Star chromatic numbers of graphs

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Abstract

We investigate the relation between the star-chromatic number $\chi^*(G)$ and the chromatic number $\chi(G)$ of a graph G. First we give a sufficient condition for graphs under which their star-chromatic numbers are equal to their ordinary chromatic numbers. As a corollary we show that for any two integers k, g, there exists a k-chromatic graph of girth at least g whose star-chromatic number is also k. The special case of this corollary with $g = 4$ answers a question of H. L. Abbott and B. Zhou. We also present an infinite family of triangle free planar graphs with star-chromatic number equal their chromatic number. We then study the star-chromatic number of critical colorable graphs. We prove that if an $(m+1)$-critical graph has large girth, then its star-chromatic number is close to m. We also consider $(m+1)$-critical graphs with high connectivity. An infinite family of graphs is constructed to show that for each $\epsilon > 0$ and each $m \geq 2$ there is an m-connected $(m+1)$-critical graph with star chromatic number at most $m + \epsilon$. This answers another question asked by H. L. Abbott and B. Zhou.

1 Introduction

Let k and d be natural numbers such that $k \geq 2d$. A (k,d)-*coloring* of a graph $G = (V, E)$ is a mapping $\gamma : V \mapsto \mathbb{Z}_k$ such that, for each edge

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$(u,v) \in E$, $|\gamma(u) - \gamma(v)|_k \geq d$, where $|x|_k := min\{|x|, k - |x|\}$. Observe that a $(k,1)$-coloring of a graph G is just an ordinary coloring of G. We say G is (k,d)-*colorable* if there is a (k,d)-coloring of G. The *star chromatic number* $\chi^*(G)$ of G is the minimum $\frac{k}{d}$ for which G is (k,d)-colorable.

The notion of the star chromatic number of a graph was introduced by A. Vince [9]. His work relies on continuous method. A pure combinatorial treatment of star-chromatic number was given by J. A. Bondy and P. Hell [3]. The following alternate definition of the star-chromatic number was given by the second author in [10], and we shall use both definitions in this paper.

Let C be a circle in R^2 of length 1, and let $r \geq 1$ be any real number. Denote by $C^{(r)}$ the set of all open intervals of C of length $1/r$. An *r-circular coloring* of a graph G is a mapping c from $V(G)$ to $C^{(r)}$ such that $c(x) \cap c(y) = \emptyset$ whenever $(x,y) \in E(G)$. If such an r-circular coloring exists, we say that G is *r-circular colorable*.

It was proved in [10] that the star-chromatic number $\chi^*(G)$ of G is equal to the minimum r for which G is r-circular colorable. (To be precise, the "minimum" in the definition of the star-chromatic number should be "infimum". However, it was proved in [9] that the infimum is attained.) Thus we obtain an alternate definition of the star-chromatic number of a graph. This alternate definition is extended by W. Deuber and X. Zhu [5] to weighted graphs and some interesting problems are discussed there.

It follows from the definition that $\chi^*(G) \leq \chi(G)$ for any graph G. There is also a lower bound for $\chi^*(G)$ in terms of $\chi(G)$, namely $\chi(G) - 1 < \chi^*(G)$ (cf. [9]). Thus the parameter $\chi^*(G)$ can be regarded as a refinement of the parameter $\chi(G)$, and it contains more information about the graph. There are graphs G for which $\chi(G) = \chi^*(G)$, and there are also graphs G for which $\chi^*(G)$ are arbitrarily close to $\chi(G) - 1$. A. Vince [9] first asked the question that what determines whether $\chi^*(G) = \chi(G)$. This has been studied by many authors, and some sufficient conditions can be found in [1, 6, 8, 9, 10] under which a graph G has its star-chromatic number equals its ordinary chromatic number. In general, it is intractable to decide whether a given graph satisfies $\chi^*(G) = \chi(G)$, (cf. [8]). In this paper we give another sufficient condition for a graph G to satisfy $\chi^*(G) = \chi(G)$. This condition is weaker (hence the result is stronger) than the previous known sufficient conditions, [1, 6, 8, 9, 10]. As a corollary, we show that uniquely colorable graphs G satisfies $\chi^*(G) = \chi(G)$.

Applying this corollary and an earlier result of B. Bollobás and N. Sauer [2], we prove that for any integers $k \geq 2$ and $g \geq 3$, there exists a graph G with girth at least g and $\chi^*(G) = \chi(G) = k$. The special case of this result with $g = 4$ gives a positive answer to a question asked by H. L. Abbott and B. Zhou in [1], namely there are triangle-free graphs G with $\chi^*(G) = \chi(G) = k$ for any k. In Section 3, we give an infinite family of triangle free planar graphs with star-chromatic number equal their ordinary chromatic number.

The "opposite" of the problem of characterizing graphs $\chi^*(G) = \chi(G)$ would be to find those graphs G for which $\chi^*(G)$ are close to $\chi(G) - 1$. There are also some work on this problem [1]. As mentioned in the previous paragraph, we shall prove that uniquely colorable graphs G have $\chi(G) = \chi^*(G)$. In some sense, the "opposite" of uniquely colorable graphs are critical graphs, i.e, those graphs G such that the deletion of any edge will decrease their chromatic numbers. Intuitively one may expect critical graphs G have star-chromatic number $\chi^*(G)$ close to $\chi(G)-1$. This intuition is supported by the fact that 3-critical graphs are odd cycles C_{2k+1} which have star-chromatic numbers $\chi^*(C_{2k+1}) = 2 + \frac{1}{k}$. Our result in Section 4 will, in some sense, confirm this intuition for $(m+1)$-critical graphs while m is an arbitrary integer. We shall show that for any $\epsilon > 0$ and for any integer m, there is an integer $g = g(m, \epsilon)$ such that every $(m+1)$-critical graph G with girth at least g has star-chromatic number $\chi^*(G) \leq m + \epsilon$.

This result implies that there are many $(m+1)$-critical graphs G for which $\chi^*(G)$ is close to m. However there is no easy constructions of such graphs. We do not have many examples of such graphs. The lack of concrete examples makes it difficult to investigate other properties of such graphs. For instance, what are their connectivities? The following question was asked by H. L. Abbott and B. Zhou [1]:

Let $\epsilon > 0$ be a positive number. Do there exist critical graphs G of high connectivity such that $\chi^*(G) \leq \chi(G) - 1 + \epsilon$?

Since critical graphs are always 2-connected, the lowest connectivity of critical graphs of interest is 3. An infinite family of 3-connected graphs G with $\chi^*(G)$ arbitrarily close to $\chi(G) - 1$ was constructed in [1]. In Section 5, we answer the above question by constructing $(m+1)$-critical graphs G of connectivity m such that $\chi^*(G) \leq m + \epsilon$ for any integer m and any $\epsilon > 0$.

2 A sufficient condition

We present in this section a sufficient condition under which a graph G satisfies $\chi^*(G) = \chi(G)$.

Theorem 1 *Suppose G is a k-chromatic graph. If there is a nonempty proper subset A of $V(G)$ such that for any k-coloring Δ of G, and for any color class S of Δ, either $S \subset A$ or $S \cap A = \emptyset$, then $\chi^*(G) = \chi(G)$.*

Proof. We use the definition of circular coloring in the proof. Suppose the result is not true. Let G be a k-chromatic graph with $\chi^*(G) = r < k$, and let A be a subset of $V(G)$ that satisfies the condition of the theorem. We will derive a contradiction by constructing a k-coloring of G which has a color class S such that $S \cap A \neq \emptyset$ and $S \cap (V(G) - A) \neq \emptyset$.

Let $c : V(G) \rightarrow C^{(r)}$ be an r-circular coloring of G. Consider the two open subsets $\cup_{x \in A} c(x)$ and $\cup_{y \in V(G)-A} c(y)$ of C. If they are disjoint, then they can not cover C, and hence there is a point $q_0 \in C$ such that $q_0 \notin c(x)$ for every $x \in V(G)$. Pick other $k-1$ points $q_1, q_2, \cdots, q_{k-1}$ on C so that the clockwise distance from q_i to q_{i-1} is equal to $1/r$. Since $r < k$, the clockwise distance from q_0 to q_{k-1} is less than $1/r$. Now we can obtain a $(k-1)$-coloring Δ of G as follows: $\Delta(x) = i$ iff either $q_i \in c(x)$ or $c(x)$ equals the open interval of C between q_{i-1} and q_i, for $i = 1, 2, \cdots, k-1$. We first note that every vertex x of G is colored. Indeed the points $q_0, q_1, \cdots, q_{k-1}$ partition the circle C into k intervals. All the intervals $(q_0, q_1), (q_1, q_2), \cdots, (q_{k-2}, q_{k-1})$ have length exactly $1/r$, and the interval (q_{k-1}, q_0) has length strictly less than $1/r$. Since $c(x)$ is an open interval of length $1/r$ which does not contain q_0, either $q_i \in c(x)$ for some $i \in \{1, 2, \cdots, k-1\}$ or $c(x)$ is one of the open intervals $(q_0, q_1), (q_1, q_2), \cdots (q_{k-2}, q_{k-1})$. Thus x is colored. It is easy to see that if $\Delta(x) = \Delta(y) = i$, then $c(x) \cap c(y) \neq \emptyset$, and hence x is not adjacent to y. So Δ is indeed a $(k-1)$-coloring of G, contrary to our assumption that $\chi(G) = k$.

Therefore $(\cup_{x \in A} c(x)) \cap (\cup_{y \in V(G)-A} c(y)) \neq \emptyset$. So there is a point q_0 on C such that $q_0 \in c(x) \cap c(y)$ for some vertices $x \in A$ and $y \in V(G) - A$. We now choose $k-1$ other points $q_1, q_2, \cdots, q_{k-1}$ on C so that the clockwise distance from q_i to q_{i-1} is equal to $1/r$. Now we can obtain a k-coloring Δ of G as

follows: $\Delta(x) = i$ iff either $q_i \in c(x)$ or $c(x)$ is the open interval of C between q_{i-1} and q_i, for $i = 1, 2, \cdots, k-2$; $\Delta(x) = k-1$ iff either $q_{k-1} \in c(x)$ and $q_0 \notin c(x)$ or $c(x)$ is the open interval between q_{k-2} and q_{k-1}; and $\Delta(x) = 0$ iff $q_0 \in c(x)$. Similar to the previous paragraph, one can verify that this is indeed a k-coloring of G. Moreover, the color class $\Delta^{-1}(0)$ contains both vertices from A and vertices from $V(G) - A$. This completes the proof of Theorem 1. ∎

Suppose G is a k-chromatic graph. If G contains a k-chromatic subgraph H with $\chi^*(H) = \chi(H) = k$, then we have $k = \chi(G) \geq \chi^*(G) \geq \chi^*(H) = k$, and hence $\chi^*(G) = \chi(G) = k$. Thus the above theorem actually applies to k-chromatic graphs which contain a k-chromatic subgraph satisfying the stated condition. However even this generalized version of Theorem 1 does not give a necessary condition. The Petersen graph does not satisfy this condition, and yet its star-chromatic number is equal to its chromatic number.

The following result was proved independently by H. L. Abbott and B. Zhou [1], and G. Gao, E. Mendelsohn and H. Zhou [6] (a short proof of this result can also be found in [10]):

Corollary 1 *If the complement of G is disconnected, then $\chi^*(G) = \chi(G)$.*

Proof. The graph G is obtained from two disjoint graphs X, Y by connecting every vertex of X to every vertex of Y. Obviously for any coloring of G, no color class can contain both vertices from $V(X)$ and vertices from $V(Y)$. Thus $V(X)$, as well as $V(Y)$, satisfies the condition of Theorem 1. Therefore $\chi^*(G) = \chi(G)$. ∎

A graph G is called a *k-uniquely colorable* graph if G is k-colorable and every k-coloring of G partitions $V(G)$ into the same k color classes. The following corollary follows immediately from the definition and the comments after the proof of Theorem 1.

Corollary 2 *If a k-chromatic graph G contains a k-uniquely colorable subgraph, then $\chi^*(G) = \chi(G) = k$.* ∎

If a k-chromatic graph G contains a clique of order k, then since K_k is obviously k-uniquely colorable, we have $\chi^*(G) = \chi(G)$ by Corollary 2. This

special case also follows easily from the definition, and was first noted in [9]. On the other hand, as observed in [1], a graph G with $\chi^*(G) = \chi(G)$ needs not have large clique number. Indeed, by using Corollary 1, one can construct K_4-free k-chromatic graphs G with $\chi^*(G) = \chi(G)$ for any integer k. H. L. Abbott and B. Zhou then asked the following question [1]: Given an integer k, does there exist a triangle-free graph G with $\chi^*(G) = k$? For $k = 3, 4$, it was noted in [9] that the Petersen graph and the Grötzsch graph are such graphs. For $k \geq 5$, the answer was unknown. By applying Corollary 2 and a result of Bollobás and Sauer [2], we obtain a much stronger result:

Corollary 3 *For any integers $k \geq 2$ and $g \geq 3$, there is a k-chromatic graph G with girth at least g and star-chromatic number k.*

Proof. B. Bollobás and N. Sauer proved in [2] that for any integers $k \geq 2$ and $g \geq 3$, there is a k-uniquely colorable graph G with girth at least g. By Corollary 2, we have $\chi^*(G) = \chi(G) = k$ for such a graph. ∎

We note that the proof in [2] of the existence of k-uniquely colorable graphs of large girth is not constructive.

3 Planar graphs

Our next result concerns planar graphs. A. Vince [9] asked for families of planar graphs whose star-chromatic numbers are strictly between 2 and 3. It seems that such graphs are abundant (see [1, 10]), while not many planar graphs are known to have star-chromatic exactly 3. Obviously if a 3-chromatic planar graph G contains a triangle, then $\chi^*(G) = 3$. The first triangle free planar graph G with $\chi^*(G) = 3$ was found by G. Gao (personal communication). Denote by W_{2n+1} the graph obtained from the cycle C_{2n+1} by adding a vertex v and connecting v to every vertex of the cycle C_{2n+1}. The graph W_{2n+1} is called the $2n+1$-*wheel*, and the edges connecting v to vertices of C_{2n+1} are called *spokes* of the wheel. Through an extensive check, G. Gao showed that the graph obtained from the 5-wheel by subdividing each of the 5 spokes has star-chromatic number 3. We prove that for all integers n, the graph obtained from W_{2n+1} by subdividing its $2n+1$ spokes

have star-chromatic number 3. This is the first (non-trivial) infinite family of triangle free planar graphs with star-chromatic number 3.

Theorem 2 *Let G_{2n+1} be the graph obtained from W_{2n+1} by subdividing each of its $2n+1$ spokes. Then $\chi^*(G) = 3$.*

Proof. We again use the definition of circular coloring in the proof. Let $V = \{v, c_0, c_1, \cdots, c_{2n}\}$ be the vertex set of W_{2n+1}, while v is connected to all the c_i's, and the set $\{c_0, c_1, \cdots, c_{2n}\}$ induces a cycle with edges (c_i, c_{i+1}). The graph G_{2n+1} is obtained from W_{2n+1} by subdividing each edge (v, c_i) into two edges. Let u_i, $i = 0, 1, \cdots, 2n$, be the vertex which subdivides the edge (v, c_i). It is easy to see that $\chi(G_{2n+1}) = 3$. Therefore $\chi^*(G_{2n+1}) \leq 3$. Suppose $\chi^*(G_{2n+1}) = r < 3$. Let $c : V(G_{2n+1}) \rightarrow C^{(r)}$ be an r-circular coloring of G_{2n+1}. As v is adjacent to all the u_i's, each of the intervals $c(u_i)$ are disjoint from $c(v)$. Since $r < 3$ and $c(c_i)$ is disjoint from $c(u_i)$, we have $c(c_i) \cap c(v) \neq \emptyset$, for $i = 0, 1, \cdots, 2n$. For any $i \in \{0, 1, \cdots, 2n\}$, we can not have $c(c_i) = c(v)$, for otherwise we would have $c(c_i) \cap c(c_{i+1}) \neq \emptyset$ while c_i is adjacent to c_{i+1}. Therefore $c(c_i)$ contains one of the end points of $c(v)$. Let p, q be the two end points of $c(v)$. Without loss of generality, we assume that $c(c_0)$ contains p. Since $c(c_1)$ is disjoint from $c(c_0)$, $c(c_1)$ must contain q. For the same reason, $c(c_2), c(c_3), \cdots, c(c_{2n})$ must alternately contain p and q. Thus $p \in c(c_{2n})$ and hence $c(c_{2n}) \cap c(c_0) \neq \emptyset$, contrary to our assumption that c is an r-circular coloring of G_{2n+1}. This completes the proof of Theorem 2. ∎

We pose two open problems:

(1): Is it true that every triangle free planar graph G with $\chi^*(G) = 3$ contains one of the graphs G_{2n+1}?

(2): It follows from Theorem 1, and also noted in [10], that W_{2n+1} has star-chromatic number 4. Presently there are no other known planar graphs with star-chromatic number 4. Is it true that every planar graph of star-chromatic number 4 contains an odd wheel W_{2n+1}?

If answers to both questions above are affirmative, and the chromatic number of a planar graph G is given, then it can be decided in time polynomial in size of G whether or not $\chi^*(G) = \chi(G)$.

We note that the argument in [8] shows that even for planar graphs G, it is NP-hard to decide whether $\chi^*(G) = \chi(G)$ if the chromatic number of G is not given. On the other hand, for general graphs G, if the chromatic number of G is given, the complexity of deciding whether or not $\chi^*(G) = \chi(G)$ is still open, as far as we know.

4 Critical graphs with large girth

A graph G is called $(m+1)$-*critical* if $\chi(G) = m+1$ and for any edge $e \in E(G)$, $\chi(G-e) = m$. We prove in this section that critical graphs G with large girth have star-chromatic numbers $\chi^*(G)$ close to $\chi(G)-1$. Indeed we shall prove the following stronger statement:

Theorem 3 *Let $m \geq 2$ and $t \geq 1$ be integers. Let G be a graph. If G has a vertex x such that $G-x$ is m-colorable and any cycle of G containing x has length at least $m(t-1)+2$, then $\chi^*(G) \leq m + \frac{1}{t}$.*

Proof. Let $\Delta : V(G) - \{x\} \mapsto \{1, 2, \cdots, m\}$ be an m-coloring of $G-x$. We define a $(tm+1, t)$-coloring γ of G as follows:

Let $V_i = \Delta^{-1}(i)$ $(i = 1, 2, \cdots, m)$ be the color classes of the m-coloring Δ. In the $(tm+1, t)$-coloring γ of G, we shall assign colors $0, 1, \cdots, t$ to vertices of $V_1 \cup \{x\}$, and colors $it+1, it+2, \cdots, (i+1)t$ to vertices of V_{i+1} for $i = 1, 2, \cdots, m-1$.

We denote by W_i the set of vertices colored i by γ, i.e., $W_i = \gamma^{-1}(i)$, for $i = 0, 1, \cdots, tm$. We construct the sets W_i inductively in the order $W_t, W_{2t}, \cdots, W_{mt}, W_{t-1}, W_{2t-1}, \cdots W_{mt-1}, \cdots, W_1, W_{t+1}, \cdots W_{(m-1)t+1}, W_0$ so that $W_i \subset V_j$ if $i = (j-1)t + q$ for some $1 \leq q \leq t$.

We first set $W_t = V_1 \cap N(x)$. Here $N(x)$ is the set of vertices adjacent to x. Suppose that W_i $(i \neq 0)$ has been determined. We construct the next set W_{i+t} (where addition modulo $tm+1$) as follows:

Suppose that $i = (j-1)t + q$ for some $1 \leq q \leq t$. Thus $W_i \subset V_j$ by our induction hypothesis. If $j < m$, then the next set to be constructed is W_{jt+q}. By the order that we construct the sets W_k's, we know that the sets

$W_{jt+q+1}, W_{jt+q+2}, \cdots, W_{(j+1)t}$ have already been constructed. If $q = 1$ then we let $W_{jt+q} = V_{j+1} - (W_{jt+q+1} \cup W_{jt+q+2} \cup \cdots \cup W_{(j+1)t})$. If $q > 1$ then we let $W_{jt+q} = N(W_i) \cap (V_{j+1} - (W_{jt+q+1} \cup W_{jt+q+2} \cup \cdots \cup W_{(j+1)t}))$. Here $N(W_i)$ is the set of those vertices of G which are adjacent to at least one vertex of W_i.

If $j = m$, then the next set to be constructed is W_{q-1}. Similarly we know that the sets $W_q, W_{q+1}, \cdots, W_t$ have already been constructed. If $q = 1$ then we let $W_0 = \{x\} \cup (V_1 - (W_1 \cup W_2 \cup \cdots \cup W_t))$. If $q > 1$ then we let $W_{q-1} = N(W_i) \cap (V_1 - (W_q \cup W_{q+1} \cup \cdots \cup W_t))$.

It is easy to see that $W_0, W_1, \cdots, W_t$ is a partition of $V_1 \cup \{x\}$ and for each $i = 1, 2, \cdots, m-1$, $W_{it+1}, W_{it+2}, \cdots, W_{(i+1)t}$ is a partition of V_{i+1}. Let $\gamma(v) = i$ if $v \in W_i$. Then γ is a mapping of $V(G)$ to $\mathbb{Z}_{tm+1}$.

We now show that γ is a $(tm+1, t)$-coloring of G, i.e., for any edge (a, b) of G, we have $t \leq |\gamma(a) - \gamma(b)| \leq t(m-1) + 1$.

Otherwise there is an edge $(a, b) \in E(G)$ such that either $|\gamma(a) - \gamma(b)| < t$ or $|\gamma(a) - \gamma(b)| > t(m-1)+1$. We assume that $\gamma(a) > \gamma(b)$. First we consider the case that $\gamma(a) - \gamma(b) < t$. Observe that the edge (a, b) can not be (y, x) for some $y \in V_1$ because $\gamma(x) = 0$ and $\gamma(y) = t$ for all $y \in N(x) \cap V_1$. Therefore we must have $b \in V_i$ and $a \in V_{i+1}$ for some $i \leq m-1$.

By our construction, $V_{i+1} = W_{it+1} \cup W_{it+2} \cup \cdots \cup W_{(i+1)t}$. Therefore $it+1 \leq \gamma(a) \leq (i+1)t$. This implies that $\gamma(b) = (i-1)t + q$ for some $q \geq 2$. However by our construction of W_{it+q} we have $N(b) \cap V_i \subset W_{it+q} \cup W_{it+q+1} \cup \cdots \cup W_{(i+1)t}$. Thus $it+q \leq \gamma(a) \leq (i+1)t$, contrary to our assumption that $\gamma(a) - \gamma(b) < t$.

If $\gamma(a) - \gamma(b) > t(m-1)+1$, then we must have $a \in V_m$ and $b \in V_1 \cup \{x\}$. Moreover $a \notin W_{(m-1)t+1}$. From the construction of W_k's, we know that for any vertex $u \in V_m - W_{(m-1)t+1}$ there is a vertex $u' \in W_t = V_1 \cap N(x)$ and a path in $G - x$ of length at most $(t-1)m$ (i.e., with $(t-1)m$ vertices) from u to u'. This implies that $b \neq x$, for otherwise we would have a cycle in G of length at most $(t-1)m + 1$ which contains x. Again by the construction of W_k's, we know that if $a \in W_{(m-1)t+q}$ for some $2 \leq q \leq t$, then $b \in W_{q-1} \cup W_q \cup \cdots \cup W_t$. This implies that $\gamma(a) - \gamma(b) \leq t(m-1) + 1$, contrary to our assumption. ∎

Corollary 4 *Suppose $m \geq 2, t \geq 1$ are integers. Then all $(m+1)$-critical graphs G of girth at least $m(t-1)+2$ have star-chromatic numbers $\chi^*(G) \leq$*

$m + \frac{1}{t}$.

We note that there are many critical graphs G for which $\chi(G) = \chi^*(G)$. Suppose G_1, G_2 are m_1-critical graph and m_2-critical graph, respectively. Let $G_1 + G_2$ be the graph obtained from the disjoint union of G_1 and G_2 by connecting every vertex of G_1 to every vertex of G_2. It is easy to see that G_1+G_2 is (m_1+m_2)-critical. However $\chi^*(G_1+G_2) = \chi(G_1+G_2) = m_1+m_2$, [1, 6, 10]. This implies that, in some sense, the condition on the girth of G in the above Corollary is necessary. In case $m = 2$, the bound $m(t-1)+2$ on the girth of G is sharp for G to have star-chromatic number $\chi^*(G) \leq 2 + \frac{1}{t}$.

On the other hand, the girth condition alone does not imply the graph G has star-chromatic number close to $\chi(G) - 1$, as shown by Corollary 3. Thus, in some sense, the condition of being critical is also necessary.

At present, all known critical graphs G with $\chi(G) = \chi^*(G)$ contain triangles. Failed in the search for triangle free $(m+1)$-critical graphs G with $\chi^*(G) = m + 1$, we propose the following conjecture:

Conjecture 1 *If G is a triangle free critical graph, then $\chi^*(G) < \chi(G)$.*

5 Critical graphs of high connectivity

In this section, we answer another question asked by H. L. Abbott and B. Zhou [1] by constructing m-connected $(m+1)$-critical graphs with star-chromatic number $\chi^*(G)$ arbitrarily close to $\chi(G) - 1$.

Theorem 4 *For all integers $m \geq 4$ and $k \geq 2$ there is an m-connected, $(m+1)$-critical graph H_m^k with $m < \chi^*(H_m^k) \leq m + \frac{1}{k}$.*

Proof. Given $m \geq 4, k \geq 2$, we construct a graph H_m^k as follows:

The vertex set V of H_m^k is $\mathbb{Z}_{mk+1}$.

Let

$$V^* = \{nk | n = 1, 2, \cdots, m\}$$

$V_1 = \{0, 1, \cdots, k-1\}$, and

$V_n = \{(n-1)k+1, (n-1)k+2, \cdots, jk-2, jk-1\}$ for $n = 2, \cdots, m$

Note that $V^*, V_1, V_2, \cdots, V_m$ is a partition of V, $|V^*| = m, |V_1| = k$ and $|V_n| = k-1$ for $n = 2, \cdots, m$.

A pair of vertices (i, j) is an edge of H_m^k if and only if

i) either $|i-j|_{mk+1} = k$,

ii) or $i \in V^*$ and $|i-j|_{mk+1} > k$.

The mapping $\gamma(i) = i$ is an $(mk+1, k)$-coloring of H_m^k. Therefore $\chi^*(H_m^k) \leq m + \frac{1}{k}$. In the following we prove that the graph H_m^k is $(m+1)$-critical and m-connected.

First we show that H_m^k is not m-colorable. Suppose to the contrary that there is an m-coloring Δ of H_m^k. Since the subset $V^* = \{nk : n = 1, 2, \cdots, m\}$ of V induces a complete graph of order m, we may assume that $\Delta(nk) = n$ for $n = 1, 2, \cdots, m$. Then the vertex $k-1$ is forced to be colored by color 1, as $k-1$ is adjacent to all the vertices nk for $n = 2, 3, \cdots, m$. This in turn forces the vertex $2k-1$ to be colored by color 2, as $2k-1$ is adjacent to $k-1$ and to all the vertices nk for $n = 3, 4, \cdots, m$. Follow the sequence $(k-1, 2k-1, \cdots, mk-1, k-2, 2k-2, \cdots, mk-2, \cdots, 2, k+2, \cdots, (m-1)k+2, 1)$ of vertices of H_m^k, we see that the color of each vertex is forced by the coloring of the previous colored vertices, and at the end there is no color available for the vertex 1. Therefore H_m^k is not m-colorable.

Next we show that $H_m^k - e$ is m-colorable for any edge $e \in E(H_m^k)$. In the following arguments, we shall frequently use the fact that any k consecutive vertices $\{i, i+1, i+2, \cdots, i+(k-1)\}$ (where addition modulo $mk+1$) is an independent set of H_m^k.

Let $e = (i, j)$ be an edge of H_m^k. First we consider the case that at least one of the two end vertices of e, say i, does not belong to V^*. Then i has degree $m-1$ in $H_m^k - e$, because i has degree m in H_m^k. There is an obvious partition of the set $V - i$ into m consecutive segments so that each of them is of order k and independent. In other words, $H_m^k - i$ is m-colorable. Since

i has degree $m-1$ in $H_m^k - e$, this m-coloring of $H_m^k - i$ can be extended to an m-coloring of $H_m^k - e$.

We assume now that both $i, j \in V^*$. If $i = nk$ and $j = (n+1)k$, then let $W = \{nk, nk+1, \cdots, (n+1)k\}$ and partition the set $V - W$ into $m-1$ consecutive segments, each of order k. This gives an m-coloring of $H_m^k - e$.

Assume now that $i = n_1 k$, $j = n_2 k$ and $n_1 \leq n_2 - 2$. Since $m \geq 4$, there is an n' such that $n_1 + 1 \leq n' \leq n_2$ and $V_{n'} \cup V_1$ is an independent set of H_m^k. There is a natural way to partition the set $V - (V_{n'} \cup V_1 \cup \{n_1 k, n_2 k\})$ into $m-2$ consecutive segments, each of them is of order k and independent. Thus we can color these $m-2$ consecutive segments by $m-2$ colors, color $V_1 \cup V_{n'}$ by 1 color, and color $n_1 k, n_2 k$ by one color. Therefore $H_m^k - e$ is m-colorable.

To finish the proof of Theorem 4, it remains to show that H_m^k are m-connected. By Menger's Theorem, we need to show for any two distinct vertices i, j of H_m^k, there are m disjoint paths connecting i and j.

Any two vertices i and j $(i < j)$ of H_m^k have either $m-4$ common neighbors in V^*, or $m-3$ common neighbors in V^*, or $m-2$ common neighbors in V^*.

Suppose i and j have exactly $m-4$ common neighbors in V^*. Then these common neighbors give $m-4$ pairwise disjoint paths connecting i and j. Also it is easy to see that we must have $i, j \in V \setminus (V^* \cup \{0, k-1\})$ and $|i-j|_{mk+1} \geq 2k$. Let $i_1 k, i_2 k$ and $j_1 k, j_2 k$ be the vertices in V^* not adjacent to i and j, respectively. Without loss of generality we may assume $i_1 k < i < i_2 k$ and $j_1 k < j < j_2 k$, and hence $i - k < i_1 k < i < i_2 k < i + k$. If $j = i + 2k$ then four additional paths between i and j are $P_1 = [i, i+k, j]$; $P_2 = [i, i-k, i_2 k, j]$; $P_3 = [i, j_2 k, i_1 k, j]$ and $P_4 = [i, j_1 k, j+k, j]$. Otherwise the four paths are $P_1 = [i, i+k, i_1 k, j]$; $P_2 = [i, i-k, i_2 k, j]$; $P_3 = [j, j+k, j_1 k, i]$ and $P_4 = [j, j-k, j_2 k, i]$. The cases that i and j have exactly $m-3$ or $m-2$ common neighbors in V^* are treated similarly, and we omit the details. [1] ∎

[1] **The following detailed argument is for the convenience of the referees:** If i and j have exactly $m-3$ common neighbors in V^* then $i \in V \setminus (V^* \cup \{0, k-1\})$, $j \in V^*$ and $|i-j|_{mk+1} > k$ or $i, j \in V \setminus V^*$, $i \in \{0, k-1\}$ and $|i-j|_{mk+1} \geq 2k - i$ or $i, j \in V \setminus V^*$ and $|j - i_2 k|_{mk+1} < k$. In the first case $(i, j) \in E$, and $P_1 = [i, j]$; $P_2 = [i, i+k, i_1 k, j]$ and $P_3 = [i, i-k, i_2 k, j]$ are three paths between i and j other than the $m-3$ paths using their common neighbors. In the second case we may assume, without loss of generality, that $i = 0$. If $j = (m-2)k + 1$ then the three other paths are:

We note that Theorem 4 remains true for $m = 2, 3$. For $m = 2$, the odd cycles are the required critical graphs. For $m = 3$ and $k \geq 3$, the graphs $H_k^3 - e^*$ where $e^* = (k, 3k)$ are the required graphs, while the verification is similar to the proof of Theorem 4.

Our result in the previous section says that critical graphs G with large girth necessarily have star-chromatic number $\chi^*(G)$ close to $\chi(G) - 1$. The example in this section shows that critical graphs with large clique also could have star-chromatic number $\chi^*(G)$ close to $\chi(G) - 1$.

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$P_1 = [0, (m-1)k+1, j]$; $P_2 = [0, (m-1)k, (m-3)k+1, j]$ and $P_3 = [0, (m-2)k, mk, j]$. In all the other cases the other paths are: $P_1 = [0, j_1 k, j+k, j]$; $P_2 = [0, j_2 k, j-k, j]$ and $P_3 = [0, (m-1)k+1, (m-2)k+1, mk, j]$. In the third case we have $i_2 = j_1$. If $j = i+k$ then the additional paths are: $P_1 = [i, j]$; $P_2 = [i, j_2 k, i_1 k, j]$ and $P_3 = [i, i-k, i_2 k, j+k, j]$. Otherwise the three pathes are $P_1 = [i, i-k, i_2 k, j+k, j]$; $P_2 = [i, i+k, i_1 k, j]$ and $P_3 = [i, j_2 k, j-k, j]$. If i and j have exactly $m-2$ common neighbors in V^* then $i = 0$ and $j = k-1$ or $i, j \in V^*$. In the first case the additional paths are $P_1 = [0, (m-1)k+1, (m-2)k, mk, k-1]$ and $P_2 = [0, k, 3k-1, 2k-1, k-1]$. In the second case $(i, j) \in E$ and they have at least one common neighbor in $V \setminus V^*$. Thus there are two additional paths between i and j.

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