FINITELY GENERATED MODULES OVER PULLBACK RINGS

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Abstract. The purpose of this paper is to outline a new approach to the classification of finitely generated indecomposable modules over certain kinds of pullback rings. If R is the pullback of two hereditary noetherian serial rings over a common semi–simple artinian ring, then this classification can be divided into the classification of indecomposable artinian modules and those modules over the coordinate rings with no non–trivial artinian submodules. The classification of the artinian modules can be reduced to the case of a finite dimensional algebra over a semi–simple ring. This approach is carried out in the case where the coordinate rings are hereditary noetherian serial rings over a common quotient which is a matrix ring over a field.

1. Introduction

A focal point of recent attempts to classify finitely generated indecomposable modules over non–artinian rings has been pullback rings. Notably, there is the monumental work of L. Levy [Le3], resulting in the classification of all finitely generated indecomposable modules over Dedekind-like rings. L. Klingler extended this classification to lattices over certain non–commutative Dedekind-like rings [Kl], and Klingler and J. Haefner [HK1, HK2] classified lattices over certain non–commutative pullback rings, which they called special quasi–triads. Common to all these classifications is the reduction to a "matrix problem" over a division ring which is then solved using techniques similar to those in [KL]. However, as the rings become more complicated, the matrix problems become rather unwieldy, thereby limiting the usefulness of this matrix approach.

At the same time, the theory of finitely generated modules over finite dimensional k–algebras, k a field, has witnessed tremendous progress with the help of new and powerful functorial techniques, like the method of functorial filtrations [CB1-3], and almost split sequences, as well as the theory of bocses, a formalization of matrix methods [CB4]. (See [Ri] for a recent survey). The matrix problems solved in [KL] are special cases of self–reproducing systems [NR], which, in turn, can be interpreted as finitely generated modules over a finite dimensional clannish k–algebra. One of the most sweeping results is Crawley-Boevey's module classification for clannish

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algebras [CB2], which subsumes many earlier results, such as the classifications in [BS, SW, WW]. His functorial filtration techniques are derived from combinatorial arguments of [GP].

There are some intriguing "coincidences" between the clannish algebra situation and the non–artinian rings with local coordinate rings considered by Levy. In both cases, indecomposable modules break up into two classes, called string and band modules in the artin algebra setting, and deleted cycle and block cycle modules in the non–artinian pullback setting. In fact, the deleted cycle and block cycle matrix solutions of [KL] can be described as strings and bands, respectively (see [AL]).

The purpose of this paper is to exploit these coincidences and outline a new approach to the classification of finitely generated indecomposable modules over certain types of pullback rings, including Dedekind–like rings. It remains to be explored how general the class of pullback rings is, to which our method applies.

Specifically, let R_1, R_2 be noetherian rings with a semi–simple artinian ring F as common quotient. Let $v_i : R_i \longrightarrow F,\ i = 1, 2$, be the canonical projections, and let

$$R = \{(r_1, r_2) \in R_1 \times R_2 | v_1(r_1) = v_2(r_2)\}$$

be the pullback ring. If R_1 and R_2 are Dedekind domains, then R is an example of a Dedekind–like ring. In turn, any artinian module over a Dedekind–like ring is a module over a pullback ring with R_1 and R_2 discrete valuation rings [Le3, Prop. 15.1]. Also, if R_1 and R_2 are finite products of hereditary orders over Dedekind domains, then R is an example of a Dedekind–like order [Kl].

For M a finitely generated R–module, define $\mathcal{A}(M)$ to be the unique maximal artinian submodule of M. After deriving some properties of the functor $\mathcal{A}$ in Section 2, and some results on serial rings in Section 3, we will show in Section 4 (Theorem 4.1) that, if R_1 and R_2 are hereditary noetherian serial rings, finitely generated over their centers, and M is indecomposable, then $\mathcal{A}(M)$ is also indecomposable, and M is determined, up to isomorphism, by $\mathcal{A}(M)$ and $M/\mathcal{A}(M)$. For instance, finite products of hereditary orders over complete discrete valuation rings are noetherian serial [Wa, Theorem 6.1]. In this case, $M/\mathcal{A}(M)$ is an R–lattice, hence easily classified up to isomorphism. Thus, classification of indecomposables is reduced to the classification of artinian indecomposables.

In Section 5 we establish a bijection between isomorphism classes of indecomposable R–modules, for a pullback ring R, and isomorphism classes of indecomposable modules over the associated graded ring $G(R)$ of R. The ring $G(R)$ is also a pullback ring.

In Section 6 we derive some properties of $G(R)$. In particular, if R_1 and R_2 are hereditary noetherian serial, and F is a matrix ring over a field k, then $G(R)$ is Morita equivalent to the Gelfand algebra $k[x, y]/\langle xy\rangle$, localized at the maximal ideal $\langle x, y\rangle$, where $k[x, y]$ is the polynomial ring in commuting variables.

Finally, in Section 7, we classify the indecomposable modules over a pullback ring of hereditary noetherian serial rings, with F a matrix ring over a field. This classification uses the complete list of indecomposables in [BS] for the truncations

$$\Lambda_{pq} = k[x, y]_{\langle x,y\rangle}/\langle xy, x^{p+1} - y^{q+1}\rangle.$$

These algebras are classical examples of Crawley-Boevey's clannish algebras, being realized as the path algebra of a quiver with one vertex and two loops x and

y, modulo the relations $xy = 0 = x^{p+1} - y^{q+1}$ [CB2]. Thus, our approach, for instance, provides an alternative derivation of the local case of Levy's classification [Le2] (Example at the end of Sect. 7).

Several problems remain, which we hope to address in a later paper:

(1) What conditions on R_1, R_2, and F guarantee that the artinian modules over the associated graded ring $G(R)$ of the pullback ring R can be classified? For instance, if $G(R)$ has suitable quotients which are clannish algebras, then the artinian modules are known from the work of Crawley-Boevey. Theorem 5.1 then leads to a classification of the artinian modules over R.

(2) Describe more general pullback rings for which an indecomposable module M can be reconstructed, up to isomorphism, from $\mathcal{A}(M)$ and $M/\mathcal{A}(M)$.

Answers to these two questions would provide conditions on the rings defining the pullback ring R, for which the above program can be carried out to classify finitely generated indecomposable modules by splitting the problem into an artinian and a non–artinian part. The artinian problem is then a problem to be settled through the representation theory of k–algebras, and the non–artinian problem reduces to classifying non–artinian modules over the coordinate rings. For instance, it seems likely that much of this program can be carried out for Dedekind–like orders [Kl], or, more generally, for certain orders defined by conductor squares.

2. Separated representations and the functor $\mathcal{A}$

In this section we study the functor which assigns to a finitely generated module over a pullback ring its maximal artinian submodule, as it relates to separated representations.

As in the previous section, let R_1 and R_2 be noetherian rings and $v_i : R_i \to F$, $i = 1,\ 2$, be ring homomorphisms onto a semi–simple artinian ring F. Let $R = \{(r_1, r_2) \in R_1 \times R_2 \mid v_1(r_1) = v_2(r_2)\}$, the *pullback ring defined by* v_1 *and* v_2. Let $I_i = ker(v_i)$, $i = 1,\ 2$, and $I = I_1 \times I_2$. Then R is a noetherian ring and $R/I \cong F \cong R_i/I_i, i = 1,\ 2$. Furthermore, for $i \neq j$, the sequence $0 \longrightarrow I_i \longrightarrow R \longrightarrow R_j \longrightarrow 0$ is an exact sequence of R-modules. Also, note that the Jacobson radical $J(R)$ is contained in I, and $J(R_i) \subset I_i$. Alternatively, one may think of R as a subdirect product of R_1 and R_2, that is, R is a subring of $R_1 \times R_2$ such that the projections $R \to R_i$ are onto. In particular, each R_i is an R–module.

If R_1 and R_2 are finite products of hereditary orders over a Dedekind domain, such that F is a finite product of fields, then R is a special case of a *Dedekind–like pullback ring*. These have been studied by Levy [Le3] (for the commutative case) and Klingler [Kl]. If R_1 and R_2 are Dedekind domains and F is a field, then the resulting pullback ring was called a *2–pullback ring* in [AL].

Now, let R be a pullback ring, and denote by $Mod(R)$ the category of finitely generated R–modules. A module S in $Mod(R)$ is called *separated* if there exist S_i in $Mod(R_i)$, $i = 1,\ 2$, such that S is a submodule of $S_1 \oplus S_2$. Alternatively, S is separated if it is isomorphic to the pullback of the diagram

$$S/I_1S \longrightarrow S/IS \longleftarrow S/I_2S$$

[Le1, Cor. 3.3].

Two separated R–modules S and S' are isomorphic if and only if there exists a commutative diagram of R-module homomorphisms

$$\begin{array}{ccccc} S/I_1S & \longrightarrow & S/IS & \longleftarrow & S/I_2S \\ \cong\big\downarrow & & \cong\big\downarrow & & \cong\big\downarrow \\ S'/I_1S' & \longrightarrow & S'/IS & \longleftarrow & S'/I_2S \end{array}$$

in which the vertical maps are isomorphisms. This follows immediately ¿from the universal property of pullback diagrams.

A *separated representation* of a module M in $Mod(R)$ is an R–module epimorphism $\varphi : S \twoheadrightarrow M$ such that S is separated and such that, if φ admits a factorization

$$\varphi : S \xrightarrow{f} S' \twoheadrightarrow M$$

with S' separated, then f is one–to–one [Le1, p. 51].

We recall some properties of separated representations. The module $K = ker(\varphi)$ is an F–module, since $F = R/I$ and $IK = 0$ [Le1, Prop. 2.4]. An exact sequence

$$0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$$

of R–modules with S separated and K an R/I–module is a separated representation of M if and only if $I_iS \cap K = 0$ for each i and $K \subset IS$ [Le1, Prop. 2.3].

Every module in $Mod(R)$ has a separated representation, which is unique up to isomorphism [Le1, Theorem 2.8]. More generally, R–homomorphisms lift to separated representations, preserving epis and monos [Le1, Theorem 2.6]. That is, separated representations are "almost functorial".

Let $\mathcal{A} : Mod(R) \longrightarrow Mod(R)$ be the functor which sends an R–module M to its (unique) maximal artinian submodule $\mathcal{A}(M)$. The following proposition shows that $\mathcal{A}$ "almost" preserves separated representations.

Lemma 2.1. *Let R be a pullback ring, and let*

$$0 \to K \to S \xrightarrow{\varphi} M \to 0$$

be a separated representation of M in $Mod(R)$. Then

(1) *the sequence*
$$0 \to K \to \mathcal{A}(S) \xrightarrow{\phi} \mathcal{A}(M) \to 0$$
is exact;

(2) *$M/\mathcal{A}(M) \cong S/\mathcal{A}(S)$ is separated, in fact, isomorphic to an R–submodule of $S_1 \oplus S_2$, such that S_i has no non–trivial artinian submodules;*

(3) *the module $\mathcal{A}(S)$ is separated;*

(4) *the sequence*
$$0 \to K \cap I\mathcal{A}(S) \to T \xrightarrow{\phi} \mathcal{A}(M) \to 0$$
is a separated representation of $\mathcal{A}(M)$, where $K = (K \cap I\mathcal{A}(S)) \oplus L$ and $\mathcal{A}(S) = T \oplus L$.

Proof. 1. Since K is a finitely generated F–module and F is artinian it follows that $K \subset \mathcal{A}(S)$. Moreover, $\mathcal{A}(S)$ is the preimage of $\mathcal{A}(M)$ in S, since extensions of artinian modules are artinian. This shows that

$$0 \longrightarrow K \longrightarrow \mathcal{A}(S) \longrightarrow \mathcal{A}(M) \longrightarrow 0$$

is exact.

2. Using the Five–Lemma it follows from Part 1 that $M/\mathcal{A}(M) \cong S/\mathcal{A}(S)$, which is separated since, if $S \subset S_1 \oplus S_2$, then $S/\mathcal{A}(S) \subset S_1/\mathcal{A}(S_1) \oplus S_2/\mathcal{A}(S_2)$.

3. Clearly, $\mathcal{A}(S) \subset \mathcal{A}(S_1) \oplus \mathcal{A}(S_2)$, hence is separated.

4. Since F is semi–simple and K is an F–module, it follows that

$$K = (K \cap I\mathcal{A}(S)) \oplus L$$

for some F–module L. Since $L \cap I\mathcal{A}(S) = 0$, it follows from [Le1, Lemma 2.2] that L is a summand of $\mathcal{A}(S)$. Let $\mathcal{A}(S) = T \oplus L$. The exact sequence in Part 1 yields the exact sequence

$$0 \longrightarrow K \cap I\mathcal{A}(S) \longrightarrow T \longrightarrow \mathcal{A}(M) \longrightarrow 0.$$

The module T is separated, being a direct summand of a separated module, and $K \cap I\mathcal{A}(S) \subset I\mathcal{A}(S) = IT$. Furthermore, $(K \cap I\mathcal{A}(S)) \cap I_iT = 0$ for each $i = 1,\ 2$. This follows from the fact that $K \cap I_iS = 0$, and

$$K \cap I\mathcal{A}(S) \cap I_iT \subset K \cap I\mathcal{A}(S) \cap I_iS = 0.$$

Assertion 4 now follows from [Le1, Prop. 2.3].

Lemma 2.2. *Let R be a pullback ring, and let S in $Mod(R)$ be separated. Assume further that each coordinate module S/I_iS is either artinian or has no non–trivial artinian submodules.*

1. *If $\mathcal{A}(S/I_iS) = S/I_iS$ for each i, then $\mathcal{A}(S) = S$, that is, if the coordinate modules of S are artinian, then so is S.*
2. *If $\mathcal{A}(S/I_iS) = 0$ for each i, then $\mathcal{A}(S) = 0$.*
3. *If $0 \neq \mathcal{A}(S) \neq S$, then $\mathcal{A}(S) = I_jS = I_j(S/I_iS)$ for some $i \neq j$.*

Proof. 1. This follows immediately, since $S \subset S/I_1S \oplus S/I_2S$, which is artinian by hypothesis.

2. Observe that $\mathcal{A}(S)$ is isomorphic to the pullback of the diagram

$$\left(\mathcal{A}(S) + I_1S\right)/I_1S \longrightarrow \left(\mathcal{A}(S) + IS\right)/IS \longleftarrow \left(\mathcal{A}(S) + I_2S\right)/I_2S,$$

because this pullback is an artinian submodule of S and contains $\mathcal{A}(S)$. Furthermore, each $\left(\mathcal{A}(S) + I_iS\right)/I_iS$ is contained in $\mathcal{A}(S/I_iS)$, which is zero by hypothesis.

3. In light of Parts 1 and 2, we may assume without loss of generality that $\mathcal{A}(S/I_1S) = 0$ and $\mathcal{A}(S/I_2S) = S/I_2S$. In particular, $\left(\mathcal{A}(S) + I_1S\right)/I_1S = 0$, hence $\mathcal{A}(S) \subset I_1S$. Conversely, I_1S is artinian, since it is contained in the artinian module S/I_2S. Thus $\mathcal{A}(S) = I_1S = I_1(S/I_2S)$. This finishes the proof.

3. Serial Rings

We next impose some conditions on the coordinate rings of R to guarantee that indecomposable separated modules in $Mod(R)$ satisfy the hypotheses of Lemma 2.2.

Let A be a ring. Recall that an A–module is *serial* if its submodules are linearly ordered under inclusion. The ring A is *serial* if it is both a finite direct sum of left serial modules and a finite direct sum of right serial modules. An A–module is *local* if it has a unique maximal submodule.

Lemma 3.1. *Let R be a pullback ring with noetherian serial coordinate rings R_i. Assume further that $I_i = J(R_i)$, the Jacobson radical of R_i. Let $S \in Mod(R)$ be separated and indecomposable.*

(1) *For each $i \neq j$, S/I_iS is a serial R_j–module, and S/IS is simple.*
(2) *If $0 \neq \mathcal{A}(S) \neq S$, then $\mathcal{A}(S) = I_j(S/I_iS)$ is indecomposable for some $i \neq j$, $S/\mathcal{A}(S) = S/I_jS$ is an indecomposable projective R_i–module, and S/I_iS is an artinian indecomposable R_j–module.*

Proof. 1. First recall that S/I_iS is an R_j–module, for $i \neq j$. Since R_j is noetherian serial, it follows from [Wa, Cor. 3.4 and Theorem 2.6] that S/I_iS is a finite direct sum of serial, hence indecomposable, modules:

$$S/I_iS \cong A_1 \oplus \cdots \oplus A_n.$$

Choose such a decomposition of minimal length. Call $x \in S/I_iS$ a *local element* if it generates a local submodule. Thus, the summands in the above decomposition are generated by local elements, and these form a minimal generating set of local elements for S/I_iS. By [Wa, Lemma 1.11] these local elements map bijectively to a minimal set of local generators for S/IS. Hence, $S/IS \cong F_1 \oplus \cdots \oplus F_n$ decomposes as a direct sum of n simple F-modules. Arguing the same way for S/I_jS, $i \neq j$, we obtain a decomposition:

$$S/I_jS \cong B_1 \oplus \cdots \oplus B_n$$

where each B_k is an indecomposable local cyclic serial R_i–module. Furthermore, after relabeling, A_k and B_k both map to F_k under reduction modulo I. Taking pullbacks of the diagrams

$$A_k \longrightarrow F_k \longleftarrow B_k,$$

we conclude that S is isomorphic to the direct sum of these pullbacks. Since S was assumed to be indecomposable, it follows that $n = 1$, which completes the proof of Part 1.

2. We verify the hypotheses of Lemma 2.2. Since S/I_iS is a serial module over the serial ring R_j, we have that $S/I_iS = P/L$ for some serial (hence indecomposable) projective R_j–module P [Wa, Cor. 3.4 and Theorem 2.6]. If S/I_iS is not artinian, then, for all $k \geq 1$, I_j^kP is not contained in L. Since P is serial, $L \subset \bigcap_k I_j^kP$. But $\bigcap_k I_j^kP = 0$ [Wa, proof of Theorem 5.11], hence $L = 0$. Thus, $S/I_iS = P$ is projective and indecomposable. A noetherian serial ring is the product of an artinian serial ring and a hereditary serial ring with no simple left or right ideals

[Wa, Theorem 5.11]. Since S/I_iS is indecomposable projective and non–artinian, it follows therefore that $\mathcal{A}(S/I_iS) = 0$.

The rest of the proof follows from Lemma 2.2 and the fact that $\mathcal{A}(S) = I_j(S/I_iS)$ is indecomposable, since it is a submodule of the serial module S/I_iS.

As mentioned above, a noetherian serial ring A is the product of an artinian serial ring and a hereditary serial ring with no simple left or right ideals [Wa, Theorem 5.11].

Lemma 3.2. *Let A be a noetherian hereditary serial ring, such that $A/J(A)$ is semi–simple artinian. Assume furthermore that A is a finitely generated module over its center. Let M and M' be artinian A–modules. Let $\phi : J(A)M \longrightarrow J(A)M'$ be an isomorphism, and $M/J(A)M \cong M'/J(A)M'$. Then ϕ extends to an isomorphism $\mu : M \longrightarrow M'$.*

Proof. We first reduce to the case where $M = M'$ is indecomposable projective. Let $J = J(A)$. Decompose M as $M = \bigoplus_{i=1}^{n} M_i$, where each M_i is a serial, hence indecomposable, module, as above. Then $JM = \bigoplus_i JM_i$ and $JM' = \bigoplus_i \phi(JM_i)$. Since $M/JM \cong M'/JM'$ is semi–simple artinian and A is serial, it follows that the decomposition $M' = \bigoplus_{i=1}^{n} M_i'$ has the same number of summands as that of M, with each M_i' serial, hence indecomposable [Wa, Lemma 1.11]. Since M and M' are artinian, the Krull–Schmidt Theorem implies that $JM_i' \cong \phi(JM_i)$ and $M_i/JM_i \cong M_i'/JM_i'$ for all i.

Thus it is sufficient to prove the lemma under the assumption that M and M' are indecomposable (and artinian), hence serial. Let

$$0 \longrightarrow L \longrightarrow P \longrightarrow M \longrightarrow 0$$

be a projective cover of M, such that $L \subset JP$ and the induced map $P/JP \longrightarrow M/JM$ is an isomorphism. Such a projective cover exists [Wa, Lemma 1.6], since serial rings are semiperfect. Since M is indecomposable, it follows by [Wa, Theorem 3.3] that P is also indecomposable. Furthermore, JP is indecomposable as well. Since A is hereditary, JP is projective, as a submodule of a projective module. We obtain a projective cover

$$0 \longrightarrow L \longrightarrow JP \longrightarrow JM \longrightarrow 0$$

of JM, which is indecomposable since M is serial. Let P' and L' be defined similarly. Then the isomorphism $\phi : JM \longrightarrow JM'$ extends to an isomorphism $\phi' : JP \longrightarrow JP'$. But, since

$$P/JP \cong M/JM \cong M'/JM' \cong P'/JP',$$

it follows that $P \cong P'$. Thus we may assume that $M = P = P' = M'$ and that ϕ is an automorphism of JP.

By [Wa, Theorem 5.15] A is a product of rings which are Morita equivalent to either an upper triangular matrix ring over a division ring (which then is artinian hereditary serial) or a $(D : J)$–upper triangular matrix ring over a discrete valuation ring D with Jacobson radical J (in which case $\mathcal{A}(A) = 0$). Since P is indecomposable, we may assume that A is a ring of one of the two types.

CASE 1: If A is Morita equivalent to an upper triangular matrix ring over a discrete valuation ring, then $\mathcal{A}(A) = 0$. Since A is assumed to be finitely generated over its center, it follows ¿from [Wa, Theorems 5.15 and 6.7] that A is actually equal to a hereditary order over a commutative discrete valuation domain D. By [Re, Theorem 18.2.3] we may assume that D is complete. Thus, we are in the situation of [Re, Theorem 39.14].

Since A is hereditary, A–lattices and A–projectives agree. By [Re, Theorem 39.23], the indecomposable A–lattices are, up to isomorphism, all of the form J^iM, $i = 0, \ldots, n$, where M is the space of column vectors of length n, with entries in D. Thus,

$$J^iM = [\underbrace{\wp,\ \wp, \ldots,\ \wp}_{i},\ D,\ \ldots,\ D]^t.$$

Hence, we may assume that our indecomposable projective P is of the form J^iM for some i and $JP = J^{i+1}M$. Our goal is to extend an automorphism ϕ of $J^{i+1}M$ to an automorphism of J^iM. The map ϕ can be expressed as an $n \times n$–matrix whose entries are D–homomorphisms of the four kinds:

$$\wp \longrightarrow \wp, \qquad \wp \longrightarrow D, \qquad D \longrightarrow \wp, \qquad D \longrightarrow D.$$

Now consider a homomorphism $f : \wp \longrightarrow D$. The ideal $\wp$ is principal, with generator π. Hence f is determined by $f(\pi) = u\pi^r$ for some $u \in R$ and some $r \geq 1$. Then the map $D \longrightarrow D$, defined by $1 \mapsto u\pi^{r-1}$, is an isomorphism, and an extension of f to D.

By extending all appropriate maps in the matrix representing ϕ in this way, we obtain an extension of ϕ to $J^iM = P$. Applying the same procedure to the inverse of ϕ it follows that this extension is also an automorphism.

CASE 2: If A is an upper triangular matrix ring over a division ring then we are in the situation of [Re, Theorem 39.7], which implies that the indecomposable projectives are of the same form as in Case 1. The proof now continues as before. This completes the proof of the lemma.

4. Indecomposable modules and the functor $\mathcal{A}$

Theorem 4.1. *Let R be a pullback ring with hereditary noetherian serial coordinate rings R_1, R_2, such that $I_i = J(R_i)$ and $I = I_1 \times I_2 = J(R)$ for $i = 1, 2$. Assume, furthermore, that R_i, $i = 1, 2$, is finitely generated as a module over its center.*

1. *If M, M' are indecomposable modules in $Mod(R)$ such that $\mathcal{A}(M) \cong \mathcal{A}(M')$ and $M/\mathcal{A}(M) \cong M'/\mathcal{A}(M')$, then $M \cong M'$.*
2. *If M in $Mod(R)$ is indecomposable, then $\mathcal{A}(M)$ is also indecomposable.*

Proof. We first give an outline of the proof to help the reader through the maze of technical details to come. The first step in the proof is to reduce the theorem to the case of separated modules via separated representations. Now, a separated module S is described via a pullback square

$$\begin{array}{ccc} S & \longrightarrow & S/I_1S \\ \downarrow & & \downarrow \\ S/I_2S & \longrightarrow & S/IS\ . \end{array}$$

To give an isomorphism between separated modules, it is sufficient to give isomorphisms of the lower left and upper right corners of the respective defining pullback diagrams, which agree when extended to the lower right corner. Under the above assumptions on the coordinate rings of R, the previous analysis of indecomposable separated R–modules imposes certain restrictions on the coordinate modules S/I_1S and S/I_2S which will be exploited to construct the desired isomorphisms.

To prove the first assertion we may clearly assume that $0 \neq \mathcal{A}(M) \neq M$, and similarly for M'. Let

$$0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$$

be a separated representation of M, and

$$0 \longrightarrow K' \longrightarrow S' \longrightarrow M' \longrightarrow 0$$

a separated representation of M'. By Lemma 2.1.4, there is a separated representation

$$0 \longrightarrow K \cap I\mathcal{A}(S) \longrightarrow T \longrightarrow \mathcal{A}(M) \longrightarrow 0$$

of $\mathcal{A}(M)$, such that $K = (K \cap I\mathcal{A}(S)) \oplus L$ and $\mathcal{A}(S) = T \oplus L$. Similarly for $\mathcal{A}(M')$. Observe that K is a module over the semi–simple ring R/I, hence so is L. Let $\phi : \mathcal{A}(M) \longrightarrow \mathcal{A}(M')$ be an isomorphism. We obtain a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K \oplus L' & \longrightarrow & T \oplus L \oplus L' & \longrightarrow & \mathcal{A}(M) & \longrightarrow & 0 \\
 & & {\scriptstyle\cong}\big\downarrow & & {\scriptstyle\cong}\big\downarrow & & {\scriptstyle\phi}\big\downarrow & & \\
0 & \longrightarrow & K' \oplus L & \longrightarrow & T' \oplus L' \oplus L & \longrightarrow & \mathcal{A}(M') & \longrightarrow & 0
\end{array}$$

The middle vertical map is the direct sum of an isomorphism $T \longrightarrow T'$, which is a lifting of ϕ [Le1, Theorem 2.6], and the obvious isomorphism $L \oplus L' \longrightarrow L' \oplus L$. Hence, the middle vertical map is an isomorphism $\mathcal{A}(S) \oplus L' \longrightarrow \mathcal{A}(S') \oplus L$. This map makes the right–hand square commute, hence induces the left vertical isomorphism. It thus suffices to extend the middle isomorphism to an isomorphism $\mu : S \oplus L' \longrightarrow S' \oplus L$, so that we obtain a commutative diagram

$$\begin{array}{ccc}
K \oplus L' & \longrightarrow & S \oplus L' \\
{\scriptstyle\cong}\big\downarrow & & {\scriptstyle\cong}\big\downarrow \\
K' \oplus L & \longrightarrow & S' \oplus L .
\end{array}$$

¿From this diagram we then obtain an induced isomorphism of quotients $M \cong M'$. If we drop the assumption that S, resp. S', is a separated representation of M, resp. M', then we can assume that L, resp. L', is a summand of S, resp. S'. The property of S and S' being separated representations is not needed for the rest of the argument. Thus we may replace S by $S \oplus L'$ and S' by $S' \oplus L$ and assume that $\mathcal{A}(S) \cong \mathcal{A}(S')$.

Let $S = \bigoplus_{k=1}^n S_k$, where each S_k is separated and indecomposable. We group the summands of S as follows:

$$\begin{aligned} A &= \bigoplus\{S_k | 0 \neq \mathcal{A}(S_k) = S_k\}; \\ U_1 &= \bigoplus\{S_k | 0 \neq \mathcal{A}(S_k) = I_1 S_k = I_1(S_k/I_2S_k)\}; \\ U_2 &= \bigoplus\{S_k | 0 \neq \mathcal{A}(S_k) = I_2 S_k = I_2(S_k/I_1S_k)\}. \end{aligned}$$

Note that, unless M is separated, $\mathcal{A}(S_k) \neq 0$ for all k, since M is indecomposable and the image of K in S_k is contained in $\mathcal{A}(S_k)$. By Lemma 2.2 and Lemma 3.1, we have $S = A \oplus U_1 \oplus U_2$. There is a similar decomposition of S'. Furthermore, $\mathcal{A}(S) = \bigoplus_k \mathcal{A}(S_k)$, with each $\mathcal{A}(S_k)$ indecomposable; similarly for $\mathcal{A}(S')$. Now,

$$\begin{aligned} U_1/\mathcal{A}(U_1) \oplus U_2/\mathcal{A}(U_2) &\cong S/\mathcal{A}(S) \cong M/\mathcal{A}(M) \\ &\cong M'/\mathcal{A}(M') \cong S'/\mathcal{A}(S') \cong U_1'/\mathcal{A}(U_1') \oplus U_2'/\mathcal{A}(U_2'). \end{aligned}$$

Since $U_1/\mathcal{A}(U_1) = U_1/I_1U_1$ is an R_2–module, and $U_2/\mathcal{A}(U_2) = U_2/I_2U_2$ is an R_1–module – similarly for U_1', U_2' – it follows that $U_i/\mathcal{A}(U_i) \cong U_i'/\mathcal{A}(U_i')$, $i = 1,\ 2$.

Observe that I_1U_1, I_1U_1' are R_1–modules, and I_2U_2, I_2U_2' are R_2–modules. Furthermore, we can decompose A as $A = A_1 \oplus A_2 \oplus A_3$, where A_i consists of those summands that are R_i–modules, for $i = 1,\ 2$, and A_3 is the sum of the remaining summands of A. Thus, the isomorphism $\phi : \mathcal{A}(S) \longrightarrow \mathcal{A}(S')$ breaks up into a sum of three isomorphisms, $\phi = \phi_1 \oplus \phi_2 \oplus \phi_3$, where

$$\begin{aligned} \phi_1 : \mathcal{A}(U_1 \oplus A_1) = I_1U_1 \oplus A_1 &\longrightarrow I_1U_1' \oplus A_1' = \mathcal{A}(U_1' \oplus A_1'), \\ \phi_2 : \mathcal{A}(U_2 \oplus A_2) = I_2U_2 \oplus A_2 &\longrightarrow I_2U_2' \oplus A_2' = \mathcal{A}(U_2' \oplus A_2'), \\ \phi_3 : A_3 &\longrightarrow A_3'. \end{aligned}$$

Since S is separated, it is described via the pullback square

$$\begin{array}{ccc} S & \longrightarrow & S/I_1S \\ \downarrow & & \downarrow \\ S/I_2S & \longrightarrow & S/IS\ , \end{array}$$

where

$$\begin{aligned} S/IS &= U_1/IU_1 \oplus U_2/IU_2 \oplus A/IA, \\ S/I_iS &= U_1/I_iU_1 \oplus U_2/I_iU_2 \oplus A_1/I_iA_1 \oplus A_2/I_iA_2 \oplus A_3/I_iA_3, \end{aligned}$$

for $i = 1,\ 2$.

By Lemma 3.1, $U_1/\mathcal{A}(U_1) = U_1/I_1U_1$ is a projective R_2–module, and $U_2/\mathcal{A}(U_2) = U_2/I_2U_2$ is a projective R_1–module. Since $I_1U_1 = I_1(U_1/I_2U_1)$ and U_1/I_2U_1, as well as A_1, are artinian R_1–modules, the isomorphism

$$\phi_1 : I_1U_1 \oplus A_1 \longrightarrow I_1U_1' \oplus A_1'$$

lifts to an isomorphism

$$\alpha_1 : U_1/I_2U_1 \oplus A_1 \longrightarrow U_1'/I_2U_1' \oplus A_1',$$

by Lemma 3.2. Passing to quotients, α_1 induces an isomorphism

$$\gamma_1 : U_1/IU_1 \oplus A_1/IA_1 \longrightarrow U_1'/IU_1' \oplus A_1'/IA_1'$$

between summands of S/IS and S'/IS'. Since A_1 is an R_1–module, it follows that $A_1/IA_1 = A_1/I_1A_1$. Also, the R_2–module U_1/I_1U_1 is a projective cover of U_1/IU_1. Therefore, the isomorphism γ_1 lifts to an isomorphism

$$\beta_1 : U_1/I_1U_1 \oplus A_1/I_1A_1 \longrightarrow U_1'/I_1U_1' \oplus A_1,/I_1A_1'$$

of summands of S/I_1S and S'/I_1S'.

Similarly, the isomorphism

$$\phi_2 : I_2U_2 \oplus A_2 \longrightarrow I_2U_2' \oplus A_2'$$

induces an isomorphism

$$\beta_2 : U_2/I_1U_2 \oplus A_2 \longrightarrow U_2'/I_1U_2' \oplus A_2'$$

between summands of S/I_1S and S'/I_1S'. This, in turn, induces an isomorphism

$$\gamma_2 : U_2/IU_2 \oplus A_2/IA_2 \longrightarrow U_2'/IU_2' \oplus A_2'/IA_2'$$

between summands of S/IS and S'/IS'. As before, the isomorphism γ_2 lifts to an isomorphism

$$\alpha_2 : U_2/I_2U_2 \oplus A_2/I_2A_2 \longrightarrow U_2'/I_2U_2' \oplus A_2'/I_2A_2'$$

between summands of S/I_2S and S'/I_2S'.

In summary, we have constructed isomorphisms

$$\alpha = \alpha_1 \oplus \alpha_2 \oplus \phi_3 : S/I_2S \longrightarrow S'/I_2S',$$

and

$$\beta = \beta_1 \oplus \beta_2 \oplus \phi_3 : S/I_1S \longrightarrow S'/I_1S',$$

both induced by $\phi : \mathcal{A}(S) \longrightarrow \mathcal{A}(S')$, which induce the same isomorphism $\gamma_1 \oplus \gamma_2 \oplus \phi_3 : S/IS \longrightarrow S'/IS'$. (By abuse of notation, ϕ_3 denotes three different isomorphisms.) It follows now from the universal property of pullback diagrams that α and β induce an isomorphism $S \longrightarrow S'$ which is an extension of ϕ, as desired. This completes the proof of statement (1).

To prove the second part, let M be an indecomposable R–module. Let

$$0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$$

be a separated representation of M. Assume that $\mathcal{A}(M) = A_1 \oplus A_2$. Then, by Lemma 2.1.4, there is a short exact sequence

$$0 \longrightarrow K = K_1 \oplus K_2 \longrightarrow \mathcal{A}(S) = B_1 \oplus B_2 \longrightarrow \mathcal{A}(M) = A_1 \oplus A_2 \longrightarrow 0,$$

which is an extension of a direct sum of separated representations of A_1 and A_2. As before, $S = \bigoplus_{k=1}^{n} S_k$ is a direct sum of indecomposable separated modules S_k. Then $\mathcal{A}(S) = \bigoplus_k \mathcal{A}(S_k)$, with each $\mathcal{A}(S_k)$ indecomposable, by Lemma 3.1. It now follows from the Krull–Schmidt Theorem for artinian modules that

$$B_i \cong C_i = \bigoplus_{k \in E_i} \mathcal{A}(S_k)$$

for $i = 1,\ 2$ and some partition $\{E_1,\ E_2\}$ of $\{1,\ \ldots,\ n\}$. Let ϕ be an automorphism of $\mathcal{A}(S)$ such that $\phi(B_i) = C_i$. Define

$$M' = S/\left(\phi(K_1) \oplus \phi(K_2)\right) = D_1/\phi(K_1) \oplus D_2/\phi(K_2),$$

with $D_i = \bigoplus_{k \in E_i} S_k$. Note that this makes sense, since $\phi(K_i) \subset C_i \subset D_i$. Then

$$\phi : \mathcal{A}(M) = \mathcal{A}(S)/K \longrightarrow \mathcal{A}(M') = \mathcal{A}(S)/\left(\phi(K_1) \oplus \phi(K_2)\right)$$

is an isomorphism and $M/\mathcal{A}(M) \cong S/\mathcal{A}(S) \cong M'/\mathcal{A}(M')$. It now follows from the first part that $M \cong M'$. (Note that, even though M' is not assumed to be indecomposable, we have that $\mathcal{A}(S_k) \neq 0$ for all k, so the proof of the first part still applies.) But M was assumed to be indecomposable, hence $D_i/\phi(K_i) = 0$ for some i, and so $C_i/\phi(K_i) \cong A_i = B_i/K_i = 0$. This shows that $\mathcal{A}(M)$ is indecomposable, completing the proof of the theorem.

Corollary 4.2. *Let R be a pullback ring, such that each coordinate ring R_i, $i = 1,\ 2$, is a finite product of hereditary orders over semi–local principal ideal domains, and let $I_i = J(R_i)$ be the Jacobson radical of R_i. Then*

(1) *if M in $Mod(R)$ is indecomposable, then $\mathcal{A}(M)$ is indecomposable;*
(2) *if $M,\ M'$ in $Mod(R)$ are indecomposable, $\mathcal{A}(M) \cong \mathcal{A}(M')$, and $M/\mathcal{A}(M) \cong M'/\mathcal{A}(M')$, then $M \cong M'$.*

Proof. Completing at maximal ideals, if necessary, we may assume [Kl, Section 2, Wa, Section 6] that each R_i is a finite product of hereditary orders over complete discrete valuation domains. But such a ring is serial [Wa, Thm. 6.1], thus Theorem 4.1 applies.

5. Reduction to the case of F–algebras

We now relate R–modules to modules over the associated graded ring $G(R)$ of R with respect to $I \subset R$. Let R be a noetherian ring and I a two-sided ideal of R. Let $G(R)$ denote the associated graded ring, that is,

$$G(R) = \bigoplus_{i=0}^{\infty} I^i/I^{i+1},$$

where $I^0 = R$. There is a functor

$$G : Mod(R) \longrightarrow Mod\left(G(R)\right),$$

defined by $G(M) = \bigoplus_i I^iM/I^{i+1}M$, for a finitely generated R-module M.

Theorem 5.1. *Let R be a noetherian ring with Jacobson radical I, such that R/I is (semi–simple) artinian. The functor G induces a bijection on isomorphism classes of objects, which preserves and reflects indecomposables.*

Proof. First of all, it is straightforward to see that G preserves finite direct sums. Next we need to show that, if $G(M) \cong G(N)$ in $Mod((G(R))$, then $M \cong N$ in $Mod(R)$. First observe that, as a consequence of the Artin–Rees Lemma [Ma, Thm. 8.5], $\bigcap_n I^n M = 0$ for all R–modules M.

So suppose we are given an isomorphism

$$\varphi = \bigoplus_i \varphi_i : \bigoplus_i I^i M/I^{i+1}M \longrightarrow \bigoplus_i I^i N/I^{i+1}N.$$

(Note that, in this case, an isomorphism of graded modules is automatically a graded isomorphism.) Let $x_1, \dots, x_n$ be a set of generators for M, such that $\bar{x}_1, \dots, \bar{x}_n$ are a minimal set of generators for M/IM. Let $\bar{y}_1 = \varphi_0(\bar{x}_1), \dots, \bar{y}_n = \varphi_0(\bar{x}_n)$ be the corresponding set of generators for N/IN. Then we can lift these to a set of generators $y_1, \dots, y_n$ of N, since I is the Jacobson radical. Now define

$$f : M \longrightarrow N$$

by $f(x_i) = y_i$, for $i = 1, \dots, n$. To see that f is well-defined, suppose that $x = \sum r_i x_i = 0$ in M. Since $\varphi_0(\bar{x}) = 0 = \sum \bar{r}_i \bar{y}_i$, it follows that $r_i \in I$ for all i, since the elements $\bar{y}_i$ are linearly independent over the semi–simple artinian ring R/I, and $y = \sum r_i y_i \in IN$. But, since φ is a graded homomorphism, we have

$$y + I^2N = \sum (r_i y_i + I^2 N) = \sum \varphi_1(r_i x_i + I^2 M) = \varphi_1(x + I^2 M) = \varphi_1(0) = 0.$$

This shows that $y \in I^2 N$. Inductively, we can push y farther down the filtration. Since $\bigcap_i I^i N = 0$, this implies that $y = 0$. This shows that the map f is well-defined, and R-linear by definition.

To see that f is one-to-one, suppose that $f(x) = 0$ for some $x \in M$. Then $\varphi_0(\bar{x}) = 0$, hence $x \in IM$. Similarly, $\varphi_1(x + I^2M) = f(x) + I^2N = 0$, hence $x \in I^2M$. Since $\bigcap_i I^i M = 0$, we can see inductively, as above, that $x = 0$.

Let $y \in N$. Then $y + IN = \varphi_0(x + IM)$ for some $x \in M$. Thus, $f(x) - y \in IN$. Therefore, $f(x) - y + I^2N = \varphi_1(x' + I^2M)$ for some $x' \in M$. So $f(x) - f(x') - y \in I^2N$. Again, since $\bigcap_i I^i N = 0$, we can see inductively that y is in the image of f, so that f is onto, hence an isomorphism. Thus, we have proven that the functor G reflects isomorphisms.

Now let $N = \bigoplus_{i=0}^n N_i$ be a graded module over $G(R)$. We need to show that $N = G(M)$ for some R–module M. Let $x_1, \dots, x_m$ be a set of generators for the R/I–module N_0. Since $(I/I^2)N_i = N_{i+1}$, we get that N_i has generators $y_1, \dots, y_m$ as an I^i/I^{i+1}–module. Let F be a free R–module with generators $z_1, \dots, z_m$, and let the submodule K of F be generated by elements $\sum r_j z_j$ such that $\sum \bar{r}_j y_j = 0$ is a relation in N_i for some i. Define $M = F/K$. Then it is straightforward to see that $G(M) \cong N$.

Now it is easy to see that the functor G reflects indecomposables. Suppose that $G(M) \cong N_1 \oplus N_2$. Then

$$G(M) \cong N_1 \oplus N_2 \cong G(M_1) \oplus G(M_2) \cong G(M_1 \oplus M_2),$$

so that $M \cong M_1 \oplus M_2$, since G reflects isomorphisms. This completes the proof.

Remarks.

(1) Define a quotient category $MOD(R)$ of $Mod(R)$ to have the same objects as $Mod(R)$ but morphism sets $Hom_R(M,N)/Hom_R(M,IN)$. It is straightforward to see that $MOD(R)$ is an additive category with finite direct sums. One observes that the canonical quotient functor $Mod(R) \longrightarrow MOD(R)$ induces a bijection on objects. The functor G induces a functor of quotient categories $MOD(R) \longrightarrow MOD(G(R))$, which can be shown to be a category equivalence. (For details, see [AL].)

(2) The preceding theorem, together with Theorem 4.1, implies that, if R is a noetherian hereditary serial pullback ring, then the classification of indecomposable R–modules is reduced to the classification of indecomposable artinian $G(R)$–modules.

Proposition 5.2. *Let R be a pullback ring of R_1, R_2. If J_i is the Jacobson radical of R_i, $i = 1,2$, then assume that the Jacobson radical of R is $J = J_1 \times J_2$. Then $G(R)$, formed with respect to J, is also a pullback ring, the Jacobson radical $J(G(R))$ is equal to $G(J)$, and*
$G(R)/J(G(R)) \cong F$. Moreover, if R_i is noetherian serial, then so is $G(R_i)$.

Proof. Using the universal property of the following pullback diagram, it is straightforward to check that $G(R)$ is isomorphic to the ring A in the pullback diagram

$$\begin{array}{ccc} A & \longrightarrow & G(R_1) \\ \downarrow & & \downarrow \\ G(R_2) & \longrightarrow & F\ . \end{array}$$

The remaining assertions of the proposition follow immediately.

6. Properties of $G(R)$

First we prove a general result about pullback rings. Throughout this section we shall assume that the Jacobson radical J of R is equal to $J_1 \times J_2$, where J_i is the Jacobson radical of R_i. Recall [Wa, Lemma–Definition 1.3] that, if R is a ring with Jacobson radical J, then R is *semiperfect* if R/J is artinian and idempotents of R/J lift to idempotents of R.

Theorem 6.1. *If R is a pullback ring with semiperfect coordinate rings, then R is semiperfect.*

Proof. Let $J = J(R)$ and $J_i = J(R_i)$, where $R_1,\ R_2$ are the coordinate rings of R. Then $R/J = F$ is semi–simple artinian.

It remains to show that an idempotent $f \in F$ lifts to an idempotent in R. But each R_i is semiperfect, hence there are idempotents $e_i \in R_i$ which lift f. Then $e = (e_1, e_2) \in R$ is an idempotent which lifts f.

We now recall the definition of basic rings and basic idempotents [AF, pp. 308–309]. Let R be a semiperfect ring with $R/J(R)$ semi–simple artinian. We can write $1 = f_1 + \cdots + f_n \in F$ as a sum of primitive orthogonal idempotents. Then $1 = e_1 + \cdots + e_n \in R$ is a sum of primitive orthogonal idempotents, with the property that $Re_i \cong Re_j$ if and only if $Ff_i \cong Ff_j$. After suitably relabeling the e_i, choose a representative $Re_1, \ldots, Re_m$ for each isomorphism class of the modules $Re_1, \ldots, Re_n$, and let

$$e = e_1 + \cdots + e_m.$$

Then e is called a *basic idempotent* of R, and eRe is a called a *basic ring* of R. If $f \in F$ is an idempotent, which lifts to $e \in R$, then f is a basic idempotent of F and fFf is a product of division rings [AF, Prop. 27.15].

Corollary 6.2. *Let R be a pullback ring, with semiperfect coordinate rings R_1, R_2. Let $e \in R$ be an idempotent such that eRe is a basic ring of R. Then*

(1) *R is Morita equivalent to eRe;*
(2) *if e_i is the image of e in R_i, and $f \in F$ is the image of e_i, then e_iRe_i is a basic ring of R_i and fFf is a product of division rings; eRe is the pullback of $e_iRe_i \longrightarrow fFf$ for $i = 1, 2$.*

Proof. 1. This is [AF, Prop. 27.14].

2. Since each R_i is semiperfect, with $R_i/J(R_i) = F$, a basic idempotent of F lifts to basic idempotents in R_i which, in turn, lift to a basic idempotent of R. This implies (2).

Corollary 6.3. *Let R be a pullback ring with semiperfect coordinate rings R_1, R_2. Then*

(1) *$G(R)$ is semiperfect;*
(2) *if F is a matrix ring over a field k, and each R_i is hereditary noetherian serial with $\mathcal{A}(R_i) = 0$, then $G(R)$ is Morita equivalent to the localization of the k-algebra $k[x, y]/\langle xy\rangle$ at the maximal ideal $\langle x, y\rangle$.*

Proof. 1. Let $J = J(R)$, and $J_i = J(R_i)$. Recall that $J(G(R)) = G(J)$, and $G(R)/G(J) = R/J = F$ (Prop. 5.2). Since R is semiperfect, a primitive idempotent f of F lifts to a primitive idempotent e of R. Thus, Re is an indecomposable summand of R. By Theorem 5.1, $G(Re)$ is an indecomposable summand of $G(R)$. It now follows that $G(Re) = G(R)e'$ for some idempotent $e' \in G(R)$, such that e' maps to $f \in F$. Thus, $G(R)$ is semiperfect.

2. By Corollary 6.2, $G(R)$ is Morita equivalent to a basic ring $\Lambda = eG(R)e$, which is a pullback of basic rings $\Lambda_i = e_iG(R_i)e_i$ over fFf. Since F is a matrix ring over a field k, $fFf = k$.

Now (identifying corresponding idempotents in R and $G(R)$), we obtain

$$\Lambda = \bigoplus_m e(J^m/J^{m+1})e,$$

with $e(R/J)e = fFf$, and

$$e(J^m/J^{m+1})e = (e_1J_1^m e_1)/(e_1J_1^{m+1}e_1) \oplus (e_2J_2^m e_2)/(e_2J_2^{m+1}e_2).$$

Furthermore, each Λ_i is a noetherian serial ring which is indecomposable. Therefore, it is serial as a left Λ_i–module, since k is a field and each R_i is noetherian serial. Thus, $J(\Lambda_i) = e_iJ_ie_i = \Lambda_ix_i$, for some $x_i \in \Lambda_i$. Furthermore,

$$J(\Lambda_i)^m/J(\Lambda_i)^{m+1} = (e_iJ_i^m e_i/e_iJ_i^{m+1}e_i)$$

is simple, hence isomorphic to k. It follows that $J(\Lambda_i)^m = \Lambda_i x_i^m$. We now have

$$\begin{aligned}\Lambda &= k \oplus (\Lambda_1x_1/\Lambda_1x_1^2 \oplus \Lambda_2x_2/\Lambda_2x_2^2) \oplus (\Lambda_1x_1^2/\Lambda_1x_1^3 \oplus \Lambda_2x_2^2/\Lambda_2x_2^3) \oplus \cdots \\ &\cong k[x_1, x_2]_{\langle x_1,x_2\rangle}/\langle x_1x_2\rangle,\end{aligned}$$

as desired.

7. Classification of Indecomposables

In this section we carry out the classification scheme for indecomposable modules based on the results in the previous section, for the case where R is a pullback ring of hereditary noetherian serial rings, finitely generated over their centers, and F is a matrix ring over a field k.

We first need to prove a result about modules over the ring Λ above. An indecomposable artinian Λ–module is also an indecomposable module over the artin k–algebra

$$\Lambda_{pq} = k[x, y]_{\langle x,y\rangle}/\langle xy, x^{p+1} - y^{q+1}\rangle,$$

for some positive integers p, q. The classification of finitely generated indecomposable modules over Λ_{pq} is surveyed in [BS, pp. 112–113]. There are two types, string modules and band modules, corresponding to the deleted cycle and block cycle indecomposables, respectively, in [Le2]. We will call an indecomposable artinian Λ–module a string or band module, if it is a string, respectively band, module over Λ_{pq} for some p, q. We first recall the definition of string and band modules in the form given in [AL, Sect. 4], since it is needed in the proof of Lemma 7.4 below.

Observe that Λ_{pq} can be described as follows: It is the k-algebra with generators a, b and relations $ab = 0 = ba$, $a^{p+1} - b^{q+1} = 0$. We now define two types of modules over Λ_{pq}. Observe that Λ_{pq} has $\{1, a^i, b^j \mid 1 \le i \le p+1, 1 \le j \le q\}$ as a k-basis.

Define a *word* to be a (non–empty) symbol

$$W = a^{i_1}b^{j_1}a^{i_2}b^{j_2}\ldots a^{i_n}b^{j_n},$$

made up of elements in the above basis of Λ_{pq}, such that $j_1, \ldots, j_{n-1}, i_2, \ldots, i_n \ge 1$. (In [BS], this is called a proper word.) We set $a^0 = 1 = b^0$. A word W is called an *irreducible band word* if $i_1, j_n \ge 1$, and it is not a power $W = (W')^m$ of another word W'. Note that a cyclic permutation of the constituent pairs a^ib^j in an irreducible band word results in another irreducible band word.

Definition 7.1. [BS, 112]

(1) Let W be a word (not necessarily irreducible) as above. Define the associated *string module* $M(W)$ to be the Λ_{pq}-module with generators $u_1, \ldots, u_n$ and relations

$$a^{i_1}u_1 = 0, \qquad b^{j_k}u_k = a^{i_{k+1}}u_{k+1} \quad (k = 1, \ldots, n-1), \qquad b^{j_n}u_n = 0\,.$$

(2) Let W be an irreducible band word, and let φ be an indecomposable automorphism of a finite dimensional k-vector space V. Define the *band module* $M(W, V, \varphi)$ to be the Λ_{pq}–module generated by n vector subspaces $u_1 \otimes V, \dots, u_n \otimes V$, subject to the relations

$$\begin{aligned} b^{j_k} u_k \otimes v &= a^{i_{k+1}} u_{k+1} \otimes v \text{ for } k = 1, \dots, n-1, \ v \in V; \\ b^{j_n} u_n \otimes v &= a^{i_1} u_1 \otimes \varphi(v), \qquad v \in V. \end{aligned}$$

Note that both $M(W)$ and $M(W, V, \phi)$ are artinian Λ–modules.

Definition 7.2. Let W be a word, as in Definition 7.1.1, and let n_1, n_2 be positive integers. Define the associated *non–artinian string module* $M(W, n_1, n_2)$ to be the Λ–module with generators $z_1, z_2, v_1, \dots, v_n$, such that z_1 or z_2 may be zero, but not both. The relations are given by:

$$b^{n_2} z_1 = a^{i_1} v_1, \ b^{j_k} v_k = a^{i_{k+1}} v_{k+1}, \ b^{j_n} v_n = a^{n_1} z_2.$$

Theorem 7.3. *([BS, p. 112]) Artinian string and band modules account for all finitely generated indecomposable Λ_{pq}–modules.*

Note that a non–artinian string module $M(W, n_1, n_2)$ is indecomposable, since $\mathcal{A}(M(W, n_1, n_2)) = M(W)$ is an (artinian) string module, which is indecomposable.

Lemma 7.4. *Let $\Lambda = k[x_1, x_2]_{\langle x_1, x_2 \rangle} / \langle x_1 x_2 \rangle$. There exists a finitely generated indecomposable Λ–module M such that $\mathcal{A}(M) \cong A$ and $M/\mathcal{A}(M) \cong L$ if and only if*

(1) *$A \cong M(W)$ is a string module with $i_1 = 1$ or $j_n = 1$ or both;*
(2) *$L = L_1 \times L_2$ is a finitely generated non-zero $\Lambda_1 \times \Lambda_2$–module, with each L_i indecomposable projective or zero (where $\Lambda_1 = \Lambda/\langle b \rangle$, and $\Lambda_2 = \Lambda/\langle a \rangle$);*
(3) *$L_1 \neq 0$ if and only if $i_1 = 1$, and $L_2 = 0$ if and only if $j_n \neq 1$. In this case, M is a string module with $z_i = 0$ if and only if $L_i = 0$.*

Proof. First suppose that M is a finitely generated non–artinian indecomposable Λ–module, with a separated representation

$$0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0.$$

Decompose S as a direct sum of indecomposable separated modules:

$$S = S_1 \oplus \cdots \oplus S_n.$$

Then some S_m is non–artinian. By Lemma 3.1, $\mathcal{A}(S_m)$ is an indecomposable (hence cyclic) summand of $\mathcal{A}(S)$ with generator z, such that either $(x_1, 0)z = 0$ and $\mathcal{A}(S_m) \subset I_2 S_m$, or $(0, x_2)z = 0$ and $\mathcal{A}(S_m) \subset I_1 S_m$.

By Lemma 2.1, there is a separated representation

$$0 \longrightarrow K' \longrightarrow T \longrightarrow \mathcal{A}(M) \longrightarrow 0,$$

with $K = K' \oplus L$, and $\mathcal{A}(S) = T \oplus L$ for some k–vector space L. Since $K \cap I_i S = 0$ for $i = 1,\ 2$, it follows that $\mathcal{A}(S_m)$ is a summand of T. Moreover, $\mathcal{A}(M)$ is indecomposable by Theorem 4.1.2. Then $\mathcal{A}(M)$ is a module over $\Lambda/ann\,(\mathcal{A}(M))$, which, in turn, is a quotient of Λ_{pq} for suitable p and q. Thus, $\mathcal{A}(M)$ is an indecomposable Λ_{pq}–module, hence is either a string or a band module, by Theorem 7.3.

We will now derive a contradiction from the assumption that $\mathcal{A}(M)$ is a band module. As a band module over Λ_{pq}, $\mathcal{A}(M)$ has a minimal presentation

$$0 \longrightarrow K'' \longrightarrow A \longrightarrow \mathcal{A}(M) \longrightarrow 0,$$

as given in Definition 7.1.2, where a and b represent the images of $(x_1, 0)$, respectively $(0, x_2)$, in Λ_{pq}. That is, $\mathcal{A}(M)$ is generated by vector subspaces $u_i \otimes V$, for $i = 1, \ldots, n$, V a finite–dimensional k–vector space, satisfying the above relations.

Now let A' be the Λ–module generated by vector subspaces $u_i \otimes V$. Then A' is a separated Λ–module, since each summand $\Lambda u_i \otimes v_j$ is separated (and indecomposable), for $v_1, \ldots, v_m$ a k–basis of V. Then clearly $A'/ann\,(\mathcal{A}(M))\,A' \cong A$ as Λ_{pq}–modules. The composite $A' \longrightarrow A \longrightarrow \mathcal{A}(M)$ is a surjection of the separated module A' onto $\mathcal{A}(M)$. The universal property of a separated representation now gives a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K' & \longrightarrow & T & \longrightarrow & \mathcal{A}(M) & \longrightarrow & 0 \\
 & & \Big\downarrow & & \phi\Big\downarrow & & =\Big\downarrow & & \\
0 & \longrightarrow & K'' & \longrightarrow & A' & \xrightarrow{\ \pi\ } & \mathcal{A}(M) & \longrightarrow & 0,
\end{array}$$

where the middle and left vertical maps are one–to–one. If $x \in T$ and $\phi(x) \in ann\,(\mathcal{A}(M))\,A'$, then $\pi\phi(x) = 0$, hence $x \in K'$. But $K' \cap \mathcal{A}(S_m) \subset K' \cap I_i T = 0$, hence $\mathcal{A}(S_m) \cap ann\,(\mathcal{A}(M))\,T = 0$. Upon passing to quotients, we therefore obtain a homomorphism

$$T/ann\,(\mathcal{A}(M))\,T \longrightarrow A'/ann\,(\mathcal{A}(M))\,A' \cong A$$

which is one–to–one on the (non–zero) summand $\mathcal{A}(S_m)/ann\,(\mathcal{A}(M))\,\mathcal{A}(S_m)$. Thus, A contains a summand C such that either $aC = 0$ or $bC = 0$. We may assume that this summand is indecomposable. Then, by the Krull–Schmidt Theorem, $C \cong \Lambda_{pq} u_i \otimes v_j$ for some $i,\ j$. But this is a contradiction since in a band module such a relation can only occur if a or b is raised to an exponent of at least 2.

So now assume that $\mathcal{A}(M)$ is a string module. The preceding argument shows that there are at most two non–artinian summands S_m in the separated representation of M. This follows because, for the artinian string module $\mathcal{A}(M)$, the only relations of the form $aC = 0$ or $bC = 0$ for an indecomposable summand C in the minimal presentation of $\mathcal{A}(M)$ are the two relations $a^{i_1}C = 0$ and $b^{j_n}C = 0$. This proves Part 1.

To prove Part 2, observe that, given M, the proof of Part 1 shows that

$$S/\mathcal{A}(S) \cong M/\mathcal{A}(M) \cong L_1 \oplus L_2,$$

with each L_i an indecomposable projective R_i–module, such that $L_1 \neq 0$ if and only if $i_1 \neq 0$, and $L_2 \neq 0$ if and only if $j_n \neq 0$, as claimed.

In the other direction, given A and L as specified, it is straightforward to construct a non–artinian string module M, via generators and relations as in Definition 7.2, such that $\mathcal{A}(M) \cong A$ and $M/\mathcal{A}(M) \cong L$.

Theorem 7.5. *Let R be a pullback ring of two hereditary noetherian serial rings R_1, R_2 over a matrix ring over a field k. Assume that each R_i is finitely generated over its center and that $\mathcal{A}(R_i) = 0$. Then the procedure 7.6 below gives a complete classification of all indecomposable finitely generated R–modules.*

Procedure 7.6.

1. Two indecomposable R–modules M and M' are isomorphic if and only if the indecomposable $G(R)$–modules $G(M)$ and $G(M')$ are isomorphic (Theorem 5.1), where $G(R)$ is formed with respect to the Jacobson radical of R.
2. The ring $G(R)$ is Morita equivalent to $\Lambda = k[x, y]_{\langle x,y \rangle}/\langle xy \rangle$ (Corollary 6.3).
3. The isomorphism class of an indecomposable Λ–module is determined by the isomorphism class of the artinian indecomposable $\mathcal{A}(M)$ and the isomorphism class of $M/\mathcal{A}(M)$ (Theorem 4.1).
4. An indecomposable artinian Λ–module is either a string or a band module over a quotient $\Lambda_{pq} = \Lambda/\langle x^{p+1} - y^{q+1} \rangle$, which is a finite–dimensional k–algebra.
5. An indecomposable non–artinian Λ–module is isomorphic to Λ, $\Lambda/\langle x \rangle$, $\Lambda/\langle y \rangle$, or a non–artinian string module.

Example. If the coordinate rings of the pullback ring R are finite products of hereditary orders over semi–local principal ideal domains, such that $I = J(R)$, then the indecomposable R–modules can be classified by applying Procedure 7.6 to each indecomposable component of R. For instance, if R_1 and R_2 are discrete valuation rings with a common residue field, then we obtain the classification in [Le2].

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