# Relaxed coloring of a graph

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**Abstract**

Let $\mathcal{P}$ be a hereditary family of graphs. A relaxed coloring of a graph $G = (V, E)$ with respect to $\mathcal{P}$ is an assignment of colors to vertices of $G$ so that each color class induces a graph which is the disjoint union of members of $\mathcal{P}$. The $\mathcal{P}$-chromatic number $\chi_{\mathcal{P}}(G)$ of $G$ is the minimum number of colors in a relaxed coloring of $G$ with respect to $\mathcal{P}$. We study the relation between the girth and the $\mathcal{P}$-chromatic number of a graph, and the $\mathcal{P}$-chromatic number of product graphs. Our results generalize some results of M. L. Weaver and D. B. West in [10], and answer some questions in that paper.

# 1 Introduction

In the last decade, various variations of graph coloring have been studied, [5, 9, 10]. Some of these generalized colorings have requirements stronger than that of the ordinary coloring. In such colorings, the pair of colors assigned to adjacent vertices are not only required to be different, they must also forbidden some patterns. Such colorings can be defined as a homomorphism of a graph $G$ to another graph $H$, and are usually referred to as $H$-coloring problems, [1, 2, 5, 6]. Other generalizations have requirements weaker than that of the ordinary coloring. In other words, adjacent vertices are allowed to be colored with the same color. However there are some restrictions on the subgraphs induced by the color classes. Such coloring are called *relaxed colorings* [10] or *improper colorings* [9]. Since this is the context of the present paper, we give a definition.

Let $\mathcal{P}$ be a so called hereditary family of graphs, which is closed under taking subgraphs. A $\mathcal{P}$-$k$-coloring of a graph $G = (V, E)$ partitions $V$ into $k$ blocks such that the subgraphs induced by the blocks are disjoint unions of members of $\mathcal{P}$. The $\mathcal{P}$-chromatic number $\chi_{\mathcal{P}}(G)$ of $G$ is the minimum number $k$ such that $G$ is $\mathcal{P}$-$k$-colorable. If $\mathcal{P}$ is the family of empty graphs, then $\chi_{\mathcal{P}}(G) = \chi(G)$. When $\mathcal{P}$ is the family of forests, then $\chi_{\mathcal{P}}(G)$ is also called the *vertex arboricity* of $G$. For a survey of results concerning $\mathcal{P}$-$k$-coloring of graphs, readers are referred to [9] and [10].

Following notions in [10], for a given family $\mathcal{H}$ of graphs we denote by $\mathcal{H}^*$ the family consisting of disjoint unions of subgraphs of members of $\mathcal{H}$, and denote by $\mathcal{H}'$ the family of graphs with no subgraph belonging to $\mathcal{H}$. We call the parameters $\chi_{\mathcal{H}^*}(G)$ and $\chi_{\mathcal{H}'}(G)$ the *$\mathcal{H}$-restricted* and the *$\mathcal{H}$-forbidden* chromatic numbers of $G$, respectively. When $\mathcal{H} = \{H\}$ consists of a single graph, then we use $H$ in place of $\{H\}$. We shall study only forbidden and restricted chromatic numbers. First we discuss the existence of graphs with large girths and large restricted and forbidden chromatic numbers. The corresponding results generalize some results of M. L. Weaver and D. B. West [10]. We then consider the restricted and forbidden chromatic numbers of the Cartesian product $G \Box G'$ and the lexicographic product $G[G']$ of graphs. For the Cartesian product, we prove in particular that the product of more than three odd cycles has *path restricted* chromatic number 3. In other words, if the Cartesian product of more than three odd cycles are partitioned into two blocks, then at least one of the blocks induce a cycle or a star $K_{1,3}$. This solves the problems posed in [10]. For lexicographic product, we will give an upper bound of the restricted (resp. forbidden) chromatic number of $G[G']$ in terms of the star-chromatic number of $G$ (see definition in Section 4) and the restricted (resp. forbidden) chromatic number of $G'$.

## 2 Girths versus restricted and forbidden chromatic numbers

A classical result of Erdös [3] says that there are graphs $G$ of arbitrary large girth and arbitrary large chromatic number. For a family $\mathcal{H}$ of graphs, we denote by $\mathcal{H}^*$ the family consisting of disjoint unions of subgraphs of members of $\mathcal{H}$, and $\mathcal{H}'$ the family of graphs with no subgraph belonging to $\mathcal{H}$. We call $\chi_{\mathcal{H}^*}(G)$ the $\mathcal{H}$-restricted chromatic number of $G$ and $\chi_{\mathcal{H}'}(G)$ the $\mathcal{H}$-forbidden chromatic number of $G$. M. L. Weaver and D. B. West [10] considered the problem whether graphs $G$ of large girths have $\mathcal{H}$-restricted chromatic number or $\mathcal{H}$-forbidden chromatic number smaller than its ordinary chromatic number $\chi(G)$. They proved constructively the following theorem [10]:

**Theorem 1** *If $\mathcal{H}$ is any finite family of bipartite graphs, then there is a triangle free $\mathcal{H}$-restricted $k$-critical graph $G_k$ with ordinary chromatic number $k$. If $\mathcal{H}$ contains any complete bipartite graph, then the same statement holds for the parameter $\chi_{\mathcal{H}'}(G)$.*

Here a graph $G$ is said to be $\mathcal{H}$-restricted $k$-critical if $\chi_{\mathcal{H}^*}(G) = k$ and $\chi_{\mathcal{H}^*}(G') < k$ for any proper subgraph $G'$ of $G$. Obviously any graph $G$ with $\chi_{\mathcal{H}^*}(G) = k$ contains a subgraph which is $\mathcal{H}$-restricted critical.

We now use probabilistic method to prove the following result:

**Theorem 2** *If $\mathcal{H}$ is any finite family of graphs, then for any integers $g, k$, there is a graph $G$ of girth at least $g$ for which $\chi_{\mathcal{H}^*}(G) = \chi(G) = k$.*

**Proof.** Let $V_1, V_2, \cdots, V_k$ be disjoint $n$-sets. Let $F$ be the complete $k$-partite graph with vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_k$, and $(x, y) \in E(G)$ if and only if $x \in V_i, y \in V_j$ and $i \neq j$. Then $F$ has $\binom{k}{2}n^2$ edges. Let $\mathcal{G}$ be the set of all subgraphs $G$ of $F$ with $m = \lfloor \binom{k}{2} n^{1+\epsilon} \rfloor$ edges, where $0 < \epsilon < 1/4g$. Then $|\mathcal{G}| = \binom{\binom{k}{2}n^2}{m}$.

In the following, $n$ is assumed to be sufficiently large. We consider $\mathcal{G}$ as a propability space in which each member occurs with the same propability $1/|\mathcal{G}|$.

First we show that most graphs $G \in \mathcal{G}$ do not contain too many short cycles. For a fixed integer $\ell$, since there are $\binom{kn}{\ell}\frac{\ell!}{2\ell}$ ways of choosing a cycle $C_\ell$ with $V(C_\ell) \subset V$, and a cycle $C_\ell$ is contained in either 0 or $\binom{\binom{k}{2}n^2-\ell}{m-\ell}$ of the graphs belonging to $\mathcal{G}$, the expected number of cycles $C_\ell$ of length $\ell$ in a graph $G \in \mathcal{G}$ is at most

$$N_\ell = \binom{kn}{\ell}\frac{\ell!}{2\ell}\binom{\binom{k}{2}n^2-\ell}{m-\ell}\binom{\binom{k}{2}n^2}{m}^{-1}.$$

Since $\binom{\binom{k}{2}n^2-\ell}{m-\ell}\binom{\binom{k}{2}n^2}{m}^{-1} < m^\ell(\binom{k}{2}n^2)^{-\ell}$, we obtain

$$N_\ell < \frac{(kn)^\ell}{2\ell}m^\ell(qn^2)^{-\ell} < (kn)^\ell n^{-\ell(1-\epsilon)} = k^\ell n^{\ell\epsilon}.$$

Therefore

$$\sum_{\ell=3}^{g-1} N_\ell < n^{-\epsilon/2}n^{g\epsilon}.$$

Let $\mathcal{G}_1$ be the set of all graphs $G \in \mathcal{G}$ with at most $\lceil n^{g\epsilon} \rceil$ cycles of length less than $g$. Then the above calculation shows that $|\mathcal{G}_1| \geq (1 - n^{-\epsilon/2})|\mathcal{G}|$.

We now show that most of graphs $G \in \mathcal{G}$ have the property that if we remove a set $E_0$ of at most $n^{g\epsilon}$ edges, then the resulting graph $G'$ has $\mathcal{H}$-restricted chromatic number $\chi_{\mathcal{H}^*}(G) = k$. Since each graph $G \in \mathcal{G}$ is a $k$-partite graph, so $\chi_{\mathcal{H}^*}(G) \leq \chi(G) \leq k$. Therefore we only need to show that $G - E_0$ is not $\mathcal{H}^*$-$(k-1)$-colorable for most graphs $G \in \mathcal{G}$.

Suppose $G \in \mathcal{G}$ and that there is a set $E_0$ of at most $n^{g\epsilon}$ edges of $G$ such that $G - E_0$ is $\mathcal{H}^*$-$(k-1)$-colorable. Let $\Delta : V(G) \to \{1, 2, \cdots, k-1\}$ be a $\mathcal{H}^*$-$(k-1)$-coloring of $G - E_0$. Then one of the $k-1$ color classes, say $\Delta^{-1}(1)$, conatins at least $\frac{kn}{k-1}$ vertices. Therefore there are $V_i, V_j (i \neq j)$ such that $|\Delta^{-1}(1) \cap V_i| \geq \frac{n}{k^2}$ and $|\Delta^{-1}(1) \cap V_j| \geq \frac{n}{k^2}$.

Let $W_i \subset V_i \cap \Delta^{-1}(1), W_j \subset V_j \cap \Delta^{-1}(1)$ be subsets of $V_i \cap \Delta^{-1}(1)$ and $V_j \cap \Delta^{-1}(1)$ respectively such that $|W_i| = |W_j| = \lceil \frac{n}{k^2} \rceil$. Let $p$ be the number of vertices in a largest connected graph in $\mathcal{H}$ (recall that $\mathcal{H}$ is a finite family of finite graphs, so there is a largest connected graph in $\mathcal{H}$). Let $G(W_i, W_j)$ be the subgraph of $G$ induced by $W_i \cup W_j$. Then $G(W_i, W_j) - E_0$ contains no connected component of size greater than $p$, because $W_i \cup W_j$ is monochromatic. As $n$ is very large (so that $\frac{n}{k^2}$ is much larger than $p$), there are subsets $W_i' \subset W_i$ and $W_j' \subset W_j$ with $|W_i'| = |W_j'| = \frac{n}{3k^2}$ such that no vertex of $W_i'$ is connected to a vertex of $W_j'$ in $G - E_0$. Therefore there are at most $n^{g\epsilon}$ edges of $G$ between vertices of $W_i'$ and $W_j'$.

For an integer $s \leq n^{g\epsilon}$, denote by $M(s)$ the expected number (in a graph $G \in \mathcal{G}$) of pairs $W_i \subset V_i, W_j \subset V_j$ such that $i \neq j$, and $|W_i| = |W_j| = \lceil \frac{n}{3k^2} \rceil$, and there are exactly $s$ edges connecting $W_i$ and $W_j$. Then

$$M(s) = \binom{k}{2} \binom{n}{\lceil \frac{n}{3k^2} \rceil}^2 \binom{\lceil \frac{n}{3k^2} \rceil^2}{s} \binom{\binom{k}{2} n^2 - \lceil \frac{n}{3k^2} \rceil^2}{m-s} \binom{\binom{k}{2} n^2}{m}^{-1}.$$

Replacing $\binom{\binom{k}{2}n^2 - \lceil \frac{n}{3k^2} \rceil^2}{m-s}$ by $\binom{\binom{k}{2}n^2 - \lceil \frac{n}{3k^2} \rceil^2}{m}$ and applying the inequality $\binom{a-x}{b}\binom{a}{b}^{-1} < e^{-bx/a}$, we obtain

$$M(s) < \binom{k}{2} (ek)^{\frac{2n}{3k^2}} \left(\frac{2n}{3k^2}\right)^{2s} e^{-n^{1+\epsilon}/3k^2} < n^{2s} e^{-\frac{1}{3k^2} n^{1+\epsilon}}.$$

Hence

$$\sum_{s < n^{g\epsilon}} M(s) < exp(-n^{1+\epsilon/2} + 3n^{g\epsilon} \log n) < e^{-n}.$$

Let $\mathcal{G}_2$ be the set of graphs $G \in \mathcal{G}$ with the property $G - E_0$ is not $\mathcal{H}^*$-$(k-1)$-colorable for any $E_0 \subset E(G)$ with $|E_0| \leq n^{g\epsilon}$. Then $|\mathcal{G}_2| \geq (1 - e^{-n})|\mathcal{G}|$. Therefore $\mathcal{G}_1 \cap \mathcal{G}_2 \neq \emptyset$. Let $G \in \mathcal{G}_1 \cap \mathcal{G}_2$. Since $G \in \mathcal{G}_1$, there is a set $E_0$ of $n^{g\epsilon}$ edges such that $G - E_0$ has girth at least $g$. On the other hand $G \in \mathcal{G}_2$ implies that $G - E_0$ is not $\mathcal{H}^*$-$(k-1)$-colorable. This completes the proof of Theorem 2. ∎

For $\mathcal{H}$-forbidden chromatic numbers, it was mentioned in [10] that there exist triangle-free graphs with $C_5$-forbidden chromatic number $k$ (observed by B. Bollobás by probabilistic methods). We now consider the problem for which graphs $H$, there exist graphs with girth at least $g$ and $H$-forbidden chromatic number at least $k$. Obviously if $H$ has girth less than $g$ then the $H$-forbidden chromatic number of any graph $G$ with girth $\geq g$ is 1. We shall prove that if $H$ has girth at least $g$, then there exist graphs with arbitrary large $H$-forbidden chromatic number and with girth at least $g$.

**Theorem 3** *Suppose $H$ is a graph with girth at least $g$. Then for any integer $k$, there exits a graph $G$ with $\chi_{H'}(G) \geq k$ and with girth at least $g$.*

**Proof.** Suppose $|V(H)| = n$. By a classical result of Erdös and A. Hajnal [3] and Lovaśz [7] there exist an $n$-uniform hypergraph $F$ of girth at least $g$ and chromatic number at least $k$. Let $G$ be a graph obtained from $F$ by (arbitrarily) embedding a copy of $H$ into each hyperedge of $F$. Then $G$ has girth at least $g$. Indeed, if $C$ is a cycle of $G$ of length less than $g$, then $C$ is not contained in any copy of $H$, because $H$ has girth at least $g$. Thus $C$ has non-empty intersection with at least two hyperedges of $F$. But then all those hyperedges which intersect $C$ must contain a cycle of $F$, which contradicts the assumption that $F$ has girth at least $g$. Now any $(k-1)$-coloring of the vertices of $G$ can be regarded as a $(k-1)$-coloring of $F$ and hence contains a monochromatic hyperedge. This hyperedge induces a copy of $H$. Therefore the $H$-forbidden chromatic number of $G$ is at lest $k$. ∎

# 3 Cartesian product

The Cartesian product $G \square H$ of two graphs $G$ and $H$ has vertex set $V(G \square H) = V(G) \times V(H)$, and $(g, h)$ is adjacent to $(g', h')$ if and only if either $g = g'$ and $(h, h') \in E(H)$ or $(g, g') \in E(G)$ and $h = h'$. It was proved in [10] that for any hereditary family $\mathcal{P}$,

$$\max\{\chi_{\mathcal{P}}(G), \chi_{\mathcal{P}}(H)\} \leq \chi_{\mathcal{P}}(G \square H) \leq \min\{\max\{\chi_{\mathcal{P}}(G), \chi(H)\}, \max\{\chi(G), \chi_{\mathcal{P}}(H)\}\}.$$

As a corrollary, the Cartesian product $G$ of odd cycles have $\chi_{\mathcal{P}}(G) \leq 3$ for any hereditary family $\mathcal{P}$. M. L. Weaver and D. B. West then asked the question whether $\chi_{P^*}(G) \leq 2$ in case $P$ is a path of certain length. One of their result implies that if $P$ is the path of length $k$, i.e., with $k$-edges, and $G$ is the product of more than $k$ odd cycles, then $\chi_{P^*}(G) = 3$. On the other hand, if $G$ is the product of two odd cycles of length at least 5, then $\chi_{P^*}(G) = 2$ if $P$ is a path of length 2. They then asked a sequence of four questions about the relation between the number of odd cycles in the product $G$ and the length of the path $P$ so that $\chi_{P^*}(G) = 2$. Our next result shows that if $G$ is the product of more than three odd cycles then the length of the path is irrelevant.

**Theorem 4** *Suppose $G$ is the Cartesian product of at least four odd cycles. Then $\chi_{P^*}(G) = 3$ for any path $P$.*

**Proof.** We only need to consider the case that $G$ is the product of exactly four odd cycles. Suppose $G = C_1 \square C_2 \square C_3 \square C_4$, where $C_i$ is an odd cycle of length $2k_i + 1$. Suppose the vertices of $C_i$ are numbers from 0 to $2k_i$ and edges are $(i, i+1), i = 0, 1, \cdots, 2k_i - 1$ and $(0, 2k_i)$. Then the vertices of $G$ are $(x_1, x_2, x_3, x_4)$ where $0 \leq x_i \leq 2k_i$.

Let $c : V(G) \to \{a, b\}$ be a two coloring of $G$. We shall show that either there is a monochromatic cycle, or a monochromatic star $K_{1,3}$.

First we consider the restriction of $c$ to the subset $\{(0,0,0,x_4) : 0 \leq x_4 \leq 2k_4\}$. Since this set induce an odd cycle, there is a monochromatic edge. Without loss of generality, we assume that $c(0,0,0,0) = c(0,0,0,1) = a$.

Next we consider the restriction of $c$ to the subset $\{(0,0,x_3,0), (0,0,x_3,1) : 0 \leq x_3 \leq 2k_3\}$. We claim that there is an index $j$ such that either $(0,0,j,0), (0,0,j,1), (0,0,j+1,0)$ are monochromatic, or $(0,0,j,0), (0,0,j,1), (0,0,j+1,0)$ are monochromatic. (Here the additions are carried out modulo $2k_3 + 1$). If one of $(0,0,1,0)$ and $(0,0,1,1)$ is colored $a$, then we are done. Otherwise both $(0,0,1,0)$ and $(0,0,1,1)$ are colored $b$. If one of $(0,0,2,0)$ and $(0,0,2,1)$ is colored $b$, again we are done. Repeating this argument we see that the claim is true unless $(0,0,j,0)$ and $(0,0,j,1)$ are both colored $a$ when $j$ is even and both are colored $b$ when $j$ is odd. But then $(0,0,0,0), (0,0,0,1), (0,0,2k_3,0)$ are monochromatic, and the claim is also true. Thus without loss of generality, we assume that $c(0,0,1,0) = c(0,0,0,0) = c(0,0,0,1) = a$.

Now we consider the subset $\{(0, x_2, 1, 0), (0, x_2, 0, 0), (0, x_2, 0, 1) : 0 \leq x_2 \leq 2k_2\}$. With an argument similar to that in the previous paragraph, we can show that either there is a monochromatic star $K_{1,3}$, or there is an index $j$ such that $(0, j, 1, 0), (0, j, 0, 0), (0, j, 0, 1), (0, j+1, 1, 0)$ are monochromatic, or $(0, j, 1, 0), (0, j, 0, 0), (0, j, 0, 1), (0, j+1, 0, 1)$ are monochromatic. If there is a monochromatic star $K_{1,3}$ then the theorem is proved. Thus we assume, without loss of generality, that $c(0, 0, 1, 0) = c(0, 0, 0, 0) = c(0, 0, 0, 1) = c(0, 1, 1, 0) = a$.

We consider the cube formed by the vertices $S_0 = \{(0, x_2, x_3, x_4) : 0 \leq x_i \leq 1\}$, which is depicted in Figure 1(a). By our assumption the four vertices $(0, 0, 1, 0), (0, 0, 0, 0), (0, 0, 0, 1), (0, 1, 1, 0)$ are colored $a$. In order to avoid a monochromatic $K_{1,3}$ or a monochromatic cycle, there are only two ways to color the remaining four vertices, as depicted in Figure 1(b) and 1(c).

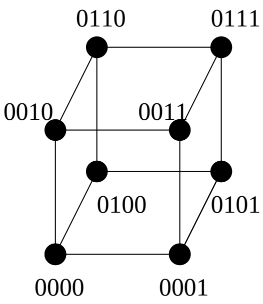


Fig. 1(a)

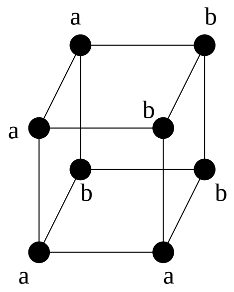


Fig. 1(b)

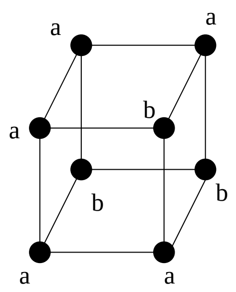


Fig. 1(c)

We consider two cases.

**Case 1** The cube $S_0 = \{(0, x_2, x_3, x_4) : 0 \leq x_i \leq 1\}$ is colored as in Figure 1(b). Consider the possible coloring of the cubes $S_j = \{(j, x_2, x_3, x_4) : 0 \leq x_i \leq 1\}$, $1 \leq j \leq 2k_1$. If for each $j$, the coloring of $S_j$ is obtained from the coloring of $S_{j-1}$ by flipping the colors of the corresponding vertices, then corresponding vertices of $S_{2k_1}$ and $S_0$ would have the same color, and hence we obtain a monochromatic cycle and the theorem is true in this case. Thus without loss of generality we assume that the coloring of $S_1$ is not obtained from the coloring of $S_0$ by flipping the colors of the corresponding vertices. By symmetry, we can assume that $c(1, 1, 1, 0) = a$. Then the remaining vertices of $S_1$ are forced to be colored as in Figure 2 in order to avoid a monochromatic $K_{1,3}$ or a monochromatic cycle, where the numbers beside the vertices indicate the forcing steps, i.e., first consider vertex with number 1, then vertex with number 2, and so on. The uncolored vertices in Figure 2 can not be colored without creating a monochromatic $K_{1,3}$ or a monochromatic cycle.

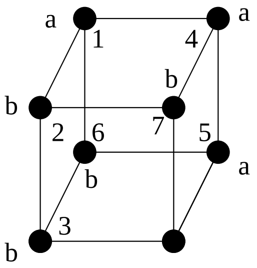


Fig. 2

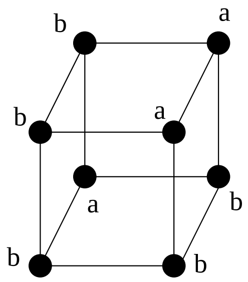


Fig. 3

**Case 2** The cube $S_0$ is colored as in Figure 1(c). Again we consider the possible coloring of the cubes $S_j = \{(j, x_2, x_3, x_4) : 0 \leq x_i \leq 1\}$, $1 \leq j \leq 2k_1$. There are essentially two possible coloring of $S_1$ which will avoid a monochromatic $K_{1,3}$ or a monochromatic cycle. One is obtained from the coloring of $S_0$ by flipping the colors of the corresponding vertices. The other is as depicted in Figure 3. In both colorings of $S_1$, there are 5 vertices colored $b$ and 3 vertices colored $a$. Repeating this argument, we see that in all possible colorings of $S_j$ that will avoid monochromatic $K_{1,3}$ and monochromatic cycle, there are 5 vertices colored $a$ if $j$ is even, and 5 vertices colored $b$ if $j$ is odd. In particular, 5 vertices in $S_{2k_1}$ are colored $a$. It is easy to see that this will creat a monochromatic cycle in the subgraph induced by the two cubes $S_0$ and $S_{2k_1}$. This completes the proof of Theorem 5. ∎

**Corollary 1** *Suppose $G$ is the product of at least five odd cycles. If the vertices of $G$ are colored by two colors, then there is a monochromatic star $K_{1,3}$.*

**Proof.** We may assume that $G = C_1 \Box C_2 \Box C_3 \Box C_4 \Box C_5$ and that the vertices of $C_i$ are $\{0, 1, 2, \cdots, 2k_i\}$. Let $\Delta : V(G) \to \{a, b\}$ be a 2-coloring of the vertices of $G$. By Theorem 5 the restriction of $\Delta$ to the subset $\{(0, x_2, x_3, x_4, x_5) : 0 \leq x_i \leq 2k_i\}$ of $V(G)$ contains either a monochromatic $K_{1,3}$ or a monochromatic cycle. If there is a monochromatic $K_{1,3}$ then we are done. Thus we assume that there is a cycle $X_0 \subset \{(0, x_2, x_3, x_4, x_5) : 0 \leq x_i \leq 2k_i\}$ such that $\Delta(X_0) = a$. Let $X_i = \{(i, x_2, x_3, x_4, x_5) : (0, x_2, x_3, x_4, x_5) \in X_0\}$. If for each $i$, the colors of the vertices of $X_i$ are obtained by flipping the colors of the corresponding vertices of $X_{i-1}$ then all the vertices of $X_{2k_1}$ are colored $a$ and hence we have a monochromatic $K_{1,3}$. Otherewise we may assume that there is a one vertex of $X_1$ is colored $a$, then again we obtain a monochromatic $K_{1,3}$. ∎

**Conjecture 1** *For any integer $m$, there is an integer $f(m)$ such that if $G$ is the product of at least $f(m)$ odd cycles then for any 2-coloring of the vertices of $G$ there is a monochromatic star $K_{1,m}$.*

# 4 Lexicographic product

The lexicographic product $G[H]$ of two graphs $G$ and $H$ has vertex set $V(G[H]) = V(G) \times V(H)$, and $(g, h)$ is adjacent to $(g', h')$ if and only if either $g = g'$ and $(h, h') \in E(H)$ or $(g, g') \in E(G)$. In this section, we derive an upper bound on the relaxed chromatic number $\chi_{\mathcal{P}}(G[H])$ for any graphs $G, H$ and any hereditary family $\mathcal{P}$ of graphs. We need the definition of the star-chromatic number of a graph.

Suppose $G$ is a graph and $k, d$ are positive integers such that $k \geq 2d$. A $(k, d)$-coloring of $G$ is a coloring $c$ of the vertices of $G$ with $k$ colors $\{0, 1, 2, \cdots, k-1\}$ such that if $(x, y) \in E(G)$, then $d \leq |c(x) - c(y)| \leq k - d$. If there is a $(k, d)$-coloring of $G$ then we say that $G$ is $(k, d)$-colorable. Note that a $(k, 1)$-coloring is just the ordinary $k$-coloring. The star-chromatic number $\chi^*(G)$ of a graph $G$ is defined as

$$\chi^*(G) = \min\{k/d : G \text{ is } (k, d) - \text{colorable}\}.$$

The existence of the "minimum" was proved in [1, 8]. It was also proved in [1, 8] that $\chi(G) - 1 < \chi^*(G) \leq \chi(G)$ for any graph $G$. An upper bound on the chromatic number of $G[H]$ in terms of the star-chromatic number of

$G$ and the chromatic number of $H$ was given in [11]. This bound is sharp for many circulant graphs [4]. The following theorem generalizes the upper bound given in [11] to relaxed colorings.

**Theorem 5** *For any graphs $G, H$ and for any hereditary family $\mathcal{P}$ of graphs, we have $\chi_{\mathcal{P}}(G[H]) \leq \lceil \chi^*(G)\chi_{\mathcal{P}}(H) \rceil$.*

**Proof.** Suppose $\chi^*(G) = k/d$ and $\chi_{\mathcal{P}}(H) = n$. Let $c : V(G) \rightarrow \{0, 1, \cdots, k-1\}$ be a $(k, d)$-coloring of $G$, and let $c' : V(H) \rightarrow \{0, 1, \cdots, n-1\}$ be a $\mathcal{P}$-$n$-coloring of $H$. Define $f : V(G[H]) \rightarrow \{0, 1, \cdots, \lceil \chi^*(G)\chi_{\mathcal{P}}(H) \rceil$ as follows:

$$f(g, h) = \lfloor \frac{c(g) + c'(h)k}{d} \rfloor.$$

As $\lfloor \frac{nk-1}{d} \rfloor \leq \lceil \frac{nk}{d} - 1 \rceil$, $f$ is a mapping of $V(G[H])$ to $\{0, 1, \cdots, \lceil \frac{nk}{d} - 1 \rceil\}$. We now show that for any color class $X = f^{-1}(i)$, $i = 0, 1, \cdots, \lceil \frac{nk}{d} - 1 \rceil$, the subgraph of $G[H]$ induced by $X$ is a member of $\mathcal{P}$. Indeed, if $f(g, h) = f(g', h')$ then $|(c(g) + c'(h)k) - (c(g') + c'(h')k)| < d$. This implies that either $c(g) = c(g')$ and $c'(h) = c'(h')$, or $0 < |c(g) - c(g')| < d$ and $c'(h) = c'(h')$, or $|c(g) - c(h)| > k - d$ and $|c'(h) - c'(h')| = 1$. In the second and third cases, $(g, h)$ is not adjacent to $(g', h')$, because $c$ is a $(k, d)$-coloring of $G$. In the first case, either $(g, h)$ and $(g', h')$ are not adjacent, or $(h, h')$ is a monochromatic edge with respect to the coloring $c'$. Therefore each connected monochromatic subgraph of $G[H]$ with respect to the coloring $f$ is contained in a copy of $H$, and is monochromatic with respect to the coloring $c'$. Since $c'$ is a $\mathcal{P}$-$n$-coloring of $H$, the mapping $f$ is indeed a $\mathcal{P}$-$\lceil \frac{nk}{d} \rceil$-coloring of $G[H]$. ∎

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