Delaunay polytopes of cut lattices

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Abstract

We continue [3], the study of the lattice L_n generated by cuts of the complete graph on a set V_n of n vertices. The lattice L_n spans an $N=\binom{n}{2}$-dimensional space of all functions defined on a set V_n^2 of all unordered pairs of the set V_n. We prove that the cut polytope, i.e. the convex hull of all cuts, is an asymmetric Delaunay polytope of L_n. Symmetric Delaunay polytopes of a lattice L are completely described by classes of the quotient $\frac{1}{2}L/L$. We show that a class of the quotient $\frac{1}{2}L_n/L_n$ is uniquely determined by a subset $S\subseteq V_n$ and a class of switching equivalent sets $A\subseteq V_n^2$. We describe minimal vectors of all classes of $\frac{1}{2}L_n/L_n$. We completely describe L-partition of 6-dimensional space into Delaunay polytopes of the lattice $L_4=\sqrt{2}D_6^{+2}$.

1 Introduction

Let V_n be a ground set of cardinality $|V_n|=n$. A *cut vector* or simply, a *cut* $\delta(S)$, $S\subseteq V_n$, is a vector of the space of all functions $d:V_n^2\to\mathbf{R}$, defined on the set V_n^2 of all unordered pairs of the set V_n. A component $\delta_{ij}(S)$ is defined as follows. For $S,T\subseteq V_n$, let

$$(S,T)=\{(ij)\in V_n^2:\ i\in S, j\in T\},\ D(S)=(S,V_n-S),\ S^2=(S,S). \tag{1}$$

Then

$$\delta_{ij}(S)=\begin{cases}1 & \text{if } (ij)\in D(S)\\ 0 & \text{otherwise}\end{cases}$$

For any map ϕ defined on subsets of the set V_n and for $i\in V_n$, we abbreviate $\phi(\{i\})$ to $\phi(i)$. In particular, notations $\delta(i)$, $D(i)$ and (i,T) are clear. Note that $(i,j)=\{(ij)\}$.

The dimension of the space spanned by all $\delta(S)$'s is equal to $N:=|V_n^2|=\frac{n(n-1)}{2}$. Since $\delta(S)=\delta(V_n-S)$, there are 2^{n-1} cuts, including zero cut $\delta(\emptyset)=0$.

Let $\mathcal{K}_n=\{\delta(S):S\subseteq V_{n-1}\}$ be the set of all cut vectors.

Let $\mathbf{Z}^N$ be the lattice of all integral N-vectors. Clearly $\mathcal{K}_n\subset\mathbf{Z}^N$. Hence the lattice L_n linearly generated by all cuts $\delta(S)\in\mathcal{K}_n$, i.e.

$$L_n=\{d:d=\sum_{\delta(S)\in\mathcal{K}_n}z_S\delta(S),\ z_S\in\mathbf{Z}\},$$

is a sublattice of $\mathbf{Z}^N$.

In this note we study an L-partition related to L_n, i.e. the partition of the space $\mathbf{R}^N$ spanned by L_n into Delaunay polytopes. This partition is normal, i.e. it is face-to-face. This means that if two Delaunay polytopes of this partition are contiguous, then they share a common face.

A *Delaunay polytope* of a lattice is a convex hull of all lattice points lying on an empty sphere of the lattice, and the lattice points on the empty sphere have full rank. A sphere is called *empty* if there is no lattice point strictly inside it. A Delaunay polytope P is called *basic* if the set of its vertices contains an affine basis of the lattice affinely generated by vertices of P. A Delaunay polytope P is called *symmetric* or *asymmetric* according to whether an antipode of each vertex of P is or not is (respectively) a vertex of P.

2 Cut polytope is an asymmetric Delaunay polytope of L_n

Let $\{e_{ij} : 1 \leq i < j \leq n\}$ be an orthonormal basis of $\mathbf{R}^N$. The lattice generated by this basis is $\mathbf{Z}^N$. For $m \in \mathbf{Z}$, let $m\mathbf{Z}^N$ be the lattice of all integral vectors divisible by m. The following simple propositions are proved in [3].

Proposition 1 *(i)* $2\mathbf{Z}^N \subset L_n$,

(ii) $L_n/2\mathbf{Z}^N = \mathcal{K}_n$, *i.e. each* $d \in L_n$ *is equivalent by mod 2 to some* $\delta(S) \in \mathcal{K}_n$.

Proposition 2 *The following set* B *of* N *cuts forms a basis of* L_n

$$B = \{\delta(i), \delta(ij) : i, j \in V_{n-1}\}.$$

Proposition 1(ii) implies that every $d \in L_n$ has the form

$$d = 2a + \delta(S), \text{ where } a \in \mathbf{Z}^N. \tag{2}$$

The equation (2) shows that unique (0,1)-vectors $d \in L_n$ are cut vectors. Let $d \in L_n$ has $(0, \pm 1)$-components. Since d (mod 2) is a cut vector, all such vectors have the following form

$$d(S; X) := \delta(S) - 2 \sum_{(ij) \in X} e_{ij}, \tag{3}$$

where $X \subseteq D(S)$, and the set $D(S)$ is defined in (1). There are $2^{k(n-k)}$, $k = |S|$, subsets $X \subseteq D(S)$.

Since any $X \subseteq D(i)$ has the form $X = (i, T)$ for some $T \subseteq V_n - \{i\}$, if $S = \{i\}$, we simplify notations setting

$$d(i, T) := d(i; (i, T)).$$

For $(ij) \in V_n^2$, let R_{ij} be a reflection of the space $\mathbf{R}^N$ in the hyperplane orthogonal to e_{ij}. Any two reflections R_{ij} and R_{kl} commute, i.e. $R_{ij}R_{kl} = R_{kl}R_{ij}$. Note that for $d(S; X)$ of (3), we have $d(S; X) = \prod_{(ij) \in X} R_{ij}\delta(S)$.

For $T \subseteq V_n - \{i\}$, set $R^i_T = \prod_{j \in T} R_{ij}$. Then $R^i_S R^i_T = R^i_{S \Delta T}$. In this notations, $d(i, T) = R^i_T \delta(i)$.

Since $\delta(S)$ is a (0,1)-vector, it is a vertex of a N-dimensional cube Q_N. Let S_N be a sphere circumscribing the cube Q_N. Squared diameter of S_N is equal to N. Hence the squared radius of S_N is equal to $\frac{1}{4}N$.

Let $PCut_n$ be the convex hull of cuts. Clearly, $PCut_n \subset Q_N$. Let j_N be the all-one vector of dimension N. Then the center of S_N is an endpoint of the vector $\frac{1}{2}j_N$.

Proposition 3 *The cut polytope $PCut_n$ is a basic asymmetric Delaunay polytope of the lattice L_n.*

Proof. By definition $PCut_n$ generates the lattice L_n. Hence it is sufficient to prove that the sphere S_N, circumscribing $PCut_n$, is empty. All integral points lying inside S_N are vertices of Q_N. These vertices lie on this sphere. Besides, by Proposition 1(ii), all (0,1)-vertices of L_n are cut vectors. Proposition 2 implies that $PCut_n$ is basic. It is easy to see that the polytope $PCut_n$ is asymmetric for $n > 2$. $\square$

Recall that all cut vectors are (0,1)-vectors. Hence they form a binary N-dimensional code. By Proposition 1, this code is linear. By Proposition 2, dimension of this code is equal to $n-1$. Minimal distance of this code is equal to $n-1$. Therefore this is $(\binom{n}{2}, n-1, n-1)$-code. The lattice L_n is the lattice generated by this code by Construction A of [2].

It is not difficult to prove that, in general, convex hull of codewords of a linear code is a Delaunay polytope of the lattice generated by this code by Construction A of [2].

3 The quotient of $\frac{1}{2}L_n/L_n$

A symmetric Delaunay polytope of a lattice L is tightly related to minimal elements of a class of the quotient $\frac{1}{2}L/L$. Let Q_{min} be a set of all minimal vectors of a class Q of the quotient $\frac{1}{2}L/L$. Let $P(Q)$ be a convex hull of endpoints of vectors of Q_{min}. Let $H(Q)$ be the space spanned by Q_{min}. Baranovskii [1] (see also [4]) proves the following

Theorem 1 *$P(Q)$ is a symmetric Delaunay polytope of the lattice $L \cap H(Q)$, and every symmetric Delaunay polytope of L is of the form $P(Q)$ for some class Q of the quotient $\frac{1}{2}L/L$.* $\square$

We want to characterize classes of the quotient $\frac{1}{2}L_n/L_n$. Since $\dim L_n = N$, there are 2^N classes of $\frac{1}{2}L_n/L_n$.

For a sets $A, B \subseteq V_n^2$, we denote by $A - B$ the set $A - A \cap B$. Then, for a symmetric difference $A\Delta B$, we have

$$A\Delta B = (A - B) \cup (B - A).$$

For $A \subseteq V_n^2$, we call the set $A\Delta D(T)$ *switching equivalent* (by a set $T \subseteq V_n$) to the set A. Since $D(T)\Delta D(R) = D(T\Delta R)$, we have

$$(A\Delta D(T))\Delta D(R) = A\Delta D(T\Delta R).$$

In particular, $(A\Delta D(T))\Delta D(T) = A$. Hence the set of all subsets of V_n^2 is partitioned into classes of switching equivalent subsets.

Remark. For $E \subseteq V_n^2$, let $G(E)$ be a graph with a set of vertices V_n and a set of edges E. By above, all graphs are partitioned into switching classes of graphs with switching equivalent sets of edges. But usually, switching classes of graphs are considered up to isomorphisms of graphs. Switching classes of subsets of V_n^2, considered by us, are, in fact, switching classes of labeled graphs. $\square$

Any switching class is uniquely determined by a set of this class. For example, the class containing an empty set $\emptyset$ consists of sets $D(S)$ for all $S \subset V_{n-1}$. Name this class by *cut class*, and denote it $\mathcal{C}_n$.

For $i \in V_n$ and $T \subseteq V_n - \{i\}$, let $\mathcal{D}_n^{i,T}$ be a class containing the set $V_n^2 - (i,T)$. Since $V_n^2 - (i, V_n - \{i\} - T) = (V_n^2 - (i,T))\Delta D(i)$ belongs to the same class $\mathcal{D}_n^{i,T}$, we suppose that $|T| \leq \lfloor \frac{n-1}{2} \rfloor$. For $T = \emptyset$, $\mathcal{D}_n^{i,\emptyset}$ does not depend on i. Denote $\mathcal{D}_n = \mathcal{D}_n^{i,\emptyset}$ and call $\mathcal{D}_n$ the *co-cut class*. For one-element set $T = \{j\}$, $\mathcal{D}_n^{i,\{j\}} = \mathcal{D}_n^{j,\{i\}}$. Denote $\mathcal{D}_n^{ij} = \mathcal{D}_n^{i,\{j\}}$.

Similarly, for $|T| \leq \lfloor \frac{n-1}{2} \rfloor$, let $\mathcal{E}_n^{i,T}$ be a class containing the set (i,T). Note that $(i, V_n - \{i\} - T) \in \mathcal{E}_n^{i,T}$, and $\mathcal{E}_n^{i,\emptyset} = \mathcal{C}_n$. Since $\mathcal{E}_n^{i,\{j\}} = \mathcal{E}_n^{j,\{i\}}$, denote $\mathcal{E}_n^{ij} = \mathcal{E}_n^{i,\{j\}}$.

Proposition 4 *(i) The switching class $\mathcal{D}_n$ consists of sets $V_n^2 - D(S)$ for all $S \subseteq V_n$.*

(ii) For $n \geq 5$, there are $\binom{n}{2}$ distinct switching classes $\mathcal{E}_n^{ij}$ different from $\mathcal{C}_n$ and $\mathcal{D}_n^{i,T}$. For $n = 3$, $\mathcal{E}_3^{ij} = \mathcal{D}_3$ for all $(ij) \in V_3^2$. For $n = 4$, $\mathcal{E}_4^{ij} = \mathcal{D}_4^{kl}$, where $\{kl\} = V_4 - \{ij\}$. $\square$

For a fixed $S \subseteq V_n$, a switching class $\mathcal{A}$ and a set $A \in \mathcal{A}$, we set

$$A_S = A \cap D(S), \tag{4}$$

$$A^S = A - D(S). \tag{5}$$

Proposition 5 *Any class Q of the quotient $\frac{1}{2}L_n/L_n$ is uniquely determined by a set $S \subseteq V_{n-1}$ and a class $\mathcal{A}$ of switching equivalent subsets $A \subseteq V_n^2$.*

Proof. Let $d \in Q$. According to (2), d is equal by mod $2\mathbf{Z}^N$ to $\frac{1}{2}\delta(S) + \sum_{(ij)\in A} e_{ij}$ for some sets $S \in V_{n-1}$ and $A \in V_n^2$. Obviously, $d \equiv \frac{1}{2}d(S;A_S) + \sum_{(ij)\in A^S} e_{ij}$, (mod $2\mathbf{Z}^N$), where A_S and A^S are given in (4) and (5). In particular, since $d(T,Y) \equiv \delta(T)$ (mod $2\mathbf{Z}^N$), the sum $\frac{1}{2}d(S;A_S) + \sum_{(ij)\in A^S} e_{ij} + d(T;Y)$ is equial by mod $2\mathbf{Z}^N$ to the vector $\frac{1}{2}d(S;B_S) + \sum_{(ij)\in B^S} e_{ij}$, where $B = A\Delta D(T)$. The assertion of the proposition follows. $\square$

Denote a class Q determined by a set $S \subseteq V_n$ and a switching class $\mathcal{A}$ by $Q(S,\mathcal{A})$. Since there are 2^{n-1} different up to complement sets $S \subseteq V_n$ and there are $2^N = 2^{\frac{1}{2}n(n-1)}$ classes of the quotient $\frac{1}{2}L_n/L_n$, we have

Corollary 1 *There are $2^{\frac{1}{2}(n-1)(n-2)} = 2^{\binom{n-1}{2}}$ switching classes.*

Note that the number of switching classes is equal to discriminant of L_n, i.e. to $\sqrt{DetL_n}$.

We call a set $A \in \mathcal{A}$ *S-minimal* if the set A^S has minimal cardinality among all sets A'^S for $A' \in \mathcal{A}$.

Now we describe minimal vectors of a class $Q(S,\mathcal{A})$.

Theorem 2 *Let $\mathcal{A}$ be a switching class containing a set $A \subseteq V_n^2$. Minimal vectors of the class $Q(S,\mathcal{A})$ are vectors*

$$d = \frac{1}{2}d(S;A_S) + \sum_{(ij)\in A^S} \varepsilon_{ij}e_{ij} \qquad (6)$$

for all S-minimal $A \in \mathcal{A}$. The sets A_S and A^S are given in (4) and (5), respectively, and ε_{ij} is an arbitrary (± 1)-function on A^S.

Proof. The proof of Proposition 5 shows that any $d \in Q(S,\mathcal{A})$ has the form $d = \frac{1}{2}d(S;A_S) + \sum_{(ij)\in A^S} e_{ij} + 2\sum_{(ij)\in V_n^2} z_{ij}e_{ij}$. Note that norm d^2 of d is equal to $d^2 = \sum_{(ij)\in D(S)-A_S}(\frac{1}{2}\delta_{ij}(S) + 2z_{ij})^2 + \sum_{(ij)\in A_S}(-\frac{1}{2}\delta_{ij}(S) + 2z_{ij})^2 + \sum_{(ij)\in A^S}(1 + 2z_{ij})^2 + \sum_{(ij)\in V_n^2 - D(S)-A^S} 4z_{ij}^2$. For d to be minimal for fixed A, it is necessarily that $z_{ij} = 0$ for $(ij) \in V_n^2 - A^S$ and $z_{ij} = 0$ or -1 for $(ij) \in A^S$.

For such z_{ij}, d has the form (6). Hence minimal vectors have the form (6). Since $d^2(S;X) = \delta^2(S)$ are the same for all $X \subseteq D(S)$, d is minimal for all S-minimal $A \in \mathcal{A}$. □

Below, we explicitely describe minimal elements of some classes $Q(S,\mathcal{A})$. For $Y \subseteq V_n$, set $A(Y) = V_n^2 - D(Y)$ and

$$d(Y,\varepsilon) = \sum_{(ij)\in A(Y)} \varepsilon_{ij}e_{ij}.$$

$$d_k(Y,\varepsilon) = \frac{1}{2}d(k,Y) + \sum_{(ij)\in A^k(Y)} \varepsilon_{ij}e_{ij}, \qquad (7)$$

where $A^k(Y) = A(Y) - D(k)$.

Proposition 6 *(i) Minimal vectors of the class $Q(S,\mathcal{C}_n)$ are $\pm\frac{1}{2}\delta(S)$;*

(ii) if $n \ge 4$, then minimal vectors of the class $Q(\emptyset,\mathcal{E}_n^{ij})$ are $\pm e_{ij}$;

(iii) minimal vectors of the class $Q(\emptyset,\mathcal{D}_n)$ are vectors $d(Y,\varepsilon)$, where ε is an arbitrary (± 1)-function on $A(Y)$, and $Y \subseteq V_n$ has cardinality $\lfloor\frac{n}{2}\rfloor$;

(iv) minimal vectors of the class $Q(k,\mathcal{D}_n^{k,T})$ are vectors $R_T^k d_k(Y,\varepsilon)$ for all $Y \subseteq V_n - \{k\}$ of cardinality $\lfloor\frac{n-1}{2}\rfloor$ or $\lceil\frac{n-1}{2}\rceil$ and all (± 1)-functions ε on $A^k(Y)$.

Proof. (i) Obviously, S-minimal sets of the class $\mathcal{C}_n$ are only sets $\emptyset$ and $D(S)$. Since $\frac{1}{2}\delta(S) + \delta(S) \equiv -\frac{1}{2}\delta(S)$ by mod 2, the result follows.

(ii) For $n \ge 4$, the switching class $\mathcal{E}_n^{ij}$ has the one-element set $\{(ij)\}$ as a unique $\emptyset$-minimal set.

(iii) It is obvious, that $\emptyset$-minimal sets of $\mathcal{D}_n$ are sets $A(Y) = V_n^2 - D(Y)$ with maximal $|D(Y)|$. Since $|D(Y)| = |Y|(n - |Y|)$, $|D(Y)|$ is maximal for Y such that $|Y| = \lfloor\frac{n}{2}\rfloor$ and $|Y| = \lceil\frac{n}{2}\rceil$. Since $D(Y) = D(V_n - Y)$, we can consider only Y with $|Y| = \lfloor\frac{n}{2}\rfloor$.

(iv) At first, we consider the case $T = \emptyset$. By Theorem 2, minimal vectors of the class $Q(k,\mathcal{D}_n)$ are vectors (6) with $A_k(Y) = (V_n^2 - D(Y)) \cap D(k)$ and $A^k(Y) = V_n^2 - D(Y) - D(k)$ for some $Y \subseteq V_n$. We can consider sets Y not containing k. Then it is easy to see that $A_k(Y) = D(k) - (k,Y)$. Since Y is an arbitrary set not containing k, we can interchange Y by $Y' = V_n - \{k\} - Y$. Then $A_k(Y') = (k,Y)$, $A^k(Y') = A^k(Y)$. For this Y', the vector (6) takes the form (7). We find Y corresponding to minimal vectors, i.e. corresponding to k-minimal sets $A^k(Y)$.

Without loss of generality, we can set $k = n$. Then $V_n - \{n\} = V_{n-1}$. Let $Y \subseteq V_n$. We can consider Y not containing n. Then $|A^n(Y)| = |(V_n^2 - D(Y)) - D(n)| = |V_{n-1}^2 - D(Y)|$.

The set $V_{n-1}^2 - D(Y)$ has minimal cardinality for Y such that $|Y| = \lfloor \frac{n-1}{2} \rfloor$ or $|Y| = \lceil \frac{n-1}{2} \rceil$. Note that, for n even, we have to consider sets Y of both cardinalities $\lfloor \frac{n-1}{2} \rfloor$ and $\lceil \frac{n-1}{2} \rceil$, since they give different intersections $(V_n^2 - D(Y)) \cap D(n)$.

Now let $T \neq \emptyset$. We show that minimal vectors of $Q(k, \mathcal{D}_n^{k,T})$ have the form $R_T^k d_k(Y, \varepsilon)$. Since $R_T^k e_{ij} = e_{ij}$ for $(ij) \in A^k(Y)$, k-minimal sets of $\mathcal{D}_n^{k,T}$ are the same as of $\mathcal{D}_n$.

Recall that the switching class $\mathcal{D}_n^{k,T}$ consists of sets $B^{k,T} = (V_n^2 - (k,T))\Delta D(Y)$. Since $V_n^2 - X = V_n^2 \Delta X$ for any $X \subseteq V_n^2$, we have $B^{k,T} = (V_n^2 - D(Y))\Delta(k,T)$. Hence $B^{k,T} - D(k) = (V_n^2 - D(Y)) - D(k) = A^{k,T}(Y)$ and $B^{k,T} \cap D(k) = D(k) - (k, T\Delta Y)$. Since $d(k, T\Delta Y) = R_T^k d(k,Y)$, we see that minimal vectors of $Q(k, \mathcal{D}_n^{k,T})$ have the form $R_T^k d_k(Y,\varepsilon)$ with $d_k(Y,\varepsilon)$ from (7). $\square$

Recall that a convex hull of all minimal vectors of a class Q of $\frac{1}{2}L_n/L_n$ is a symmetric Delaunay polytope $P(Q)$ of dimension equal to or less than N. For $Q = Q(S, \mathcal{A})$, denote the polytope $P(Q)$ by $P(S, \mathcal{A})$. Now we describe some symmetric Delaunay polytopes of L_n of dimension N.

Proposition 7 *(i) The polytope $P(\emptyset, \mathcal{D}_n)$ has dimension N.*

(ii) The polytope $P(k, \mathcal{D}_n^{k,T}) = R_T^k P(k, \mathcal{D}_n)$ has dimension N if n is even, and dimension $N-1$ if n is odd.

Proof. (i) By Proposition 6(iii), all minimal vectors of the class $Q(\emptyset, \mathcal{D}_n)$ are $d(Y,\varepsilon)$ for $Y \subseteq V_n$, $|Y| = \lfloor \frac{n}{2} \rfloor$, and (± 1)-function ε on $V_n^2 - D(Y)$.

Fix a pair $(kl) \in V_n^2$. Consider the vectors $d(Y,\varepsilon)$ and $d(Y,\varepsilon')$, where $\varepsilon_{ij} = \varepsilon'_{ij}$ for $(ij) \neq (kl)$, and $\varepsilon_{kl} = -\varepsilon'_{kl} = 1$. Then $d(Y,\varepsilon) - d(Y,\varepsilon') = 2e_{kl}$. Since for every $(kl) \in V_n^2$ there exists a set Y such that $(kl) \in V_n^2 - D(Y))$, the vectors $d(Y,\varepsilon)$ for all sets Y of cardinality $\lfloor \frac{n}{2} \rfloor$ and functions ε generate space $\mathbf{R}^N$. This means that $\dim P(\emptyset, \mathcal{D}_n) = N$.

(ii) Since minimal vectors of $Q(k, \mathcal{D}_n^{k,T})$ are obtained by reflections R_T^k from minimal vectors of $Q(k, \mathcal{D}_n)$, we have $P(k, \mathcal{D}_n^{k,T}) = R_T^k P(k, \mathcal{D}_n)$. Therefore it is sufficient to prove assertion of (ii) for $P(k, \mathcal{D}_n)$.

By Proposition 6(iv), minimal vectors of the class $Q(k, \mathcal{D}_n)$ are $d_k(Y,\varepsilon)$. Without loss of generality, we set $k = n$ and $n \notin Y$. Then $A^n(Y) \subseteq V_{n-1}^2$, and as in proof of (i), we can show that, for any pair $(ml) \in V_{n-1}^2$, the vector $2e_{ml}$ is a linear combination of minimal vectors of the class $Q(n, \mathcal{D}_n)$.

Note that $|(n,Y)| = |Y|$. Recall that $|D(n)| = n-1$. Since $|Y| = \lfloor \frac{n-1}{2} \rfloor$ or $\lceil \frac{n-1}{2} \rceil$, we have $|(n,Y)| = \frac{1}{2}(n-1) = \frac{1}{2}|D(n)|$ if n is odd, and $|(n,Y)| = \lceil \frac{n-1}{2} \rceil$ or $\lfloor \frac{n-1}{2} \rfloor$ if n is even.

Recall that an (in)-component of $d(n,Y)$ is equal to -1 if $i \in Y$, and it is equal to $+1$ otherwise. Hence, for odd n, we have $\delta(n)d(n,Y) = 0$, and therefore $\delta(n)d_n(Y,\varepsilon) = 0$ for all minimal vectors of $Q(n, \mathcal{D}_n)$ if n is odd. This implies that the vectors $d_n(Y,\varepsilon)$ span a space of dimension at most $N-1$.

Consider a matrix M_n with $n-1$ columns, row of which are ± 1-vectors $d(n,Y)$. We show that the rank of this matrix is equal to $n-1$ for even n, and to $n-2$ for odd n. This implies that the vectors e_{in}, $1 \leq i \leq n-1$, can be represented as linear combinations of vectors $d(n,Y) = d_n(Y,\varepsilon) - \sum_{(ml)\in A^n(Y)} \varepsilon_{ml} e_{ml}$ for even n. Since the vectors e_{ml} can be represented as linear combinations of minimal vectors of the class $Q(n, \mathcal{D}_n)$, we obtain that, for even n, the polytope $P(n, \mathcal{D}_n)$ spans the space $\mathbf{R}^N$, i.e. it has dimension N. Similarly, we obtain that, for odd n, the polytope $P(n, \mathcal{D}_n)$ has dimension $N-1$.

Let $n = 2m$ be even. Consider a square $(n-1) \times (n-1)$ submatrix T_{2m-1} formed by $2m-1$ rows $d(n,Y)$ of M_n corresponding to the following $2m-1$ sets Y of cardinality

$|Y| = m$. Let $W_{m-1} = V_{2m-1} - V_m$. Recall that $V_k = \{1, 2, ..., k\}$. Set $Y_i = \{i\} \cup W_{m-1}$ for $i \in V_m$, and $Y_i = (V_m - \{1\}) \cup \{i\}$ for $i \in W_{m-1}$. The matrix T_{2m-1} with rows $d(n, Y)$, $i \in V_{2m-1}$, has the following form

$$\begin{pmatrix} -1 & j^T_{m-1} & -j^T_{m-1} \\ j_{m-1} & -2I_{m-1} + J_{m-1} & -J_{m-1} \\ j_{m-1} & -J_{m-1} & -2I_{m-1} + J_{m-1} \end{pmatrix},$$

where j_{m-1} is the all-one column of order $m-1$, J_{m-1} is the all-one matrix of order $m-1$, and I_{m-1} is the unit matrix of order $m-1$. It is easy to verify that the matrix T_{2m-1} has rank $n-1 = 2m-1$.

Let now $n = 2m+1$ be odd. We know that rank of M_n is $\leq n-2 = 2m-1$. We give a submatrix of M_n of rank $n-2 = 2m-1$. Recall that now $|Y| = m$ for all Y. If we add the column j_{2m-1} to the matrix T_{2m-1}, we obtain a desired submatrix of rank $2m-1 = n-2$ of M_n for odd n. $\square$

4 Contiguities of Delaunay polytopes of L_n

Let x be a center of a Delaunay polytope P of L_n whose set of vertices $V(P)$ contains origin $0 = \delta(\emptyset)$. Then, for all lattice points $d \in L_n$, we have

$$(d-x)^2 \geq r^2,$$

with equality for $d \in V(P)$. Here r is radius of the spere circumscribing P. Since $0 \in V(P)$, we have $r^2 = x^2$, and the above inequality takes the form

$$2dx \leq d^2, \text{ for all } d \in L_n. \tag{8}$$

We say that x is of *full rank* if a system of inequalities (8) satisfied by x as equalities uniquely determines x. Clearly, x of full rank is a center of a Delaunay polytope $P(x)$ of the lattice L_n.

Proposition 8 *Let $x \in \mathbf{R}^N$ satisfies (8) for $d = 2e_{ij}$ for all $(ij) \in V_n^2$, and for $d = \delta(S)$ for all $S \subseteq V_n$. Then*
(i) $|x_{ij}| \leq 1$ for all $(ij) \in V_n^2$,
(ii) $x_{ij} = 1$ if $2e_{ij}$ satisfies (8) as equality,
(iii) if $x \geq 0$, then x satisfies (8) for all $d \in L_n$.

Proof. (i) is implied by the inequality (8) for $d = \pm 2e_{ij} \in L_n$.

(ii) If $d = 2e_{ij}$ satisfies (8) as equality, then this equality takes the form $x_{ij} = 1$.

(iii) If $d \in L_n$, then, according to (2), $d = 2a + \delta(S)$ for $a \in \mathbf{Z}^N$ and $S \subseteq V_n$. Recall that (8) is equivalent to $(d-x)^2 \geq x^2$. Set $y = \delta(S) - x$. Since $x \geq 0$ and $x_{ij} \leq 1$, $|y_{ij}| \leq 1$ for all (ij). We have $(d-x)^2 = (2a+y)^2 = 4a(a+y) + y^2$. But $a(a+y) = \sum_{(ij):a_{ij}\neq 0} |a_{ij}||a_{ij} + y_{ij}| \geq 0$. Hence $(d-x)^2 \geq y^2 = (\delta(S) - x)^2 \geq x^2$. $\square$

Recall that by Proposition 3 the cut polytope $PCut_n$ is a Delaunay polytope of L_n. We consider Delaunay polytopes of L_n which are contiguous to $PCut_n$. A type of a new polytope depends on a facet by which it is contiguous to $PCut_n$.

Let $P_n(F)$ denote a Delaunay polytope which is contiguous to $PCut_n$ by a facet F. Let W be a support of the facet F, i.e. coefficients of an inequality defining F are nonzero only for (ij) such that $i, j \in W$. Let $\mathcal{S}(F)$ be a set of all $S \subseteq W$ such that $\delta(S)$ belongs to the facet F. We suppose that $\emptyset \in \mathcal{S}(F)$. Note that $W - S \in \mathcal{S}(F)$ for $S \in \mathcal{S}(F)$. Then a cut $\delta(X)$ belongs to the facet F if and only if $X \in \mathcal{T}(F)$, where $\mathcal{T}(F) = \{S \cup S' : S \in \mathcal{S}(F) \text{ and } S' \subseteq V_n - W\}$.

Let x be a center of $P_n(F)$. Then the inequality (8) holds as equality for $\delta(X)$ for all $X \in \mathcal{T}(F)$, i.e.

$$\delta(X)x = \frac{1}{2}\delta^2(X), \;\; X \in \mathcal{T}(F). \tag{9}$$

Since F is a facet, rank of the matrix $\|\delta_{ij}(X)\|$, $(ij) \in V_n^2$, $X \in \mathcal{T}(F)$, of order $|\mathcal{T}(F)| \times N$ is equal to $N - 1$. Hence the equations (9) determine all coordinates x_{ij} of the vector x up to a parameter λ. A value of λ is uniquely determined by a vertex of $P_n(F)$ not belonging to $PCut_n$.

Proposition 9 *The system of equations (9) is equivalent to the following system*

$$\begin{array}{ll} x_{ij} = \frac{1}{2} & \textit{for } i, j \in V_n - W, \\ \sum_{i \in S} x_{ip} = \frac{1}{2}s & \textit{for } p \in V_n - W, S \in \mathcal{S}(F), \\ \sum_{(ij) \in (S, W-S)} x_{ij} = \frac{1}{2}s(w - s) & \textit{for } S \in \mathcal{S}(F), \end{array}$$

where $s = |S|$, $w = |W|$.

Proof. Note that $\delta(X) \in \mathcal{T}(F)$ for all $X \subseteq V_n - W$, since $\emptyset \in \mathcal{S}(F)$. Recall that $\delta^2(X) = |X|(n - |X|)$. Hence, summing the equations (9) for $X = \{i\}, \{j\}$, subtracting the equation (9) for $\{ij\}$, $i, j \in V_n - W$, and using the equality $2e_{ij} = \delta(i) + \delta(j) - \delta(ij)$, we obtain

$$2x_{ij} = (\delta(i) + \delta(j) - \delta(ij))x = \frac{1}{2}(\delta^2(i) + \delta^2(j) - \delta^2(ij)) = 1,$$

i.e. the first system of equations.

Now consider the equation (9) for $X = S$ and $X = S \cup \{p\}$, where $S \in \mathcal{S}(F)$ and $p \in V_n - W$. We have $\delta^2(S) = s(n - s)$ and $\delta^2(S \cup \{p\}) = (s + 1)(n - s - 1)$. Therefore

$$\delta(S)x = \sum_{(ij) \in D(S)} x_{ij} = \frac{1}{2}s(n - s), \tag{10}$$

$$\delta(S \cup \{p\})x = \sum_{(ij) \in D(S)} x_{ij} - \sum_{i \in S} x_{ip} + \sum_{j \in V_n - S - \{p\}} x_{jp} = \frac{1}{2}(s + 1)(n - s - 1).$$

Substituting the first equality in the second, and using the equality $x_{jp} = \frac{1}{2}$ for $j \in V_n - (W \cup \{p\})$, we obtain the second system of equations of this proposition.

We can rewrite the equation (10) as follows

$$\sum_{(ij) \in (S, W-S)} x_{ij} + \sum_{i \in S, p \in V_n - W} x_{ip} = \frac{1}{2}[s(w - s) + s(n - w)].$$

Using the second system of equations of the proposition, we obtain the third system. $\square$

Example. Now, as an example, we consider a facet $F = F_q$, $q \in \mathbf{Z}_+$, $1 \le q \le \frac{n}{2}$, defined by the equation $\sum_{1 \le i < j \le w} b_i b_j d_{ij} = 0$. (This facet is called *pure* $(2q+1)$*-gonal* facet of $PCut_n$.) Here $W = W^+ \cup W^-$, $|W| = w = 2q+1$, $|W^+| = q+1$, $|W^-| = q$, and $b_i = 1$ for $i \in W^+$, $b_i = -1$ for $i \in W^-$. Similarly to W, all sets of $\mathcal{S}(F_q)$ are partitioned into two parts, i.e.

$$\mathcal{S}(F_q) = \{S^+ \cup S^-, W - (S^+ \cup S^-) : S^+ \subseteq W^+, S^- \subseteq W^-, |S^+| = |S^-| = s, 0 \le s \le q\}.$$

We describe new Delaunay polytopes contiguous to $PCut_n$ by this facet F_q.

The second system of equations of Proposition 9 implies $x_{ip} = \frac{1}{2}$ for all $i \in W$, $p \in V_n - W$. Using symmetry of the third system under permutations of W^+ and W^-, we can suppose that

$$x_{ij} = u \text{ for all } i,j \in W^+, x_{ij} = y \text{ for all } i,j \in W^-, x_{ij} = z \text{ for all } i \in W^+, j \in W^-.$$

Note that if $q = 1$, then $|(W^+)^2| = 1$ and $|(W^-)^2| = 0$. (See definition of S^2 given in (1).) Hence there is no y if $q = 1$. (We can suppose that $y = 0$ for $q = 1$.)

Then the third system takes the form

$$s(q-s+1)u + s(q-s)y + s(2(q-s)+1)z = s(2(q-s)+1), \ 0 \le s \le q.$$

A solution of this system is $u = y = \lambda$ and $z = 1 - \lambda$, where λ is a parameter.

Suppose that $2e_{ij} \in V(P_n(F_q))$ for some $(ij) \in V_n^2$. Then according to Proposition 8(ii), $x_{ij} = 1$. Hence $i,j \in W$. The case $i \in W^+$, $j \in W^-$ is impossible. In fact, then $z = 1$, $w = y = 0$ and, for $j \in W^-$, we have $2\delta(j)x = 2(q+1) + (n-w) > \delta^2(j) = 2q + (n-w)$. This contradicts to (8).

Let $i,j \in W^+$. Then $u = y = 1$, $z = 0$. In this case, $x = \sum_{(ij) \in W^{+2}} e_{ij} + \sum_{(ij) \in W^{-2}} e_{ij} + \frac{1}{2}\sum_{(ij) \in V_n^2 - W^2} e_{ij}$. Note that $V_n^2 - W^2 = D(W) \cup (V_n - W)^2$. Hence

$$x = \frac{1}{2}\delta(W) + \frac{1}{2}\sum_{(ij) \in (V_n - W)^2} e_{ij} + \sum_{(ij) \in W^{+2}} e_{ij} + \sum_{(ij) \in W^{-2}} e_{ij}. \tag{11}$$

Since $0 \le x_{ij} \le 1$ for all (ij), Proposition 8(ii) implies that the inequality $(d-x)^2 \ge x^2$ holds for all $d \in L_n$, i.e. x is a center of a Delaunay polytope that we have denoted as $P_n(F_q)$.

Proposition 10 *The Delaunay polytope $P_n(F_q)$, $n > 2$, is symmetric if and only if $n = 2q+1$ or $n = 2q+2$. For $n = 2q+1$, $P_n(F_q) = P(\emptyset, \mathcal{D}_n)$. For $n = 2q+2$ and $W = V_n - \{i\}$, $P_n(F_q) = P(i, \mathcal{D}_n)$.*

Proof. Note that $(V_n - W)^2 \ne \delta(S)$ for any S if $(V_n - W)^2 \ne \emptyset$, and $(V_n - W)^2 = \emptyset$ if $|V_n - W| \le 1$. Comparing (2) and (11), we see that $x \in \frac{1}{2}L_n/L_n$ if and only if $(V_n - W)^2 = \emptyset$. This implies that a Delaunay polytope $P_n(F_q)$ is symmetric if and only if $n = 2q+1$ or $n = 2q+2$.

Taking in attention Proposition 6, we see that x is a minimal vector of the class $Q(\emptyset, \mathcal{D}_n)$ for odd n, and of the class $Q(i, \mathcal{D}_n)$ for even n. This implies assertion of this proposition. □

5 The lattices L_n for small n.

In this section we consider classes of the quotient $\frac{1}{2}L_n/L_n$ for small n in details.

n=2, $N=1$, $L_2=\mathbf{Z}$. There are only two cuts, $\delta(\emptyset)=0$ and $\delta(1)=\delta(2)$, and one switching class, the cut class $\mathcal{C}_2$. Hence there are only two classes of $\frac{1}{2}L_2/L_2$, namely, zero class $\mathcal{C}_2$ and the class $\frac{1}{2}\delta(1)+\mathcal{C}_2$. Minimal vectors of these classes are 0 and $\pm\frac{1}{2}\delta(1)$, respectively. The cut polytope, a unit segment, is, in this case, symmetric and coincides with $P(1,\mathcal{C}_2)$.

n=3, $N=3$, $L_3=A_3=D_3$ is a unique 3-dimensional root lattice (face-centered cubic lattice). There are 4 cuts, $\delta(\emptyset)$, $\delta(i)$, $1\le i\le 3$, and two switching classes, the cut class $\mathcal{C}_3$ and the co-cut class $\mathcal{D}_3$. The co-cut class contains 3 one-element sets $\{(12)\}$, $\{(13)\}$, $\{(23)\}$. There are the following $8=2^3$ classes of $\frac{1}{2}L_3/L_3$:

0-class=$\mathcal{C}_3$,

3 classes $\frac{1}{2}\delta(i)+\mathcal{C}_3$, $1\le i\le 3$. Minimal vectors of these classes are $\pm\frac{1}{2}(e_{ij}+e_{ik})$, where $\{ijk\}=\{123\}=V_3$.

The co-cut class $\mathcal{D}_3$ with minimal vectors $\pm e_{12}$, $\pm e_{13}$, $\pm e_{23}$.

3 classes $\frac{1}{2}\delta(i)+\mathcal{D}_3$ with minimal vectors $\pm\frac{1}{2}(e_{ij}-e_{ik})$, $\{ijk\}=V_3$, $1\le i\le 3$.

Proposition 7(i) gives a symmetric Delaunay polytope $P(\emptyset,\mathcal{D}_3)$ which is 3-dimensional cross polytope (octahedron) β_3. The cut polytope $PCut_3$ is a regular 3-dimensional simplex α_3. These are known Delaunay polytopes of the root lattice $A_3=D_3$. The norm (squared length) of edges of polytopes $P(\emptyset,\mathcal{D}_3)$ and $PCut_3$ is equal to 2.

n=4, $N=6$, $L_4=\sqrt{2}D_6^+=\Lambda_6\{3\}$. The lattice D_6^+ is a union of the root lattice D_6 with a translated copy of it (see [2], ch.4, p.119). The lattice $\Lambda_6\{3\}$ is an integral laminated lattice of minimal norm 3 (see [2], ch.6, p.179).

There are $2^3=8$ cuts: $\delta(\emptyset)$, $\delta(i)$, $1\le i\le 4$, $\delta(i4)$, $1\le i\le 3$. Since $\lfloor\frac{n-1}{2}\rfloor=1$, $|T|\le 1$ in $\mathcal{D}_4^{i,T}$ and $\mathcal{E}_4^{i,T}$. By Proposition 4(ii), there are 8 switching classes: the cut class $\mathcal{C}_4$, the co-cut class $\mathcal{D}_4$, and 6 classes $\mathcal{E}_4^{ij}$, $1\le i<j\le 4$. Combining cuts and switching classes, we obtain $2^6=64$ classes of the quotient $\frac{1}{2}L_4/L_4$. Proposition 7 gives two types of symmetric Delaunay polytopes, $P(\emptyset,\mathcal{D}_4)$ and $P(i,\mathcal{D}_4)$.

A description of classes of $\frac{1}{2}L_4/L_4$ is given in Table 1 below. The set of all classes is partitioned into orbits of the group of all permutations of V_4. The indices i,j,k,l in the table are all distinct and such that $\{i,j,k,l\}=V_4$.

Note that minimal vectors of classes $Q(i,\mathcal{D}_4)$ and $Q(i,\mathcal{D}_4^{il})$ multiplied by 2 have norm 7 and mutual inner products ±1. Hence they span 6 equiangular lines at angle $\arccos\frac{1}{7}$.

Let e_0 be a unit vector orthogonal to the space spanned by L_4. If we add two pairs of vectors

$$\pm\frac{1}{2}(e_{ij}+e_{ik}+e_{il})\pm e_0$$

to the set of minimal vectors of $Q(i,\mathcal{D}_4)$, we obtain 8 pairs of vectors spanning in 7-dimensional space 8 equiangular lines at angle $\arccos\frac{1}{7}$. The convex hull of all these vectors is, up to the multiple $\sqrt{2}$, a unique (basic) Delaunay polytope of the lattice E_7^*. Hence the lattice $\frac{1}{\sqrt{2}}L_4=D_6^+$ is a section of E_7^* by a hyperplane orthogonal to e_0 .

Besides the cut polytope $PCut_4$, the cross-polytope $\beta_6=P(\emptyset,\mathcal{D}_4)$ and the "twisted" cross-polytope $P(k,\mathcal{D}_4)$, there is a fourth polytope of the lattice L_4, a 6-dimensional simplex S_6. Norms of its edges are equal to 3 and 4. There are 6 edges of norm 3. All these edges are adjacent to the same vertex.

Note that, for all Delaunay polytopes of L_4, norms of its edges (i.e. one-dimensional faces) take only two values, 3 and 4. Let P be a Delaunay polytope of the lattice L_4. Since edges of P have only two norms, 1-skeleton of P can be described by a graph $G(P)$. The set of vertices of $G(P)$ is the set of vertices of P, and two vertices are adjacent in $G(P)$ if and only if they are vertices of an edge of P of norm 3.

Facets (i.e. 5-dimensional faces) of all four Delaunay polytopes are simplexes, i.e. all Delaunay polytopes of L_4 are simplicial. Similarly to above each facet can be described by a graph. All graphs corresponding to facets are bipartite graphs $K_{i,j}$ with parts of cardinalities i and j such that $i+j=6$. (Recall that a 5-dimensional simplex has 6 vertices.) Of course, $K_{0,6}$ is the complement of the complete graph K_6.

Table 1

class $Q(S,\mathcal{A}) =$ $\frac{1}{2}\delta(S)+\mathcal{A}$	number of classes in orbit	minimal vectors of $Q(S,\mathcal{A})$	norm of minimal vectors	$\dim P(S,\mathcal{A})$
$\mathcal{C}_4$, cuts	1	0	0	0
$\frac{1}{2}\delta(i)+\mathcal{C}_4$	4	$\pm\frac{1}{2}\delta(i)$	$\frac{3}{4}$	1
$\frac{1}{2}\delta(ij)+\mathcal{C}_4$	3	$\pm\frac{1}{2}\delta(ij)$	1	1
$\mathcal{D}_4$, anticuts	1	$\pm(e_{12}\pm e_{34})$ $\pm(e_{13}\pm e_{24})$ $\pm(e_{14}\pm e_{23})$	2	6
$\frac{1}{2}\delta(i)+\mathcal{D}_4$	4	$\pm\frac{1}{2}(-e_{ij}+e_{ik}+e_{il})\pm e_{kl}$ $\pm\frac{1}{2}(e_{ij}-e_{ik}+e_{il})\pm e_{jl}$ $\pm\frac{1}{2}(e_{ij}+e_{ik}-e_{il})\pm e_{jk}$	$\frac{7}{4}$	6
$\frac{1}{2}\delta(ij)+\mathcal{D}_4$	3	$\pm\frac{1}{2}d(ij;ik,jl)$ $\pm\frac{1}{2}d(ij;il,jk)$	1	2
$\mathcal{E}_4^{ij}=\mathcal{D}_4^{kl}$	6	$\pm e_{ij}$	1	1
$\frac{1}{2}\delta(i)+\mathcal{E}_4^{ij}$	12	$\pm\frac{1}{2}d(i;ij)$	$\frac{3}{4}$	1
$\frac{1}{2}\delta(i)+\mathcal{E}_4^{jk}=$ $\frac{1}{2}\delta(i)+\mathcal{D}_4^{il}$	12	$\pm\frac{1}{2}(e_{ij}-e_{ik}+e_{il})\pm e_{kl}$ $\pm\frac{1}{2}(-e_{ij}+e_{ik}+e_{il})\pm e_{jl}$ $\pm\frac{1}{2}(e_{ij}+e_{ik}+e_{il})\pm e_{jk}$	$\frac{7}{4}$	6
$\frac{1}{2}\delta(ij)+\mathcal{E}_4^{ik}=$ $\frac{1}{2}\delta(kl)+\mathcal{E}_4^{ik}$	6	$\pm\frac{1}{2}d(ij;ik)=$ $\pm\frac{1}{2}d(kl;ik)$	1	1
$\frac{1}{2}\delta(ij)+\mathcal{E}_4^{ij}=$ $\frac{1}{2}\delta(kl)+\mathcal{E}_4^{ij}$	6	$\pm\frac{1}{2}d(ij;jk,il)$ $\pm\frac{1}{2}d(ij;il,jk)$	1	2

We call Delaunay polytopes P and P' *lattice equivalent* if either $P'=-P$ or $P'=P+a$ for some lattice vector a. Note that if P is symmetric and $0\in V(P)$, then $-P=P-2x$, where x is a center of P. Hence two symmetric Delaunay polytopes P and P' are lattice equivalent if and only if its centers $x(P)$ and $x(P')$ belong to the same class of the quotient of $\frac{1}{2}L_n/L_n$.

For a Delaunay polytope, we denote

a center of P by $x(P)$, and set $r^2(P)=x^2(P)$;

a number of vertices of P by $v(P)$;

a ratio of volume of P to volume V_0 of a basic simplex ($V_0=\frac{8}{6!}$) by $V(P)$;

a number of lattice nonequivalent Delaunay polytopes congruent to P in the star at 0 by $N(P)$;

a graph of the 1-skeleton of P by $G(P)$.

Let $s(P)=1$ if P is symmetric, and $s(P)=2$ if P is asymmetric. Then the number of Delaunay polytopes congruent to P in a star is equal to $s(P)v(P)N(P)$. Besides we have $\sum_P s(P)V(P)N(P)=N!$, where $N=\frac{n(n-1)}{2}=\dim L_n$.

The star at $0\in L_4$ (i.e. a set of all Delaunay polytopes having 0 as a vertex) and corresponding Voronoi polytope of L_4 are described by Table 2 below. All indices in Table 2 are distinct. In this table and Table 3 below, we describe, in fact, a special Delaunay tesselation of $\mathbf{R}^6$ into simplicial Delaunay polytopes of the lattice L_4.

Table 2

P	$s(P)$	$N(P)$	$v(P)$	$V(P)$	$x^2(P)$	$r^2(P)$	$G(P)$
$PCut_4$	2	10	8	4	$\frac{1}{2}\sum_{(ij)\in V^2}\pm e_{ij}$, $\pm e_{ij}+\frac{1}{2}(\pm e_{ik}\pm e_{jk})$	$\frac{3}{2}$	$K_{4,4}$
S_6	2	16	7	2	$\pm\frac{3}{4}e_{ij}\pm e_{kl}+\frac{1}{4}(\pm e_{ik}\pm e_{jk})$, $\frac{3}{4}(\pm e_{ij}\pm e_{jk}\pm e_{ik})$	$\frac{27}{16}$	$K_{1,6}$
$P(k,\mathcal{D}_4)$	1	16	12	32	$\pm e_{ij}+\frac{1}{2}(\pm e_{ik}\pm e_{jk}\pm e_{kl})$	$\frac{7}{4}$	$K_{6,6}$
$P(\emptyset,\mathcal{D}_4)$	1	1	12	64	$\pm e_{ij}\pm e_{kl}$	2	$K_{2\times 6}$

In Table 2, the graph $K_{2\times 6}$ is a cocktail-party graph.

The centers of all Delaunay polytopes congruent to P can be obtained from $x(P)$ of Table 2 by taking $i,j,k,l\in\{1,2,3,4\}$ and taking signs $+$ and $-$ in $\pm$ independently. For example, $x(P)=\pm e_{ij}\pm e_{kl}$ has 4 distinct patterns of signs and 3 distinct partitions of $V_4=\{1,2,3,4\}$ into equal parts. Hence there are 3×4 cross-polytopes $\beta_6=P(\emptyset,\mathcal{D}_4)$ in a star.

Note that the 16 lattice nonequivalent Delaunay polytopes of type $P(k,\mathcal{D}_4)$ are represented by 4 polytopes $P(k,\mathcal{D}_4)$, $1\le k\le 4$, and 12 polytopes $P(k,\mathcal{D}_4^{kl})=R_l^kP(k,\mathcal{D}_4)$, $l\in V_4-\{k\}$, $1\le k\le 4$.

In Table 3 below we describe adjacencies of the Delaunay polytopes of L_4. We denote a polytope P by its graph $G(P)$. A graph describing a facet by which a polytope P is contiguous to a polytope P' stays in an intersection of the row $G(P)$ with the column $G(P')$. A number in parantheses shows how many polytope congruent to P' is adjacent to P. Note that Table 3 is asymmetric with respect to numbers in parantheses.

Table 3

$G(P)$	$K_{4,4}$	$K_{1,6}$	$K_{6,6}$	$K_{2\times 6}$
$K_{4,4}$	$-$	$-$	$(20)K_{3,3}$	$-$
$K_{1,6}$	$-$	$-$	$(12)K_{1,5}$	$(32)K_{0,6}$
$K_{6,6}$	$(16)K_{3,3}$	$(6)K_{1,5}$	$(30)K_{2,4}$	$(32)K_{0,6}$
$K_{2\times 6}$	$-$	$(1)K_{0,6}$	$(2)K_{0,6}$	$-$

Note that there are $\sum_P s(P)v(P)N(P)=588$ Delaunay polytopes in a star, and therefore the Voronoi polytope P_V has 588 vertices.

The Voronoi polytope P_V has $60+32=92$ facets. The 60 facets of P_V with its center in $0\in L_4$ are orthogonal to all 60 vectors of norm 4 of L_4 and 32 facets are orthogonal to all 32 vectors of norm 3. Each facet contains the middle point of the corresponding vector. A facet orthogonal to a vector of norm 4 contains 42 vertices. A facet orthogonal to a vector of norm 3 contains 56 vertices.

References

[1] E.P.Baranovskii, *Subdivision of Euclidean spaces into L-polytopes of certain perfect lattices*, Trudy Mat.Inst. Steklov **196** (1991) 27–46 [translated in: Steklov Inst.Math. **196** (1992)]

[2] J.H.Conway, N.J.A.Sloane, *Sphere Packings, Lattices and Groups*, vol.290 of *Grundlagen der mathematischen Wissenschaften*, Springer-Verlag, Berlin, 1987.

[3] M.Deza, V.P.Grishukhin, *Cut polytope and its lattices*, Rapport de Recherche LIENS-94-8, 1994.

[4] M.Deza, V.P.Grishukhin, *Bounds on the covering radius of a lattice*, Rapport de Recherche LIENS-94-17, 1994.