O} \in \mathcal{O}(c) \cup \mathcal{O}(-d)$ and q is a root of 1, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{I}^1$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{I}^1_{\mathcal{O},w}, \mathcal{I}^1_{\mathcal{O},z,f}\}$.

If either $\mathcal{O} \neq \mathcal{O}(c)$, $\mathcal{O} \neq \mathcal{O}(-d)$, q is a root of 1, or $\mathcal{O} = \mathcal{O}(0)$, then $\mathcal{W}(\mathcal{O}) \simeq \mathcal{C}$-mod, Ind $\mathcal{W}(\mathcal{O}) = \{\mathcal{C}_{\mathcal{O},f,n}\}$.

Generalized weight modules. If $\mathcal{O} \neq \mathcal{O}(c)$, $\mathcal{O} \neq \mathcal{O}(-d)$, $\mathcal{O} \neq \mathcal{O}(0)$ and q is not a root of 1, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{B}$-mod, Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{B}_{\mathcal{O},n} \,|\, n \in \mathbf{N}\}$.

If $\mathcal{O} \in \mathcal{O}(c) \cup \mathcal{O}(-d)$ and q is not a root of 1, then $\mathcal{GW}(\mathcal{O}) \simeq \mathcal{D}$-mod, Ind $\mathcal{GW}(\mathcal{O}) = \{\mathcal{D}_{\mathcal{O},n,i} \,|\, n \in \mathbf{N},\ i = 1,2\}$. In the remaining cases $\mathcal{GW}(\mathcal{O})$is wild.

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References

[B] V.V.Bavula, Generalized Weyl algebras and their representations, Algebra i Analiz, **4** (1992), no. 1, 75–97; English transl. in St. Petersburg Math. J. **4** (1993), no. 1, 71–92.

[B1] V.V.Bavula, The finite-dimensionality of Ext^n and Tor_n of simple modules over a class of algebras, Function. Anal. i Priloz. **25** (1991), no. 3, 80–82.

[B2] V.V.Bavula, Generalized Weyl algebras, kernel and tensor-simple algebras, their simple modules, Representations of algebras. Sixth International Conference, August 19–22, 1992. CMS Conference Proceedings (V.Dlab and H.Lenzing Eds.), **14** , 83–107.

[BB] A.Beilinson and I.N.Bernstein, Localization de $\mathcal{G}$-modules, C. R. Acad. Sci. Paris, Sér. A, **292** (1981), 15–18.

[Be] V.I.Bekkert, Tame two-point quivers with relations, Izvest. Vuz. Math. **12** (1986), 62–64 .

[Bo] V.M.Bondarenko, Representations of bunches of semichain posets and their applications, Algebra i Analiz, **3** (1992), no. 5, 38–61.

[CB1] W.W.Crawley-Boevey, Functorial filtrations II: clans and the Gelfand problem, J. London Math. Soc. **40** (1989), no. 2, 9–30.

[CB2] W.W.Crawley-Boevey, On tame algebras and bocses, Proc. London Math. Soc. **56** (1988), 451–483.

[D1] Yu.A.Drozd, Tame and wild matrix problem, In: The representation theory II, Springer LMN **832** (1980), 242–258 .

[D2] Yu.A.Drozd, On the representations of the Lie algebra $sl(2)$, Visnyk Kiev Univ. **25** (1983), 70–77 (Ukrainian).

[DGO] Yu.A.Drozd, B.Guzner and S.A.Ovsienko, Weight modules over generalized Weyl algebras, preprint, Bielefeld.

[DOF] Yu.A.Drozd, B.Yu.Furchin and S.A.Ovsienko, Categorical constructions in the representation theory, In: Algebraic structures and their applications, Kiev, (1988), 17–43 .

[G] I.M.Gelfand, The cohomology of infinite dimensional Lie algebras: some questions of the integral geometry, Actes du Congress International des Mathematician, Nice, 1970.

[Ge] Ch.Geiβ, Tame distributive 2-point algebras, Representations of algebras. Sixth International Conference, August 19–22, 1992. CMS Conference Proceedings (V.Dlab and H.Lenzing Eds.), **14** , 193–204 .

[GP] I.M.Gelfand and V.A.Ponomarev, Indecomposable representations of the Lorenz group, Russian Math. Surveys **23** (1968), 1-58 .

[Hod] T.J.Hodges, Noncommutative deformation of type-A Kleinian singularities, J. Algebra, **161** (1993), no. 2, 271–290.

[HoM] M.Hoshino and J.Miyachi, Tame two-point algebras, Tsukuba Math. J. **12** (1988), no. 1, 65–96 .

[Kh] S.M.Khoroshkin, Irreducible representations of the Lorenz group, Funct. Anal. Appl. **15** (1981), 114–122 .

[Na] L.A.Nazarova, Representations of quivers of infinite type, Izv.Akad.Nauk SSSR **37** (1973), 752–791 (in Russian).

[NR] L.A.Nazarova and A.V.Roiter, The Gelfand problem, Funct. Anal. Appl. **7** (1973), 299–312 .

[RR] I.Reiten and Ch.Riedtmann, Skew group algebras in the representation theory of Artin algebras, J. Algebra **92** (1985), 224–282 .

[WW] B.Wald and J.Waschbüsch, Tame biserial algebras, J. Algebra **95** (1985), 480–500 .

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