Pseudo-LYM inequalities and AZ identities

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Abstract

We give pseudo-LYM inequalities in some posets and give a new restriction in this way for their antichains. Typically these posets fail the LYM inequality and some of them are known not to be Sperner. In the latter class our inequality yields an upper bound for the maximum size of an antichain.

1 Introduction

Let us be given a ranked partially ordered set $\mathcal{P}$, in which the set of elements of rank r are denoted by $\mathcal{P}_r$. (Following tradition and natural notation, the smallest rank in a poset is either 0 or 1.) The *profile* of an antichain $\mathcal{A}$ in $\mathcal{P}$ is the sequence of cardinalities $f_r = |\mathcal{A} \cap \mathcal{P}_r|$. The poset $\mathcal{P}$ satisfies the *LYM inequality* ([6], [12], [13] and [17]) if for every antichain $\mathcal{A}$

$$\sum_r \frac{f_r}{|\mathcal{P}_r|} \leq 1.$$

A vast amount of literature investigates which posets have the LYM property. The LYM property has become a central issue due to two facts: (a) the LYM property seems to be the most straightforward mean to prove that a poset has the *Sperner property*, i.e. the largest antichain has size $\max_r |\mathcal{P}_r|$; (b) the LYM property is equivalent to some other, perhaps more illuminating properties of the poset (see e.g. [8]). Recently Ahlswede and Zhang [3] extended the "classical" LYM inequality on the inclusion partial order of the power set into an elegant identity, and LYM inequalities of several other posets has been extended into a corresponding AZ type identity ([1], [2], [4], [16]).

One important poset that fails the LYM inequality is the *partition lattice* of a set ([15]). Contrary to a longstanding conjecture of Rota ([14]), it does not even satisfy the Sperner property ([7]).

The purpose of the present paper is to prove *pseudo-LYM inequalities* for five posets that fail (with one exception) the LYM inequality. The merit of the pseudo-LYM inequality is that it still describes a severe restriction on the profile of an antichain. We also give the AZ identity form of the inequalities and characterize how equality may hold.

Three posets studied in this paper—the partition lattice, the poset of chains of a Boolean algebra, and the poset of chains of subspaces of a finite vector space—all share the property that a poset element can be defined as a set of subsets of a universe. The divisor lattice of a given number and the generalized Boolean algebra exhibit anology to the above phenomenon. This property expectedly implies pseudo-LYM inequalities in other posets as well.

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2 The partition lattice

Let S be an n-element set and let $\mathcal{P}$ be its partition lattice. We say a partition α is less than partition γ iff γ is a refinement of α. For a partition family $\mathcal{A}$ in $\mathcal{C}$ and for $k \geq 1$, $0 < i_1 \leq i_2 \leq ... \leq i_k$, let $f_{k;i_1,i_2,...,i_k}(\mathcal{A})$ denote the number of elements of $\mathcal{A}$ of k classes, with class sizes $i_1, i_2, ..., i_k$.

Theorem 2.1 *Let $\mathcal{A} \neq \emptyset$ denote an antichain in the partition lattice $\mathcal{P}$ of an $n \geq 2$ element set, i.e. no element of $\mathcal{A}$ is refinement of another. We have*

$$\sum_{\substack{k\geq 1;\\ i_1,i_2,...,i_k}} \frac{f_{k;i_1,i_2,...,i_k}(\mathcal{A})k!}{\binom{n}{i_1, i_2, ..., i_k}\binom{n-1}{k-1}} \leq 1. \tag{1}$$

Theorem 2.2 *Let $\mathcal{A} \neq \emptyset$ denote a set of partitions of an $n \geq 2$ element set. Assume that $\mathcal{A}$ does not contain the coarsest partition consisting of a single* *class. We have*

$$\sum_{\gamma\in\mathcal{P}} \frac{N(\gamma)k!}{\binom{n}{i_1, i_2, ..., i_k}\binom{n-1}{k-1}\binom{k}{2}} = 1, \tag{2}$$

where the $\gamma \in \mathcal{P}$ summation variable has $k \geq 2$ classes of sizes $0 < i_1, i_2, ..., i_k$, and $N(\gamma)$ counts unordered pairs of classes $A, B \in \gamma$ such that for all $\alpha \in \mathcal{A}$, of which γ is a refinement, A and B are subsets of different classes of α.

Proof of Theorem 2.2 Consider $\mathcal{A}^f$, which consists of all possible refinements of partitions taken from $\mathcal{A}$. (With other words, let $\mathcal{A}^f$ be the filter generated by $\mathcal{A}$.) Let us call a sequence of partitions a *cat* (from catena), if it starts with the coarsest partition, the following partitions are derived from the previous one by splitting a class into two non-empty further classes, and it ends with the finest partition into singletons. Clearly, the number of cats is $\binom{n}{2}\binom{n-1}{2}\cdots\binom{2}{2}$, since one may count cats starting from the finest partition and joining two classes at any step. Any cat $(\gamma_1, \gamma_2, ..., \gamma_n)$ has a smallest $i > 1$ with $\gamma_i \in \mathcal{A}^f$. We claim

$$\sum_{\gamma\in\mathcal{P}} N(\gamma)\binom{k-1}{2}\binom{k-2}{2}\cdots\binom{2}{2} \times \frac{(n-k)!}{(i_1-1)!(i_2-1)!\cdots(i_k-1)!} \times$$
$$\prod_{j=1}^{k}\left\{\binom{i_j}{2}\binom{i_j-1}{2}\cdots\binom{2}{2}\right\} = \binom{n}{2}\binom{n-1}{2}\cdots\binom{2}{2}, \tag{3}$$

where γ runs through the partitions into at least 2 classes. Note that $\binom{1}{2}\cdots\binom{2}{2} = 1$, since the value of the empty product is 1. Formula (3) implies formula (2) through simple algebra. We show that both sides of (3) count all cats. We have already concluded it about the RHS. On the LHS, for every cat we specify the partition γ, with which the cat enters $\mathcal{A}^f$. For a given γ, we count the number of cats entering $\mathcal{A}^f$ in γ. There are

$$\frac{(n-k)!}{(i_1-1)!(i_2-1)!\cdots(i_k-1)!}\prod_{j=1}^{k}\left\{\binom{i_j}{2}\binom{i_j-1}{2}\cdots\binom{2}{2}\right\}$$

ways to build the part of the cat between γ and the finest partition. (Any i_j element classes can be built in $\binom{i_j}{2}\binom{i_j-1}{2}\cdots\binom{2}{2}$ ways from singletons independently, and one even has freedom which class to make a step in.) Connecting γ and the coarsest partition, we have to join two classes of γ such that the coarser partition obtained is not in $\mathcal{A}^f$, and then finish the cat in $\binom{k-1}{2}\binom{k-2}{2}\cdots\binom{2}{2}$ ways. Observe that the number of choices for the coarser partition is exactly $N(\gamma)$. $\Box$

Observe that Theorem 2.2 implies Theorem 2.1. If an antichain $\mathcal{A}$ contains the coarsest partition consisting of one class, then $\mathcal{A} = \mathcal{P}_1$, and (1) is straightforward to check. If $\mathcal{A}$ is an antichain, $\gamma \in \mathcal{A}$ and γ has $k \geq 2$ classes, then $N(\gamma) = \binom{k}{2}$. Consider the terms from $\gamma \in \mathcal{A}$ only, since the others have a non-negative contribution.

Theorem 2.3 *Equality holds in (1) iff $\mathcal{A} = \mathcal{P}_k$ for a certain $k \geq 1$.*

Proof. It is an easy exercise to show that the choice $\mathcal{A} = \mathcal{P}_k$ yields equality in (1) and we leave this exercise to the reader. Assume that equality holds. If the coarsest partition is in $\mathcal{A}$, then $\mathcal{A} = \mathcal{P}_1$. Hence we may assume that the coarsest partition is not in $\mathcal{A}$. The antichain $\mathcal{A}$ is maximal for inclusion, otherwise we could further increase the LHS of (1). Assume k to be the largest number of classes for any $\gamma \in \mathcal{A}$. We show that $\mathcal{P}_k \subset \mathcal{A}$, and since $\mathcal{A}$ is an antichain, we are at home. Assume for contrary, that $\delta = (A_1, A_2, ..., A_k) \in \mathcal{A}$ and for a certain element x, $\delta' = (A_1 - \{x\} \neq \emptyset, A_2 \cup \{x\}, A_3, ..., A_k) \notin \mathcal{A}$. (We can assume it becase it is easy to see that any element of $\mathcal{P}_k$ can be obtained from any other element of $\mathcal{P}_k$ iterating the following step: move an element from a non-singleton class to another class.) We show that δ' has a positive contribution to the LHS of (1). There exists an $\epsilon \in \mathcal{A}$ which is coarser than δ', due to the maximality of $\mathcal{A}$. ϵ is not coarser than δ, since $\mathcal{A}$ is an antichain. Hence, in ϵ the sets $A_1 - \{x\}$ and $A_2 \cup \{x\}$ are subsets of different classes. The last conclusion holds for all $\epsilon \in \mathcal{A}$ which is coarser than δ', therefore in $N(\delta')$ the pair $(A_1 - \{x\}, A_2 \cup \{x\})$ has been counted. $\square$

Theorem 2.1 provides a bound for the largest antichain in the partition lattice. Namely,

Corollary 2.4 *Let $\mathcal{A}$ be an antichain in the partition lattice $\mathcal{P}$. Then*

$$|\mathcal{A}| \leq \max_{k \geq 1} \max_{\{i_1, i_2, ..., i_k\}} \frac{1}{k!} \binom{n}{i_1, ..., i_k} \binom{n-1}{k-1}.$$

$\square$

The second maximization problem is easily solved since the multinomial coefficient is maximized if the k numbers in lower position differ from each other by at most one.

The partition lattice is not self-dual, i.e. turning it upside down, the new lattice $\mathcal{P}^*$ is not isomorphic to the original. Most texts call $\mathcal{P}^*$ as partition lattice. The pseudo-LYM inequality can be extended in $\mathcal{P}^*$ into an AZ identity, which is different from the AZ identity in $\mathcal{P}$.

Let γ denote a partition of the universe into $k < n$ non-empty classes of size $i_1, i_2, ..., i_k$. Let $\mathcal{A} \neq \emptyset$ denote a set of partitions. Also assume that $\mathcal{A}$ is not consisting of a single partition of n classes.

Although the classes of a partition are unlabelled by definition, it is easy to introduce a natural order on them: for example we can consider the lexicographic order of the subsets of S. Therefore we may speak about the j^{th} class of a partition γ.

Let $N(\gamma, j, a)$ $(1 \leq j \leq k,\ 1 \leq a \leq i_j - 1)$ denote the number of partitions γ' which can be obtained from γ by splitting the j^{th} partition class into classes of size a and $i_j - a$, such that γ' is not in $\mathcal{A}^f$. Then we have

Theorem 2.5

$$\sum_{\gamma} \sum_{j=1}^{k} \sum_{a=1}^{i_j-1} \frac{2 \cdot k! \cdot N(\gamma, j, a)}{\binom{n}{i_1, i_2, ..., i_k} \binom{n-1}{k-1} \binom{i_j}{a} (n-k)} = 1, \tag{4}$$

where γ denotes again a partition of the universe into $k < n$ non-empty classes of size $i_1, i_2, ..., i_k$. It is not difficult to derive the pseudo-LYM inequality (1) from this AZ identity.

The proof consists of counting all cats like in (3)

$$\sum_{\gamma} \sum_{j=1}^{k} \sum_{a=1}^{i_j-1} N(\gamma, j, a) \binom{k}{2} \cdots \binom{2}{2} \Big\{ \prod_{l:l \neq j} \binom{i_l}{2} \cdots \binom{2}{2} \Big\} \times$$
$$\binom{a}{2} \cdots \binom{2}{2} \binom{i_j - a}{2} \cdots \binom{2}{2} \frac{(n-k-1)!}{(a-1)!(i_j-a-1)! \prod_{l:l \neq j} (i_l - 1)!} \tag{5}$$

and simple algebra.

3 Multisets

Let us be given $n_1, n_2, ..., n_q$ balls of colour $1, 2, ..., q$. Balls of the same colour are not distinguishable. The multiset $\mathbf{k} = (k_1, ..., k_q)$ is a selection of $k_1, k_2, ..., k_q$ balls, $0 \leq k_i \leq n_i$, of colour i. There are different generalizations of the set inclusion to multiset inclusion. One generalization is that a multiset $\mathbf{k} = (k_1, k_2, ..., k_q)$ is a subset of the multiset $\mathbf{t} = (t_1, t_2, ..., t_q)$, iff $k_i \leq t_i$ for all i. There is a largest multiset $\mathbf{n} = (n_1, n_2, ..., n_q)$. Take q different primes, $p_1, p_2, ..., p_q$, and establish a one-to-one correspondence between the subsets of the multiset $\mathbf{n}$ and the divisors of $p_1^{n_1} p_2^{n_2} \ldots p_q^{n_q}$. Since $\mathbf{k} \leq \mathbf{t}$ if and only if $p_1^{k_1} p_2^{k_2} \cdots p_q^{k_q}$ divides $p_1^{t_1} p_2^{t_2} \cdots p_q^{t_q}$, the emerging poset is isomorphic to the divisor lattice of the number $p_1^{n_1} p_2^{n_2} \ldots p_q^{n_q}$. We recall this model as the *divisor lattice* of $\mathcal{P}$.

Another generalization is that $\mathbf{k} = (k_1, k_2, ..., k_q) \leq \mathbf{t} = (t_1, t_2, ..., t_q)$ iff $\{k_i = 0 \text{ or } t_i = k_i\}$ hold for all i. This model is known as *generalized Boolean algebra* (e.g. [9] or *space of integer sequences* (e.g. [11])).

3.1 Divisor lattices

Let $\mathcal{P}$ denote the divisor lattice of the integer $p_1^{n_1}p_2^{n_2}\dots p_q^{n_q}$. The *rank function* $\mathrm{r}(\mathbf{k})$ of this poset is defined as $k_1+k_2+\dots+k_q$. So $\mathrm{r}(\mathbf{n})=n_1+\dots+n_q$. This poset is known to be LYM (e.g. Theorem 4.2.3 in [5]). Here we prove another inequality for antichains.

Theorem 3.1 *Let $\mathcal{A}\neq\emptyset$ denote an antichain in the poset $\mathcal{P}$. Then we have*

$$\sum_{\mathbf{k}\in\mathcal{A}}\frac{\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_q}{k_q}}{\binom{\mathrm{r}(\mathbf{n})}{\mathrm{r}(\mathbf{k})}}\leq 1. \tag{6}$$

Equality holds in (6) iff $\mathcal{A}$ is the set of all multisets of rank l for a certain l.

Theorem 3.2 *Let $\mathcal{A}\neq\emptyset$ denote a subset of $\mathcal{P}$. Assume that $\mathcal{A}\neq\{(0,0,...,0)\}$. Then we have*

$$\sum_{\mathbf{k}\in\mathcal{P}}\left(\sum_{\substack{i:\ k_i\geq 1\\ (k_1,\dots,k_i-1,\dots,k_q)\notin\mathcal{A}^f}}\frac{k_i}{\mathrm{r}(\mathbf{k})}\right)\frac{\prod_{j=1}^{r}\binom{n_j}{k_j}}{\binom{\mathrm{r}(\mathbf{n})}{\mathrm{r}(\mathbf{k})}}=1. \tag{7}$$

Proof. Consider the set $\mathcal{A}^f$ which consists of all possible elements of $\mathcal{P}$ containing some member taken from $\mathcal{A}$, that is the filter generated by $\mathcal{A}$. Define a *cat* as a maximal sequence of multisets $\mathbf{t}_1,\mathbf{t}_2,...,\mathbf{t}_{\mathrm{r}(\mathbf{n})}$ such that for every index i multiset $\mathbf{t}_{i+1}$ includes and not equal to $\mathbf{t}_i$. It is easy to see that every cat has a smallest index i such that $\mathbf{t}_i\in\mathcal{A}^f$. We can count all cats like in (2):

On the one hand, the total number of cats is $\mathrm{r}(\mathbf{n})!/(n_1!\cdots n_q!)$, since any cat can be described as a permutation with repetition of the coloured balls of the model. On the other hand, grouping the cats by the multiset $\mathbf{k}$ with which they first enter $\mathcal{A}^f$, the multiset $\mathbf{t}$ in this cat, just below $\mathbf{k}$, does not contain any element in $\mathcal{A}$. Therefore we have

$$\sum_{\mathbf{k}\in\mathcal{P}}\sum_{\substack{i:\ k_i\geq 1\\ (k_1,\dots,k_i-1,\dots,k_q)\notin\mathcal{A}^f}}\frac{(\mathrm{r}(\mathbf{k})-1)!(\mathrm{r}(\mathbf{n})-\mathrm{r}(\mathbf{k}))!}{k_1!\cdots(k_i-1)!\cdots k_q!(n_1-k_1)!\cdots(n_q-k_q)!}=$$

$$=\frac{\mathrm{r}(\mathbf{n})!}{n_1!\cdots n_q!}. \tag{8}$$

Now simple algebra gives (7).

Turning to the proof of Theorem 3.1 let $\mathcal{A}$ be an antichain. If $(0, ..., 0)$ belongs to $\mathcal{A}$, then (6) is straightforward to check. Otherwise if $\mathbf{k} \in \mathcal{A}$ then the inner sum in (7) runs for every i, therefore the value of this sum is 1. Consider the terms from $\mathbf{k} \in \mathcal{A}$ only, since the others have a non-negative contribution.

Finally, if (6) holds with equality, then antichain $\mathcal{A}$ is maximal for inclusion. It is easy to see that the choice $\mathcal{A} = \mathcal{P}_l$ yields equality for each l. If $(0, ..., 0) \in \mathcal{A}$ then $\mathcal{A} = \mathcal{P}_0$. Otherwise we can apply Theorem 3.2. Assume that the maximum rank among the elements of $\mathcal{A}$ is l. Then we show that $\mathcal{P}_l \subset \mathcal{A}$. Otherwise we can find multisets $\mathbf{k} \in \mathcal{A}$ and $\mathbf{t} \notin \mathcal{A}$ of rank l such that there are indices i and j for which $k_i - 1 = t_i$, $k_j + 1 = t_j$ and all other coordinates are equal. Now, like in the proof of Theorem 2.3, it is easy to see that the contribution of multiset $\mathbf{t}$ to the LHS of (7) is positive, a contradiction. $\square$

Finally note that, as we have already mentioned, the divisor lattice satisfies the 'real' LYM inequality. That is

Theorem 3.3 (Theorem 4.2.3 in [5]) *Let $\mathcal{A}$ be an antichain in the divisor lattice $\mathcal{P}$. Then*

$$\sum_{\mathbf{k} \in \mathcal{A}} \frac{1}{N_{\mathrm{r}(\mathbf{k})}} \leq 1.$$

$\square$

The l^{th} Whitney number N_l of the divisor lattice is

$$N_l = \sum_{k_1+k_2+...+k_q=l} \binom{n_1}{k_1}\binom{n_2}{k_2}...\binom{n_k}{k_q}.$$

It seems to be that none of Theorem 3.1 and Theorem 3.3 is a consequence of the other.

3.2 Generalized Boolean algebra

We recall that in the generalized Boolean algebra $\mathcal{P}$ multiset $\mathbf{k} = (k_1, k_2, ..., k_q) \leq \mathbf{t} = (t_1, t_2, ..., t_q)$ iff $\{k_i = 0 \text{ or } t_i = k_i\}$ holds for all i. The set $s(\mathbf{k})$ of the nonzero coordinates is called the *support* of the multiset $\mathbf{k}$. The rank function r is defined as $\mathrm{r}(\mathbf{k}) = |s(\mathbf{k})|$. Now one can state the following results:

Theorem 3.4 *Let $\mathcal{A} \neq \emptyset$ be an antichain in $\mathcal{P}$. Then we have*

$$\sum_{\mathbf{k} \in \mathcal{A}} \frac{1}{\binom{q}{\mathrm{r}(\mathbf{k})} \prod_{i \in s(\mathbf{k})} n_i} \leq 1. \tag{9}$$

Equality holds in (9) iff $\mathcal{A}$ consists of all the element of rank l for a certain l.

Before we state the AZ identity for generalized Boolean algebra $\mathcal{P}$ we need a notation: let A and B be arbitrary subsets of a poset. If no element of A is comparable to any element of B then we write $A >\!|\!< B$. If A has just a unique element a then we write $a >\!|\!< B$.

Theorem 3.5 *Let $\mathcal{A} \neq \emptyset$, $\mathcal{A} \neq \{(0,0,...,0)\}$ denote an arbitrary family of multisets in $\mathcal{P}$. We have*

$$\sum_{\mathbf{k}\in\mathcal{P}} \frac{\left|\bigcap_{\mathbf{t}\in\mathcal{A}:\mathbf{t}\leq\mathbf{k}} s(\mathbf{t})\right|}{\mathrm{r}(\mathbf{k})\binom{q}{\mathrm{r}(\mathbf{k})}\prod_{i\in s(\mathbf{k})} n_i} + \frac{|\{\mathbf{k}\in\mathcal{P} : (\mathrm{r}(\mathbf{k})=q)\ \&\ (\mathbf{k} >\!|\!< \mathcal{A})\}|}{\prod_{i=1}^{q} n_i} = 1. \tag{10}$$

Proof. The proof is very similar to the previous one. Let $\mathcal{A}^f$ the filter generated by $\mathcal{A}$ in $\mathcal{P}$. Define a *cat* as a strictly increasing sequence of multisets $\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, ..., \mathbf{t}_q$. The rank of the i^{th} element in a cat is i. It is easy to see that every cat has a smallest index i such that $\mathbf{t}_i \in \mathcal{A}^f$. We count all cats:

On the one hand, the total number of cats is $q!n_1 \cdots n_q$, since any cat is described with the entering order of the non-zero coordinates, and with values of those coordinates. In coordinate i this value can be $1, 2, \ldots, n_i$. On the other hand, let us group the cats by the multiset $\mathbf{k}$ with which they first enter $\mathcal{A}^f$. Let $\mathbf{t}$ be the multiset in this cat just below $\mathbf{k}$ (that is $\mathrm{r}(\mathbf{t}) = \mathrm{r}(\mathbf{k}) - 1$). Now $s(\mathbf{k}) \setminus s(\mathbf{t})$ must belong to the support of every element of $\mathcal{A}$ which is smaller then $\mathbf{k}$. Finally those cats which never enter to $\mathcal{A}^f$ have maximal elements $\mathbf{k}$ which are uncomparable to $\mathcal{A}$, that is $\mathbf{k} >\!|\!< \mathcal{A}$. Therefore it is easy to check that the LHS of the next equality counts all the cats as well:

$$\sum_{\mathbf{k}\in\mathcal{P}} \left|\bigcap_{\mathbf{t}\in\mathcal{A}:\mathbf{t}\leq\mathbf{k}} s(\mathbf{t})\right| (\mathrm{r}(\mathbf{k})-1)!\,(q-\mathrm{r}(\mathbf{k}))! \prod_{i\notin s(\mathbf{k})} n_i + \sum_{\substack{\mathbf{k}:\ \mathrm{r}(\mathbf{k})=q \\ \mathbf{k} >\!|\!< \mathcal{A}}} q! = q! \prod_{i=1}^{q} n_i. \tag{11}$$

Now simple algebra gives (10). $\square$

We remark, that if $\mathcal{A}$ is a generator of the entire poset, that is for no $x \in \mathcal{P}$ holds $x >\!|\!< \mathcal{A}$, then the second item in the LHS of (10) is 0.

The proof of Theorem 3.4 goes like the proof of Theorem 3.1. It is enough to notice, that if $\mathcal{A}$ is an antichain in (10), and $\mathbf{k} \in \mathcal{A}$, then the cardinality of the intersection is exactly $\mathrm{r}(\mathbf{k})$.

We remark that in the case of $n_i = n$ (for all i) Theorem 3.4 and 3.5 become the proper LYM inequality and AZ identity. Finally, if $n = 1$ then we get back the original LYM inequality and AZ identity.

4 The chain poset

Let us be given an n element set X. For $k \geq 1$, a *k-chain* in the power set of X is a sequence $\lambda = (L_1 \subset L_2 \subset \ldots \subset L_k)$, where L_i is a subset of X and $L_i \neq L_{i+1}$. We say that a k-chain contains an l-chain, if the first as a k-element set contains the second as an l-element subset. In this way all chains in the power set of X make a poset $\mathcal{C}$ for inclusion, this is the *chain poset*. We note here that P. L. Erdős, Seress, and Székely proved an Erdős-Ko-Rado theorem for the chain poset ([10]). If, in addition, one forbids $\emptyset$ and X as elements in chains, then the chain poset can be thought of as an "ordered partition version" of the partition lattice, which is no longer lattice at all. Namely, for a k-chain $\lambda = (L_1 \subset L_2 \subset \ldots \subset L_k)$ with $L_1 \neq \emptyset$ and $L_k \neq X$, $(L_1, L_2 - L_1, \ldots, L_k - L_{k-1}, X - L_k)$ is an ordered partition of X. The counterexample already quoted [15] easily can be adapted to the chain poset to show that it fails the LYM property.

For a chain family $\mathcal{A}$ in $\mathcal{C}$, let $f_{k;i_1,i_2,\ldots,i_k}(\mathcal{A})$ denote the number of k-chains in $\mathcal{A}$, in which the elements of the chain have sizes $i_1 < i_2 < \ldots < i_k$.

Theorem 4.1 *Let $\mathcal{A} \neq \emptyset$ be an antichain in $\mathcal{C}$. Then*

$$\sum_{k;\ i_1,i_2,\ldots,i_k} \frac{f_{k;i_1,i_2,\ldots,i_k}(\mathcal{A})}{\binom{n}{i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k}\binom{n+1}{k}} \leq 1. \tag{12}$$

Equality holds in (12) iff $\mathcal{A}$ is the set of all k-chains for a certain k.

Theorem 4.2 *Let $\mathcal{A} \neq \emptyset$ denote a set of chains in $\mathcal{C}$. We have*

$$\sum_{\lambda \in \mathcal{C}} \frac{\left| \bigcap_{\mathcal{A} \ni \lambda' \subset \lambda} \lambda' \right|}{k\binom{n}{i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k}\binom{n+1}{k}} +$$
$$+ \frac{|\{\lambda \in \mathcal{C} : (|\lambda| = n+1) \ \&\ (\lambda >|< \mathcal{A})\}|}{n!} = 1. \tag{13}$$

Proof. Consider the set $\mathcal{A}^f = \{\delta \in \mathcal{C} : \exists \alpha \in \mathcal{A} \text{ with } \alpha \subset \delta\}$. Define a *cat* as a maximal sequence of chains for proper inclusion, $\lambda_1 \subset \lambda_2 \subset \ldots \subset \lambda_{n+1}$. It is easy to see that λ_i in a cat has to be an i-chain. It is also easy to see that every cat has a smallest index i such that $\lambda_i \in \mathcal{A}^f$. One counts all cats like in (2). On the one hand, the total number of cats is $(n+1)!n!$, since there are $n!$ n-chain of subsets and each gives rise to $(n+1)!$ cats. On the other hand, grouping the cats by the chain λ with which they first enter $\mathcal{A}^f$ and handling separately those cats which don't enter at all, one has

$$\sum_{\lambda\in\mathcal{C}}\left|\bigcap_{\mathcal{A}\ni\lambda'\subset\lambda}\lambda'\right|(k-1)!(n+1-k)!i_1!(i_2-i_1)!...(i_k-i_{k-1})!(n-i_k)!+$$

$$+\sum_{\substack{\lambda:\ |\lambda|=n+1\\ \lambda\ast\mathcal{A}}}(n+1)!=(n+1)!n!. \quad (14)$$

It is obvious that (14) implies (13) and the missing details of the proof are easy to fill. □

5 Chain poset of subspaces of a vector space

Let us be given an n-dimensional vector space V over the finite field GF(q). For $k\geq 1$, a *k-chain* of subspaces of X is a sequence $\lambda=(L_1\subset L_2\subset ...\subset L_k)$, where L_i is a subspace of X and $L_i\neq L_{i+1}$. We say that a k-chain contains an l-chain, if the first as a k-element set contains the second as an l-element subset. In this way all chains of subspaces of V make a poset $\mathcal{V}$ for inclusion, this is the *subspace chain poset.*

Recall the following definitions of the q-generalization of factorials and multinomial coefficients:

$$[n]_q!=\prod_{i=1}^{n}\left(1+q+q^2+...+q^{i-1}\right),$$

$$\begin{bmatrix}a_1+a_2+...+a_m\\ a_1,a_2,...,a_m\end{bmatrix}_q=\frac{[a_1+a_2+...+a_m]_q!}{[a_1]_q![a_2]_q!\cdots[a_m]_q!}.$$

For a family $\mathcal{A}$ in $\mathcal{V}$, let $f_{k;i_1,i_2,...,i_k}(\mathcal{A})$ denote the number of k-chains in $\mathcal{A}$, in which the dimensions of the elements of the chain have sizes $i_1<i_2<...<i_k$.

Theorem 5.1 *Let $\mathcal{A}\neq\emptyset$ be an antichain in $\mathcal{C}$. Then*

$$\sum_{k;\ i_1,i_2,...,i_k}\frac{f_{k;i_1,i_2,...,i_k}(\mathcal{A})}{\begin{bmatrix}n\\ i_1,i_2-i_1,...,i_k-i_{k-1},n-i_k\end{bmatrix}_q\binom{n+1}{k}}\leq 1. \quad (15)$$

Equality holds in (15) iff $\mathcal{A}$ is the set of all k-chains for a certain k.

Theorem 5.2 *Let $\mathcal{A}\neq\emptyset$ denote a set of chains in $\mathcal{V}$. We have*

$$\sum_{\lambda\in\mathcal{V}}\frac{\left|\bigcap_{\mathcal{A}\ni\lambda'\subset\lambda}\lambda'\right|}{k\begin{bmatrix}n\\ i_1,i_2-i_1,...,i_k-i_{k-1},n-i_k\end{bmatrix}_q\binom{n+1}{k}}+$$

$$+\frac{|\{\lambda\in\mathcal{V}:(|\lambda|=n+1)\ \&\ (\lambda>|<\mathcal{A})\}|}{[n]_q!}=1. \quad (16)$$

Proof. Consider the set $\mathcal{A}^f = \{\delta \in \mathcal{V} : \exists \alpha \in \mathcal{A} \text{ with } \alpha \subset \delta\}$. Define a *cat* as a maximal sequence of chains for proper inclusion, $\lambda_1 \subset \lambda_2 \subset \ldots \subset \lambda_{n+1}$. It is easy to see that λ_i in a cat has to be an i-chain. Furthermore every cat has a smallest index i such that $\lambda_i \in \mathcal{A}^f$. One counts all cats like in (14). On the one hand, the total number of cats is $(n+1)![n]_q!$, since there are

$$\frac{q^n-1}{q-1} \cdot \frac{q^n-q}{q^2-q} \cdots \frac{q^n-q^{n-1}}{q^n-q^{n-1}} = [n]_q!$$

$(n+1)$-chains of subspaces, and each gives rise to $(n+1)!$ cats. On the other hand, grouping the cats by the chain λ with which they first enter $\mathcal{A}^f$ and handling separately those which don't enter at all, one has

$$\sum_{\lambda \in \mathcal{V}} \left| \bigcap_{\mathcal{A} \ni \lambda' \subset \lambda} \lambda' \right| (k-1)!(n+1-k)![i_1]_q![i_2-i_1]_q!\ldots[i_k-i_{k-1}]_q![n-i_k]_q! + \\ + \sum_{\substack{\lambda:\ |\lambda|=n+1 \\ \lambda \not\ast \mathcal{A}}} (n+1)! = (n+1)![n]_q!. \quad (17)$$

The end of the proof goes like before. □

References

1. R. Ahlswede - N. Cai: A generalization of the AZ identity, *Combinatorica* **13** (1993), 241–247.

2. R. Ahlswede - N. Cai: Incomparability and intersection properties of Boolean interval lattices and chain posets, *Preprint 93-037* SDS 343, Diskerete Strukturen in der Mathematik, University of Bielefeld, (1993), 1–16.

3. R. Ahlswede - Z. Zhang: An identity in combinatorial set theory, *Advances in Mathematics* **80** (1990), 137–151.

4. R. Ahlswede - Z. Zhang: On cloud-antichains and related configurations, *Discrete Math.* **85** (1990), 225–245.

5. I. Anderson: *Combinatorics of finite sets,* Oxford Science Publication, Oxford, 1987.

6. B. Bollobás: On generalized graphs, *Acta Math. Hung.* **16** (1965), 447-452.

7. E. R. Canfield: On a problem of Rota, *Advances in Mathematics* **29** (1978), 1–10.

8. K. Engel - H.-D. O. F. Gronau: *Sperner theory in partially ordered sets,* Teubner Texte zur Mathematik, Band 78, Teubner Verlagsgesellschaft, Leipzig, 1985.

9. P. L. Erdős - U. Faigle - W. Kern: A group-theoretic setting for some intersecting Sperner families, *Combinatorics, Probability and Computing* **1** (1992), 323–334.

10. P. L. Erdős - Á. Seress - L. A. Székely: On intersecting chains in Boolean algebras, *Combinatorics, Probability and Computing* **3** (1994), 57–62.

11. P. Frankl - Z. Füredi: The Erdős-Ko-Rado theorem for integer sequences, *SIAM J. Alg. Disc. Math.* **1** (1980), 376–381.

12. D. Lubell: A short proof of Sperner's lemma, *J. Combinatorial Theory* **1** (1966), 299.

13. L. D. Meshalkin: A generalization of Sperner's theorem on the number of subsets of a finite set, *Teor. Verojatnost i Primen.* **8** (1963), 219–220, (in Russian)

14. G.-C. Rota: A generalization of Sperner's theorem, *J. Comb. Theory* **2** (1967), 104.

15. J. H. Spencer: A generalized Rota conjecture for partitions, *Stud. Appl. Math.* **53** (1974), 239–241.

16. T.D. Thu: An inductive proof of the Ahlswede-Zhu identity, *J. Combinatorial Theory (A)* **62** (1993), 168–169.

17. K. Yamamoto: Logarithmic order of free distributive lattices, *J. Math. Soc. Japan* **6** (1954), 343–353.