Proximal elements in p-adic groups

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Let K be a field with valuation v. An element $g \in GL_n(K)$ is called proximal if g has a unique eigenvalue λ – counting multiplicities – of maximal value $v(\lambda)$ (We use the same notation v for some extension of v). Proximal elements play an important role in the study of linear groups [T] and of dynamics of linear transformations ([F], [GM], [GR]).

Recently in [GM], [AMS] different properties of proximal elements for $K = \mathbb{R}$ are considered. For example, if G is a simple algebraic group defined over $\mathbb{R}$, such that the group $G(\mathbb{R})$ is irreducible as a linear group and contains a proximal element, then every Zariski dense subgroup $H \subset G(\mathbb{R})$ contains a proximal element, too (in fact, one can find more general results in [AMS]). In this form the analogous result does not hold for p-adic groups even for $K = \mathbb{Q}_p$. For example, $H = SL_n(\mathbb{Z}_p)$ is Zariski dense in $SL_n(\mathbb{Q}_p)$, but $SL_n(\mathbb{Z}_p)$ has no proximal elements. The group $SL_n(\mathbb{Z}_p)$ is compact and therefore bounded in the p-adic topology.

In this connection G. Margulis formulated the following question. Let G be a simple algebraic group defined over $\mathbb{Q}_p$, let $H \subset G(\mathbb{Q}_p)$ be an unbounded Zariski dense subgroup and suppose $G(\mathbb{Q}_p)$ contains a proximal element. Does H contain a proximal element?

We prove in this paper that it is true if either H is a non-discrete subgroup of $G(\mathbb{Q}_p)$ or H is a lattice in $G(\mathbb{Q}_p)$. If H is discrete (but not a lattice) we give the example of Margulis of a subgroup $H \subset SL_4(\mathbb{Q}_p)$ without proximal elements.

We need the following simple lemmas.

Lemma 1 *If $G(\mathbb{Q}_p)$ has a proximal element then the set of proximal elements in $G(\mathbb{Q}_p)$ is open in the p-adic topology.*

Proof. If $g \in G(\mathbb{Q}_p)$ is proximal then for a sufficiently small neighbourhood U of the identity in $G(\mathbb{Q}_p)$ the set gU consists of proximal elements. This follows from the well-known property that the roots of polynomials over $\mathbb{Q}_p$ depend continuously on the coefficients, applied to the characteristic polynomials $\chi_t(x)$ for $t \in gU$.

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Lemma 2 *Let G be a connected algebraic group defined over a complete field K, with non-trivial absolute value, and assume that $G(K)$ is irreducible as a linear group (i.e. the $G(K)$-module K^n is irreducible). If $G(K)$ has a proximal element then $G(K)$ is absolutely irreducible.*

Proof. Let $\overline{K}$ be the algebraic closure of K (we will assume that K is a field of characteristic 0).

If $G(K)$ is not absolutely irreducible then $\overline{K}^n$ is a direct sum of at least two irreducible $G(\overline{K})$-submodules: $\overline{K}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_d$. All V_i are defined over some Galois extension L/K. Then $V_i \simeq V_1^{\sigma_i}$ for suitable $\sigma_i \in \mathrm{Gal}(L/K)$ and the group $G(K)$ is conjugate in $GL_n(L)$ with the group G' having at least two absolutely irreducible blocks, A and A^{σ_2}. Therefore $G(K)$ has no proximal elements.

Lemma 2 reduces the study of proximal elements essentially to the case of an absolutely irreducible $G(K)$.

An algebraic group G is called K-simple if G is defined over K and G has no proper connected normal subgroups defined over K. Of course, any simple group defined over K is K-simple.

Theorem 1 *Let G be a $\mathbb{Q}_p$-simple algebraic group and H be an unbounded Zariski dense non-discrete subgroup of $G(\mathbb{Q}_p)$. If $G(\mathbb{Q}_p)$ has a proximal element then H has a proximal element, too.*

Proof. Let $\overline{H}$ be the p-adic closure of H in $G(\mathbb{Q}_p)$. Then $\overline{H}$ is a p-adic Lie group. Let $L(\overline{H})$ be the Lie algebra of $\overline{H}$ over $\mathbb{Q}_p$. Since $h\overline{H}h^{-1} = \overline{H}$ for any $h \in H$ we have $hL(\overline{H})h^{-1} = L(\overline{H})$. But H is Zariski dense in $G(\mathbb{Q}_p)$, so $g(L(\overline{H}))g^{-1} = L(\overline{H})$ for any $g \in G(\mathbb{Q}_p)$. Then $L(\overline{H})$ is an ideal of $L(G(\mathbb{Q}_p))$ (see [B], p. 347), and, since H is non-discrete, $L(\overline{H}) \neq 0$. The Lie algebra $L(G(\mathbb{Q}_p))$ is simple, therefore $L(G(\mathbb{Q}_p)) = L(\overline{H})$. This means that $\overline{H}$ is an open subgroup of $G(\mathbb{Q}_p)$.

H is an unbounded subgroup of $G(\mathbb{Q}_p)$, therefore $G(\mathbb{Q}_p)$ is not compact and G is isotropic over $\mathbb{Q}_p$. Let $G^+(\mathbb{Q}_p)$ be the subgroup of $G(\mathbb{Q}_p)$ generated by its unipotent elements. It is a non-central normal subgroup, therefore $G(\mathbb{Q}_p)/G^+(\mathbb{Q}_p)$ is finite ([PR], § 3.3). The finiteness of $G(\mathbb{Q}_p)/G^+(\mathbb{Q}_p)$ implies the finiteness of $\overline{H}/G^+(\mathbb{Q}_p) \cap \overline{H}$. Therefore $G^+(\mathbb{Q}_p) \cap \overline{H}$ is an open non-compact subgroup of $G^+(\mathbb{Q}_p)$. Any open unbounded subgroup of $G^+(\mathbb{Q}_p)$ coincides with $G^+(\mathbb{Q}_p)$ by Tits' theorem. Thus, $G^+(\mathbb{Q}_p) \cap \overline{H} = G^+(\mathbb{Q}_p)$, that is $\overline{H} \supseteq G^+(\mathbb{Q}_p)$. If $g \in G(\mathbb{Q}_p)$ is a proximal element then for some $m \in \mathbb{N}^+$ we have $g^m \in G^+(\mathbb{Q}_p)$, and g^m is a proximal element of $\overline{H}$. Then H also has proximal elements by lemma 1.

Theorem 1 is proven.

It is natural to ask whether it is possible to generalize theorem 1 to some finite extensions of the field $\mathbb{Q}_p$. It is interesting that for any extension $K/\mathbb{Q}_p$ of degree m, $m > 1$, the generalization of theorem 1 is not true even for $G(K) = SL_n(K)$. I will give now the corresponding construction.

For any m there exists a finite-dimensional division algebra D with center $\mathbb{Q}_p$, such that $\dim_{\mathbb{Q}_p} D = m^2$. If $[K : \mathbb{Q}_p] = m$ then K is a maximal subfield of D. Let $SL_r(D)$ be the subgroup of $GL_r(D)$ with reduced norm equal to 1. We have the natural embedding $\varphi : SL_r(D) \to SL_{rm}(K)$ which is induced by the isomorphism $D \otimes_{\mathbb{Q}_p} K \simeq M_m(\mathbb{Q}_p)$.

Theorem 2 *Let $r > 1, m > 1$. Then $\varphi(SL_r(D))$ is an unbounded Zariski dense non-discrete subgroup of $SL_{rm}(K)$ without proximal elements.*

Proof. The density in the Zariski topology follows immediately from the construction. The subgroup $H = \varphi(SL_r(D))$ is unbounded and non-discrete because H has non-trivial unipotent elements ($r > 1$).

We need to prove now that $H = \varphi(SL_r(D))$ has no proximal elements.

Let $g \in H$ be a proximal element. Then $g = \varphi(h), h \in SL_r(D)$. Let L be the Galois extension of $\mathbb{Q}_p$ generated by K and all eigenvalues of g. All the eigenvalues of g are roots of the reduced characteristic polynomial $f_h(x)$ which is defined over $\mathbb{Q}_p$.

If λ is a root of $f_h(x)$ of maximal value $v(\lambda)$ (v denotes the extension of the p-adic valuation of $\mathbb{Q}_p$ to L), then for any $\sigma \in \mathrm{Gal}(L/\mathbb{Q}_p)$ we have $f_h(\sigma(\lambda)) = 0$ and $v(\sigma(\lambda)) = v(\lambda)$. Since λ is the unique eigenvalue of maximal value $v(\lambda)$ we have $\sigma(\lambda) = \lambda$ for all $\sigma \in \mathrm{Gal}(L/\mathbb{Q}_p)$. This implies that $\lambda \in \mathbb{Q}_p$. Then (using eigenvectors), up to conjugation in $GL_r(D)$, we can assume that

$$h = \begin{bmatrix} \lambda & \\ 0 & \\ \vdots & * \\ 0 & \end{bmatrix} .$$

Therefore

$$g = \varphi(h) = \begin{bmatrix} \left.\begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}\right\} m & * \\ 0 & \end{bmatrix} ,$$

and $f_h(x)$ is divisible by $(x - \lambda)^m$, a contradiction. We proved that H has no proximal element.

H.Abels and A.Potapchik constructed a similar example for quaternion division algebra.

Remark 1 *The main reason that the proof of theorem 1 cannot be extended to the fields K with $[K:\mathbb{Q}_p]=m>1$, is the following deep fact: the v-adic closure $\overline{H}$ does not have a structure of a Lie group over K.*

Remark 2 *The condition "H is unbounded" is automatically true for Zariski dense H over $K=\mathbb{R}$ (because every compact subgroup of $G(\mathbb{R})$ is closed in the Zariski topology).*

Let now H be a discrete Zariski dense subgroup of an algebraic group G defined over K, a finite extension of $\mathbb{Q}_p$. If $G(K)$ has a proximal element then G has a non-trivial $\mathbb{Q}_p$-split subtorus. Let T be a maximal K-split subtorus of G containing S. The following lemma gives a criterion when H has a proximal element.

Lemma 3 *Let $G(K)$ have a proximal element and the intersection $T(K)\cap H$ is cocompact in $T(K)$. Then H has a proximal element.*

Proof. T is the almost-direct product of S and the maximal anisotropic subtorus T_a. Then $S(K)\cdot T_a(K)$ has finite index in $T(K)$. If $g\in G(K)$ is a proximal element then we can assume that g is semisimple and g is contained in some maximal subtorus $F\subset G$ defined over K. It follows that the maximal K-split subtorus $F_s\subset F$ is non-trivial. Since $F=F_s\cdot F_a$, where F_a is the anisotropic subtorus of F, for a suitable $m\in\mathbb{N}^+$ we have $g^m\in F_s(K)F_a(K)$. But all eigenvalues of elements from $F_a(K)$ have absolute value equal to 1. Therefore, if $g^m=h\cdot f, h\in F_s(K), f\in F_a(K)$, then h is a proximal element. F_s is conjugate over K to some subtorus of S. It means that $S(K)$ contains a proximal element. Since $T_a(K)$ is compact, $S(O_v)\cdot T_a(K)=U$ is a compact open subgroup of $T(K)$ (O_v is the ring of integers in K). We have that $P=H\cap T(K)$ is cocompact in $T(K)$. Then the subgroup PU has finite index in $T(K)$. Therefore the intersection $S(K)\cap PU$ has finite index in $S(K)$. We proved that $S(K)$ has a proximal element. Then PU also has a proximal element.

Let $t\in PU$ be a proximal element, $t=\theta u, \theta\in P, u\in U$. Since all eigenvalues of u have the absolute value equal to 1, $\theta=tu^{-1}$ is a proximal element. We proved that H contains a proximal element.

Theorem 3 *Let G be a semisimple group defined over K where $[K:\mathbb{Q}_p]<\infty$. If $G(K)$ has a proximal element then any lattice H of $G(K)$ has a proximal element, too.*

Proof. There exists a maximal torus T in G defined over K such that T contains a maximal K-split subtorus S and $H\cap T(K)$ is cocompact in $T(K)$ ([Pr], Proposition 2.14). Then H has a proximal element by lemma 3.

If H is discrete in $G(K)$ but H is not a lattice, then H does not necessarily contain a proximal element. The following example was given by G. Margulis for $G(\mathbb{Q}_p) = SL_4(\mathbb{Q}_p)$.

Example. We will assume (for simplicity) that p is a prime number $\neq 2, 3$. Consider the following four 2-dimensional subspaces in $\mathbb{Q}_p^4$:

$$\begin{aligned} U_1 = &\ \{x = (x_1, x_2, x_3, x_4) \,|\, x_3 = x_4 = 0\}, \\ U_2 = &\ \{x = (x_1, x_2, x_3, x_4) \,|\, x_1 = x_2 = 0\} \\ V_1 = &\ \{x = (x_1, x_2, x_3, x_4) \,|\, x_1 = x_3, x_2 = x_4\}, \\ V_2 = &\ \{x = (x_1, x_2, x_3, x_4) \,|\, x_1 = -x_3, x_2 = -x_4\} \end{aligned}$$

Let S_1 (resp. S_2) denote the projection of $\mathbb{Q}_p^4$ onto U_1 (resp. U_2) along U_2 (resp. U_1) and let T_1 (resp. T_2) denote the projection of $\mathbb{Q}_p^4$ onto V_1 (resp. V_2) along V_2 (resp. V_1).

Let $B = pS_1 + p^{-1}S_2$ and $C = pT_1 + p^{-1}T_2$. Further let Y and Z be two unimodular isometric linear transformations of $\mathbb{Q}_p^4$ such that $Y(U_1) = U_1, Y(U_2) = U_2, Z(V_1) = V_1$ and $Z(V_2) = V_2$. Let now $B_1 = YB, C_1 = ZC$. It is possible to prove that there exist Y and Z such that the subgroup $H = \langle B_1, C_1 \rangle$ generated by B_1, C_1 is Zariski dense in $SL_4(\mathbb{Q}_p)$ and contains no proximal elements.

Since $\mathrm{rank}_K(SL_4) = 3$, it is natural to ask what happens for G when $\mathrm{rank}_K G < 3$ (here K is a finite extension of $\mathbb{Q}_p$). If $\mathrm{rank}_K G = 1$ we have the following result for arbitrary K.

Theorem 4 *Let G be a semisimple group defined over K, $\mathrm{rank}_K G = 1$. If $G(K)$ has a proximal element then any unbounded Zariski dense subgroup $H \subset G(K)$ has a proximal element, too.*

Proof. There exists a non-trivial rational irreducible representation ρ of G defined over K such that $\rho(H)$ contains a proximal element. Since the set of all proximal elements of $\rho(H)$ is dense in the Zariski topology (see [T]) and the set of all semisimple regular elements is open, there exists a semisimple proximal element $\rho(h), h \in H$. We have $\mathrm{rank}_K G = \mathrm{rank}_K \rho(G) = 1$. Therefore, $\mathrm{rank}_K Ker\rho = 0$. Let $h = h_s h_u$ be the Jordan decomposition of h. Since $\rho(h)$ is semisimple, $\rho(h_u) = 1$. This implies $h_u \in (Ker\rho)(K)$ and $h_u = 1$. Then $h = h_s$ and not all eigenvalues of h have absolute value equal to 1. Therefore h is contained in a K-isotropic maximal subtorus T defined over K. Moreover $T = T_s \cdot T_a$, where $\dim T_s = 1$.

If B_s is the maximal compact subgroup of $T_s(K)$ then $T_s(K)/B_s$ is cyclic. It implies that the intersection of $\langle h \rangle$ and $T(K)$ is cocompact in $T(K)$. Therefore h is a proximal element by lemma 3.

Let G be a semisimple group defined over K such that $G(K)$ is irreducible as a linear group. When does $G(K)$ contain a proximal element? The group $G(K)$ should be absolutely irreducible by lemma 2. In general an absolutely irreducible group $G(K)$ does not necessarily contain a proximal element. But for K-split groups we have the following result.

Theorem 5 *Let $G \subset GL_n$ be an irreducible K-split semisimple group. Then $G(K)$ contains a proximal element.*

The proof of theorem 5 is similar to the proof of theorem 6.3 in [AMS].

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