Hammocks and the algorithm of Nazarova-Roiter

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1. *Introduction*

Hammocks have been considered by Brenner [**1**] in order to give a numerical criterion for a finite translation quiver to be the Auslander-Reiten quiver of some representation-finite algebra. Ringel and Vossieck [**11**] gave a combinatorial definition of left hammocks, which generalizes the concept of hammocks in the sense of Brenner, as a translation quiver H and an additive function h on H (called the hammock function) satisfying some conditions, and they showed that a thin left hammock with finitely many projective vertices is just the preprojective component of the Auslander-Reiten quiver of the category of $\mathcal{S}$-spaces, where $\mathcal{S}$ is a finite partially ordered sets (abbreviated: poset). An impotant role in representation theory of poset is played by two differentiation algorithms. One of the algorithms is due to Nazarova and Roiter [**8**] which reduce a poset $\mathcal{S}$ with a maximal element a to a new poset $\mathcal{S}' = {}_a\partial\mathcal{S}$. The second is due to Zavadskij [**13**] which reduce a poset $\mathcal{S}$ with a suitable pair (a, b) of elements a, b to a new poset $\mathcal{S}' = \partial_{(a,b)}\mathcal{S}$. The main purpose of the present paper is to construct new left hammocks from a given one, and to show the relationship between these new left hammocks and the algorithm of Nazarova-Roiter. Our operation on hammocks is a generalization of their algorithm. In a forthcoming paper ([**6**]), We will discuss the relationshipe between hammocks and the algorithm of Zavadskij.

In §2, we recall some basic definitions and facts. Let H be a left hammock with hammock function h_H, $p(a)$ a thin projective vertex of H different from the source. In §3, we construct a new left hammock ${}_aH$ and determine its hammock function $h_{({}_aH)}$. We prove that the subquiver, denote by $H/{}_aH$, consisting of all vertices x satisfying $h_H(x) - h_{({}_aH)}(x) \neq 0$ is also a left hammock. We show that if H is infinite, then ${}_aH$ is also infinite. In case H is a hammock, the relation between the hammock induced by a projective vertex of H and the algorithm of Nazarova-Roiter is stated in §6. The case when $p(a)$ is a non-thin projective vertex of H is discussed in §4. In §5, we consider the hammock induced by some injective vertex of H. The corresponding results concerning $\ell(\mathcal{S})$, the category of $\mathcal{S}$-spaces, are also described.

Throughout this paper, all algebras are assumed to be finite-dimensional (associative) basic algebras with unit over an algebraically closed field k, all modules are finitely generated right modules. We denote by A-mod the category of A-modules. The composition of two morphisms $f : M_1 \to M_2$ and $g : M_2 \to M_3$ is denoted by fg. All posets are assumed to be finite. We denote by $\mathbb{N}, \mathbb{N}_1, \mathbb{Z}$ the set of natural numbers, positive integers and integers, respectively. For all unexplained notation we refer to [**11**] and [**12**].

2. *Preliminaries*

2.1 *Left hammocks and hammocks.* Let $H = (H_0, H_1, \tau)$ be a proper translation quiver. We define inductively the full subquivers ${}^dH$ of H. First of all, ${}^{-1}H$ is the empty quiver, and z belongs to ${}^dH$ if and only if $z^- \subseteq {}^{d-1}H$. Also, ${}^\infty H = \bigcup_{d\in\mathbb{N}} {}^dH$. Thus, for all $d \in \mathbb{N} \cup \{\infty\}$, we see that ${}^dH$ is a predeccessor closed subquiver, and we may consider it as a translation quiver, using the restriction of τ. Suppose H has a unique source ω, and $H = {}^\infty H$, then we define $h_H : H_0 \to \mathbb{Z}$ inductively as follows. By abuse of notation, let $h_H(\tau x) = 0$ for x projective (note that in this case, τx is not defined). Now, let $h_H(\omega) = 1$, and, for $x \neq \omega$, with h_H already defined on all proper predecessors of x, let $h_H(x) = \sum_{y\to x} h_H(y) - h_H(\tau x)$, where y runs over all starting vertices of all arrows ending at x. (note: the multiplicity of arrows $y \to x$ should be calculated in.) With these preparations, we are able to recall the main definition: the translation quiver H is said to be a *left hammock* provided

(1) $H = {}^\infty H$;

(2) H has a unique source ω and $h_H(\omega) = 1$;

(3) h_H takes values in $\mathbb{N}_1$;

(4) if q is an injective vertex then $h_H(q) \geq \sum_{q\to y} h_H(y)$.

When H is a left hammock, the function h_H is said to be its *hammock function*. In fact, $h_H(x)$ is just $\dim_k \operatorname{Hom}_{k(H)}(\omega, x)$ for each $x \in H$.

A vertex x of H is called *thin* if $h_H(x) = 1$. A left hammock H is said to be *thin* provided $h_H(p) = 1$ for any projective vertex p of H. A left hammock H is called a *hammock* if $|H_0| < \infty$. A hammock is always thin and has a unique sink, say ω'.

2.2 *S-spaces.* Fix some field k. Given a poset $\mathcal{S}$, an *$\mathcal{S}$-space* $V = (V_\omega; V_s)_{s\in\mathcal{S}}$ is given by a vector space V_ω over k, and subspaces V_s of V_ω, for $s \in \mathcal{S}$, such that $V_s \subseteq V_t$ for $s \leq t$. We call V_ω the *total space* of V, and define its k-dimemsion by $\underline{\dim}_\omega V = \dim_k V_\omega$. Given two $\mathcal{S}$-spaces V, W, a map $\varphi : V \to W$ is given by a k-linear map $\varphi_\omega : V_\omega \to W_\omega$ satisfying $\varphi_\omega(V_\mathcal{S}) \subseteq W_s$ for all $s \in \mathcal{S}$; the induced map $V_s \to W_s$ will be denoted by φ_s. The posets we will consider are always assumed to be finite. We denote the category of

$\mathcal{S}$-spaces V with $\dim_k V_\omega < \infty$ by $\ell(\mathcal{S})$. For convenience, we denote $\mathrm{Hom}_{\ell(\mathcal{S})}(V,W)$ for two $\mathcal{S}$-spaces V and W by $\mathrm{Hom}_{\mathcal{S}}(V,W)$. Recall that we denote by $\mathcal{S}^+$ the poset obtained from $\mathcal{S}$ by adding an element ω with $s < \omega$ for all $s \in \mathcal{S}$. For $s \in \mathcal{S}^+$, we recall *projective objects* in $\ell(\mathcal{S})$, denoted by $P_{\mathcal{S}}(s)$ for $s \in \mathcal{S}^+$, are defined by putting $P_{\mathcal{S}}(s)_t = k$ for $t \geq s$, $= 0$ for $t \ngeq s$, where $t \in \mathcal{S}^+$. In a similar way, we denote by $\mathcal{S}^-$ the poset obtained from $\mathcal{S}$ by adding an element ω' with $s > \omega'$ for all $s \in \mathcal{S}$. The *injective objects* in $\ell(\mathcal{S})$, denoted by $Q_{\mathcal{S}}(s)$ with $s \in \mathcal{S}^-$, is defined by putting $Q_{\mathcal{S}}(s)_t = k$ for $t \nleq s$, $= 0$ for $t \leq s$, where $t \in \mathcal{S}^+$. For $t \in \mathcal{S}$ and $V \in \ell(\mathcal{S})$, we have $\dim_k \mathrm{Hom}_{\mathcal{S}}(P_{\mathcal{S}}(t), V) = \dim_k V_t$ and $\dim_k \mathrm{Hom}_{\mathcal{S}}(V, Q_{\mathcal{S}}(t)) = \dim_k V_\omega - \dim_k V_t$. Recall that a $\mathcal{S}$-space V is *thin* if its total space V_ω is one-dimensional. We denote by $\tau_{\mathcal{S}}$ the *Auslander-Reiten translation* in $\ell(\mathcal{S})$. It is well-known that the Auslander-Reiten quiver of $\ell(\mathcal{S})$ always has a unique preprojective component, denoted by $\mathcal{P}_{\mathcal{S}}$, which is standard.

Given a Krull-Schmidt category Λ, let us recall the definition of the full subcategories ${}_d\Lambda$. First of all, ${}_{-1}\Lambda$ contains only the zero object. Second, an indecomposable object X of Λ belongs to ${}_d\Lambda$ if and only if any indecomposable object Y of Λ with $\mathrm{rad}(Y,X) \neq 0$ belongs to ${}_{d-1}\Lambda$. Finally, ${}_\infty\Lambda = \bigcup_{d\in\mathbb{N}} {}_d\Lambda$. Let $\mathcal{S}$ be a poset, we observe that ${}_\infty\ell(\mathcal{S})$ is just the full subcategory of $\ell(\mathcal{S})$ whose indecomposable objects occur in $\mathcal{P}_{\mathcal{S}}$. So ${}_\infty\ell(\mathcal{S}) \cong \mathrm{add}\, k(\mathcal{P}_{\mathcal{S}})$.

There is a strong relationship between thin left hammocks and the representation theory of posets as described as follows, which is due to Ringel and Vossieck (see [**11**]).

Theorem 2.1 *Let $\mathcal{S}$ be a finite poset, and k a field. Then the preprojective component $\mathcal{P}_{\mathcal{S}}$ of the Auslander-Reiten quiver of $\ell(\mathcal{S})$ is a thin left hammock with finitely many projective vertices. The hammock function on $\mathcal{P}_{\mathcal{S}}$ is $\underline{\dim}_\omega$. Conversely, given a thin left hammock H with n projective vertices, there exists a unique poset $\mathcal{S} := \mathcal{S}(H)$ with $n-1$ elements such that* $\mathrm{add}\, k(H) \cong {}_\infty\ell(\mathcal{S})$ *as categories, and $H \cong \mathcal{P}_{\mathcal{S}}$ as translation quivers.*

From now on we will take any thin left hammock H as the preprojective component $\mathcal{P}_{\mathcal{S}}$ for $\mathcal{S} = \mathcal{S}(H)$. Accordingly we have a bijective map $p : \mathcal{S}^+ \to \{$ projective vertex of $H\}$ where $p(s)$ is the vertex corresponding to $P_{\mathcal{S}}(s)$. Let $(\mathcal{S}^-)^0$ be the subset of $\mathcal{S}^-$ such that injective object $Q_{\mathcal{S}}(s)$ occurs in ${}_\infty\ell(\mathcal{S})$ for $s \in (\mathcal{S}^-)^0$. Then we have a bijective map $q : (\mathcal{S}^-)^0 \to \{$injective vertex of $H\}$, where $q(s)$ is the vertex corresponding to $Q_{\mathcal{S}}(s)$. In particular, we obtain that $\mathrm{Hom}_{k(H)}(p(s), p(t)) \neq 0$ if and only if $s \geq t$ in $\mathcal{S}^+$ and $\mathrm{Hom}_{k(H)}(q(s), q(t)) \neq 0$ if and only if $s \geq t$ in $\mathcal{S}^-$. By abuse of language, we denote $p(\omega)$ by ω, denote $q(\omega')$ by ω'.

2.3 *Incidence algebras and socle-projective modules.* Given a Krull-Schmidt category Λ, a Λ-*module* M is a finitely presented functor $\Lambda^{op} \to k$-mod. We denote by Λ-mod the category of all Λ-modules and by Λ-spmod the full subcategory of Λ-mod generated by all modules $M \in \Lambda$-mod which have a projective socle. Clearly, Λ-spmod is closed under direct sums, direct summands, submodules and extensions. A module M in Λ-spmod is said to be *thin* if M has a simple socle. We will use the following easy result.

Lemma 2.1 *Let Λ be a Krull-Schmidt category, $M, N \in \Lambda$-spmod.*

(1) *Assume that $0 \neq \psi \in \mathrm{Hom}_\Lambda(M, N)$ and M is thin. Then ψ is a monomorphism.*

(2) *Assume that $0 \neq \varphi \in \mathrm{Hom}_\Lambda(M, N)$, $0 \neq \phi \in \mathrm{Hom}_\Lambda(N, L)$ and M, N are thin. Then $\varphi\phi \neq 0$.*

Proof. Suppose that ψ is not a monomorphism, then $\mathrm{soc}(\ker(\psi)) = \mathrm{soc}\, M$ since M is thin. As a consequence, $\mathrm{soc}(\mathrm{Im}(\psi)) \cong \mathrm{soc}(M/\ker(\psi))$ is not projective. A contradiction to the fact that Λ-spmod is closed under submodules. □

Let A be an algebra over an algebraically closed field k with projective left socle W. It is clear to see that the projective A-modules are socle-projective. So a relative projective module in A-spmod is just a projective module in A-mod. A module E is relative injective module in A-spmod if and only if $E \in \mathrm{add}(\tau_A B \oplus I_A(W))$, where $I_A(W)$ is the injective hull of the socle of A and $B := A/W$ (see [**3**]).

Lemma 2.2 *Let A be an algebra over an algebraically closed field with projective socle, let $P_A(\omega)$ be the unique simple projective module.*

(1) *Let $P_A(a)$, where $a \neq \omega$, be an indecomposable projective module and $I_A(a)$=$DHom_A(P_A(a), A)$. Then $\mathrm{Hom}_A(I_A(a), M) = 0$ for any module $M \in A$-spmod.*

(2) *We have $\mathrm{Hom}_A(X, Q_A(\omega')) \neq 0$ for each $X \in A$-spmod.*

Proof. The socle of M is generated by the simple projective module $P_A(\omega)$, but $I_A(a)$ has no composition factor which is isomorphic to $P_A(\omega)$. This proved (1). For (2), $X \in A$-spmod means that $\mathrm{Hom}_A(P_A(\omega), X) \neq 0$. So $\mathrm{Hom}_A(X, Q_A(\omega')) \neq 0$. □

Let $\mathcal{S}$ be a poset, k an algebraically closed field. By $A(\mathcal{S}) := k\mathcal{S}^+$ we mean the *k-incidence algebra* of the enlarged poset $\mathcal{S}^+$. Note that $P_A(\omega)$ is the unique simple projective $A(\mathcal{S})$-module. The following Theorem is due to Ringel and Vossieck (see [**11**]).

Theorem 2.2 *Let H be a left hammock with source ω, and k a field. Let $\mathcal{P}(H, k)$ be the full additive subcategory of $k(H)$ whose indecomposable objects are just the projective vertices of H. Define the functor $\mathbf{M} : k(H) \to \mathcal{P}(H, k)$-mod by $\mathbf{M}(x) = \mathrm{Hom}_{k(H)}(-, x)|\mathcal{P}(H, k)$. Then*

(1) *there is a unique simple projective object in $\mathcal{P}(H, k)$-mod, namely $\mathbf{M}(\omega)$. An object X of $\mathcal{P}(H, k)$-mod belongs to $\mathcal{P}(H, k)$-spmod if and only if its socle is generated by $\mathbf{M}(\omega)$;*

(2) *${}_\infty(\mathcal{P}(H, k)$-spmod) has Auslander-Reiten sequences;*

(3) *$\mathbf{M}$ induces the equivalence $k(H) \cong_\infty(\mathcal{P}(H, k)$-spmod) (as categories);*

(4) *$H \cong \Gamma_{\infty(\mathcal{P}(H,k)\text{-}spmod)}$ (as translation quivers), where $\Gamma_{\infty(\mathcal{P}(H,k)\text{-}spmod)}$ is the Auslander-Reiten quiver of ${}_\infty(\mathcal{P}(H, k)$-spmod).*

For convenience, we put $\mathcal{F} := \mathcal{P}(H, k)$-spmod, thus we write ${}_\infty(\mathcal{P}(H, k)$-spmod) as

${}_\infty\mathcal{F}$, and we write instead of $\mathrm{Hom}_{\mathcal{P}(H,k)\text{-spmod}}(X,Y)$ just $\mathrm{Hom}_{\infty\mathcal{F}}(X,Y)$. If the left hammock H has only finitely many projective vertices, $\mathcal{P}(H,k)$ is a finite category; therefore $\mathcal{P}(H,k)$-mod $\cong$ $A(H)$-mod for some finite dimensional algebra $A(H)$, and $k(H) \cong_\infty(A\text{-spmod})$, $H \cong \Gamma_{\infty(A\text{-spmod})}$. We call $A(H)$ *the finite dimensional algebra corresponding to* H. Note that if H is a thin left hammock with finitely many projective vertices, then $A(H)$ is just the incidence algebra of the poset $\mathcal{S}(H)$.

3. *Hammocks induced by a thin projective object*

Let Λ be a Krull-Schmidt category, Ξ a class of objects of Λ. For $x, y \in \Lambda$, we denote by $\mathrm{Hom}_\Lambda(x,y)_\Xi$ the subspace of the all maps in $\mathrm{Hom}_\Lambda(x,y)$ which factor through some object of Ξ.

Theorem 3.1 *Let k be a field. Let H be a left hammock with source ω, h_H the hammock function of H. Assume that $p(a) \neq \omega$ is a thin projective vertex of H. Then*

(1) ${}_aH = \{x \in H \mid \mathrm{Hom}_{k(H)}(p(a),x) \neq 0\}$ *is a left hammock with source $p(a)$. The hammock function on ${}_aH$ is* $h_{({}_aH)} = \dim_k \mathrm{Hom}_{k(H)}(p(a),-) = \dim_k \mathrm{Hom}_{k(H)}(\omega,-)_{\{p(a)\}}$.

(2) $H/{}_aH = \{x \in H \mid h_H(x) - h_{({}_aH)}(x) \neq 0\}$ *is a left hammock with source ω. The hammock function on $H/{}_aH$ is* $h_{(H/{}_aH)} = h_H - h_{({}_aH)} = \dim_k \mathrm{Hom}_{k(H)}(\omega,-) - \dim_k \mathrm{Hom}_{k(H)}(\omega,-)_{\{p(a)\}}$.

In order to prove Theorem 3.1, we need the following Lemma.

Lemma 3.1 *Let H be a left hammock with source ω and h_H the hammock function of H. Let k be a field. Assume that $p(a) \neq \omega$ is a thin projective vertex of H. Then* $\dim_k \mathrm{Hom}_{k(H)}(p(a),x) = \dim_k \mathrm{Hom}_{k(H)}(\omega,x)_{\{p(a)\}}$ *for $x \in k(H)$.*

Proof. It is clear that $\mathrm{Hom}_{k(H)}(\omega,x)_{\{p(a)\}} = \mathrm{Hom}_{k(H)}(\omega,p(a))\,\mathrm{Hom}_{k(H)}(p(a),x)$. So $\dim_k \mathrm{Hom}_{k(H)}(\omega,x)_{\{p(a)\}} \leq \dim_k \mathrm{Hom}_{k(H)}(p(a),x)$ since $\dim_k \mathrm{Hom}_{k(H)}(\omega,p(a)) = 1$. On the other hand, we choose, then fix some $0 \neq \phi \in \mathrm{Hom}_{k(H)}(\omega,p(a))$. Assume that $\dim_k \mathrm{Hom}_{k(H)}(p(a),x) = r$ and $\varphi_1,\cdots,\varphi_r$ is a basis of $\mathrm{Hom}_{k(H)}(p(a),x)$ as k-vector space, then Lemma 2.1 implies that $\phi\varphi_1,\cdots,\phi\varphi_r$ are linear independent. So we have $\dim_k \mathrm{Hom}_{k(H)}(p(a),x) \leq \dim_k \mathrm{Hom}_{k(H)}(\omega,x)_{\{p(a)\}}$. $\square$

Remark. The assumption that $p(a)$ is thin in Lemma 3.1 cannot be omitted. Otherwise it is not difficult to find a counterexample.

Proof of Theorem 3.1. In the case $x \in {}_aH$ is not a projective vertex of H. Let $0 \longrightarrow \mathbf{M}(\tau x) \longrightarrow \bigoplus_{y\to x}\mathbf{M}(y) \longrightarrow \mathbf{M}(x) \longrightarrow 0$ be an Auslander-Reiten sequence in $\mathcal{F}$, where $\mathbf{M}$ is the equivalence functor $\mathbf{M} : k(H) \cong_\infty\mathcal{F}$. Apply $\mathrm{Hom}_{\mathcal{P}(H,k)}(\mathbf{M}(p(a)),-)$ to the sequence above, we obtain

$$0 \to \mathrm{Hom}_{\mathcal{P}(H,k)}(\mathbf{M}(p(a)),\mathbf{M}(\tau x)) \to \sum_{y\to x} \mathrm{Hom}_{\mathcal{P}(H,k)}(\mathbf{M}(p(a)),\mathbf{M}(y))$$

$$\to \mathrm{Hom}_{\mathcal{P}(H,k)}(\mathbf{M}(p(a)), \mathbf{M}(x)) \to 0.$$

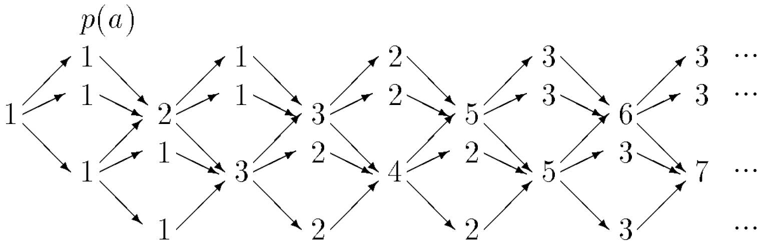
This sequence is exact, since x is not a projective vertex. So $\dim_k \mathrm{Hom}_{\mathcal{P}(H,k)}(\mathbf{M}(p(a)), \mathbf{M}(\tau x)) + \dim_k \mathrm{Hom}_{\mathcal{P}(H,k)}(\mathbf{M}(p(a)), \mathbf{M}(x)) = \sum_{y\to x} \dim_k \mathrm{Hom}_{\mathcal{P}(H,k)}(\mathbf{M}(p(a)), \mathbf{M}(y))$. and therefore $h_{({}_aH)}(x)+h_{({}_aH)}(\tau x) = \sum_{y\to x} h_{({}_aH)}(y)$. In case $x \in {}_aH$ is a projective vertex of H different from $p(a)$. Then $\bigoplus_{y\to x} \mathbf{M}(y) = \mathrm{rad}\,\mathbf{M}(x)$. It is obvious $\dim_k \mathrm{Hom}_{\mathcal{P}(H,k)}(\mathbf{M}(p(a)), \bigoplus_{y\to x} \mathbf{M}(y)) = \dim_k \mathrm{Hom}_{\mathcal{P}(H,k)}(\mathbf{M}(p(a)), \mathbf{M}(x))$. So $h_{({}_aH)}(x) = \sum_{y\to x} h_{({}_aH)}(y)$. Thus $h_{({}_aH)}$ is an additive function on ${}_aH$. (In fact, $h_{({}_aH)}$ is an additive function on H if we put $h_{({}_aH)}(x) = 0$ for $x \notin {}_aH$.) It is clear that ${}_aH$ has the unique source $p(a)$.

Now let q be an injective vertex of ${}_aH$. If q is also an injective vertex of H and not a sink, then $|q^+| = 1$ and $h_H(q) = h_H(y) = 1$, where $q^+ = \{y\}$. From Lemma 3.1, we know $h_H(y) \geq h_{({}_aH)}(y)$. So $h_{({}_aH)}(q) = h_H(q) = h_H(y) \geq h_{({}_aH)}(y)$. If q is not an injective vertex in H, then there exists the Auslander-Reiten sequence $o \to \mathbf{M}(q) \to \sum_{q\to y} \mathbf{M}(y) \to \mathbf{M}(\tau^- q) \to 0$. Apply the functor $\mathrm{Hom}_{\mathcal{P}(H,k)}(\mathbf{M}(p(a)), -)$ to this exact sequence, we get the exact sequence $0 \to \mathrm{Hom}_{\mathcal{P}(H,k)}(M(p(a)), M(q)) \to \sum_{q\to y} \mathrm{Hom}_{\mathcal{P}(H,k)}(M(p(a)), M(y)) \to 0$. So $h_{({}_aH)}(q) \geq \sum_{q\to y} h_{({}_aH)}(y)$. Thus we have (1).

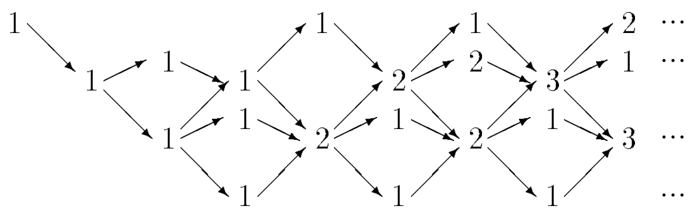
In order to prove (2), assume that x is not the source, then we have $h_H(x)+h_H(\tau x) = \sum_{y\to x} h_H(y)$ and $h_{({}_aH)}(x) + h_{({}_aH)}(\tau x) = \sum_{y\to x} h_{({}_aH)}(y)$ for $x \in H$ (we put $h_{({}_aH)}(x) = 0$ for $x \notin {}_aH$). So $h_{(H/{}_aH)}(x) + h_{(H/{}_aH)}(\tau x) = \sum_{y\to x} h_{(H/{}_aH)}(y)$. Thus, $H/{}_aH$ is a proper translation quiver with source ω and there is the additive function $h_{(H/{}_aH)}$ on it such that $h_{(H/{}_aH)}(\omega) = 1$ and $h_{(H/{}_aH)}(p) = \sum_{y\to p} h_{(H/{}_aH)}(y)$ for any projective vertex $p \neq \omega$ of $H/{}_aH$. We claim that ω is the unique source in $H/{}_aH$. If not, assume that ω_0 is another source. Then we have $h_{(H/{}_aH)}(\omega_0) = \sum_{y\to\omega_0} h_{(H/{}_aH)}(y) - h_{(H/{}_aH)}(\tau\omega_0) = 0$, contradiction. Now, let q be an injective vertex in $H/{}_aH$. If q is also an injective vertex in H, then $|q^+| = 1$ and $h_H(q) = h_H(y) = 1$ for $q^+ = \{y\}$. This implies that $h_{(H/{}_aH)}(q) = 1$ and $h_{(H/{}_aH)}(y) \leq 1$. As a result, $h_{(H/{}_aH)}(q) \geq h_{(H/{}_aH)}(y)$. If q is not an injective vertex in H, it is clear that $h_{(H/{}_aH)}(q) \geq \sum_{q\to y} h_{(H/{}_aH)}(y)$. This finishes the proof. $\square$

Remark. Let H be a thin left hammock, $p(a)$ a projective vertex. Then it is possible that the left hammocks ${}_aH$ and $H/{}_aH$ are not thin.

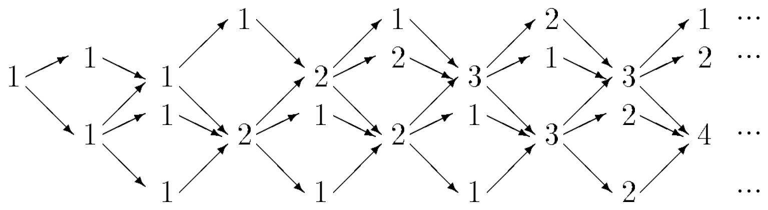
Example. Consider the following thin left hammock H



Then ${}_aH$ is as follows



and $H/{}_aH$ is as follows



we see that ${}_aH$ and $H/{}_aH$ are not thin.

For a poset $\mathcal{S}$ and a field k, we know that $\mathcal{P}_{\mathcal{S}}$, the preprojective component of the Auslander-Reiten quiver of $\ell(\mathcal{S})$, is a thin left hammock with finitely many projective vertices, with hammock function $h_H(X) = \dim_k X_\omega$. Thus, we have the following

Corollary 3.1 *Let $\mathcal{S}$ be a poset, k a field, h_H the hammock function of $\mathcal{P}_{\mathcal{S}}$. Then for $a \in \mathcal{S}$ we have*

(1). ${}_aH = \{X \in \mathcal{P}_{\mathcal{S}} \mid \mathrm{Hom}_{\mathcal{S}}(P_{\mathcal{S}}(a), X) \neq 0\} = \{X \in \mathcal{P}_{\mathcal{S}} \mid X_a \neq 0\}$ *is a left hammock with source $P_{\mathcal{S}}(a)$. The hammock function on $h_{({}_aH)}$ is $h_{({}_aH)}(X) = \dim_k X_a$.*

(2). $H/{}_aH = \{X \in \mathcal{P}_{\mathcal{S}} \mid h_H(X) - h_{({}_aH)}(X) \neq 0\} = \{X \in \mathcal{P}_{\mathcal{S}} \mid \dim_k X_\omega - \dim_k X_a \neq 0\}$ *is a left hammock with source ω. The hammock function on $H/{}_aH$ is $h_{(H/{}_aH)}(X) = h_H(X) - h_{({}_aH)}(X) = \dim_k X_\omega - \dim_k X_a$.*

Proposition 3.1 *Let k be a field. Let H be a left hammock, a and b thin projective vertices of H different from ω. Assume that $\mathrm{Hom}_{k(H)}(p(a), p(b)) \neq 0$. Then we have ${}_aH \supseteq {}_bH$ and ${}_b({}_aH) = {}_bH$, where ${}_aH \supseteq {}_bH$ means that ${}_bH$ is a translation subquiver of ${}_aH$.*

Proof. Since $\mathrm{Hom}_{k(H)}(p(a), p(b)) \neq 0$, we can choose some $0 \neq \varphi \in \mathrm{Hom}_{k(H)}(p(a), p(b))$. Then φ is monomorphism, since $p(a)$ is thin by Lemma 3.1. Now, for any $x \in {}_bH$, we take $0 \neq \phi \in \mathrm{Hom}_{k(H)}(p(b), x)$, then ϕ is monomorphism and $0 \neq \varphi\phi \in \mathrm{Hom}_{k(H)}(p(a), x)$, again using Lemma 3.1. As a consequence, ${}_aH \supseteq {}_bH$.

For given $p(a)$, we denote by ${}_a\mathcal{M}$ the object class in $k(H)$ whose objects x satisfy $\mathrm{Hom}_{k(H)}(p(a), x) = 0$. It is easy to see that $k({}_aH) = k(H)/{}_a\mathcal{M}$. Note that $\mathrm{Hom}_{k(H)}(p(b), x)_{{}_a\mathcal{M}} = 0$ for any vertex $x \in H$, since $k(H)$ is directing and $p(a)$ is a predecessor of $p(b)$. Thus we have

$$\begin{aligned}
{}_b({}_aH) &= \{x \in {}_aH \mid \operatorname{Hom}_{k({}_aH)}(p(b),x) \neq 0\} \\
&= \{x \in H \mid \operatorname{Hom}_{k(H)/{}_a\mathcal{M}}(p(b),x) \neq 0\} \\
&= \{x \in H \mid \operatorname{Hom}_{k(H)}(p(b),x)/\operatorname{Hom}_{k(H)}(p(b),x)_{{}_a\mathcal{M}} \neq 0\} \\
&= \{x \in H \mid \operatorname{Hom}_{k(H)}(p(b),x) \neq 0\} \\
&= {}_bH.
\end{aligned}$$

□

For a representation-finite poset $\mathcal{S}$ and $t \in \mathcal{S}$, we denote by ${}_tH$ the left hammock induced by $P_{\mathcal{S}}(t)$ as in Corollary 3.1. Now, we define a poset $\bar{\mathcal{S}} = \{\, {}_tH \mid t \in \mathcal{S}\}$, the poset formed by left hammocks ${}_tH, t \in \mathcal{S}$, ordered ${}_sH \geq {}_tH$ if and only if ${}_sH \supseteq {}_tH$. Then we have the following corollary.

Corollary 3.2 *There is an isomorphism of posets $\theta : \mathcal{S} \longrightarrow \bar{\mathcal{S}}$ given by sending t to ${}_tH$.*

Proof. Clearly θ is a bijective map. We need only to prove that $s \geq t$ if and only if ${}_sH \supseteq {}_tH$. Suppose that $s \geq t$ in $\mathcal{S}$. This means $\operatorname{Hom}_{k(H)}(P_{\mathcal{S}}(s), P_{\mathcal{S}}(t)) \neq 0$. Therefore Proposition 3.1 implies that ${}_sH \supseteq {}_tH$. On the other hand, if ${}_sH \supseteq {}_tH$, then $P_{\mathcal{S}}(t) \in {}_sH$, therefore $\operatorname{Hom}_{k(H)}(P_{\mathcal{S}}(s), P_{\mathcal{S}}(t)) \neq 0$, so $s \geq t$. □

The following Theorem extends the result shown as (1) in Theorem 3.1.

Theorem 3.2 *Let k be a field. Let H be a thin left hammock, z a thin vertex of H. Then ${}_zH = \{x \in H \mid \operatorname{Hom}_{k(H)}(z,x) \neq 0\}$ is a left hammock with source z. The hammock function is $h_{({}_zH)} = \dim_k \operatorname{Hom}_{k(H)}(z,-)$.*

Proof. Since z is thin, $\dim_k \operatorname{Hom}_{k(H)}(\omega, z) = 1$. So we can choose a nonzero path in $k(H)$ of the form $x_0 = \omega \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \rightarrow x_{n-1} \xrightarrow{\alpha_n} x_n = z$, where $x_i, 1 \leq i \leq n$, are vertices of H, and $\alpha_i, 1 \leq i \leq n$, are arrows in H. Now, x_1 is a direct successor of the source of H. Since H is thin, it follows that x_1 is a thin projective vertex in H. By Theorem 3.1 we obtain a left hammock ${}_{x_1}H$ with source x_1. Clearly, $x_1 \xrightarrow{\alpha_2} \cdots \rightarrow x_{n-1} \xrightarrow{\alpha_n} x_n = z$ is a non-zero path in $k({}_{x_1}H)$.

Since H is thin, we also know that there are no multiple arrows in $k(H)$. So there are no multiple arrows in $k({}_{x_1}H)$. Thus, x_2, as a direct successor of the source x_1 in ${}_{x_1}H$, is a thin projective vertex of ${}_{x_1}H$. By Theorem 3.1 again, we obtain a left hammock ${}_{x_2}({}_{x_1}H)$, and ${}_{x_2}({}_{x_1}H)$ is just ${}_{x_2}H = \{x \in H \mid \operatorname{Hom}_{k(H)}(x_2, x) \neq 0\}$, this follows from the proof of Proposition 3.1, and the hammock function is $h_{({}_{x_2}H)} = \dim_k \operatorname{Hom}_{k(H)}(x_2, -)$. Again, z is a thin vertex in ${}_{x_2}H$ and $x_2 \xrightarrow{\alpha_3} \cdots \rightarrow x_{n-1} \xrightarrow{\alpha_n} x_n = z$ is a nonzero path in $k({}_{x_2}H)$.

By induction on the length of the path from ω to z, we obtain a left hammock ${}_zH$ with source z. The hammock function is $h_{({}_zH)} = \dim_k \operatorname{Hom}_{k(H)}(z,-)$. □

Remark. Let H be a thin left hammock, z a thin vertex of H. Assume that z is not a projective vertex. Then clearly $\{x \in H \mid h_H(x) - h_{({}_zH)}(x) \neq 0\}$ is not a left hammock.

The following result is analogous to Proposition 3.1.

Proposition 3.2 *Let H be a thin left hammock, y and z thin vertices of H different from ω. Assume that* $\mathrm{Hom}_{k(H)}(y,z) \neq 0$. *Then we have* ${}_yH \supseteq {}_zH$ *and* ${}_z({}_yH) = {}_zH$.

Theorem 3.3 *Let H be a infinite left hammock. Assume that $p(a)$ is a thin projective vertex of H. Then the left hammock ${}_aH$ is also infinite.*

Proof. Let $\mathcal{P}(d)$ be the full subcategory of $\mathcal{P}(H,k)$ given by all projective vertices p with $p \in {}^dH$. We may identify $\mathcal{P}(d)$-mod with the full subcategory of $\mathcal{P}(H,k)$-mod given by all modules X with $X(p') = 0$ for all projective vertices $p' \notin {}^dH$. Note that we have the inclusions $\mathcal{P}(0)$-mod $\subseteq \mathcal{P}(1)$-mod $\subseteq \ldots \subseteq \mathcal{P}(d)$-mod $\subseteq \ldots$ and we have ${}_\infty\mathcal{P}(H,k)$-mod $= \bigcup_{d\in\mathbf{N}} \mathcal{P}(d)$-mod, since any object in $\mathcal{P}(H,k)$-mod is of finite length. Note that $\mathbf{M}(\omega)$ belongs to $\mathcal{P}(0)$-mod, and thus to all $\mathcal{P}(d)$-mod, where $\mathbf{M}$ is the equivalence functor $\mathbf{M} : k(H) \cong {}_\infty\mathcal{F}$. We denote by $\mathcal{F}(d)$ the set of all objects in $\mathcal{P}(d)$-mod with socle generated by $\mathbf{M}(\omega)$; thus we also have $\mathcal{F}(d) = \mathcal{F} \cap \mathcal{P}(d)$-mod. Let $Q_\omega(d)$ be the injective hull of $\mathbf{M}(\omega)$ in $\mathcal{P}(d)$-mod, we have $Q_\omega(d) \in \mathcal{F}(d)$-mod and the inclusions $Q_\omega(0) \subseteq Q_\omega(1) \subseteq \ldots \subseteq Q_\omega(d) \subseteq \ldots$ in $\mathcal{F}$.

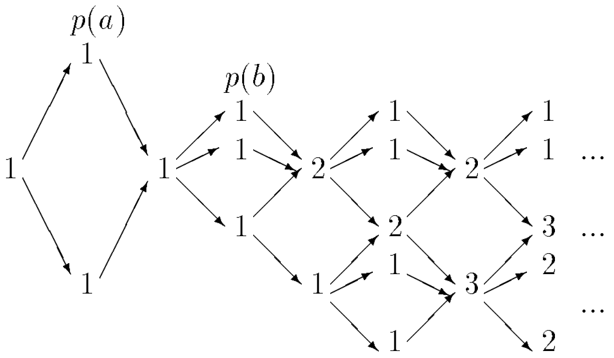
Consider a projective vertex $p(a) \neq \omega$, and assume that $p(a) \in {}^{d_0}H \backslash {}^{d_0-1}H$. In order to prove the result, it is sufficent to prove the following assertion:

$(*)$ For every $d_0 < d \in \mathbf{N}$, there is $x_d \in {}^dH\backslash{}^{d-1}H$, and $\psi_d \in \mathrm{Hom}_{\mathcal{F}(d)}(\mathbf{M}(p(a)), \mathbf{M}(x_d))$ and $\phi_d \in \mathrm{Hom}_{\mathcal{F}}(\mathbf{M}(x_d), Q_\omega(d))$ such that $\psi_d\phi_d \neq 0$.

We can show it by induction on d. In the case $d = d_0$, we only need to put $x_{d_0} = p(a)$. Now, assume the assertion holds for d. That is, there is $x_d \in ({}^dH)\backslash({}^{d-1}H), \psi_d \in \mathrm{Hom}_{\mathcal{F}(d)}(\mathbf{M}(p(a)), \mathbf{M}(x_d))$ and $\phi_d \in \mathrm{Hom}_{\mathcal{F}}(\mathbf{M}(x_d), Q_\omega(d))$ such that $\psi_d\phi_d \neq 0$. We denote by ι the embedding map $\iota : Q_\omega(d) \to Q_\omega(d+1)$ in $\mathcal{P}(d+1)$-mod. Let $\phi'_d = \phi_d\iota$, then $\psi_d\phi'_d \neq 0$. Let (f_{y_d}) be the maps $(f_{y_d}) : x_d \to \oplus_{y_d\in x_d^+} y_d$ in $k(H)$, then by [?], $\mathbf{M}(f_{y_d}) : \mathbf{M}(x_d) \to \oplus_{y_d\in x_d^+}\mathbf{M}(y_d)$ in $\mathcal{F}$, in fact in $\mathcal{F}(d+1)$, is a source map. So ϕ'_d can be lifted. That is, there are $(g_{y_d}) \in \mathrm{Hom}_{\mathcal{F}}(\bigoplus_{y_d\in x_d^+}\mathbf{M}(y_d), Q_\omega(d+1))$ such that $(f_{y_d})(g_{y_d}) = \phi'_d$. Thus $\psi_d(f_{x_d})(g_{x_d}) \neq 0$. As a result, there is some indecomposable summand of $\bigoplus_{y_d\in x_d^+}\mathbf{M}(y_d^+)$, say $\mathbf{M}(y_d^0)$, and a corresponding map, say $f_{y_d^0}$, such that $\psi f_{y_d^0} g_{y_d^0} \neq 0$. We put $x_{d+1} := y_d^0, \psi_{d+1} := \psi_d f_{y_d^0}$ and $\phi_{d+1} := g_{y_d^0}$, then it is clear $x_{d+1} \in {}^{d+1}H\backslash{}^dH, \psi_{d+1} \in \mathrm{Hom}_{\mathcal{F}(d+1)}(\mathbf{M}(p(a)), \mathbf{M}(x_{d+1}))$ and $\phi_{d+1} \in \mathrm{Hom}_{\mathcal{F}}(\mathbf{M}(x_{d+1}), Q_\omega(d+1))$. Thus the assertion is proved. $\square$

Remark. Let H be a infinite thin left hammock and $p(a)$ a projective vertex of H. Then $H/{}_aH$ may be finite.

Example. In the following infinite thin left hammock H,



we can see that $H/_aH$ is finite and $H/_bH$ is infinite.

4. *The case when a projective object is not thin*

Let H be a left hammock with finitely many projective vertices, and $A(H)$ the finite dimensional algebra corresponding to H. Let $\{e_\omega, e_a, e_b, \cdots\}$ be the set of primitive idempotents of $A(H)$. We would like to mention the fact that any indecomposable relative injective module in A-spmod is of the form $Q_A(a) = \tau_A B_A(a)$ or $Q_A(\omega') = I_A(\omega)$, where $B_A(a) = P_A(a)/\operatorname{soc}(P_A(a))$. An indecomposable projective A-module $P_A(a)$ is called *last projective module* if $\operatorname{Hom}_A(P_A(a), P_A(b)) = 0$ for $b \neq a$.

Lemma 4.1 *Let H be a left hammock with finitely many projective vertices, $A(H)$ the finite dimensional algebra corresponding to H. Assume that $P_A(a)$ is a last projective A-module. Then there is an irreducible map $Q_A(a) \xrightarrow{\chi} Q_A(\omega')$.*

Proof. Suppose that the result is false. Let us consider the Auslander-Reiten sequence $0 \to \tau_A Q_A(\omega') \to \bigoplus_{i\in I} M_i \to Q_A(\omega') \to 0$ in A-spmod, where $M_i, i \in I$ are indecoposable modules. Obviously, the M_i are relative injective modules by Lemma 2.2. Now, we know $\operatorname{Hom}_A(Q_A(a), Q_A(\omega')) \neq 0$ by Lemma 2.2 again. Let $0 \neq \psi \in \operatorname{Hom}_A(Q_A(a), Q_A(\omega'))$. Since ψ is a non-zero non-split epimorphism, ψ can be lifted to $\bigoplus_{i\in I} M_i$. This means that there is some relative injective module $Q_A(b)$ with $a \neq b$ such that $\operatorname{Hom}_A(Q_A(a), Q_A(b)) \neq 0$. So $\operatorname{Hom}_A(B_A(a), B_A(b)) \neq 0$. We choose $0 \neq \zeta \in \operatorname{Hom}_A(B_A(a), B_A(b))$. Let $P_A(a) \xrightarrow{\eta_a} B_A(a)$ and $P_A(b) \xrightarrow{\eta_b} B_A(b)$ be projective covers in A-mod. Then $\eta_a\zeta$ is not zero and can be lifted to $P_A(b)$. This implies that $\operatorname{Hom}_A(P_A(a), P_A(b)) \neq 0$. Thus the assumption that there is no irreducible map $Q_A(a) \to Q_A(\omega')$ leads to a contradiction, and the Lemma is proved. $\square$

Proposition 4.1 *Let H be a left hammock with finitely many projective vertices, $A(H)$ the finite dimensional algebra corresponding to H. Assume that $P_A(a)$ is a non-thin projective A-module. Then $Q_A(a)$ does not occur in the preprojective component $\mathcal{P}_{A\text{-}spmod}$.*

Proof. First we consider the case when $P_A(a)$ is a last projective module. If $Q_A(a) \in \mathcal{P}_{A\text{-spmod}}$, then $Q_A(\omega') \in \mathcal{P}_{A\text{-spmod}}$ follows from Lemma 4.1. Since $k(H) \cong \mathcal{P}_{A\text{-spmod}}$, we see that $\mathcal{P}_{A\text{-spmod}}$ is finite and $P_A(a)$ is thin. A contradiction.

Now we consider the case when $P_A(a)$ is not a last projective module. Let $e = \sum_{b\in J} e_b$, $J = \{b \mid \operatorname{Hom}_A(P_A(a), P_A(b)) \neq 0\}$. We put $D = A/AeA$. Then D-mod is naturally a full subcategory of A-mod and D-spmod is a full subcategory of A-spmod. Note that $P_A(a)$ is a non-thin last projective D-module and $B_A(a)$ is also a D-module. By induction, we see that $\tau_D B_A(a)$ does not occur in $\mathcal{P}_{D\text{-spmod}}$. Let $0 \to \tau_D B_A(a) \xrightarrow{f_1} N \xrightarrow{g_1} B_A(a) \to 0$ be the Auslander-Reiten sequence in D-mod, and $0 \to \tau_A B_A(a) \xrightarrow{f} M \xrightarrow{g} B_A(a) \to 0$ be the Auslander-Reiten sequence in A-mod. Then g_1 can be lifted to M. We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \tau_D B_A(a) & \xrightarrow{f_1} & N & \xrightarrow{g_1} & B_A(a) & \to & 0 \\ & & \downarrow \psi & & \downarrow \xi & & \| & & \\ 0 & \to & \tau_A B_A(a) & \xrightarrow{f} & M & \xrightarrow{g} & B_A(a) & \to & 0 \end{array}$$

It is easy to see that $\psi \neq 0$. This shows that $Q_A(a) = \tau_A B_A(a)$ does not occur in $\mathcal{P}_{A\text{-spmod}}$, since otherwise, there are only finitely many predecessors of $\tau_D B_A(a)$ in $\mathcal{P}_{A\text{-spmod}}$ and hence in $\mathcal{P}_{D\text{-spmod}}$. $\square$

Corollary 4.1 *Let H be a left hammock with finitely many projective vertices, $A(H)$ the finite dimensional algebra corresponding to H. Assume that $P_A(a)$ is a non-thin projective A-module and* $\operatorname{Hom}_A(P_A(a), P_A(b)) \neq 0$. *Then $Q_A(b)$ does not occur in the preprojective component $\mathcal{P}_{A\text{-}spmod}$.*

Proof. Since $\operatorname{Hom}_A(P_A(a), P_A(b)) \neq 0$, it follows that $\operatorname{Hom}_A(P_A(a)/\operatorname{soc} P_A(a), P_A(b)/\operatorname{soc} P_A(b)) \neq 0$ since $\operatorname{soc} P_A(a)$ and $\operatorname{soc} P_A(b)$ are generated by the simple projective module $P_A(\omega)$. In other words, $\operatorname{Hom}_A(B_A(a), B_A(b)) \neq 0$. So $\operatorname{Hom}_A(Q_A(a), Q_A(b)) \neq 0$ since $Q_A(a) = \tau_A B_A(a)$ and $Q_A(b) = \tau_A B_A(b)$. This shows that $Q_A(b)$ does not occur in $\mathcal{P}_{A-\mathrm{spmod}}$. $\square$

Theorem 4.1 *Let H be a left hammock with source ω, let h_H be the hammock function of H. Assume that $p(a)$ is a non-thin projective vertex of H. Then ${}_aH = \{x \in H \mid \operatorname{Hom}_{k(H)}(p(a), x) \neq 0\}$ is a left hammock with source $p(a)$. The hammock function on ${}_aH$ is $h_{({}_aH)} = \dim_k \operatorname{Hom}_{k(H)}(p(a), -)$.*

Proof. (1) It is clear that ${}_aH$ has a unique source $p(a)$;

(2) Let $0 \longrightarrow \mathbf{M}(\tau x) \longrightarrow \bigoplus_{y \to x} \mathbf{M}(y) \longrightarrow \mathbf{M}(x) \longrightarrow 0$ be a Auslander-Reiten sequence in $\mathcal{F}$, where $\mathbf{M}$ is the equivalence functor $\mathbf{M} : k(H) \cong_\infty \mathcal{F}$. We apply $\operatorname{Hom}_{\mathcal{P}(H,k)}(\mathbf{M}(P(a)), -)$ to this sequence and obtain $h_{({}_aH)}(x) + h_{({}_aH)}(\tau x) = \sum_{y \to x} h_{({}_aH)}(y)$.

(3) Now let q be an injective vertex of ${}_aH$. If q is not an injective of H, then there exists an exact sequence $o \to \mathbf{M}(q) \to \sum_{q \to y} \mathbf{M}(y) \to \mathbf{M}(\tau^- q) \to 0$. Apply the functor $\operatorname{Hom}_{\mathcal{P}(H,k)}(\mathbf{M}(p(a)), -)$ to this exact sequence. Note that q is an injective vertex of ${}_aH$, thus we have the exact sequence $0 \to \operatorname{Hom}_{\mathcal{P}(H,k)}(M(p(a)), M(q)) \to \sum_{q \to y} \operatorname{Hom} \mathcal{P}(H,k)(M(p(a)), M(y)) \to 0$. So $h_{({}_aH)}(q) \geq \sum_{q \to y} h_{({}_aH)}(y)$.

Assume that q is also an injective vertex of H, not a sink, and let $q \in {}^dH$. We consider the full subcategory $\mathcal{P}(d+2)$ of $\mathcal{P}(H,k)$ given by all projective vertices p with

$p \in {}^{d+2}H$. We can consider $\mathcal{P}(d+2)$-modules as $\mathcal{P}(H,k)$-modules. Since $\mathcal{P}(d+2)$ is a finite category, there is a finite dimensional algebra A with A-mod $\cong \mathcal{P}(d+2)$-mod, and $k({}^{d+2}H) \cong {}_{d+2}\mathcal{F} \cong {}_{d+2}(A\text{-}spmod)$. For this equivalence, we denote by $\mathbf{M}$ the corresponding equivalence functor. We can write $\mathbf{M}(p(a)) = P_A(a), \mathbf{M}(q(b)) = Q_A(b)$ with $a \neq b$ by Proposition 4.1. Using Corollary 4.1, it follows that $\mathrm{Hom}_A(P_A(a), P_A(b)) = 0$. Let $0 \to Q_A(b) \to X \to B_A(b) \to 0$ be the Auslander-Reiten sequence in A-mod and $Q_A(b) \to \bigoplus_{q\to y} \mathbf{M}(y)$ the source map. Then we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & Q_A(b) & \to & X & \to & B_A(b) & \to & 0 \\ & & \| & & \downarrow \chi & & & & \\ & & Q_A(b) & \to & \bigoplus_{y\in q^+} \mathbf{M}(y) & & & & \end{array}$$

Observe that $\mathrm{Hom}_A(P_A(a), B_A(b)) = 0$, since $\mathrm{Hom}_A(P_A(a), P_A(b)) = 0$, and χ is epimorphism. Thus we have $\dim_k \mathrm{Hom}_A(P_A(a), Q_A(b)) = \dim_k \mathrm{Hom}_A(P_A(a), X)$ $\geq \dim_k \mathrm{Hom}_A(P_A(a), \bigoplus_{q\to y} \mathbf{M}(y))$. In other words, $h_{({}_aH)}(q) \geq \sum_{q\to y} h_{({}_aH)}(y)$. So the Theorem is proved. $\square$

remark. Let H be a left hammock with hammock function of H. Let $p(a)$ is a projective vertex of H, let $H/{}_aH$ be the full translation subquiver $\{x \in H \mid h_H(x) - h_{({}_aH)}(x) \neq 0\}$. Then $H/{}_aH$ is a left hammock if and only if $p(a)$ is non-thin. In fact the sufficy follows from Theorem 3.1. For necessity, assume that $p(a)$ is non-thin. It is easy to see that $p(a)$ occurs in $H/{}_aH$ as a projective vertex, but $h_H(p(a)) - h_{({}_aH)}(p(a)) < \sum_{y\in p(a)^-}(h_H(y) - h_{({}_aH)}(y))$.

5. *Hammocks induced by an injective object*

Lemma 5.1 *Let k be a field. Let H be a left hammock with source ω, h_H the hammock function of H. Assume that $q(a) \neq \omega'$ is an injective vertex of H. Then we have* $\dim_k \mathrm{Hom}_{k(H)}(\omega, x) \geq \dim_k \mathrm{Hom}_{k(H)}(x, q(a))$ *for all $x \in H$.*

Proof. Assume that $q \in {}^dH$. We consider the full subcategory $\mathcal{P}(d+2)$ of $\mathcal{P}(H,k)$ given by all projective vertices p with $p \in {}^{d+2}H$. We can consider $\mathcal{P}(d+2)$-modules as $\mathcal{P}(H,k)$-modules. Since $\mathcal{P}(d+2)$ is a finite category, there is a finite dimensional algebra A with A-mod $\cong \mathcal{P}(d+2)$-mod, and $k({}^{d+2}H) \cong {}_{d+2}(\mathcal{P}(H,k)\text{-spmod}) \cong {}_{d+2}(A\text{-spmod})$. For this equivalence, we denote by $\mathbf{M}$ the corresponding equivalence functor. We can write $\mathbf{M}(p(a)) = P_A(a), \mathbf{M}(q(a)) = Q_A(b)$, $B_A(a) = P_A(a)/\operatorname{soc} P_A(a)$. Since $Q_A(a)$ is an injective vertex of H, $P_A(a)$ is thin according to Proposition 4.1. Let $0 \to P_A(\omega) \to P_A(a) \to B_A(a) \to 0$ be a minimal projective resolution. We have the following exact sequence $0 \to Q_A(a) \to I_A(\omega) \to I_A(a) \to 0$. Thus for $X \in {}_{d+2}(A\text{-spmod})$, we have $0 \to \mathrm{Hom}_A(X, Q_A(a)) \to \mathrm{Hom}_A(X, I_A(\omega)) \to \mathrm{Hom}_A(X, I_A(a)) \to 0$. Therefore $\dim_k \mathrm{Hom}_A(P_A(\omega), X) = \dim_k \mathrm{Hom}_A(X, I_A(\omega)) \geq \dim_k \mathrm{Hom}_A(X, Q_A(a))$. In other words, $\dim_k \operatorname{Hom}_{k(H)}(\omega, x) \geq \dim_k \operatorname{Hom}_{k(H)}(x, q(a))$ for $x \in {}^{d+2}H$. The inequality holds naturally for $x \in H \setminus {}^{d+2} H$, since $\operatorname{Hom}_{k(H)}(x, q(a)) = 0$. $\square$

Theorem 5.1 *Let k be a field. Let H be a left hammock with hammock function h_H. Assume that $q(a) \neq \omega'$ is an injective vertex of H. Then*

(1) *$H_a = \{x \in H \mid \operatorname{Hom}_{k(H)}(x, q(a)) \neq 0\}$ is a hammock with source ω and sink $q(a)$. The hammock function on H_a is $h_{(H_a)} = \dim_k \operatorname{Hom}_{k(H)}(-, q(a))$.*

(2) *$H/H_a = \{x \in H \mid h_H(x) - h_{(H_a)}(x) \neq 0\}$ is a left hammock with source $p(a)$. The hammock function on H/H_a is $h_{(H/H_a)} = h_H - h_{(H_a)}$.*

(3) *$H_a = H/{}_aH$ and ${}_aH = H/H_a$.*

Proof. Assume that $q \in {}^dH$. We consider the full subcategory $\mathcal{P}(d+2)$ of $\mathcal{P}(H, k)$ given by all projective vertices p with $p \in {}^{d+2}H$. Naturally, $\mathcal{P}(d+2)$-modules are $\mathcal{P}(H, k)$-modules. We have $\mathcal{P}(d+2)$-mod $\cong$ A-mod for some finite dimensional algebra A. Since $\mathcal{P}(d+2)$ is a finite category, we have $k({}^{d+2}H) \cong {}_{d+2}(\mathcal{P}(H, k)\text{-spmod}) \cong_{d+2} (A\text{-spmod})$. For this equivalence, we denote by $\mathbf{M}$ the corresponding equivalence functor. We can write $\mathbf{M}(p(a)) = P_A(a), \mathbf{M}(q(a)) = Q_A(b)$, $B_A(a) = P_A(a)/\operatorname{soc}(P_A(a))$. Since $Q_A(a)$ is an injective vertex of H, $P_A(a)$ is thin by Proposition 4.1. Let $0 \to P_A(\omega) \to P_A(a) \to B_A(a) \to 0$ be the minimal projective resolution. So we have the following exact sequence for $X \in A$-spmod $0 \to \operatorname{Hom}_A(B_A(a), X) \to \operatorname{Hom}_A(P_A(a), X) \to \operatorname{Hom}_A(P_A(\omega), X) \to \operatorname{Ext}^1_A(B_A(a), X) \to 0$. Note that $\operatorname{soc} X$ is generated by the simple projective module $P_A(\omega)$ and $B_A(a)$ has no composition factor which is isomorphismic to $P_A(\omega)$, so we see $\operatorname{Hom}_A(B_A(a), X) = 0$. The Auslander-Reiten formula and Lemma 2.2 imply that $\mathrm{D}\overline{\operatorname{Hom}}_A(X, \tau_A B_A(a)) = \mathrm{D}\operatorname{Hom}_A(X, \tau_A B_A(a))$. Hence $\dim_k \operatorname{Hom}_A(P_A(\omega), X) = \dim_k \operatorname{Hom}_A(P_A(a), X) + \dim_k \operatorname{Hom}_A(X, \tau_A B_A(a))$. In other word, $\dim_k \operatorname{Hom}_{k(H)}(p(\omega), x) = \dim_k \operatorname{Hom}_{k(H)}(p(a), x) + \dim_k \operatorname{Hom}_{k(H)}(x, q(a))$. This shows $H_a = H/{}_aH$, $h_{(H_a)} = h_{(H/{}_aH)}$ and ${}_aH = H/H_a$, $h_{({}_aH)} = h_{(H/H_a)}$. We have proved (3). Then (3) implies (1) and (2). $\square$

Corollary 5.1 *Let $\mathcal{S}$ be a poset, k a field, h_H the hammock function of $P_{\mathcal{S}}$. Then, for $Q_{\mathcal{S}}(a) \in P_{\mathcal{S}}$, we have:*

(1) *$H_a = \{X \in \mathcal{P}_{\mathcal{S}} \mid \operatorname{Hom}_{\mathcal{S}}(X_\omega, Q_{\mathcal{S}}(a)) \neq 0\} = \{X \in \mathcal{P}_{\mathcal{S}} \mid X_\omega / X_a \neq 0\}$ is a hammock with source $P_{\mathcal{S}}(\omega)$ and sink $Q_{\mathcal{S}}(a)$. The hammock function on H_a is $h_{(H_a)}(X) = \dim_k \operatorname{Hom}_{\mathcal{S}}(X, Q_{\mathcal{S}}(a)) = \dim_k X_\omega - \dim_k X_a$.*

(2) *$H/H_a = \{X \in \mathcal{P}_{\mathcal{S}} \mid h_H(X) - h_{(H_a)}(X) \neq 0\} = \{X \in \mathcal{P}_{\mathcal{S}} \mid \dim_k X_a \neq 0\}$ is a left hammock with source $P(a)$. The hammock function on H/H_a is $h_{(H/H_a)}(X) = h_H(X) - h_{(H_a)}(X) = \dim_k X_a$.*

(3) *$H_a = H/{}_aH$ and ${}_aH = H/H_a$.*

Proposition 5.1 *Let H be a left hammock, $q(a)$ and $q(b)$ injective vertices of H different from ω'. Assume that $\operatorname{Hom}_{k(H)}(q(a), q(b)) \neq 0$. Then we have $H_a \subseteq H_b$ and $(H_b)_a = H_a$.*

Proof. The proof of this Proposition is analogous to the proof of Proposition 3.1. $\square$

For a given representation-finite poset $\mathcal{S}$ and $t \in \mathcal{S}$, we denote by H_t the hammock induced by $Q_{\mathcal{S}}(t)$, as in Corollary 5.1. We define the poset $\tilde{\mathcal{S}} = \{H_t \mid t \in \mathcal{S}\}$ formed by the left hammocks $H_t, t \in \mathcal{S}$, with $H_s \geq H_t$ if and only if $H_s \subseteq H_t$. Then the following corollary is analogous to Corollary 3.2.

Corollary 5.2 *There is an isomorphisme of posets $\theta : \mathcal{S} \longrightarrow \tilde{\mathcal{S}}$ given by sending t to H_t.*

6. *Hammocks and the algorithm of Nazarova-Roiter*

Let $\mathcal{S}$ be a poset. Let a be a maximal element of $\mathcal{S}$. Following [**2**], one can construct a new poset ${}_a\partial\mathcal{S}$, whose vertices consist of (1) all elements of $\mathcal{S}$ except a; (2) new elements $s+t$, for any pair s, t such that s, t and a are mutually incomparable. The order relations on ${}_a\partial\mathcal{S}$ are the following (1) for $s, t \in \mathcal{S}$, the order follows from the order on $\mathcal{S}$; (2) if $x \leq s$ or $x \leq t$, then $x \leq s+t$; (3) if $x \geq s$ and $x \geq t$, then $x \geq s+t$; (4) $s+t \leq u+v$ if and only if first, $s \leq u$ or $s \leq v$, and second, $t \leq u$ or $t \leq v$. We say that ${}_a\partial\mathcal{S}$ is obtained from $\mathcal{S}$ by *differentiation with respect to the maximal element a.*

Theorem 6.1 *Let H be a hammock, $\mathcal{S}(H)$ the poset corresponding to H. Assume that $p(a)$ is a projective vertex of H different from ω. Then $\mathcal{S}({}_aH)$ is obtained from $\mathcal{S}(H)$ by a finite sequence of differentiations with respect to maximal elements in the sense of Nazarova-Roiter.*

More precisely, there is a finite sequence of vertices $x_1, x_2, \cdots, x_l = p(a)$ and a finite sequence of posets $\mathcal{S}_1 = \mathcal{S}(H), \mathcal{S}_2, \cdots, \mathcal{S}_l$ such that

(1) x_i is a maximal element of $\mathcal{S}_i$;

(2) $\mathcal{S}_i = {}_{x_{i-1}}\partial\mathcal{S}_{i-1}$ for $i = 2, \cdots, l$, that is, $\mathcal{S}_i$ is obtained from $\mathcal{S}_{i-1}$ by differentiation with respect to x_{i-1};

(3) $\mathcal{S}_l \cong \mathcal{S}({}_aH)$.

Let $\mathcal{S}$ be a poset with a maximal element a, let ${}_a\partial\mathcal{S}$ be the poset obtained from $\mathcal{S}$ by differentiation with respect to a. We recall the *differentiation functor* ${}_a\partial : \ell(\mathcal{S}) \to \ell({}_a\partial\mathcal{S})$, it is defined by $({}_a\partial(V))_\omega = V_a$, $({}_a\partial(V))_s = V_a \cap V_s$, $({}_a\partial(V))_{s+t} = V_a \cap (V_s + V_t)$, and ${}_a\partial(\phi) = \phi_a$. The following Theorem is due to Gabriel (see [2]).

Theorem 6.2 *Let $\mathcal{S}$ be a poset, let $a \in \mathcal{S}$ be a maximal element such that* $\mathrm{width}(\mathcal{S} \setminus a_\wedge) \leq 2$. *Assume that ${}_a\partial\mathcal{S}$ is obtained from $\mathcal{S}$ by differentiation with respect to a. Then ${}_a\partial$ induces an equivalence $\ell(\mathcal{S})/{}_a\mathcal{N} \cong \ell({}_a\partial\mathcal{S})$, where ${}_a\mathcal{N}$ is the object class of $\ell(\mathcal{S})$ consisting of all $\mathcal{S}$-spaces V with $V_a = 0$.*

Proof of Theorem 6.1. Since H is a hammock, we have $\mathrm{width}(\mathcal{S}(H)) \leq 3$. So $\mathrm{width}(\mathcal{S}(H) \setminus a_\wedge) \leq 2$ for each maximal element a.

In case the vertex a is a maximal element of $\mathcal{S}(H)$, we have $\ell(\mathcal{S}({}_aH)) \cong k({}_aH) \cong k(H)/{}_a\mathcal{M} \cong \ell(\mathcal{S})/{}_a\mathcal{N} \cong \ell({}_a\partial\mathcal{S})$. The hammock function is uniquely determined, so $\mathcal{S}({}_aH) \cong {}_a\partial\mathcal{S}$.

In case the vertex a is not a maximal element of $\mathcal{S}$, we can choose some vertex x_1 in H with (i) x_1 is a maximal element of $\mathcal{S}(H)$; (ii) $\mathrm{Hom}_{k(H)}(x_1, p(a)) \neq 0$. Then we consider ${}_{x_1}H$. Of coucse, $p(a)$ is a projective vertex of ${}_{x_1}H$. If a is not a maximal element in $\mathcal{S}({}_{x_1}H)$, we can choose some vertex x_2 in ${}_{x_1}H$ with (i) x_2 is a maximal element of $\mathcal{S}({}_{x_1}H)$; (ii) $\mathrm{Hom}_{k({}_{x_1}H)}(x_2, p(a)) \neq 0$.

By Proposition 3.1, ${}_{x_2}({}_{x_1}H) = {}_{x_2}H$. We consider the hammock ${}_{x_2}H$. In this way, we obtain a sequence of vertices: $x_1, x_2, \cdots, x_l = p(a)$ and a sequence of posets $\mathcal{S}_1 = \mathcal{S}(H), \mathcal{S}_i = \mathcal{S}({}_{x_i}H), i = 2, \cdots, l$, such that (i) x_i is a maximal element of $\mathcal{S}_i$, for $i = 1, \cdots, l$; (ii) $\mathrm{Hom}_{k({}_{x_{i-1}}H)}(x_i, p(a)) \neq 0$ for $i = 2, \cdots, l$. In fact, we obtain a non-zero path in $k(H)$ $\omega \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{l-1}} x_{l-1} \xrightarrow{\alpha_l} x_l = p(a)$ where $p(a) \in {}^lH \setminus {}^{l-1}H$. Thus ${}_{x_i}H = {}_{x_i}({}_{x_{i-1}}H)$ and $\mathcal{S}_i = {}_{x_{i-1}}\partial\mathcal{S}_{i-1}$, for $i = 2, \cdots, l$. In particular, ${}_aH = {}_a({}_{x_{l-1}}(\cdots {}_{x_2}({}_{x_1}H)\cdots))$ and $\mathcal{S}({}_aH) = {}_a\partial(\mathcal{S}({}_{x_{l-1}}H)) = ({}_a\partial)({}_{x_{l-1}}\partial)\mathcal{S}({}_{x_{l-2}}H) = \cdots = ({}_a\partial)({}_{x_{l-1}}\partial)\cdots({}_{x_2}\partial)({}_{x_1}\partial)\mathcal{S}(H)$. This completes the proof. $\square$

The differentiation algorithm with respect to minimal elements is analogous to the differentiation algorithm with respect to maximal elements. (see in [**7**] the remark following Lemma 2, also see [**12**]). We have following result which is analogous to Theorem 6.1.

Theorem 6.3 *Let H be a hammock, $\mathcal{S}(H)$ the poset corresponding to H. Assume that $q(a)$ is an injective vertex of H different from ω'. Then $\mathcal{S}(H_a)$ is obtained from $\mathcal{S}(H)$ by a finite sequence of differentiations with respect to minimal elements.*

More precisely, there is a finite sequence of vertices $x_1, x_2, \cdots, x_l = q(a)$ and a finite sequence of posets $\mathcal{S}_1 = \mathcal{S}(H), \mathcal{S}_2, \cdots, \mathcal{S}_l$ such that

(1) x_i is a minimal element of $\mathcal{S}_i$;

(2) $\mathcal{S}_i = \partial_{x_{i-1}}\mathcal{S}_{i-1}$ for $i = 2, \cdots, l$, that is, $\mathcal{S}_i$ is obtained from $\mathcal{S}_{i-1}$ by a differentiation with respect to the minimal element x_{i-1};

(3) $\mathcal{S}_l \cong \mathcal{S}(H_a)$.

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