A BERRY-ESSÉEN BOUND FOR STUDENT'S STATISTIC IN THE NON-I.I.D. CASE

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Abstract. We establish the Berry – Esséen bound for Student's statistic for independent nonidentically distributed random variables. If the third moments of these r.v. exist this bound yields an estimate which coincides (up to an absolute constant) with Esséen's bound for nonstudentized sums. If Lindeberg's condition for these r.v. holds, the bound yields the Central Limit Theorem for Student's statistics.

1. Introduction and Results

Let $X_1, ..., X_N$ be independent random variables. Let

$$\mathbf{t} = \overline{X}/\hat{\sigma}$$

denote the Student statistic, where

$$\overline{X} = N^{-1}(X_1 + \cdots + X_N), \qquad \hat{\sigma}^2 = N^{-1}\sum_{i=1}^{N}(X_i - \overline{X})^2.$$

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We shall investigate the rate of the normal approximation of $\sqrt{N}\mathbf{t}$ when the random variables $X_1, ..., X_N$ are nonidentically distributed. Define

$$\delta_N := \sup_x |\mathbf{P}(\sqrt{N}\ \mathbf{t} < x) - \Phi(x)|,$$

where $\Phi(x)$ is the standard normal distribution function. Denote

$$B_N := \sum_{i=1}^N \mathbf{Var} X_i \qquad \text{and} \qquad L_N := B_N^{-3/2} \sum_{i=1}^N \mathbf{E}|X_i|^3.$$

Theorem 1.1. *Let $X_1, ..., X_N$ denote independent random variables with mean zero. There exists an absolute constant $c > 0$ such that*

$$\delta_N \le c L_N, \tag{1.1}$$

whenever B_N is positive.

Inequality (1.1) is an extension to the case of studentized sums of the classical bound by Esséen (1945)

$$\sup_x |\mathbf{P}\left(X_1 + \cdots + X_N < x\sqrt{B_N}\right) - \Phi(x)| \le c L_N.$$

Theorem 1.1 is a consequence of a general result formulated as Theorem 1.2.

For a given number $V > 0$ define

$$Z_i = V^{-1/2} X_i \mathbb{I}\{X_i^2 \le V\}, \qquad 1 \le i \le N, \quad \text{and} \qquad M = \sum_{i=1}^N \mathbf{Var} Z_i, \tag{1.2}$$

where $\mathbb{I}\{A\}$ denotes the indicator function of event A.

Theorem 1.2. *Let $X_1, X_2, ..., X_N$ denote independent random variables. There exists an absolute constant $c > 0$ such that*

$$\begin{aligned} \delta_N \le c \sum_{i=1}^N \mathbf{P}(X_i^2 > V) + c M^{-1/2} |\sum_{i=1}^N \mathbf{E} Z_i| \\ + c M^{-1} \sum_{i=1}^N (\mathbf{E} Z_i)^2 + c M^{-3/2} \sum_{i=1}^N \mathbf{E}|Z_i - \mathbf{E} Z_i|^3, \end{aligned} \tag{1.3}$$

for any $V > 0$.

Theorem 1.2 is an extension of a Berry-Esséen bound for Student's statistic in the i.i.d. case due to Bentkus and Götze (1994) (see Corollary 1.2 below).

In Example 2.1 (see the end of Section 2) we show that the term $M^{-1}\sum_{i=1}^N(\mathbf{E}Z_i)^2$ in the bound (1.3) is optimal in the sense that it can not be replaced by

$$\sum_{i=1}^N (M^{-1/2}\mathbf{E}Z_i)^{2+\delta} \quad \text{or by} \quad (M^{-1}\sum_{i=1}^N(\mathbf{E}Z_i)^2)^{1+\delta},$$

with some $\delta > 0$.

Corollary 1.1. *Let $X_1, X_2, ..., X_N$ denote independent random variables. There exists an absolute constant $c > 0$ such that*

$$\delta_N \le c\sum_{i=1}^N \mathbf{P}(X_i^2 > V) + cM^{-1/2}\sum_{i=1}^N |\mathbf{E}Z_i| + cM^{-3/2}\sum_{i=1}^N \mathbf{E}|Z_i|^3,$$

for any $V > 0$.

Let a denote the largest solution of the equation

$$a^2 = \mathbf{E}X_1^2 \mathbb{I}\{X_1^2 \le a^2 N\}. \tag{1.4}$$

Corollary 1.2 (Bentkus and Götze (1994)). *Let $X_1, ..., X_N$ be i.i.d. random variables. Assume that the largest solution a of the equation (1.4) is positive. There exists an absolute constant $c > 0$ such that*

$$\delta_N \le cN\mathbf{P}(X_1^2 \ge a^2 N) + cN|\mathbf{E}Z_1| + cN\mathbf{E}|Z_1|^3, \tag{1.5}$$

where

$$Z_1 = a^{-1}N^{-1/2}X_1\mathbb{I}\{X_1^2 \le a^2 N\}.$$

Notice, that the right-hand side of (1.5) tends to zero as $N \to \infty$ if and only if X_1 belongs to the domain of attraction of the standard normal law, see Bentkus and Götze (1994). Thus in the case when the random variables $X_1, X_2, ...$ are independent and identically distributed (1.5) yields the weakest known sufficient condition for the Central Limit Theorem for Student's $\mathbf{t}$ statistic (Maller (1981), Csörgő and Mason (1987)).

Let X_i, $i \ge 1$ be a sequence of independent random variables with mean zero. Assume that $\sigma_i^2 := \mathbf{E}X_i^2 < \infty$, for each $i \ge 1$. It is well known (see Feller (1971)) that the Lindeberg condition

$$\forall \varepsilon > 0 \qquad \frac{1}{B_N}\sum_{i=1}^N \mathbf{E}X_i^2\mathbb{I}\{X_i^2 \ge \varepsilon B_N^2\} \to 0 \tag{1.6}$$

is sufficient for

$$\mathcal{L}\left(\frac{X_1 + \cdots + X_N}{\sqrt{B_N}}\right) \to N(0,1), \tag{1.7}$$

and it is necessary for (1.7) provided that

$$\frac{\max_{1\le i\le N}\sigma_i^2}{B_N} \to 0. \tag{1.8}$$

Corollary 1.3. *Let $X_i, i \ge 1$ be a sequence of independent (nonidentically distributed) mean zero random variables. Assume that the sequence satisfies the Lindeberg condition (1.6) Then $\delta_N \to 0$, that is, $\mathcal{L}(\sqrt{N}\mathbf{t})$ converge to the standard normal distribution.*

Remark 1.1. Corollary 1.3 tells us that the Lindeberg condition is sufficient for

$$\mathcal{L}(\sqrt{N}\ \mathbf{t}) \to N(0,1). \tag{1.9}$$

It is interesting to note that the Lindeberg condition (1.6) is not necessary for (1.9) even if (1.8) holds. More specifically, there exists a sequence (see Example 1.1 below) of centered independent random variables which satisfies (1.8) and (1.9) and which fails to satisfy (1.6).

Example 1.1 (cf. Feller (1971), p. 521). Let $\tau_i,\ i \ge 1$, be a sequence of independent symmetric random variables with

$$\mathbf{P}(|\tau_i| = i^2) = 1 - \mathbf{P}(|\tau_i| = 0) = i^{-2}, \quad i \ge 1.$$

Let $\xi_i,\ i \ge 1$ be a sequence of independent standard normal random variables independent of $\tau_i,\ i \ge 1$. It is easy to see that the sequence of independent centered random variables $X_i := \tau_i + \xi_i$, $i \ge 1$ satisfies (1.7) and (1.9) and fails to satisfy the Lindeberg condition. In this case

$$\frac{X_1 + \cdots + X_N}{\sqrt{B_N}} \to_P 0.$$

Notice that (1.9) follows from the Borel-Cantelli lemma or Corollary 1.1 (with $V = V(N) = N$).

Thus we conclude that in case of finite variance and centered and asymptotically negligible summands in sense of (1.8) the studentization of the sum leads to better results than the standardization by means of $\sqrt{B_N}$. However in general a different situation arises. In the following example we construct a sequence $X_i,\ i \ge 1$ of independent random variables such that (1.7) holds and $\mathcal{L}(\sqrt{N}\mathbf{t}(X_1,...,X_N)) \nrightarrow N(0,1)$. Here $\mathbf{t}(X_1,...,X_N)$ denotes the Student statistic based on the observations $X_1,...,X_N$.

Example 1.2. Let $\xi_i,\ i \ge 1$ be the sequence of i.i.d. standard normal random variables. Let $p > 0$. Put

$$X_i = \frac{1}{\sqrt{i}}\,\xi_i + \frac{(-1)^i}{i^p}. \tag{1.10}$$

It is easy to see that $B_N \sim \ln(N+2)$, and

$$\mathcal{L}\left(\frac{X_1 + \cdots + X_N}{\sqrt{\ln(N+2)}}\right) = \mathcal{L}\left(\sqrt{\frac{\sum_{i=1}^N i^{-1}}{\ln(N+2)}} \cdot \xi_1 + \frac{\sum_{i=1}^N (-1)^i i^{-p}}{\sqrt{\ln(N+2)}}\right) \to \mathcal{L}(\xi_1)$$

converges to the standard normal distribution.

Let $V = V(N) = \log(N+2)$. In what follows we show that, for the sequence (1.10),

$$\mathcal{L}(\sqrt{N}\mathbf{t}(X_1,...,X_N)) \to N(0,1) \tag{1.11}$$

if and only if

$$\sum_{i=1}^{N}(\mathbf{E}Z_i)^2 \to 0, \tag{1.12}$$

where Z_i, $1 \le i \le N$ are defined in (1.2). A calculation shows that all terms other than $\sum_{i=1}^{N}(\mathbf{E}Z_i)^2$ in the right-hand side of (1.3) tend to 0 as $N \to \infty$, for each $p > 0$. Hence (1.12) implies (1.11). Notice that (1.12) holds if and only if $p > 1/2$. Therefore, in order to prove the converse implication, (1.11) $\Rightarrow$ (1.12) it suffices to show that

$$\sqrt{N}\mathbf{t}(X_1,...,X_N) \to_P 0, \qquad \text{for} \quad 0 < p < 1/2, \tag{1.13}$$

and that

$$\mathcal{L}(\sqrt{N}\mathbf{t}(X_1,...,X_N)) \to N(0,1/2), \qquad \text{for} \quad p = 1/2. \tag{1.14}$$

Proofs of (1.13) and (1.14) are given in Section 2.

Remark 1.2. Example 1.2 demonstrates that the limiting behaviour of the distribution of the studentized sum crucially depends on $\sum_{i=1}^{N}(\mathbf{E}Z_i)^2$. This observation can be generalized. It is well known, see Petrov (1975), that if X_i, $i \ge 1$ is a sequence of independent random variables and $V(N)$, $N \ge 1$ is a sequence of positive numbers such that,

$$\text{for each} \qquad \varepsilon > 0, \qquad \max_{1\le k\le N} \mathbf{P}\left(|X_k| > \varepsilon\sqrt{V(N)}\right) \to 0, \tag{1.15}$$

and

$$\mathcal{L}\left(\frac{X_1+\cdots+X_N}{\sqrt{V(N)}}\right) \to N(0,1) \tag{1.16}$$

then, for each $\varepsilon > 0$,

$$\sum_{i=1}^{N}\mathbf{P}(X_i^2 \ge \varepsilon^2 V(N)) \to 0, \quad \sum_{i=1}^{N}\mathbf{E}Z_i \to 0, \quad M \to 1, \quad \sum_{i=1}^{N}\mathbf{E}\,|Z_i - \mathbf{E}Z_i|^3 \to 0$$

as $N \to \infty$. Hence if (1.15) and (1.16) hold then (1.12) implies (1.11). A simple analysis shows that the converse implication (1.11) $\Rightarrow$ (1.12) is also true provided (1.15) and (1.16) holds. A similar effect was noticed by Robbins (1948). He considered the distribution of the Student statistic based on the independent normal observations $X_i \sim N(\mu_i, \sigma^2)$, $\sigma^2 > 0$, $1 \le i \le N$ with means μ_i, $1 \le i \le N$

satisfying $\sum_{i=1}^N \mu_i = 0$. In particular Robbins (1948) showed that in this case the distribution function depends only on N and $\lambda = \sum_{i=1}^N (\mu_i)^2/(2\sigma^2)$. In his results λ plays a role similar to that of $\sum_{i=1}^N (\mathbf{E} Z_i)^2$ in our results.

Remark 1.3. All the results stated above remain valid if instead of the standardized Student statistic $\sqrt{N}\mathbf{t}$ one considers the selfnormalized sums

$$\frac{X_1 + \dots + X_N}{\sqrt{X_1^2 + \dots + X_N^2}}.$$

In particular Theorems 1.1, 1.2 and Corollaries 1.1, 1.2, 1.3 hold with δ_N replaced by δ_N', where

$$\delta_N' := \sup_x \Big|\mathbf{P}\Big(\frac{X_1 + \dots + X_N}{\sqrt{X_1^2 + \dots + X_N^2}} < x\Big) - \Phi(x)\Big|.$$

Indeed, the uniform distance

$$\sup_x \Big|\mathbf{P}(\sqrt{N}\mathbf{t} < x) - \mathbf{P}\Big(\frac{X_1 + \dots + X_N}{\sqrt{X_1^2 + \dots + X_N^2}} < x\Big)\Big|$$

is bounded from above by the right-hand side of (1.3) (see the argument in the proof of Lemma 2.3 below).

The convergence of selfnormalized sums was investigated by a number of authors, see, for instance, Efron (1969), Logan, Mallows, Rice and Shepp (1973), LePage, Woodroofe and Zinn (1981), Griffin and Kuelbs (1989, 1991), Hahn, Kuelbs and Weiner (1990), Griffin and Mason (1991).

Convergence rates and Edgeworth expansions for Student's and related statistics were considered by Chung (1946), Chibisov (1980), Helmers and van Zwet (1982), van Zwet (1984), Helmers (1985), Slavova (1985), Bhattacharya and Ghosh (1986), Hall (1987, 1988), Praskova (1989), Friedrich (1989), Bhattacharya and Denker (1990), Bentkus, Götze and van Zwet (1994), Bentkus and Götze (1994), etc.

In the case when random variables $X_1, ..., X_N$ are identically distributed the rate $O(N^{-1/2})$ under the 3-rd moment condition was obtained by Slavova (1985). The bound $\delta_N \le cN^{-1/2}\beta_3/\sigma^3$, where $\beta_s = \mathbf{E}|X_1|^s$ and $\sigma^2 = \mathbf{E}X_1^2$, follows from a general result of Bentkus and Götze (1994). A related result $O(N^{-1/2})$ under the additional condition: there exists $y \in \mathbf{R}$ such that

$$\mathbf{E}p(y)(1 - p(y)) > 0, \quad \text{with} \quad p(y) = \mathbf{P}\{X - y|\,|X - y|\}$$

was proved by Hall (1988) (see Bai and Shepp (1994) for a discussion of such condition).

Most of the papers mentioned above deal with the case of independent and *identically distributed* random variables. Friedrich (1989) constructed a Berry-Esséen bound for a general class of statistics without the i.i.d. assumption. In the simplest case, when the random variables are i.i.d. the result of Friedrich (1989) yields $\delta_N = O(N^{-1/2})$ provided $\beta_{10/3}$ is finite, whereas our result implies $\delta_N \le cN^{-1/2}\beta_3/\sigma^3$.

2. Proofs

The proofs are close to those given in Bentkus and Götze (1994). By means of truncation we replace the Student statistic $\mathbf{t}(X_1, ...X_N)$ by a smooth function $S = S(X_1, ..., X_N)$ of the sample (see Lemma 2.3 below). Then we apply Esséen's smoothing inequality. When estimating the difference between the characteristic functions $\varphi(t) = \exp\{-t^2/2\}$ and $f(t) = \mathbf{E}\exp\{itS\}$ we split f into a conditional product of two characteristics functions (different for different values of t) of conditionally independent random variables. This type of approach has been used by Helmers and van Zwet (1982), van Zwet (1984) and was extended in Götze and van Zwet (1992), Bentkus, Götze and van Zwet (1994), to the case of general symmetric statistics of i.i.d. random variables. The crucial step of the proof in the nonidentically distributed case is to bound the characteristic function of the non i.i.d. sum (which may contain, for example, only one summand) by a characteristic function which splits into a product. For this purpose we use a randomization by means of Bernoulli random variables.

Proof of Theorem 1.1. Choose $V = B_N$. Then

$$\sum_{i=1}^{N} \mathbf{P}(X_i^2 > B_N) \le L_N, \quad \sum_{i=1}^{N} |\mathbf{E}Z_i| \le L_N, \quad \sum_{i=1}^{N} \mathbf{E}|Z_i|^3 \le L_N.$$

Without loss of generality we may assume that $L_N \le 1/4$. Then

$$\begin{aligned} 1 \ge M &= \sum_{i=1}^{N} (\mathbf{E}Z_i^2 - (\mathbf{E}Z_i)^2) \\ &\ge 1 - B_N^{-1} \sum_{i=1}^{N} \mathbf{E}X_i^2 \mathbb{I}\{X_i^2 \ge B_N\} - \sum_{i=1}^{N} |\mathbf{E}Z_i| \ge 1 - 2L_N \ge 1/2. \end{aligned}$$

Now Theorem 1.1 follows from Corollary 1.1.

Proof of Corollary 1.1. We may and shall assume that $\sum_{i=1}^{N} |M^{-1/2}\mathbf{E}Z_i| \le 1/2$. Then

$$\sum_{i=1}^{N} (M^{-1/2}|\mathbf{E}Z_i|)^3 \le \sum_{i=1}^{N} M^{-1/2}|\mathbf{E}Z_i|$$

and

$$\sum_{i=1}^{N} (M^{-1/2}\mathbf{E}Z_i)^2 \le \sum_{i=1}^{N} M^{-1/2}|\mathbf{E}Z_i|.$$

Hence, Corollary 1.1 follows from Theorem 1.2.

Proof of Corollary 1.2. Here we may assume that $\sum_{i=1}^N |\mathbf{E}Z_i| \le 1/2$. Then

$$1 \ge M = \sum_{i=1}^N (\mathbf{E}Z_i^2 - (\mathbf{E}Z_i)^2) = 1 - \sum_{i=1}^N (\mathbf{E}Z_i)^2 \ge 1 - \sum_{i=1}^N |\mathbf{E}Z_i| \ge 1/2.$$

Hence, Corollary 1.2 follows from Corollary 1.1.

Proof of Corollary 1.3. Choose $V = B_N$. It follows from (1.6) that

$$\sum_{i=1}^N |\mathbf{E}Z_i| \to 0, \qquad \sum_{i=1}^N \mathbf{P}(X_i^2 \ge B_N) \to 0, \quad \text{as} \quad N \to +\infty$$

and, for given $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{i=1}^N \mathbf{E}|Z_i|^3 \\
&\quad = \sum_{i=1}^N B_N^{-3/2}\mathbf{E}|X_i|^3 \mathbb{I}\{X_i^2 \le \varepsilon^2\ B_N\} + B_N^{-3/2}\mathbf{E}|X_i|^3 \mathbb{I}\{\varepsilon^2 B_N \le X_i^2 \le B_N^2\} \\
&\quad \le \varepsilon B_N^{-1} \sum_{i=1}^N \mathbf{E}X_i^2 + B_N^{-1} \sum_{i=1}^N \mathbf{E}X_i^2 \mathbb{I}\{X_i^2 \ge \varepsilon^2 B_N\} \\
&\quad = \varepsilon + o(1), \quad \text{as} \quad N \to +\infty.
\end{aligned}$$

In order to prove the corollary it suffices to show that $M \to 1$. Arguing in the same way as in the proof of Theorem 1.1 we have

$$1 \ge M \ge 1 - B_N^{-1} \sum_{i=1}^N \mathbf{E}X_i^2 \mathbb{I}\{X_i^2 \ge B_N\} - \sum_{i=1}^N |\mathbf{E}Z_i| = 1 - o(1)$$

as $N \to +\infty$. Now Corollary 1.3 follows from Corollary 1.1.

It remains to prove Theorem 1.2. Define

$$Y_i = M^{-1/2}(Z_i - \mathbf{E}Z_i), \quad 1 \le i \le N.$$

Note that $\sum_{i=1}^N \mathbf{E}Y_i^2 = 1$.

We may assume throughout the rest of the paper that for a small absolute constant $c_0 > 0$

$$\sum_{i=1}^N \mathbf{E}|Y_i|^3 \le c_0, \quad M^{-1}\sum_{i=1}^N (\mathbf{E}Z_i)^2 \le c_0, \quad M^{-1/2}|\sum_{i=1}^N \mathbf{E}Z_i| \le c_0, \tag{2.1}$$

since otherwise the result follows from the obvious estimate $\delta_N \le 1$. Define

$$Y := Y_1 + \cdots + Y_N, \qquad \eta := \eta_1 + \cdots + \eta_N, \qquad \text{where} \qquad \eta_i := Y_i^2 - \mathbf{E}Y_i^2.$$

Let $\alpha = (\alpha_1, ..., \alpha_N)$ denote a sequence of i.i.d. Bernoulli random variables such that $\mathbf{P}(\alpha_1 = 1) = 1 - \mathbf{P}(\alpha_1 = 0) = m$, with some $m \le 1$. Assume that the random variables α and Y_i, $1 \le i \le N$ are independent.

Lemma 2.1. *Assume that (2.1) holds. Let $\kappa_i := \alpha_i(\eta_i + M^{-1/2} Y_i \mathbf{E} Z_i)$, for $1 \le i \le N$. There exists an absolute constant $c > 0$ such that*

$$\mathbf{E}|\sum_{i=1}^{N} \alpha_i Y_i \kappa_i| \le cm \cdot \left(\sum_{i=1}^{N} \mathbf{E}\,|Y_i|^3 + M^{-1} \sum_{i=1}^{N} (\mathbf{E} Z_i)^2 \right),$$

$$\mathbf{E}|\sum_{i=1}^{N} \kappa_i|^{3/2} \le cm \cdot \left(\sum_{i=1}^{N} \mathbf{E}\,|Y_i|^3 + M^{-1} \sum_{i=1}^{N} (\mathbf{E} Z_i)^2 \right).$$

If

$$U = \sum_{\{1 \le i,j \le N,\ i \ne j\}} \alpha_i Y_i \kappa_j$$

then

$$\mathbf{E}|U|^{3/2} \le cm^{1+3/4} \cdot \left(\sum_{i=1}^{N} \mathbf{E}|Y_i|^3 + M^{-1} \sum_{i=1}^{N} (\mathbf{E} Z_i)^2 \right).$$

Proof of Lemma 2.1. We shall apply the following well known inequality. If $T_1, ..., T_N$ are independent mean zero random variables, then

$$\mathbf{E} \left| \sum_{i=1}^{N} T_i \right|^p \le c(p) \sum_{i=1}^{N} \mathbf{E}\,|T_i|^p, \qquad \text{for} \quad 1 \le p \le 2.$$

We have

$$\mathbf{E}|\sum_{i=1}^{N} \alpha_i Y_i \kappa_i| \le cm \cdot \sum_{i=1}^{N} \left(\mathbf{E}|Y_i \eta_i| + \mathbf{E} Y_i^2 |M^{-1/2} \mathbf{E} Z_i| \right)$$

$$\le cm \cdot \left(\sum_{i=1}^{N} \mathbf{E}\,|Y_i|^3 + M^{-1} \sum_{i=1}^{N} (\mathbf{E} Z_i)^2 \right).$$

Here we applied the inequality $a^{2/3} b^{1/3} \le a + b$ with $\mathbf{E} Y_i^2 \le (\mathbf{E}|Y_i|^3)^{2/3} =: a^{2/3}$ and $M^{-1/2}|\mathbf{E} Z_i| =: b^{1/3}$ and the bound $M^{-1/2}|\mathbf{E} Z_i| \le 1$ which follows from (2.1).
Furthermore,

$$\mathbf{E}|\sum_{i=1}^{N} \alpha_i \kappa_i|^{3/2}$$

$$\le cm \cdot \left(\sum_{i=1}^{N} \mathbf{E}\,|\eta_i|^{3/2} + \sum_{i=1}^{N} |\mathbf{E} Y_i|^{3/2} |M^{-1/2} \mathbf{E} Z_i|^{3/2} \right).$$

$$\le cm \cdot \left(\sum_{i=1}^{N} \mathbf{E}\,|Y_i|^3 + M^{-1} \sum_{i=1}^{N} (\mathbf{E} Z_i)^2 \right).$$

In the last step we applied the inequality $a \cdot b \le a^2 + b^2$ with $a := \mathbf{E}|Y_i|^{3/2}$ and $b := |M^{-1/2}\mathbf{E}Z_i|^{3/2}$ and the bound $|M^{-1/2}\mathbf{E}Z_i| \le 1$ which follows from (2.1).

Let $f(u) = |u|^{3/2}$. We have

$$\mathbf{E}|U|^{3/2} = \mathbf{E}f(U) = \mathbf{E}f(U_1 + U_2) \le 2^{3/2}\mathbf{E}f(U_1) + 2^{3/2}\mathbf{E}f(U_2),$$

where

$$U_1 = \sum_{1 \le i < j \le N} \alpha_i Y_i \kappa_j, \qquad U_2 = \sum_{1 \le j < i \le N} \alpha_i Y_i \kappa_j.$$

Let us estimate $\mathbf{E}f(U_1)$. One may write

$$U_1 = U_{1,2} + \cdots + U_{1,N}, \quad \text{where} \quad U_{1,j} = \kappa_j(\alpha_1 Y_1 + \cdots + \alpha_{j-1}Y_{j-1}), \quad 2 \le j \le N.$$

Since, for some absolute constant $c > 0$, $\quad |f'(s) - f'(t)| \le c|t - s|^{1/2}$ we have

$$\begin{aligned} f(U_{1,2} &+ \cdots + U_{1,N-1} + U_{1,N}) \\ &= f(U_{1,2} + \cdots U_{1,N-1}) + f'(U_{1,2} + \cdots + U_{1,N})U_{1,N} + R \end{aligned}$$

with $|R| \le c|U_{1,N}|^{3/2}$. Taking the expectation we obtain

$$|\mathbf{E}f(U_1) - \mathbf{E}f(U_{1,2} + \cdots + U_{1,N-1})| \le \mathbf{E}|R| \le c\mathbf{E}|U_{1,N}|^{3/2},$$

since

$$\mathbf{E}f'(U_{1,2} + \cdots + U_{1,N-1})U_{1,N} = \mathbf{E}f'(U_{1,2} + \cdots + U_{1,N-1})\mathbf{E}_{\eta_N}U_{1,N} = 0,$$

where $\mathbf{E}_{\eta_N}$ denotes the conditional expectation given all random variables but η_N. Proceeding in the same way we arrive at

$$\begin{aligned} |\mathbf{E}f(U_1)| &\le c\sum_{j=2}^{N} \mathbf{E}|U_{1,j}|^{3/2} \\ &= c\sum_{j=2}^{N} \mathbf{E}|\kappa_j|^{3/2}\mathbf{E}|\alpha_1 Y_1 + \cdots + \alpha_{j-1}Y_{j-1}|^{3/2} \\ &\le c\sum_{j=2}^{N} \mathbf{E}|\kappa_j|^{3/2} \left(\mathbf{E}(\alpha_1 Y_1 + \cdots + \alpha_{j-1}Y_{j-1})^2\right)^{3/4} \\ &\le cm^{1+3/4}\sum_{j=1}^{N} \mathbf{E}|\kappa_j|^{3/2}. \end{aligned}$$

Similarly,

$$|\mathbf{E}f(U_2)| \le cm^{1+3/4} \sum_{j=1}^{N-1} \mathbf{E}|\kappa_j|^{3/2}.$$

Hence,

$$\mathbf{E}|U|^{3/2} \le cm^{1+3/4} \sum_{j=1}^{N} \mathbf{E}|\kappa_j|^{3/2} \le cm^{1+3/4} \left(\sum_{j=1}^{N} \mathbf{E}|Y_j|^3 + M^{-1} \sum_{j=1}^{N} (\mathbf{E}Z_i)^2 \right).$$

Lemma 2.2. *Let β be a random variable with finite third moment. Then*

$$\left|\mathbf{E}\exp\{it\beta\}\right|^2 \le 1 - t^2\mathbf{E}\beta^2 + \frac{4}{3}\,|t|^3\mathbf{E}|\beta|^3. \tag{2.2}$$

Let α_1 be a Bernoulli random variable with $\mathbf{P}(\alpha_1 = 1) = 1 - \mathbf{P}(\alpha_1 = 0) = m$. Then

$$\mathbf{E}\big(1 - \alpha_1 t^2\mathbf{E}\beta^2 + \alpha_1|t|^3\,\frac{4}{3}\,\mathbf{E}|\beta|^3\big)^{1/2} \le \exp\big\{-mt^2\,\frac{1}{2}\,\mathbf{E}\beta^2 + |t|^3 m\,\frac{2}{3}\,\mathbf{E}|\beta|^3\big\}. \tag{2.3}$$

Proof of Lemma 2.2. Proof of (2.2) is easy and can be found in Petrov (1975). Formula (2.3) is a simple consequence of Jensen's inequality.

Let $g : R \to R$ denote a function which is infinitely many times differentiable with bounded derivatives and such that

$$\frac{1}{8} \le g(x) \le 2, \quad \text{for all} \quad x \in R, \quad \text{and} \quad g(x) = \frac{1}{\sqrt{|x|}}, \quad \text{for} \quad \frac{1}{4} \le x \le \frac{7}{4}.$$

Define the statistic $S = Yg(1 + \eta + 2A + B)$, where

$$A := M^{-1/2} \sum_{i=1}^{N} Y_i\mathbf{E}Z_i \qquad \text{and} \qquad B := M^{-1} \sum_{i=1}^{N} (\mathbf{E}Z_i)^2.$$

Let ξ denote a standard normal random variable.

Lemma 2.3. *Assume that (2.1) holds. There exists an absolute constant $c > 0$ such that*

$$\delta_N \le \sup_x |\mathbf{P}(S < x) - \mathbf{P}(\xi < x)| + R, \tag{2.4}$$

where

$$R = c\sum_{i=1}^{N} \mathbf{P}(X_i^2 > V) + c\sum_{i=1}^{N} \mathbf{E}|Y_i|^3 + cM^{-1/2}|\sum_{i=1}^{N} \mathbf{E}Z_i| + cM^{-1} \sum_{i=1}^{N} (\mathbf{E}Z_i)^2.$$

Proof of Lemma 2.3. The event

$$\sqrt{N}\mathbf{t}(X_1,...,X_N) \neq \sqrt{N}\mathbf{t}(Z_1,...,Z_N)$$

has probability less than $\sum_{i=1}^N \mathbf{P}(X_i^2 > V)$. It is easy to verify that

$$\sqrt{N}\mathbf{t}(Z_1,...,Z_N) = \frac{Y+C}{\sqrt{1+\eta+2A+B+N^{-1}(Y+C)^2}},$$

where A and B are defined above, $C = M^{-1/2}\sum_{i=1}^N \mathbf{E}Z_i$. The event

$$\sqrt{N}\mathbf{t}(Z_1,...,Z_N) \neq (Y+C)g\big(1+\eta+2A+B+N^{-1}(Y+C)^2\big)$$

has probability less than

$$\begin{aligned}
&\mathbf{P}\big(|\eta+2A+B+N^{-1}(Y+C)^2|>1/4\big)\\
&\le \mathbf{P}\big(|\eta+2A+2N^{-1}Y^2|>1/4-B-2N^{-1}C^2\big)\\
&\le \mathbf{P}\big(|\eta+2A+2N^{-1}Y^2|>1/8\big).
\end{aligned} \tag{2.5}$$

Here we applied (2.1) using $|B-2N^{-1}C^2| \le \frac{1}{8}$. The probability in (2.5) does not exceed

$$\mathbf{P}(|\eta|>1/24)+\mathbf{P}(|2A|>1/24)+\mathbf{P}(2N^{-1}Y^2>1/24).$$

We have

$$\mathbf{P}(|\eta| > \frac{1}{24}) \le (24)^{3/2}\mathbf{E}|\eta|^{3/2} \le c\sum_{i=1}^N \mathbf{E}|Y_i|^3;$$

$$\mathbf{P}(|2A| > \frac{1}{24}) \le (48)^2\mathbf{E}A^2 \le cM^{-1}\sum_{i=1}^N(\mathbf{E}Z_i)^2;$$

$$\mathbf{P}(2N^{-1}Y^2 > \frac{1}{24}) \le 48N^{-1}\mathbf{E}Y^2 = \frac{48}{N} \le 48\sum_{i=1}^N \mathbf{E}|Y_i|^3.$$

In the last step we used the fact that

$$\sum_{i=1}^N \mathbf{E}|Y_i|^3 \ge \frac{1}{\sqrt{N}} \qquad \text{provided} \qquad \sum_{i=1}^N \mathbf{E}Y_i^2 = 1. \tag{2.6}$$

Denote
$$S' := (Y+C)\cdot g\big(1+\eta+2A+B+\frac{1}{N}(Y+C)^2\big).$$
We see that the event $\sqrt{N}\mathbf{t}\neq S'$ has probability less than
$$c\sum_{i=1}^{N}\mathbf{P}(X_i^2>V)+c\sum_{i=1}^{N}\mathbf{E}\,|Y_i|^3+cM^{-1}\sum_{i=1}^{N}(\mathbf{E}Z_i)^2.$$
Hence we obtain (2.4) but with S' instead of S. Expanding g in powers of $N^{-1}(Y+C)^2$ we obtain

$$S' =$$
$$= Yg\left(1+\eta+2A+B+N^{-1}(Y+C)^2\right)+Cg\left(1+\eta+2A+B+N^{-1}(Y+C)^2\right)$$
$$= Yg(1+\eta+2A+B)+R=S+R,$$
where
$$|R|\le R_1+R_2,\quad R_1=c\cdot|C|,\quad R_2=\frac{c}{N}|Y|\cdot(Y+C)^2.$$
Further,
$$\sup_x|\mathbf{P}(S'<x)-\mathbf{P}(\xi<x)|\le\cdot\sup_x|\mathbf{P}(S<x)-\mathbf{P}(\xi<x)|$$
$$+\sup_x\mathbf{P}(\xi\in[x,x+2R_1+2N^{-1/2}])+\mathbf{P}(R_2\ge N^{-1/2}).$$
Chebyshev's inequality, (2.1) and lemma 2.1 give
$$\mathbf{P}(R_2>N^{-1/2})\le cN^{-1/2}\big(\mathbf{E}|Y|\cdot C^2+\mathbf{E}|Y|^3\big)\le cN^{-1/2}.$$
Estimating
$$\sup_x\mathbf{P}(\xi\in[x,x+2R_1+\frac{2}{\sqrt{N}}])\le cR_1+\frac{c}{\sqrt{N}}=cM^{-1/2}|\sum_{i=1}^{N}\mathbf{E}Z_i|+\frac{c}{\sqrt{N}}$$
and using (2.6) we obtain the lemma.

Let $\xi,\ \xi_1^0,\ \xi_2^0,\ \dots$ denote a sequence of independent standard normal random variables. Define
$$\xi_i=(\mathbf{E}Y_i^2)^{1/2}\cdot\xi_i^0,\ 1\le i\le N.$$
In what follows let $\theta,\theta_1,\theta_2,...$ denote real numbers which satisfy $|\theta|\le 1$, $|\theta_i|\le 1$ and which can be different in different places. For a vector valued differentiable function H we will use the following mean value formula $H(b)-H(a)=\mathbf{E}H'\left(a+\hat\theta(b-a)\right)(b-a)$, $a,\ b\in\mathbf{R}$, where $\hat\theta$ is a random variable uniformly distributed in $[0,1]$. We shall use the notation
$$\mathbf{E}_\zeta F(\zeta,\beta)=\mathbf{E}(F(\zeta,\beta)|\beta),\qquad \mathbf{E}^\zeta F(\zeta,\beta)=\mathbf{E}(F(\zeta,\beta)|\zeta),$$
for conditional expectations. Furthermore, if Q denotes the sum $q_1+\cdots+q_N$ then $Q^{i,j}:=Q-q_i-q_j$. Similarly, $Q^i:=Q-q_i$.

Lemma 2.4. *Assume that (2.1) holds. Let* $H : \mathbf{R} \to \mathbf{C}$ *denote an infinitely many times differentiable function with bounded derivatives. Then, for* $1 \le k \le N$

$$
\begin{aligned}
&|\mathbf{E}H((Y_1 + \cdots + Y_k)g(1 + \kappa + B)) - \mathbf{E}H(\xi_1 + \cdots + \xi_k)| \\
&\le c \cdot c_H \left(\sum_{i=1}^{k} \mathbf{E}|Y_i|^3 + M^{-1} \sum_{i=1}^{k} \mathbf{E}Z_i^2 \right)
\end{aligned}
$$

where

$$
\kappa = \sum_{i=1}^{k} \kappa_i, \quad \kappa_i = \eta_i + 2M^{-1/2} Y_i \mathbf{E}Z_i, \quad 1 \le i \le N,
$$

$$
c_H = \|H'\|_\infty + \|H''\|_\infty + \|H'''\|_\infty \quad \text{and} \quad \|H\|_\infty = \sup_x |H(x)|.
$$

Proof of Lemma 2.4. We shall prove the lemma for $k = N$ only. It suffices to prove that

$$
|\mathbf{E}H(Yg(1 + \kappa + B)) - \mathbf{E}H(Yg(1 + \kappa)| \le c c_H M^{-1} \sum_{i=1}^{N} (\mathbf{E}Z_i)^2,
$$

$$
|\mathbf{E}H(Yg(1 + \kappa)) - \mathbf{E}H(Y)| \le c \cdot c_H \sum_{i=1}^{N} \mathbf{E}|Y_i|^3 + c c_H M^{-1} \sum_{i=1}^{N} (\mathbf{E}Z_i)^2
$$

and

$$
|\mathbf{E}H(Y) - \mathbf{E}H(\xi)| \le c \cdot c_H \sum_{i=1}^{N} \mathbf{E}|Y_i|^3.
$$

Expanding g in powers of B and then expanding H we obtain the first inequality. The proof of the third inequality is easy and does not differ from that of given in Bentkus et al. (1990) where the i.i.d. case was considered.

Let us prove the second inequality. Expanding g in powers of κ we have

$$
\begin{aligned}
Yg(1 + \kappa) &= Yg(1) + Y\kappa g'(1 + \theta_1 \kappa) \\
&= Y + Dg'(1 + \theta_1 \kappa) + Ug'(1 + \theta_1 \kappa)
\end{aligned}
$$

Here we split $Y\kappa$ into the sum $D + U$, where

$$
D = \sum_{i=1}^{N} Y_i \kappa_i, \qquad U = \sum_{1 \le i,j \le N, \ i \ne j} Y_i \kappa_j.
$$

Expanding H in powers of $Dg'(1+\theta_1\kappa)$ we obtain

$$\mathbf{E}H(Yg(1+\kappa)) = \mathbf{E}H(Y+Ug'(1+\theta_1\kappa)) + R$$

with $|R| \le cc_H \mathbf{E}\,|D|$. An application of Lemma 2.1 gives

$$\mathbf{E}|R| \le cc_H \sum_{i=1}^N \mathbf{E}|Y_i|^3 + cc_H M^{-1} \sum_{i=1}^N (\mathbf{E}Z_i)^2.$$

Expanding

$$Ug'(1+\theta_1\kappa) = Ug'(1) + U\theta_1\kappa g''(1+\theta\theta_1\kappa)$$

and applying the inequality $|H(s)-H(t)| \le c_H|t-s|^{3/4}$ we get

$$\mathbf{E}H(Y+Ug'(1+\theta_1\kappa)) = \mathbf{E}H(Y+Ug'(1)) + R$$

with

$$|R| \le c\mathbf{E}(|U|\cdot|\kappa|)^{3/4} \le c\mathbf{E}|U|^{3/2} + c\mathbf{E}|\kappa|^{3/2} \le cc_H \sum_{i=1}^N \mathbf{E}|Y_i|^3 + cc_H M^{-1} \sum_{i=1}^N (\mathbf{E}Z_i)^2.$$

In the last step we applied the inequality $ab \le a^2+b^2$ with $a = |U|^{3/4}$ and $b = |\kappa|^{3/4}$ and Lemma 2.1. Since H, H', H'' are uniformly bounded one may write

$$\mathbf{E}H(Y+Ug'(1)) = \mathbf{E}H(Y) + \mathbf{E}H'(Y)Ug'(1) + R,$$

with

$$|R| \le cc_H \mathbf{E}|U|^{3/2} \le cc_H \sum_{i=1}^N \mathbf{E}\,|Y_i|^3 + cc_H M^{-1} \sum_{i=1}^N (\mathbf{E}Z_i)^2.$$

Further, expanding $H'(Y) = H'(Y^i + Y_i)$ in powers of Y_i we get

$$\begin{aligned}
\mathbf{E}H'(Y)U &= \mathbf{E} \sum_{1\le i,j\le N,\ i\ne j} Y_i\kappa_j H'(Y^i+Y_i) \\
&= \mathbf{E} \sum_{1\le i,j\le N,\ i\ne j} Y_i\kappa_j [H'(Y^i) + H''(Y^i)Y_i + H'''(Y^i + \hat\theta_2\hat\theta_1 Y_i)\hat\theta_1 Y_i^2] \\
&= R_1 + R_2,
\end{aligned}$$

where $\hat\theta_1$ and $\hat\theta_2$ are mutually independent (and independent of $Y_1, ..., Y_N$) random variables uniformly distributed in $[0,1]$,

$$|R_1| = |\mathbf{E} \sum_{1\le i,j\le N,\ i\ne j} Y_i^2 \kappa_j H''(Y^i)|,$$

$$|R_2| \le cc_H \sum_{1\le i,j\le N} \mathbf{E}\,|Y_i|^3\,\mathbf{E}\,|\kappa_j| \le cc_H \sum_{i=1}^N \mathbf{E}\,|Y_i|^3.$$

Here we used

$$\sum_{j=1}^{N} \mathbf{E}\,|\kappa_j| \le \sum_{j=1}^{N} \mathbf{E}\,|\eta_j| + \sum_{j=1}^{N} \mathbf{E}\,|Y_j|\,M^{-1/2}\,|\mathbf{E}Z_j|$$
$$\le c\sum_{j=1}^{N} \mathbf{E}(Y_j)^2 + \sum_{j=1}^{N} (\mathbf{E}Y_j^2 + (M^{-1/2}\mathbf{E}Z_j)^2) \le c.$$

Let us write $H''(Y^i) = H''(Y^{i,j} + Y_j)$. Expanding H'' in powers of Y_j we have

$$|R_1| = |\mathbf{E} \sum_{1\le i,j\le N,\ i\ne j} Y_i^2 \kappa_j [H''(Y^{i,j}) + Y_j H'''(Y^{i,j} + \theta Y_j)]|$$
$$= |\mathbf{E} \sum_{1\le i,j\le N,\ i\ne j} Y_i^2 \kappa_j Y_j H'''(Y^{i,j} + \hat\theta Y_j)| \le c c_H \sum_{i=1}^{N} \mathbf{E}Y_i^2 \cdot \sum_{j=1}^{N} \mathbf{E}\,|\kappa_j Y_j|$$
$$\le c c_H \sum_{j=1}^{N} \mathbf{E}\,|\kappa_j Y_j| \le c c_H \sum_{i=1}^{N} \mathbf{E}\,|Y_i|^3 + c c_H M^{-1} \sum_{i=1}^{N} (\mathbf{E}Z_i)^2.$$

Thus proving the lemma.

Proof of Theorem 1.2.
Without loss of generality we shall assume that (2.1) is fulfilled. An application of Lemma 2.3 reduces the proof to the verification of the inequality

$$\sup_x |\mathbf{P}(S < x) - \mathbf{P}(\xi < x)| \le c\mathcal{R}, \quad \text{with} \quad \mathcal{R} = \sum_{i=1}^{N} \mathbf{E}|Y_i|^3 + M^{-1} \sum_{i=1}^{N} (\mathbf{E}Z_i)^2, \tag{2.7}$$

assuming that (2.1) holds.
In order to prove (2.7) let us apply the Berry -Esséen inequality for characteristic functions. Write

$$f(t) = \mathbf{E}\exp\{itS\} = \mathbf{E}\exp\{itYg(1+\eta)\}, \quad \varphi(t) = \mathbf{E}\exp\{it\xi\} = \exp\{-t^2/2\}.$$

Estimating

$$\int_{|t|\le C_1} |f(t) - \varphi(t)|/|t|dt \le c\mathcal{R}$$

by Lemma 2.4 we see that (2.7) is a consequence of

$$\int_{C_1 \le |t| \le T} |f(t) - \varphi(t)|/|t|dt \le c\mathcal{R}, \tag{2.8}$$

where $T = c_1/(\sum_{i=1}^N \mathbf{E}|Y_i|^3)$. Here we may choose the absolute constant C_1 sufficiently large, and the absolute constant c_1 sufficiently small. We may assume that the interval (C_1, T) is nonempty since otherwise the theorem follows from (2.1).

Define the function

$$m(t) = C_2 \frac{\ln t}{t^2} < \frac{1}{2}, \quad \text{for} \quad C_1 \le |t| \le T,$$

where C_2 is a sufficiently large absolute constant. Throughout the proof we shall write $h \simeq g$ if

$$\int_{C_1 \le |t| \le T} |h(t) - g(t)|/|t| dt \le c\mathcal{R}.$$

In particular, (2.8) is equivalent to $f \simeq \varphi$. We shall prove the inequality (2.8) in two steps. In the first step using randomization by means of Bernoulli random variables with the parameter $m(t)$ we shall split f into a product of two conditionally independent characteristics functions. The first characteristics function will respond the contribution of the $m(t)$–th part of the sum $Y_1 + \cdots + Y_N$ and will ensure the convergence of the integral, cf. Bentkus and Götze (1994).

Step 1. Let $(\overline{Y}_1, ..., \overline{Y}_N)$ denote an independent copy of $(Y_1, ..., Y_N)$. Recall that $\alpha_1, ..., \alpha_N$ denotes a sequence of i.i.d. Bernoulli random variables, assume that $\mathbf{P}(\alpha_1 = 1) = 1 - \mathbf{P}(\alpha_1 = 0) = m(t)$ and assume that all these random variables are independent. Then

$$f(t) = \mathbf{E} \exp\{it(X + Z)g(1 + \gamma + \rho)\},$$

where

$$\begin{aligned}
X &= \alpha_1 Y_1 + \cdots + \alpha_N Y_N, \quad Z = (1 - \alpha_1)\overline{Y}_1 + \cdots + (1 - \alpha_N)\overline{Y}_N, \\
\gamma &= \sum_{i=1}^N \alpha_i \gamma_i, \quad \text{with} \quad \gamma_i = \eta_i + 2M^{-1/2} Y_i \mathbf{E} Z_i, \\
\rho &= \sum_{i=1}^N (1 - \alpha_i)\rho_i + B, \quad \text{with} \quad \rho_i = \overline{\eta}_i + 2M^{-1/2} \overline{Y}_i \mathbf{E} Z_i, \\
\overline{\eta}_i &= \overline{Y}_i^2 - \mathbf{E}\overline{Y}_i^2, \quad 1 \le i \le N.
\end{aligned}$$

Let us show that

$$f(t) \simeq f_2(t) = \mathbf{E} \exp\{it[Xg(1 + \rho) + Ug'(1 + \rho) + Zg(1 + \rho) + Z\gamma g'(1 + \rho)]\},$$

where

$$U = \sum_{\{1 \le j, k \le N, \ j \ne k\}} \alpha_j Y_j \alpha_k \gamma_k.$$

Expanding $g(1+\gamma+\rho)$ in powers of γ

$$\begin{aligned}(X+Z)g(1+\gamma+\rho) &= Xg(1+\rho)+X\gamma g'(1+\rho+\theta_1\gamma)\\ &\quad + Zg(1+\rho)+Zg'(1+\rho)\gamma+Zg''(1+\rho+\theta_2\gamma)\theta_3\gamma^2;\\ X\gamma g'(1+\rho+\theta_1\gamma) &= (D+U)g'(1+\rho+\theta_1\gamma)\\ &= Dg'(1+\rho+\theta_1\gamma)+Ug'(1+\rho)+U\theta_1\gamma g''(1+\rho+\theta_4\gamma)\end{aligned}$$

where $D=\sum_{i=1}^N \alpha_i Y_i\gamma_i$, and then expanding the exponent in powers of $itDg'(1+\rho+\theta_1\gamma)$ we have $f(t)=f_1(t)+R$. Here

$$\begin{aligned}&f_1(t)=\mathbf{E}\exp\{it[Xg(1+\rho)+Ug'(1+\rho)+Zg(1+\rho)+Z\gamma g'(1+\rho)]+itr\}+R,\\ &\text{with}\quad r=Zg''(1+\rho+\theta_2\gamma)\theta_3\gamma^2+U\theta_1\gamma g''(1+\rho+\theta_4\gamma)\end{aligned}$$

and

$$|R|\le c|t|\mathbf{E}|D|\le c|t|m(t)\mathcal{R}.$$

In the last estimate we applied Lemma (2.1). Since the function $u\to\exp\{iu\}$ and it's derivatives are uniformly bounded there exists an absolute constant $c>0$ such that

$$f_1(t)=f_2(t)+R,$$

and $|R|\le c\mathbf{E}|t\cdot r|^{3/4}$. It follows from the inequality $a\cdot b\le a^2+b^2$ applied with $a=|U|^{3/4}$ and $b=|\gamma|^{3/4}$ that

$$\begin{aligned}|R|&\le c|t|^{3/4}\mathbf{E}|\gamma|^{3/2}\mathbf{E}_{\overline{Y}}|Z|^{3/4}+c|t|^{3/4}\mathbf{E}|U\gamma|^{3/4}\le c|t|^{3/4}\mathbf{E}|\gamma|^{3/2}+c|t|^{3/4}\mathbf{E}|U|^{3/2}\\ &\le c|t|^{3/4}m(t)\mathcal{R}+c|t|^{3/4}m^{1+3/4}\mathcal{R}.\end{aligned}$$

In the last step we estimated $\mathbf{E}_{\overline{Y}}|Z|^{3/4}\le c$ uniformly over α and applied lemma (2.1). We have

$$|f(t)-f_2(t)|\le c(|t|+|t|^{3/4})m(t)\mathcal{R}.$$

Hence $f\simeq f_2$ because the factor $m(t)$ ensures the convergence of the integral with respect to the measure $dt/|t|$ as $|t|\to+\infty$.

Since the function $u\to\exp\{iu\}$ and it's derivatives are bounded there exists an absolute constant $c>0$ such that

$$|f_2(t)-f_3(t)-f_4(t)|\le R$$

where

$$\begin{aligned}f_3(t)&=\mathbf{E}\exp\{it[Xg(1+\rho)+Zg(1+\rho)+Z\gamma g'(1+\rho)]\},\\ f_4(t)&=\mathbf{E}\exp\{it[Xg(1+\rho)+Zg(1+\rho)+Z\gamma g'(1+\rho)]\}\cdot itUg'(1+\rho)\end{aligned}$$

and $|R| \le c|t|^{3/2}\mathbf{E}|U|^{3/2}$. Applying Lemma 2.1 we get

$$|R| \le c|t|^{3/2} m^{1+3/4}(t)\mathcal{R}.$$

Hence $f_2 \simeq f_3 + f_4$. Furthermore, denoting

$$f_5(t) = \mathbf{E}\exp\{it[Xg(1+\rho) + Zg(1+\rho)]\} \cdot itUg'(1+\rho),$$

we have

$$|f_4(t) - f_5(t)| \le R, \quad \text{with} \quad R = \mathbf{E}\left(c|t||U||tZ\gamma|^{1/2}\right).$$

Estimating $\mathbf{E}_{\overline{Y}}|Z|^{1/2} \le (\mathbf{E}_{\overline{Y}}|Z|^2)^{1/4} \le c$ uniformly with respect to α we obtain

$$\begin{aligned}|R| &\le c\mathbf{E}|t||U| \cdot |tZ\gamma|^{1/2} \le c|t|^{3/2}\mathbf{E}|U| \cdot |\gamma|^{1/2}\mathbf{E}_{\overline{Y}}|Z|^{1/2} \\ &\le c|t|^{3/2}\mathbf{E}|U| \cdot |\gamma|^{1/2} \le c|t|^{3/2}(\mathbf{E}|U|^{3/2} + \mathbf{E}|\gamma|^{3/2}) \le c|t|^{3/2}m(t)\mathcal{R}.\end{aligned}$$

In the last step we used the inequality $a^{2/3}b^{1/3} \le a + b$ with $a^{2/3} = |U|$ and $b^{1/3} = |\gamma|^{1/2}$ and applied Lemma 2.1. Since the integral with respect to the measure $|t|^{3/2}m(t)\,\frac{dt}{|t|}$ converges we conclude that $f_4 \simeq f_5$. We rewrite $f_5(t)$ in a more convenient form such that some independent random variables get separated. Write

$$P_{k,l} := \exp\{it[X^{k,l}g(1+\rho) + Zg(1+\rho)]\}g'(1+\rho), \quad 1 \le k, l \le N, k \neq l.$$

Recall that $X^{k,l}$ denotes the sum $\alpha_1 Y_1 + \cdots + \alpha_N Y_N$ with the summands $\alpha_k Y_k$ and $\alpha_l Y_l$ removed. Then

$$f_5(t) =$$
$$it\mathbf{E}_\alpha \sum_{\{1\le k,l\le N,\ k\neq l\}} \mathbf{E}^\alpha \alpha_k Y_k \exp\{it\alpha_k Y_k g(1+\rho)\} \cdot \alpha_l \eta_l \exp\{it\alpha_l Y_l g(1+\rho)\} \times P_{k,l}.$$

Expanding the exponents in powers of $it\alpha_k Y_k g(1+\rho)$ and $it\alpha_l Y_l g(1+\rho)$ we have

$$\begin{aligned}&|f_5(t)| \\ &\le c|t|\mathbf{E}_\alpha \sum_{\{1\le k,l\le N,\ k\neq l\}} \alpha_k c|t|\mathbf{E}Y_k^2 \cdot \alpha_l|t|\mathbf{E}|\gamma_l Y_l| \\ &\le c|t|^3 m^2(t) \sum_{i=1}^N \mathbf{E}|\gamma_l Y_l| \\ &= c|t|^3 m^2(t)\mathcal{R}\end{aligned}$$

Here we used the fact that Y_k, γ_l and $P_{k,l}$ are independent, for $k \neq l$, $|P_{k,l}| \leq c$ and $\mathbf{E}Y_k = 0$, $\mathbf{E}\gamma_l = 0$. Hence $f_5 \simeq 0$ and we conclude that $f \simeq f_3$. We shall show that $f_3 \simeq f_6 + f_7$, where

$$\begin{aligned} f_6(t) =& \mathbf{E} \exp\{it[Xg(1+\rho) + Zg(1+\rho)]\}, \\ f_7(t) =& \mathbf{E} itZ\gamma g'(1+\rho) \exp\{it[Xg(1+\rho) + Zg(1+\rho)]\}. \end{aligned}$$

Using the inequality

$$|\exp\{ia\} - 1 - ia| \leq |a|^{3/2}$$

with $a = tZ\gamma g'(1+\rho)$ we obtain $f_3(t) = f_6(t) + f_7(t) + R$, where

$$|R| \leq c|t|^{3/2}\mathbf{E}|\gamma|^{3/2}\mathbf{E}_{\overline{Y}}|Z|^{3/2} \leq c|t|^{3/2}\mathbf{E}|\gamma|^{3/2} \leq c|t|^{3/2} m(t)\mathcal{R}.$$

Thus $f_3 \simeq f_6 + f_7$. Next we will prove that $f_7 \simeq 0$. Introduce for brevity

$$T_j := \mathbf{E}_{X^j} \exp\{itX^j g(1+\rho)\}, \quad Q := \exp\{itZg(1+\rho)\} \cdot Zg'(1+\rho).$$

Then

$$f_7(t) = it\mathbf{E} \sum_{j=1}^{N} \alpha_j \gamma_j \exp\{it\alpha_j Y_j g(1+\rho)\}\, T_j\, Q.$$

Expanding the exponent and estimating $|\alpha_j| \leq 1$ we have

$$|f_7(t)| \leq c|t|^2 \mathbf{E}_\alpha \sum_{i=1}^{N} \mathbf{E}_{Y_j} |\gamma_j Y_j| \mathbf{E}^\alpha |T_j| \cdot |Q|.$$

Using inequality (2.2) with $\beta = Y_l \alpha_l g(1+\rho)$ and with expectation taken with respect to Y_l as well as $1/8 \leq g \leq 2$ we get

$$|T_j| \leq T_j' := \prod_{1 \leq l \leq N,\ l \neq j} (1 - \frac{1}{64} t^2 \alpha_l \mathbf{E}Y_l^2 + \frac{32}{3} |t|^3 \alpha_l \mathbf{E}|Y_l|^3)^{1/2}.$$

Using the simple bound $\mathbf{E}_{\overline{Y}}|Q| \leq c$ and inequality (2.3) for $\mathbf{E}_\alpha T_j'$ we get

$$\begin{aligned} \mathbf{E}|Q| \cdot T_j' \leq& c\mathbf{E}T_j' \\ \leq& c\big(\exp\{-\frac{1}{64} t^2 m(t) \sum_{\{1 \leq l \leq N,\ l \neq j\}} \mathbf{E}Y_l^2 + \frac{32}{3}|t|^3 m(t) \sum_{i=1}^{N} \mathbf{E}|Y_l|^3\}\big)^{1/2} \\ \leq& \exp\{-\frac{1}{1000} t^2 m(t)\}. \end{aligned}$$

The last inequality is true for $C_1 \le |t| \le T$ provided that $\mathbf{E}|Y_j|^2 \le 1/2$ which is satisfied due to (2.1) (otherwise $\mathbf{E}|Y_i|^3 \ge (\mathbf{E}|Y_i|^2)^{3/2} \ge c_0$). Thus we have

$$|f_7(t)| \le c|t|^2 \exp\{-C_3 t^2 m(t)\} \sum_{i=1}^{N} \mathbf{E}|\gamma_i Y_i| \le c|t|^2 \exp\{-C_3 t^2 m(t)\}\mathcal{R},$$

where C_3 is an absolute positive constant. Hence, $f_7 \simeq 0$.

It remains to prove that $f_6 \simeq \varphi$. Let $\tau = (\tau_1, ..., \tau_N)$ be a sequence of i.i.d. Bernoulli random variables with $\mathbf{P}(\tau_1 = 1) = \mathbf{P}(\tau_1 = 0) = 1/2$. Let $(Y_1', ..., Y_N')$ be independent copy of $(Y_1, ..., Y_N)$. Assume that τ, α, $(Y_1, ..., Y_N)$, $\overline{Y}_1, ..., \overline{Y}_N$ and $(Y_1', ..., Y_N')$ are independent. Then

$$f_6(t) = \mathbf{E} \exp\{itZg(1+\rho)\} \exp\{it(X(\tau) + X(1-\tau))g(1+\rho)\},$$

where

$$X(\tau) = \alpha_1 \tau_1 Y_1 + \cdots + \alpha_N \tau_N Y_N$$

and

$$X(1-\tau) = \alpha_1 (1-\tau_1) Y_1' + \cdots + \alpha_N (1-\tau_N) Y_N'$$

are conditionally independent given α and τ. Write

$$\begin{aligned} Q_0 &:= \exp\{itZg(1+\rho)\}, \quad Q_1 := \exp\{itX(\tau)g(1+\rho)\} \\ Q_2 &:= \exp\{itX(1-\tau)g(1+\rho)\}. \end{aligned}$$

Expanding $g(1+\rho)$ in powers of ρ we obtain

$$g(1+\rho) = g(1) + g'(1)\rho + \theta_1 g''(1+\theta_2\rho)\rho^2$$

and expanding the exponents

$$\begin{aligned} &\exp\{itX(\tau)\theta_1 g''(1+\theta_2\rho)\rho^2\} = 1 + r_1, \quad \text{with} \quad |r_1| \le c|tX(\tau)\rho^2|^{3/4}, \\ &\exp\{-\tfrac{1}{2}\, itX(\tau)\rho\} = 1 - \tfrac{1}{2}\, itX(\tau)\rho + r_2, \quad \text{with} \quad |r_2| \le c|tX(\tau)\rho|^{3/2} \end{aligned}$$

we get

$$\begin{aligned} Q_1 &= \exp\{itX(\tau)(1 + g'(1)\rho + \theta_1 g''(1+\theta_2\rho)\rho^2)\} \\ &= \exp\{itX(\tau)\} \exp\{-\tfrac{1}{2}\, itX(\tau)\rho\} \exp\{itX(\tau)\theta_1 g''(1+\theta_2\rho)\rho^2\} \\ &= F(\tau) + P(\tau) + R(\tau), \end{aligned}$$

where

$$F(\tau) = \exp\{itX(\tau)\}, \qquad P(\tau) = \frac{-it}{2} X(\tau)\rho \exp\{itX(\tau)\},$$

and

$$|R(\tau)| \le c|tX(\tau)|^{3/4}|\rho|^{3/2} + c|tX(\tau)\rho|^{3/2}.$$

Similarly,

$$Q_2 = F(1-\tau) + P(1-\tau) + R(1-\tau).$$

On the other hand one may write

$$Q_2 = \exp\{itX(1-\tau)(1+g'(1+\theta_1\rho)\rho)\} = F(1-\tau) + R_0(1-\tau),$$

where

$$|R_0(1-\tau)| \le c|tX(1-\tau)\rho|^{1/2}.$$

We have

$$\begin{aligned} f_6 =& \mathbf{E}Q_0(F(\tau)+P(\tau)+R(\tau))\cdot Q_2 \\ =& \mathbf{E}Q_0F(\tau)(F(1-\tau)+P(1-\tau)+R(1-\tau)) \\ &+ \mathbf{E}Q_0P(\tau)(F(1-\tau)+R_0(1-\tau)) + \mathbf{E}Q_0R(\tau)Q_2. \end{aligned}$$

In what follows we shall prove that

$$f_6 \simeq \mathbf{E}Q_0 \cdot F(\tau)F(1-\tau). \tag{2.9}$$

First, let us prove that $\mathbf{E}Q_0R(\tau)Q_2 \simeq 0$. The random variables $R(\tau)$ and Q_2 are conditionally independent given $\alpha, \tau, \overline{Y}$ and $|Q_0| \le 1$. Hence

$$|\mathbf{E}Q_0R(\tau)Q_2| \le |\mathbf{E}Q_0\mathbf{E}_Y R(\tau)\mathbf{E}_{Y'}Q_2| \le \mathbf{E}\mathbf{E}_Y|R(\tau)|\cdot|\mathbf{E}_{Y'}Q_2|.$$

Proceeding in the same way as above while estimating T_j, we get

$$|\mathbf{E}_{Y'}Q_2| \le \prod_{l=1}^{N}\Big(1 - \frac{1}{64}t^2(1-\tau_l)\alpha_l\mathbf{E}(Y_l')^2 + \frac{32}{3}|t|^3(1-\tau_l)\alpha_l\mathbf{E}|Y_l'|^3\Big)^{1/2}.$$

The estimate

$$\mathbf{E}_Y|R(\tau)| \le c(|t|^{3/4} + |t|^{3/2})|\rho|^{3/2}$$

which holds uniformly with respect to τ together with integration with respect to τ yields

$$\mathbf{E}_\tau\mathbf{E}_Y|R(\tau)|\cdot|\mathbf{E}_{Y'}Q_2| \le c(|t|^{3/4}+|t|^{3/2})|\rho|^{3/2}\prod_{l=1}^{N}T_l, \tag{2.10}$$

where

$$T_l = (1 - \frac{1}{128}\, t^2 \alpha_l \mathbf{E}(Y_l')^2 + \frac{16}{3}\, |t|^3 \alpha_l \mathbf{E}|Y_l'|^3)^{1/2}, \qquad 1 \le l \le N.$$

Now we may integrate the product obtained in the right-hand side of (2.10) firstly with respect to $\overline{Y}$ and afterwards with respect to α. Applying Lemma 2.1 and (2.1) we get

$$\mathbf{E}_{\overline{Y}}|\rho|^{3/2} \le 2^{3/2}\mathbf{E}|\sum_{i=1}^{N} \rho_i|^{3/2} + 2^{3/2}B^{3/2} \le c\mathcal{R}.$$

Hence,

$$\mathbf{E}(|t|^{3/4} + |t|^{3/2})|\rho|^{3/2} \prod_{l=1}^{N} T_l \le c(|t|^{3/4} + |t|^{3/2}) \cdot \mathbf{E}_\alpha(\prod_{l=1}^{N} T_l)\mathcal{R}.$$

The expectation $\mathbf{E}_\alpha(\prod \ldots)$ does not exceed

$$\exp\{-\frac{1}{256}\, m(t)t^2 + \frac{16}{3}\, m(t)|t|^3 \sum_{i=1}^{N} \mathbf{E}|Y_i|^3\} \le \exp\{-C_3 m(t)t^2\},$$

for $C_1 \le |t| \le T$. We conclude that $\mathbf{E}Q_0R(\tau)Q_2 \simeq 0$. Proceeding in a similar way we get $\mathbf{E}Q_0F(\tau)R(1-\tau) \simeq 0$.

Let us prove that $\mathbf{E}Q_0P(\tau)R_0(1-\tau) \simeq 0$. Expanding the exponent in powers of $it\alpha_j\tau_jY_j$ we get

$$\begin{aligned} |\mathbf{E}_Y P(\tau)| =& \left| -\frac{it}{2}\, \rho \mathbf{E}_Y \sum_{j=1}^{N} \alpha_j\tau_jY_j \exp\{it\alpha_j\tau_jY_j + itX^j(\tau)\} \right| \\ \le& c|t||\rho| \sum_{j=1}^{N} G_j \cdot H_j, \end{aligned}$$

where $G_j = |t| \cdot \mathbf{E}Y_j^2$ and

$$\begin{aligned} H_j :=& |\mathbf{E}_Y \exp\{itX^j(\tau)\}| \\ \le& \prod_{1\le l\le N, l\ne j} (1 - t^2\alpha_l\tau_l\mathbf{E}Y_l^2 + \frac{4}{3}\, |t|^3\alpha_l\tau_l\mathbf{E}|Y_l|^3)^{1/2}. \end{aligned}$$

An application of Lemma 2.2 leads to

$$\begin{aligned} \mathbf{E}_{\alpha,\tau}H_j \le& \exp\{-\frac{1}{4}\, t^2 m(t) \sum_{1\le l\le N,\ l\ne j} \mathbf{E}Y_l^2 + \frac{1}{3}\, |t|^3 m(t) \sum_{l=1}^{N} \mathbf{E}|Y_l|^3\} \\ \le& \exp\{-C_3 m(t)t^2\}, \qquad C_1 \le |t| \le T \qquad\qquad (2.11) \end{aligned}$$

Here we assumed that $\sum_{1\le l\le N,\ l\ne j} \mathbf{E}Y_l^2 > 1/2$, since otherwise the theorem follows from (2.1). Estimating

$$|Q_0| \le 1, \quad \mathbf{E}_{Y'}|X(1-\tau)|^{1/2} \le c, \quad \mathbf{E}_{\overline{Y}}|\rho|^{3/2} \le c\mathcal{R}, \quad \sum_{j=1}^N G_j \le c|t|.$$

and using (2.11) we get

$$\begin{aligned}|\mathbf{E}Q_0P(\tau)R_0(1-\tau)| &= \mathbf{E}_{\alpha,\tau,\overline{Y}}|Q_0|\cdot|\mathbf{E}_Y P(\tau)|\cdot E_{Y'}|R_0(1-\tau)|\\ &\le c|t|^{3/2}\sum_{j=1}^N G_j\mathbf{E}_{\alpha,\tau}H_j\mathbf{E}_{\overline{Y}}|\rho|^{3/2} \le c|t|^{3/2}\exp\{-C_3m(t)t^2\}\mathcal{R}.\end{aligned}$$

Thus we conclude that $\mathbf{E}Q_0P(\tau)R_0(1-\tau)\simeq 0$.

Let us prove that $\mathbf{E}Q_0P(\tau)F(1-\tau)\simeq 0$. Expanding the exponent as above we get

$$|\mathbf{E}_Y X(\tau)\exp\{itX(\tau)\}| \le c\sum_{j=1}^N G_j\cdot H_j.$$

Observe that

$$|\mathbf{E}_{Y'}F(1-\tau)| \le |\mathbf{E}_{Y'}\exp\{itX^j(1-\tau)\}|, \quad \text{for each} \quad j.$$

Hence,

$$\begin{aligned}&|\mathbf{E}^\tau X(\tau)\exp\{itX(\tau)\}F(1-\tau)|\\ \le&\sum_{j=1}^N G_j\mathbf{E}^\tau|\mathbf{E}_{Y,Y'}\exp\{itX^j(\tau)+itX^j(1-\tau)\}|\\ \le&\sum_{j=1}^N G_j\exp\{-C_3m(t)t^2\}\\ \le& c|t|\exp\{-C_3m(t)t^2\},\end{aligned}$$

for $C_1\le|t|\le T$. The relation $\mathbf{E}Q_0P(\tau)F(1-\tau)\simeq 0$ now follows, provided the following estimate holds

$$|\mathbf{E}_{\overline{Y}}Q_0\rho| \le c(1+|t|^{1/2}+|t|^{3/4}+|t|)\mathcal{R} \tag{2.12}$$

uniformly in α. Define $\hat\rho := \sum_{l=1}^N \rho_l$. We have

$$\mathbf{E}_{\overline{Y}}\rho Q_0 = \mathbf{E}_{\overline{Y}}\hat\rho Q_0 + R, \quad |R| \le |\mathbf{E}_{\overline{Y}}BQ_0| \le cB \le c\mathcal{R}.$$

Write $g_B(u) := g(u+B)$ and

$$\begin{aligned}\hat\rho Q_0 :&= \sum_{j=1}^N \rho_j \exp\{itZ g_B(1+\hat\rho)\}\\ &= \sum_{j=1}^N \rho_j \exp\{it(1-\alpha_j)\overline{Y}_j g_B(1+\hat\rho)\}\exp\{itZ^j g_B(1+\hat\rho)\}.\end{aligned}$$

Expanding the first exponent we get

$$\hat\rho Q_0 = f_8 + R$$

where

$$f_8(t) = \sum_{j=1}^N \rho_j \exp\{itZ^j g_B(1+\hat\rho)\}$$

and

$$\mathbf{E}_{\overline{Y}}|R| \le c|t|\sum_{j=1}^N \mathbf{E}_{\overline{Y}}|\rho_j\overline{Y}_j| \le c|t|\mathcal{R}.$$

In the last inequality we applied Lemma 2.1. Expanding

$$g_B(1+\hat\rho) = g_B(1) + g_B'(1+\theta\hat\rho)\cdot(\hat\rho^j + \rho_j),$$

and then expanding the exponent in f_8 as follows

$$\exp\{itZ^j g_B(1+\hat\rho)\} = \exp\{itZ^j g_B(1) + itZ^j g_B'(1+\theta\hat\rho)\cdot\hat\rho^j\} + r,$$

with $|r| \le c|tZ^j\rho_j|^{1/2}$, we get $f_8 = f_9 + R$, where

$$f_9 = \sum_{j=1}^N \rho_j \exp\{itZ^j g_B(1) + itZ^j g_B'(1+\theta\hat\rho)\cdot\hat\rho^j\}.$$

and

$$|\mathbf{E}_{\overline{Y}}R| \le c|t|^{1/2}\sum_{j=1}^N \mathbf{E}_{\overline{Y}}|\rho_j|^{3/2}|Z^j|^{1/2} \le c|t|^{1/2}\sum_{j=1}^N \mathbf{E}_{\overline{Y}}|\rho_j|^{3/2} \le c|t|^{1/2}\mathcal{R}.$$

Here we used the independence of ρ_j and Z^j as well as the bound

$$\mathbf{E}_{\overline{Y}}|Z^j|^{1/2} \le (\mathbf{E}_{\overline{Y}}|Z^j|^2)^{1/4} \le c$$

and Lemma 2.1. Consider the second summand in the argument of the exponent in f_{10}. Let $D^{(j)}$ denote the diagonal part of the product $Z^j \hat{\rho}^j$,

$$D^{(j)} = \sum_{\{1 \le k \le N, \ k \ne j\}} (1-\alpha_k)\overline{Y}_k \rho_k.$$

and let $U^{(j)}$ denote the rest, i.e., $U^{(j)} + D^{(j)} = Z^j \cdot \hat{\rho}^j$. Expanding the exponent in powers of $itD^{(j)} g'_B(1+\theta\hat{\rho})$ we obtain $f_9 = f_{10} + R$, where

$$f_{10}(t) = \sum_{j=1}^{N} \rho_j \exp\{itZ^j + itU^{(j)} g'_B(1+\theta\hat{\rho})\}$$

and

$$\mathbf{E}_{\overline{Y}}|R| \le c|t|\mathbf{E}_{\overline{Y}} \sum_{j=1}^{N} |\rho_j| \sum_{\{1\le l \le N, l\ne j\}} |\overline{Y}_l \rho_l| \le c|t| \mathbf{E}_{\overline{Y}} \sum_{j=1}^{N} |\rho_j| \cdot \sum_{l=1}^{N} |\overline{Y}_l \rho_l|.$$

Estimating $\mathbf{E}_{\overline{Y}} \sum_{j=1}^{N} |\rho_j| \le c$ as it was done in proof of Lemma 2.4 but with κ_j instead of ρ_j and estimating $\sum_{l=1}^{N} \mathbf{E}_{\overline{Y}} |\overline{Y}_l \rho_l| \le c\mathcal{R}$ we get $\mathbf{E}_{\overline{Y}}|R| \le c|t|\mathcal{R}$.

Let us consider the second summand in the argument of the exponent in f_{10}. We may write

$$U^j g'_B(1+\theta\hat{\rho}) = U^j g'_B(1) + U^j \hat{\rho}\theta g''_B(1+\theta_1\theta\hat{\rho}), \quad \text{and} \quad U^j\hat{\rho} = U^j \hat{\rho}^j + U^j \rho_j.$$

Applying simple inequality $|\exp\{i(a+b)\} - 1| \le c(|a|^{1/2} + |b|^{3/4})$, where $c > 0$ is an absolute constant, we have

$$\begin{gathered} \exp\{itZ^j + itU^{(j)} g'_B(1+\theta\hat{\rho})\} = \exp\{itZ^j + itU^j g'(1)\} + r_1 + r_2, \\ |r_1| \le |tU^j \rho_j|^{1/2}, \qquad |r_2| \le |tU^j \hat{\rho}^j|^{3/4}. \end{gathered}$$

Hence,

$$f_{10} = f_{11} + R,$$

where

$$\begin{gathered} f_{11}(t) = \sum_{j=1}^{N} \rho_j \exp\{itZ^j + itU^j g'(1)\}, \\ |R| \le \sum_{j=1}^{N} |\rho_j| \cdot \left(|tU^j \rho_j|^{1/2} +, |tU^j \hat{\rho}^j|^{3/4} \right). \end{gathered}$$

Observe that $\mathbf{E}_{\overline{Y}} f_{11} = 0$, since ρ_j and (Z^j, U^j) are independent and $\mathbf{E}_{\overline{Y}} \rho_j = 0$. Furthermore,

$$\begin{aligned}
\mathbf{E}_{\overline{Y}}|R| \leq & c|t|^{1/2} \sum_{j=1}^{N} \mathbf{E}_{\overline{Y}_j} |\rho_j|^{3/2} \cdot \mathbf{E}_{\overline{Y}} |U^j|^{1/2} + c|t|^{3/4} \sum_{j=1}^{N} \mathbf{E}_{\overline{Y}_j} |\rho_j| \mathbf{E}_{\overline{Y}} (|U^j||\hat{\rho}^j|)^{3/4} \\
\leq & c|t|^{1/2} \sum_{j=1}^{N} \mathbf{E}_{\overline{Y}_j} |\rho_j|^{3/2} + c|t|^{3/4} \sum_{j=1}^{N} \mathbf{E}_{\overline{Y}_j} |\rho_j| \mathbf{E}_{\overline{Y}} (|U^j|^{3/2} + |\hat{\rho}^j|^{3/2}) \\
\leq & c(|t|^{1/2} + |t|^{3/4}) \mathcal{R}.
\end{aligned}$$

Here we used $\sum_{j=1}^{N} \mathbf{E}|\rho_j| \leq c$ (see the estimate $\sum_{i=1}^{N} \mathbf{E}|\kappa_i| \leq c$ in proof of Lemma 2.4) and applied Lemma 2.1.

Thus (2.12) is proved. We have $\mathbf{E} Q_0 P(\tau) F(1-\tau) \simeq 0$. Similarly, $\mathbf{E} Q_0 F(\tau) P(1-\tau) \simeq 0$. We arrive at (2.9).

Step 2. Observe that $\mathbf{E} Q_0 F(\tau) F(1-\tau) = \mathbf{E} Q_0 \exp\{itX\} =: f_{12}(t)$. We shall show that $f_{12} \simeq \varphi$. Recall that $\xi_i, 1 \leq i \leq N$ denote independent centered Gaussian random variables with variances $\mathbf{E}\xi_i^2 = \mathbf{E} Y_i^2, \quad 1 \leq i \leq N$. Given α we may apply Lemma 2.4 conditionally,

$$|\mathbf{E}_\xi \exp\{it \sum_{i=1}^{N} (1-\alpha_i)\xi_i\} - \mathbf{E}_{\overline{Y}} Q_0| \leq c|t|^3 \mathcal{R}$$

and use

$$\mathbf{E}_\alpha |\mathbf{E}_Y \exp\{itX\}| \leq \exp\{-C_3 m(t) t^2\}$$

in order to get

$$f_{12} \simeq \mathbf{E} \exp\{it \sum_{i=1}^{N} (1-\alpha_i)\xi_i\} \exp\{itX\}.$$

Furthermore,

$$f_{12} \simeq \mathbf{E} \exp\{it \sum_{i=1}^{N} (1-\alpha_i)\xi_i\} \exp\{it \sum_{i=1}^{N} \alpha_i \xi_i'\} = \varphi.$$

Here $\xi_1', .., \xi_N'$ denotes independent copies of $\xi_i, \quad 1 \leq i \leq N$. Thus Theorem 1.2 is proved.

Proof of (1.13) and (1.14). Assume that $0 < p < 1/2$. We have

$$\begin{aligned}
&\mathbf{P}(X_1^2 + \cdots + X_N^2 > N^{1-2p}) \\
=&\mathbf{P}\left(\sum_{i=1}^N \frac{1}{i}\,\xi_i^2 + 2\sum_{i=1}^N \frac{(-1)^i}{i^{1/2+p}}\,\xi_i + \sum_{i=1}^N \frac{1}{i^{2p}} > N^{1-2p}\right) \\
\geq&\mathbf{P}\left(2\sum_{i=1}^N \frac{(-1)^i}{i^{1/2+p}}\,\xi_i + \sum_{i=1}^N \frac{1}{i^{2p}} > N^{1-2p}\right) \to 1,
\end{aligned}$$

because the series $\sum_{i=1}^N \frac{(-1)^i}{i^{1/2+p}}\,\xi_i$ converges a.s. by Kolmogorov's Three Series Theorem. Since $\mathcal{L}\left((X_1 + \cdots + X_N)/\sqrt{\ln(N+2)}\right) \to N(0,1)$, it follows that

$$\sqrt{N}\mathbf{t}(X_1, ..., X_N) = \frac{X_1 + \cdots + X_N}{\sqrt{X_1^2 + \cdots + X_N^2 + N^{-1}(X_1 + \cdots + X_N)^2}} \to_P 0.$$

Assume that $p = 1/2$. In order to prove (1.14) it suffices to show that

$$\frac{X_1^2 + \cdots + X_N^2}{\sum_{i=1}^N i^{-1}} \to_P 2.$$

For each $\varepsilon > 0$, we have

$$\begin{aligned}
&\mathbf{P}\left(\left|\frac{X_1^2 + \cdots + X_N^2}{\sum_{i=1}^N i^{-1}} - 2\right| > 2\varepsilon\right) \\
&= \mathbf{P}\left(\left|\frac{\sum_{i=1}^N i^{-1}(\xi_i^2 - 1)}{\sum_{i=1}^N i^{-1}} + \frac{2\sum_{i=1}^N (-1)^i i^{-1}\xi_i}{\sum_{i=1}^N i^{-1}}\right| > 2\varepsilon\right) \\
&\leq \varepsilon^{-2}\,\frac{\mathbf{E}\left(\sum_{i=1}^N i^{-1}(\xi_i^2 - 1)\right)^2}{\left(\sum_{i=1}^N i^{-1}\right)^2} + \mathbf{P}\left(|2\sum_{i=1}^N (-1)^i i^{-1}\xi_i| > \varepsilon\sum_{i=1}^N i^{-1}\right).
\end{aligned}$$

In the last step we applied the Chebyshev inequality. A simple calculation shows that the first summand tends to zero as $N \to 0$. The second summand tends to zero because the series $\sum_{i=1}^N (-i)^{-1}\xi_i$ converges a.s. by Kolmogorov's Three Series Theorem. Hence (1.14) is proved.

Example 2.1. Let ξ_i, $i \geq 1$ be a sequence of i.i.d. standard normal random variables. Fix $0 < \varepsilon < 1$ and define $X_i = \xi_i + (-1)^i\varepsilon$, $i \geq 1$. Chose $V(N) = N$. A simple calculation shows that the bound in (1.3) implies

$$\limsup_{N\to\infty} \delta_N \leq \varepsilon^2 \quad \text{and} \quad \lim_{N\to\infty} M^{-1}\sum_{i=1}^N (\mathbf{E}Z_i)^2 = \varepsilon^2.$$

On the other hand we have

$$\mathbf{P}(\sqrt{N}\mathbf{t}(X_1, ..., X_N) < 1) = \mathbf{P}\left(\xi_1 < \frac{1 + \varepsilon((-1)^i - 1)/(2\sqrt{N})}{\sqrt{1 + \varepsilon^2 + R}}\right), \tag{2.13}$$

where R is a random variable such that $R \to_P 0$ as $N \to \infty$. Hence, there exists an absolute constant $0 < c < \infty$ such that

$$\liminf_{N\to\infty} \delta_N \geq \liminf_{N\to\infty} |\mathbf{P}(\sqrt{N}\mathbf{t}(X_1, ..., X_N) < 1) - \mathbf{P}(\xi_1 < 1)| \geq c\varepsilon^2. \tag{2.14}$$

The relations (2.13) and (2.14) contradict to any estimate of the type (1.3) with

$$\sum_{i=1}^{N} (M^{-1/2}\mathbf{E}Z_i)^{2+\delta} \quad \text{or} \quad (M^{-1}\sum_{i=1}^{N}(\mathbf{E}Z_i)^2)^{1+\delta},$$

instead of $M^{-1}\sum_{i=1}^{N}(\mathbf{E}Z_i)^2$, since

$$\lim_{N\to\infty} \sum_{i=1}^{N}(M^{-1/2}\mathbf{E}Z_i)^{2+\delta} = 0 \quad \text{and} \quad \lim_{N\to\infty}(M^{-1}\sum_{i=1}^{N}(\mathbf{E}Z_i)^2)^{1+\delta} = \varepsilon^{2+2\delta}.$$

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