Spectral Variation Bounds for Diagonalisable Matrices

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Abstract

This note is related to an earlier paper by Bhatia, Davis, and Kittaneh [4]. For matrices similar to Hermitian, we prove an inequality complementary to the one proved in [4, Theorem 3]. We also disprove a conjecture made in [4] about the norm of a commutator.

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§**1.** Let A be an $n\times n$ (complex) matrix with real eigenvalues $\alpha_1\geq\alpha_2\geq\cdots\geq\alpha_n$. Let $\mathrm{Eig}^{\downarrow}A$ and $\mathrm{Eig}^{\uparrow}A$ denote the diagonal matrices whose diagonal entries are $(\alpha_1,\alpha_2,\ldots,\alpha_n)$ and $(\alpha_n,\alpha_{n-1},\ldots,\alpha_1)$, respectively. Let $|||\cdot|||$ be any *unitarily invariant norm* on the space of matrices.

If A and B are Hermitian we have the following inequalities that provide lower and upper bounds for $|||A-B|||$ in terms of the distance between their spectra:

$$|||\mathrm{Eig}^{\downarrow}A-\mathrm{Eig}^{\downarrow}B|||\leq|||A-B|||\leq|||\mathrm{Eig}^{\downarrow}A-\mathrm{Eig}^{\uparrow}B|||. \tag{1}$$

See [2, Theorem 9.7]. The first inequality in (1) is a consequence of a theorem of Lidskii and was noted in this form by Mirsky [11], the second inequality is easier to prove and was noted by Sunder [14] and by Bhatia [3].

Now suppose A, B are not necessarily Hermitian but are similar to real diagonal matrices; i.e., there exist invertible matrices S and T such that SAS^{-1} and TBT^{-1} are real diagonal matrices. Let $c(S)$ be the *condition number* of S defined as $c(S)=\|S\|\,\|S^{-1}\|$ where $\|\cdot\|$ is the operator bound norm. Then it was shown in [4] that

$$|||\mathrm{Eig}^{\downarrow}A-\mathrm{Eig}^{\downarrow}B|||\leq c(S)\,c(T)\,|||A-B|||. \tag{2}$$

When A,B are Hermitian, S,T are unitary and the inequality (2) reduces to the first inequality in (1). The corresponding generalisation of the second inequality in (1) is provided below.

Theorem 1 *Let A,B be two matrices such that $A=S\Lambda S^{-1}, B=TMT^{-1}$, where S,T are invertible matrices and Λ, M are real diagonal. Then*

$$|||A-B|||\leq c(S)\,c(T)\,|||\mathrm{Eig}^{\downarrow}A-\mathrm{Eig}^{\uparrow}B|||, \tag{3}$$

for every unitarily invariant norm.

Proof: Note that

$$\begin{aligned}|||A-B||| &= |||S\Lambda S^{-1}-TMT^{-1}|||\\ &= |||S(\Lambda S^{-1}T-S^{-1}TM)T^{-1}|||\\ &\leq \|S\|\,\|T^{-1}\|\,|||\Lambda S^{-1}T-S^{-1}TM|||, \end{aligned}\tag{4}$$

by a familiar property of unitarily invariant norms [2]. So, to prove (3) it would suffice to show that

$$|||\Lambda X-XM|||\leq\|X\|\,|||\Lambda^{\downarrow}-M^{\uparrow}|||, \tag{5}$$

for every matrix X. This is equivalent to the assertion

$$|||\Lambda X - XM||| \leq |||\Lambda^{\downarrow} - M^{\uparrow}|||, \tag{6}$$

for every matrix X with $\|X\| = 1$. It is an easy consequence of the singular value decomposition that if $\|X\| = 1$ then there exist two unitary matrices U, V such that $X = \frac{1}{2}(U + V)$. So, it suffices to prove (6) for unitary X. But if X is unitary then

$$|||\Lambda X - XM||| = |||\Lambda - XMX^*||| \leq |||\Lambda^{\downarrow} - M^{\uparrow}|||$$

because of the second inequality in (1). ■

Remark 1 When A, B are normal matrices an inequality as simple as (1) is known to be true only for the special case of the Frobenius norm $\|A\|_F := (\mathrm{tr} A^* A)^{1/2}$. Here we have a theorem of Hoffman and Wielandt [9] which says that if A, B are normal matrices with eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$, respectively, then

$$\min_{\sigma} \left(\sum_i |\lambda_i - \mu_{\sigma(i)}|^2 \right)^{1/2} \leq \|A - B\|_F \leq \max_{\sigma} \left(\sum_i |\lambda_i - \mu_{\sigma(i)}|^2 \right)^{1/2}, \tag{7}$$

where σ varies over all permutations on n indices. A generalisation of this to all diagonalisable matrices has been proved. If $A = S\Lambda S^{-1}$, $B = TMT^{-1}$, where Λ, M are diagonal matrices with diagonal entries $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$, respectively, then

$$\min_{\sigma} \left(\sum_i |\lambda_i - \mu_{\sigma(i)}|^2 \right)^{1/2} \leq c(S)\, c(T)\, \|A - B\|_F, \tag{8}$$

$$\|A - B\|_F \leq c(S)\, c(T) \max_{\sigma} \left(\sum_i |\lambda_i - \mu_{\sigma(i)}|^2 \right)^{1/2}. \tag{9}$$

The inequality (8) was proved by Sun [13] and (9) by Elsner and Friedland [8].

Remark 2 For normal matrices, if $\|\cdot\|_F$ is replaced by the operator norm $\|\cdot\|$ then the analogue of (7) is a little more complicated. We have

$$\frac{1}{2.91} \min_{\sigma} \max_{i} |\lambda_i - \mu_{\sigma(i)}| \leq \|A - B\| \leq \sqrt{2} \max_{\sigma} \max_{i} |\lambda_i - \mu_{\sigma(i)}|. \tag{10}$$

The first inequality in (10) is due to Bhatia, Davis, McIntosh, and Koosis [6], [5]. It was shown by Holbrook [10] that the factor 1/2.91 occuring here can not be replaced by 1. The second inequality in (10) was proved by Ando and Nakamara [1] and by Omladič and Šemrl [12]. Here the constant $\sqrt{2}$ is sharp. Using the argument given above one can extend the second inequality in (10) to the case when $A = S\Lambda S^{-1}$, $B = TMT^{-1}$ with Λ and M diagonal, once again picking up an additional factor $c(S)\, c(T)$.

§**2** The proof of (2) in [4] and that of (8) in [13] relied on a common ingredient. Let Γ be a positive definite matrix. It was shown in [13] that for any normal matrices A, B we have the inequality

$$\|A - B\|_F \leq \|\Gamma^{-1}\|\, \|A\Gamma - \Gamma B\|_F. \tag{11}$$

It was shown in [4] that when A, B are both Hermitian more is true: we then have

$$|||A - B||| \leq \|\Gamma^{-1}\| |||A\Gamma - \Gamma B|||. \tag{12}$$

for every unitarily invariant norm. The inequality (11) led to (8) and the inequality (12) led to (2).

It was proved in [4] that the inequality (12) is also true when A, B are both unitary. This result has been recently used to derive good perturbation bounds for the generalised eigenvalue problem [7].

It was conjectured in [4] that the inequality (12) would remain true for all normal matrices A, B. We present below examples to refute this conjecture. For normal matrices A, B the inequality (12) breaks down both for the operator norm $\|\cdot\|$ and for the trace norm $\|\cdot\|_{\mathrm{tr}}$. Our examples were discovered by a directed computer search with an optimisation program.

Example 1 Given the positive definite matrix

$$\Gamma = \begin{pmatrix} 0.6384 & 0 & 0 \\ 0 & 0.6384 & 0 \\ 0 & 0 & 1.0000 \end{pmatrix}$$

and the normal matrices

$$A = \begin{pmatrix} -0.5205 - 0.1642i & 0.1042 - 0.3618i & -0.1326 - 0.0260i \\ -0.1299 + 0.1709i & 0.4218 + 0.4685i & -0.5692 - 0.3178i \\ 0.2850 - 0.1808i & -0.3850 - 0.4257i & -0.2973 - 0.1715i \end{pmatrix},$$

$$B = \begin{pmatrix} -0.6040 + 0.1760i & 0.5128 - 0.2865i & 0.1306 + 0.0154i \\ 0.0582 + 0.2850i & 0.0154 + 0.4497i & -0.5001 - 0.2833i \\ 0.4081 - 0.3333i & -0.0721 - 0.2545i & -0.2686 + 0.0247i \end{pmatrix},$$

we have

$$\frac{\|\Gamma^{-1}\| \, \|A\Gamma - \Gamma B\|}{\|A - B\|} = 0.8763.$$

Example 2 Given the positive definite matrix

$$\Gamma = \begin{pmatrix} 1.0000 & 0 & 0 \\ 0 & 0.6544 & 0 \\ 0 & 0 & 0.6544 \end{pmatrix}$$

and the normal matrices

$$A = \begin{pmatrix} 0.3596 + 0.1128i & 0.4341 - 0.2250i & -0.0036 - 0.1897i \\ -0.3788 + 0.0642i & 0.2661 - 0.4767i & 0.3584 - 0.4866i \\ -0.3016 - 0.1909i & -0.4722 - 0.2255i & 0.3174 - 0.0802i \end{pmatrix},$$

$$B = \begin{pmatrix} 0.4148 + 0.1336i & 0.3457 - 0.1404i & -0.1374 - 0.1275i \\ -0.3537 - 0.1098i & 0.4415 - 0.7470i & 0.0283 + 0.1332i \\ -0.1918 - 0.0191i & -0.1272 - 0.0176i & -0.4703 - 0.4669i \end{pmatrix},$$

we have

$$\frac{\|\Gamma^{-1}\| \, \|A\Gamma - \Gamma B\|_{\mathrm{tr}}}{\|A - B\|_{\mathrm{tr}}} = 0.8734.$$

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