ON THE LATTICE POINT PROBLEM FOR ELLIPSOIDS

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Abstract. For k-dimensional ellipsoids E with $k \geq 9$ we show that the number of lattice points in rE is approximated by the volume of rE up to an error of order $\mathcal{O}(r^{k-2})$, as r tends to infinity. The estimate is best possible and uniform in the class of ellipsoids with bounded ratios of lengths of axes. It improves the classical estimate $\mathcal{O}(r^{k-2k/(k+1)})$ due to Landau (1915).

1. Introduction and results

Let $\mathbf{R}^k$, $1 \leq k < \infty$ denote a real k-dimensional Euclidean space with scalar product $(\cdot,\cdot)$ and norm $|x|^2 = x_1^2 + \cdots + x_k^2$, for $x = (x_1, \ldots, x_k) \in \mathbf{R}^k$. Denote by $|x|_\infty = \max\{|x_j| : \ 1 \leq j \leq k\}$ the maximum-norm. Let $\mathbb{Z}^k$ be the standard lattice of points with integer coordinates in $\mathbf{R}^k$.

For a (measurable) set $B \subset \mathbf{R}^k$ let $\mathrm{vol} B$ denote the Lebesgue measure of B, and let $\mathrm{vol}_{\mathbb{Z}} B$ denote the lattice volume of B, that is the number of points in B with integer coordinates.

Let us consider a quadratic form,

$$Q(x) \overset{\text{def}}{=} (Qx, x), \qquad \text{for} \quad x \in \mathbf{R}^k,$$

where $Q : \mathbf{R}^k \to \mathbf{R}^k$ denotes a symmetric positive linear bounded operator, that is $Q(x) > 0$, for $x \neq 0$. Define the ellipsoid

$$E_s \overset{\text{def}}{=} \{x \in \mathbf{R}^k : \ Q(x) \leq s\}, \qquad \text{for} \quad s \geq 0.$$

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We shall prove the following

Theorem 1.1. *For $k \geq 9$,*

$$\Delta(s,Q) \overset{\text{def}}{=} \sup_{a\in\mathbf{R}^k} \Big| \frac{\operatorname{vol}_{\mathbb{Z}}(E_s+a)-\operatorname{vol}E_s}{\operatorname{vol}E_s}\Big| = \mathcal{O}\Big(\frac{1}{s}\Big), \qquad \textit{for} \quad s\geq 1.$$

The constant in $\mathcal{O}(s^{-1})$ may depend on k and the lengths of axes of the ellipsoid E_1 only.

An inspection of the proof of this result yields an explicit bound for the remainder term. Let

$$0 < q_1^2 \leq q_2^2 \leq \cdots \leq q_k^2$$

denote the eigenvalues of the operator Q.

Theorem 1.2. *For $k \geq 9$,*

$$\Delta(s,Q) \leq c\Gamma\Lambda s^{-1}, \qquad \textit{for} \quad s \geq q_1^2,$$

where $\Lambda = q_1^2(q_k/q_1)^k$ and $\Gamma = (q_k/q_1)^4$, and where the constant c may depend on the dimension k only.

Remark. For the dimension $k = 8$ we can prove that

$$\Delta(s,Q) \leq c q_8^{12} q_1^{-10} s^{-1} \ln^2(s+1), \qquad \textit{for} \quad s \geq q_1^2$$

with an absolute constant c.

For the class of *general* ellipsoids described above Landau (1915) obtained the estimate $\mathcal{O}(s^{-1+1/(1+k)})$. This result has been extended by Hlawka (1950) to convex bodies with smooth boundary and strictly positive Gaussian curvature. Hlawka's estimate has been recently improved by Krätzel and Nowak (1991,1992) to $\mathcal{O}(s^{-1+\lambda})$, where $\lambda = 5/6k+2$, for $k \geq 8$, and $\lambda = 12/14k+8$, for $3 \leq k \leq 7$.

Since the lower bound $\Delta(s,Q) = \Omega(s^{-1})$ holds for spheres (Q =Identity) (see, e.g. Fricker (1982)), Theorem 1.2 solves the problem of *uniform* error bounds for ellipsoids with arbitrary center provided that the dimension k is sufficiently large, i.e. $k \geq 9$.

The bound of Theorem 1.2 shows that the number of lattice points in an ellipsoid depends asymptotically only on the size of ellipsoid, i.e. only on radii of the largest inscribed and the smallest circumscribed spheres, and does not depend on assumptions like "rationality" or "orientation" of the ellipsoid that is on the conjugation class of Q under the action of $SL(k,\mathbb{Z})$.

For special ellipsoids a number of particular results is available. For example, the error bound $\mathcal{O}(s^{-1})$ holds for $k \geq 5$ for a fixed rational form Q (see Walfisz (1924), $k \geq 9$, and Landau (1924), $k \geq 5$). Jarnik (1928) proved the same bound for diagonal forms Q with arbitrary (non zero) real coefficients. Related results are due to Novák (1968), Diviš and Novák (1974). For a discussion see the monographs by Walfisz (1957), Fricker (1982) and Krätzel (1988).

Since our results were obtained by extending the methods for proving optimal rates of convergence in the Central Limit Theorem (CLT) for ellipsoids (Bentkus and Götze (1994)) we kept the probabilistic setup. Bounds for rates of convergence in the multivariate CLT for convex bodies seem to correspond to bounds in the lattice point problem for these bodies interpreting s as the number, say N, of random vector summands in the CLT. This fact is mentioned by Esseen (1945), who proved the rate $\mathcal{O}(s^{-1+1/(1+k)})$ for balls around the origin and random vectors with Identity covariance, a result similar to the result of Landau (1915) mentioned above. For sums taking values in a *lattice* and *special* ellipsoids the relation of these error bounds for the lattice point problem and the CLT has been made explicit in Yarnold (1972).

The results of Esseen have been extended to convex bodies by Matthes (1970), a result similar to that of Hlawka (1950).

The bound $\mathcal{O}(N^{-1})$, for $k \geq 5$ of Bentkus and Götze (1994) for ellipsoids with diagonal Q and random vectors with independent components (and with arbitrary distribution) is comparable to the results of Jarnik (1928). The proofs of these probabilistic results seem to be technically more involved compared to the number theoretic ones.

The basic steps of our proof consist of

(1) rewriting in Section 2 the lattice point approximation error as a difference of distribution functions of quadratic forms of sums of uniformly distributed random variables taking values in $\mathbb{Z}^k$ and $\mathbf{R}^k$
(2) using Fourier transforms and regularization (Lemma 8.1) to bound the error by Fourier *integrals* over characteristic functions (ch.f.)
(3) replacing the ch.f. of the quadratic form by the ch.f. of a bilinear form using the symmetrization inequality of Section 7 (Götze (1979)), which extends the inequality of Weyl (1915/16) and van der Corput (see, e.g. Graham and Kolesnik (1991))
(4) estimating the size and separating the location of maxima of the resulting ch.f. for $k \geq 9$ by means of inequality (5.1) in Section 5

Once the problem has been reformulated in terms of distribution functions in step (1), it is sufficient to assume that the quadratic form Q is nondegenerated (see the Remark in Section 2).

The symmetrization inequality in step (3) has been quite useful in the investigation of the convergence rates and Edgeworth expansions in the CLT in Hilbert and Banach spaces (see, for example the review paper Bentkus, Götze, Paulauskas and Račkauskas (1990)).

Let $A \geq 0$. Denote $\mu(\varepsilon) = (\varepsilon A^2)^{-k}$, for $\varepsilon \leq 1/A$, and $\mu(\varepsilon) = \varepsilon^k$, for $\varepsilon \geq 1/A$.

The inequality (5.1) in step (4) represents the essential tool of our proof. In conjunction with symmetrization Lemma 7.1 it yields the following bound

$$|S(t)S(t+\varepsilon)|^2 \le c(k)\Lambda^2\mu(\varepsilon), \qquad \text{for all} \quad t \in \mathbf{R} \quad \text{and} \quad \varepsilon \ge 0, \tag{1.1}$$

for trigonometric sums defined as

$$S(t) = \frac{1}{(2A+1)^{2k^2+2k}} \sum_{|m_1|_\infty \le A} \cdots \sum_{|m_{2k+2}|_\infty \le A} \exp\{itQ(m_1 + \cdots + m_{2k+2})\}, \tag{1.2}$$

for $t \in \mathbf{R}$, where the sums are taken over $m_j \in \mathbb{Z}^k$, for all $1 \le j \le 2k+2$.

Taking $t = 0$ the inequality (1.1) yields so called 'double large sieve' estimate for distributions on the lattice, see, e.g. Bombieri and Iwaniec (1986), which extends the sieve methods introduced by Brun (1920), Selberg (1947), Linnik (1941) and Rényi (1962).

We shall use the following notation.

By c (resp. C) with or without indices we shall denote generic constants (resp. sufficiently large generic constants) which may depend on the dimension k only. Dependence of constants on other parameters, say α, we indicate by $c(\alpha)$. We shall write $B \ll D$ instead of $B \le cD$. By $[B]$ we denote the integer part of a real number B. We shall denote $r = \sqrt{s}$ and $A = [r]$. If $r \ge 1$ then the natural number $A \ge 1$, and

$$A \le r \le 2A.$$

The random variables (r.v.) we introduce in this paper (all in capital letters) are: r.v. uniformly distributed on lattice points in a box in $\mathbf{R}^k$ (lattice r.v. for short); r.v. uniformly distributed in a box in $\mathbf{R}^k$; Gaussian r.v. or sums of independent copies of such r.v. The use of this probabilistic notion will significantly simplify the exposition of the proofs. For example, let M_1 denote a lattice r.v. uniformly distributed in the box $\{x \in \mathbf{R}^k : \ |x|_\infty \le A\}$, and let $M_2, M_3, \dots$ be independent identically distributed (i.i.d.) copies of M_1. Then we may rewrite the trigonometric sum (1.2) as

$$S(t) = \mathbf{E} \exp\{itQ(M_1 + \cdots + M_{2k+2})\},$$

where $\mathbf{E}$ denotes the expectation.

The distribution (or law) $L(X)$ of a r.v. X is defined as Borel measure on $\mathbf{R}^k$ such that

$$L(X)(B) = \mathbf{P}\{X \in B\} = \int_B L(X)(dx), \qquad \text{for measurable} \quad B \subset \mathbf{R}^k.$$

Let $\bar{X}$ denote an independent copy of X. This means that distributions of X and $\bar{X}$ are equal, and the independence condition is equivalent to

$$\mathbf{E}\, f(X, \bar{X}) = \int_{\mathbf{R}^k} \int_{\mathbf{R}^k} f(x,y) L(X)(dx) L(\bar{X})(dy),$$

for any measurable function f such that the Lebesgue integral exists.

We say that $\tilde{X}$ is symmetrization of X if the distribution of $\tilde{X}$ is equal to the distribution of $X - \bar{X}$. For example, we may choose $\tilde{X} = X - \bar{X}$.

All random variables are supposed to be independent, if the contrary is not clear.

By

$$\mathbf{E}_X f(X, Y, Z, \dots)$$

we shall denote the conditional expectation given all random variables $Y, Z, \dots$ but X. In other words, if random variables X and $Y, Z, \dots$ are independent, then the r.v.

$$\mathbf{E}_X f(X, Y, Z, \dots) = \int f(x, Y, Z, \dots) L(X)(dx).$$

Similarly, $\mathbf{P}_X A$ denotes the conditional probability of an event A given all random variables but X. Finally, by $\mathbf{I}\{A\}$ we denote the indicator function of event A.

2. Reduction to a probability setting; The Fourier transform

The ellipsoid $E_s + a = E(s, a, Q) = \{x \in \mathbf{R}^k : \ Q(x - a) \le s\}$ satisfies

$$E(s, a, Q) = E(\lambda s, a, \lambda Q), \qquad \text{for any} \quad \lambda > 0.$$

Therefore

$$\Delta(s, a, Q) = \left| \frac{\mathrm{vol}_{\mathbb{Z}} E(s, a, Q) - \mathrm{vol} E(s, a, Q)}{\mathrm{vol} E(s, a, Q)} \right|$$

satisfies

$$\Delta(s, a, Q) = \Delta(\lambda s, a, \lambda Q), \qquad \text{for any} \quad \lambda > 0, \tag{2.1}$$

as well.

Thus without loss of generality we may assume that $q_1 = 1$. Under this assumption we shall prove that

$$\Delta(s, a, Q) \ll s^{-1} \Gamma \Lambda, \qquad \text{for} \quad s \ge 1, \tag{2.2}$$

where now and henceforth $\Gamma = q_k^4$ and $\Lambda = q_k^k$. For operators Q with $q_1 \ne 1$ this estimate and (2.1) together imply Theorem 1.2.

Furthermore, we may and shall assume that in (2.2) $|a|_\infty \le 1/2$. Indeed, $\mathrm{vol}_{\mathbb{Z}}(E_s + a) = \mathrm{vol}_{\mathbb{Z}}(E_s + a - m)$, for any $m \in \mathbb{Z}^k$, and we may replace a in (2.2) by $a - m$, with some $m \in \mathbb{Z}^k$ such that $|a - m|_\infty \le 1/2$.

For a real $D \ge 0$ consider the cube ("box") $B(D) = [-D, D]^k$, and the box $B([D] + 1/2)$, where $[D]$ denotes the integer part of D. Then

$$(2[D] + 1)^k = \mathrm{vol}_{\mathbb{Z}} B([D] + 1/2) = \mathrm{vol}_{\mathbb{Z}} B(D) = \mathrm{vol} B([D] + 1/2).$$

We have as well that

$$\mathrm{vol}B\,(D) \le 2^k \mathrm{vol}B\,([D]+1/2) \quad \text{and} \quad \mathrm{vol}B\,(D) \ge \left(\tfrac{2}{3}\right)^k \mathrm{vol}B\,([D]+1/2)\,,$$

for $D \ge 1$. Thus different volumes of a given box are equivalent, up to a constant, provided the size of the box is larger than 1.

Recall that $q_1^2 \le q_2^2 \le \cdots \le q_k^2$ denote the eigenvalues of the operator Q, or in other words, $1/q_1 \ge 1/q_2 \ge \cdots \ge 1/q_k > 0$ denote the lengths of half-axes of the ellipsoid E_1. Due to the assumption $q_1 = 1$ the longest half-axis of E_1 has length 1. Obviously $B_E\,(r/q_k) \subset E_s \subset B_E\,(r)$, where $B_E(\rho) = \{x \in \mathbf{R}^k : \ |x| \le \rho\}$ denotes the Euclidean ball, and $B\left(r/(\sqrt{k}q_k)\right) \subset E_s \subset B\,(r)$.

Let $m = 6k + 10$, and let $X_* \in \mathbf{R}^k$ be a lattice random variable uniformly distributed in the box $B\,([mr]+1/2)$, that is

$$\mathbf{P}\{X_* \in C\} = \frac{\mathrm{vol}_{\mathbb{Z}} C \bigcap B\,([mr]+1/2)}{\mathrm{vol}B\,([mr]+1/2)}\,, \qquad \text{for} \quad C \subset \mathbf{R}^k.$$

Similarly, let Y_* denote a random variable uniformly distributed in the same box,

$$\mathbf{P}\{Y_* \in C\} = \frac{\mathrm{vol} C \bigcap B\,([mr]+1/2)}{\mathrm{vol}B\,([mr]+1/2)}\,, \qquad \text{for measurable} \quad C \subset \mathbf{R}^k.$$

We assume that $r \ge 1$. Thus $\mathrm{vol}B([mr]+1/2)/\mathrm{vol}E_s \ll m^k \ll 1$. The set $E_s + a \subset B([mr]+1/2)$ since $|a|_\infty \le 1/2$. Therefore we can rewrite (2.2) (and the result of Theorem 1.2) as

$$|\mathbf{P}\{X_* \in E_s + a\} - \mathbf{P}\{Y_* \in E_s + a\}| \ll \Gamma\Lambda s^{-1}, \qquad \text{for} \quad s \ge 1. \tag{2.3}$$

Notice that the distributions of X_* and Y_* depend on s.

Let $X_1 \in \mathbf{R}^k$ denote a lattice random variable uniformly distributed in the box $B\,(r)$, that is

$$\mathbf{P}\{X_1 \in C\} = \frac{\mathrm{vol}_{\mathbb{Z}} C \bigcap B\,(r)}{\mathrm{vol}B\,(A+1/2)}\,, \qquad \text{for} \quad C \subset \mathbf{R}^k.$$

Let $X_2, X_3, \ldots$ denote independent identically distributed (i.i.d.) copies of X_1, and define $X_{**} = X_1 + \cdots + X_{6k+6}$. Then, for any set $C \subset B\,(4r)$,

$$\mathbf{P}\{X_* \in C\} = \mathbf{P}\{X_* + X_{**} \in C\}. \tag{2.4}$$

Indeed,

$$\mathbf{P}\{X_* + X_{**} \in C\} = \mathbf{E}\,\mathbf{P}_{X_*}\{X_* \in C - X_{**}\}.$$

For $x \in C$, the triangle inequality and $|X_j|_\infty \le A \le r$ imply

$$|x - X_{**}|_\infty \le (6k + 10)r,$$

which means that for fixed $X_1, \ldots, X_{6k+6}$, both C and the shifted (random) set $C - X_{**}$ are subsets of the box $B\,(mr)$ with $m = 6k + 10$. The (random) shift X_{**} assumes integer values only (with probability 1). Therefore, due to the invariance of the distribution of X_* under shifts by *integer vectors* in the box $B\,(mr)$,

$$\mathbf{E}\,\mathbf{P}_{X_*}\{X_* \in C - X_{**}\} = \mathbf{E}\,\mathbf{P}\{X_* \in C\} = \mathbf{P}\{X_* \in C\},$$

which implies (2.4).

Similarly

$$\mathbf{P}\{Y_* \in C\} = \mathbf{P}\{Y_* + X_{**} \in C\}. \tag{2.5}$$

Due to (2.4) and (2.5) we may rewrite inequality (2.3) as

$$|\Delta(s)| \ll \Gamma\Lambda s^{-1}, \qquad\qquad \text{for} \quad s \ge 1, \tag{2.6}$$

where

$$\begin{aligned}\Delta(s) &\overset{\mathrm{def}}{=} \mathbf{P}\{X_* + X_{**} \in E_s + a\} - \mathbf{P}\{Y_* + X_{**} \in E_s + a\}\\ &= \mathbf{P}\{Q(X_* + X_{**} - a) \le s\} - \mathbf{P}\{Q(Y_* + X_{**} - a) \le s\}.\end{aligned}$$

This suggests to consider the distribution functions

$$F(x) = \mathbf{P}\{Q(X_* + X_{**} - a) \le x\} \qquad \text{and} \qquad G(x) = \mathbf{P}\{Q(Y_* + X_{**} - a) \le x\}.$$

Let

$$f(t) = \mathbf{E}\,\exp\{itQ(X_* + X_{**} - a)\} \quad \text{and} \quad g(t) = \mathbf{E}\,\exp\{itQ(Y_* + X_{**} - a)\}$$

denote the characteristic functions of F and G respectively.

Applying the smoothing Lemma 8.1 we have

$$|\Delta(s)| = |F(s) - G(s)| \le \int_{-1}^{1} |f(t) - g(t)|\,\frac{dt}{|t|} + R, \tag{2.7}$$

where

$$R \overset{\mathrm{def}}{=} \int_{-1}^{1} \bigl|f(t)\bigr|\,dt + \int_{-1}^{1} \bigl|g(t)\bigr|\,dt.$$

Relation (2.7) reduces the proof of (2.6) to the estimation of characteristic functions. In the next Section 3 we shall show that $R \ll \Lambda s^{-1}$, which is simpler than the estimation of the integral in (2.7). In Section 4 we shall conclude the proof of Theorem 1.2 by showing that a similar estimate holds for the integral in (2.7).

Throughout we shall write

$$\varphi(t) \overset{\text{def}}{=} \mathbf{E} \exp\{2it(QX, Y)\}, \tag{2.8}$$

where $X = \tilde{X}_2 + \cdots + \tilde{X}_{2k+1}$ and $Y = \tilde{X}_1$, and

$$\mu(t) \overset{\text{def}}{=} \begin{cases} (tA^2)^{-k}, & \text{for} \quad t \le A^{-1}, \\ t^k, & \text{for} \quad t \ge A^{-1}. \end{cases} \tag{2.9}$$

Remark. Once our problem we have reformulated in (2.6) as a problem of the estimation of distribution functions, we can consider the general case of conic sections instead of ellipsoids, that is we only need the assumption that the operator Q is invertible. Furthermore, we can consider the case of polynomials of second order, $P(x) = (Qx, x) + (L, x) + C$ with some $L \in \mathbf{R}^k$ and $C \in \mathbf{R}$, instead of quadratic forms; this leads to certain natural changes in estimating the integral in (2.7) for $|t| \le s^{-1+2/k}$.

3. An estimate of concentration functions of F and G

In this section we shall show that the remainder term R in (2.7) satisfies

$$R \ll \Lambda s^{-1}, \qquad \text{for} \quad s \ge 1. \tag{3.1}$$

Before proving (3.1) let us note that this estimate implies the following concentration inequality for the number of lattice points in any shifted "ring" $E_{s+1} \setminus E_s = \{x \in \mathbf{R}^k : \ s < Q(x) \le s+1\}$,

$$\sup_{a \in \mathbf{R}^k} \left| \frac{\operatorname{vol}_{\mathbb{Z}}\big((E_{s+1} \setminus E_s) + a\big)}{\operatorname{vol} E_s} \right| \ll \frac{\Lambda}{s}, \qquad \text{for} \quad s \ge 1.$$

This is a consequence of (3.1) and the following well-known estimate of the concentration of a distribution function, say F,

$$\sup_x \left\{ F\left(x + \tfrac{1}{T}\right) - F(x) \right\} \le \frac{2}{T} \int_{-T}^{T} |f(t)| dt, \qquad \text{for any} \quad T > 0,$$

where f denotes the characteristic function of F (see, e.g. Petrov (1975)).

Let us return to the estimation of R. We may split $R \le R_1 + R_2$, where

$$R_1 \le \int_{|t| \le s^{-2/k}} \Big(|f(t)| + |g(t)| \Big) dt$$

and

$$R_2 \le \int_{s^{-2/k} \le |t| \le 1} \Big(|f(t)| + |g(t)| \Big) dt.$$

To prove (3.1) it is sufficient to show that $R_1 \ll \Lambda s^{-1}$ and $R_2 \ll \Lambda s^{-1}$.

We shall use the symmetrization Lemma 7.1. Write

$$X_* + X_{**} - a = Y_1 + Y_2 + Y_3 + W_X,$$
$$Y_* + X_{**} - a = Y_1 + Y_2 + Y_3 + W_Y,$$

where $Y_1 = X_1 + \cdots + X_{2k}$, $Y_2 = X_{2k+1}$, $Y_3 = X_{2k+2}$, and where W_X and W_Y are defined as

$$W_X = X_* + X_{2k+3} + \cdots + X_{6k+6} - a,$$
$$W_Y = Y_* + X_{2k+3} + \cdots + X_{6k+6} - a.$$

Applying Lemma 7.1 (with $L = 0$ and $C = 0$), we obtain

$$|f(t)|^2 \le \varphi(t) \qquad \text{and} \qquad |g(t)|^2 \le \varphi(t), \tag{3.2}$$

where $\varphi(t)$ is defined by (2.8) (notice that $X_1, X_2, \dots$ are i.i.d. r.v. and therefore we may interchange X_1 with X_{2k+1} in the definition of φ).

Estimation of R_1. Let us apply (3.2). Then

$$R_1 \le 2 \int_{|t| \le s^{-2/k}} \sqrt{\varphi(t)}\, dt = 4 \int_0^{s^{-2/k}} \sqrt{\varphi(t)}\, dt. \tag{3.3}$$

Put $t = 0$ in the estimate of Lemma 5.1. Then

$$\varphi(t) \ll \Lambda^2 \mu(t), \qquad \text{for all} \quad t \ge 0, \tag{3.4}$$

where $\mu(t)$ is defined in (2.9). Splitting of the interval

$$[0, s^{-2/k}] = [0, A^{-2}] \cup [A^{-2}, A^{-1}] \cup [A^{-1}, s^{-2/k}],$$

the trivial bound $\varphi(t) \le 1$, for $0 \le t \le M^{-2}$, and the bound (3.4), for $t \ge M^{-2}$, allow to deduce from (3.3) that

$$R_1 \ll \frac{1}{s} + \int_{A^{-2}}^{A^{-1}} \frac{\Lambda\, dt}{(tA^2)^{k/2}} + \int_{A^{-1}}^{s^{-2/k}} \Lambda t^{k/2} dt \ll \frac{\Lambda}{s}.$$

Remark. One could derive the bound (3.4) using a modification of double large sieve method (see, for example, Bombieri and Iwaniec (1986), Graham and Kolesnik (1991)). This would be considerably simpler than to prove Lemma 5.1.

Estimation of R_2. Using (3.2) we get

$$R_2 \le 4 \int_{s^{-2/k}}^{1} \sqrt{\varphi(t)}dt \le 4 \int_{s^{-2/k}}^{1} \sqrt{\varphi(t)}\,\frac{dt}{t}. \tag{3.5}$$

We show in Section 5 that $\varphi(t)\varphi(t+\varepsilon) \ll \Lambda^2\mu(\varepsilon)$, for all $t \ge 0$ and $\varepsilon \ge 0$, with μ defined in (2.9). In Section 6 we show that this inequality implies that

$$\int_{s^{-2/k}}^{1} \sqrt{\varphi(t)}\,\frac{dt}{t} \ll \frac{\Lambda}{s}, \tag{3.6}$$

and the desired estimate of R_2 follows.

4. An estimate of the integral in (2.7)

In this section we shall show that the integral

$$I = \int_{-1}^{1} |f(t) - g(t)|\,\frac{dt}{|t|}$$

in (2.7) may be bounded as $I \ll \Gamma\Lambda/s$. The proof of this bound for $s^{-1+2/k} \le |t| \le 1$ is similar to considerations in Section 3. For $|t| \le s^{-1+2/k}$ we shall use Taylor expansions and symmetrization in order to reduce the problem to estimation of integrals like those in Section 3.

We have $I \ll J + R_3 + R_4$, where

$$J = \int_{|t|\le s^{-1+2/k}} |f(t) - g(t)|\,\frac{dt}{|t|},$$

and

$$R_3 = \int_{s^{-1+2/k}\le|t|\le s^{-2/k}} (|f(t)| + |g(t)|)\,\frac{dt}{|t|},$$

$$R_4 = \int_{s^{-2/k}\le|t|\le 1} (|f(t)| + |g(t)|)\,\frac{dt}{|t|}.$$

Estimation of R_3 is similar to the estimation of R_1 in Section 3. Proceeding as by the proof of (3.3) and using (3.4) we obtain

$$R_3 \ll \int_{s^{-1+2/k}\le t\le s^{-2/k}} \Lambda\sqrt{\mu(t)}\,\frac{dt}{t} \ll \frac{\Lambda}{s}.$$

Estimation of R_4 is similar to the estimation of R_2 in Section 3. Proceeding as by the proof of (3.5) and (3.6), we obtain

$$R_4 \ll \int_{s^{-2/k}}^{1} \sqrt{\varphi(t)}\,\frac{dt}{t} \ll \frac{\Lambda}{s}.$$

We shall show that

$$|f(t) - g(t)| \ll \Gamma(|t| + st^2)\sqrt{\varphi(t)}. \tag{4.1}$$

Therefore, using the bound (3.4), we get

$$J \ll \int_0^{s^{-1+2/k}} (1 + s|t|)\sqrt{\varphi(t)}dt \ll \frac{\Gamma\Lambda}{s},$$

and it remains to prove only (4.1).

Recall that X_* is a random variable uniformly distributed *on the lattice points* in the box $B([mr] + 1/2)$, where $m = 6k + 10$, and that Y_* is a random variable uniformly distributed in the same box. Introducing a random variable U uniformly distributed in the box $\{x \in \mathbf{R}^k : \ |x|_\infty \le 1/2\}$ we have (this relation is obvious and basic for our purposes)

$$Y_* \qquad \textit{has the same distribution as} \qquad X_* + U.$$

As before we assume independence of all random variables introduced, in particular, X_* and U are independent. Therefore we may write

$$f(t) = \mathbf{E}\,\exp\{itQ(X_* + X_{**} - a)\},$$
$$g(t) = \mathbf{E}\,\exp\{itQ(U + X_* + X_{**} - a)\}.$$

We shall expand into the Taylor series in powers of U. Introduce the (random) functions u and v by

$$u(x) \overset{\text{def}}{=} \exp\{v(x)\}, \quad \text{where} \quad v(x) \overset{\text{def}}{=} it\big(Q(x + X_* + X_{**} - a), x + X_* + X_{**} - a\big).$$

The function u is infinitely many times differentiable. We may write the Taylor series

$$u(x) = u(0) + u'(0)x + \theta(x), \tag{4.2}$$

where $\theta(x) = \mathbf{E}_\tau(1-\tau)u''(\tau x)x^2$, and where τ denotes a random variable uniformly distributed on the interval $[0, 1]$. Notice that

$$f(t) = \mathbf{E}\,u(0), \qquad\qquad g(t) = \mathbf{E}\,u(U),$$

and $\mathbf{E}_U u'(0)U = 0$ due to the symmetry of U. Thus (4.2) yields

$$|f(t) - g(t)| \ll |\mathbf{E}\, \theta(U)|. \tag{4.3}$$

Since $v(x)$ is a quadratic form in x, we have

$$\theta(x) = \mathbf{E}_\tau (1-\tau) u(\tau x)\big(b^2(x) + d(x)\big), \tag{4.4}$$

where

$$b(x) = v'(\tau x)x = 2it\big(Q(\tau x + X_* + X_{**} - a), x\big),$$
$$d(x) = v''(\tau x)x^2 = 2it(Qx, x).$$

Due to (4.3) and (4.4) the proof of (4.1) is reduced to showing that

$$|\mathbf{E}\,(1-\tau)u(\tau U)b^2(U)| \ll q_k^4 s t^2 \sqrt{\varphi(t)}, \tag{4.5}$$

and that

$$|\mathbf{E}\,(1-\tau)u(\tau U)d(U)| \ll q_k^2 |t| \sqrt{\varphi(t)}. \tag{4.6}$$

Let us prove (4.6). Recall that $X_{**} = X_1 + \cdots + X_{6k+6}$. As in Section 3 in estimation of R_1 we may write

$$\tau U + X_* + X_{**} = Y_1 + Y_2 + Y_3 + W,$$

where $Y_1 = X_1 + \cdots + X_{2k}$, $\quad Y_2 = X_{2k+1}$, $\quad Y_3 = X_{2k+2}$, and where W is defined as

$$W = \tau U + X_* + X_{2k+3} + \cdots + X_{6k+6} - a.$$

Using conditional expectations we have by Fubini's Theorem

$$\begin{aligned} &|\mathbf{E}\,(1-\tau)d(U)u(\tau U)| = |\mathbf{E}\,(1-\tau)d(U)\mathbf{E}_{Y_1,Y_2,Y_3} u(\tau U)| \\ &\le \mathbf{E}\,|d(U)||\mathbf{E}_{Y_1,Y_2,Y_3} \exp\{itQ(Y_1+Y_2+Y_3+W)\}| \\ &\ll |t|\sqrt{\varphi(t)} \end{aligned}$$

if we apply the symmetrization Lemma 7.1 and use $|d(U)| \ll q_k^2 |t||U|^2 \ll q_k^2 |t|$.

It remains to prove (4.5). Let us write

$$\tau U + X_* + X_{**} - a = Z_1 + Z_2 + Z_3 + W,$$

where $W = \tau U + X_* - a$, and where

$$Z_j = X_{j(2k+2)-2k-1} + \cdots + X_{j(2k+2)}, \qquad 1 \le j \le 3.$$

Notice that $b(U) = b(U, Z_1 + Z_2 + Z_3)$ is linear in Z_j, $1 \le j \le 3$, and that W is independent of Z_1, Z_2, Z_3. Therefore

$$|\mathbf{E}\,(1-\tau)b^2(U)v(\tau U)| \ll \sum_{1\le i_1,i_2\le 3} |\mathbf{E}\,(1-\tau)u(\tau U)b(U, Z_{i_1})b(U, Z_{i_2})|. \tag{4.7}$$

Each summand in the sum in (4.7) depends on two of the random variables Z_1, Z_2, Z_3 only. For notational simplicity, let Z_1 be the remaining one. Using conditional expectations we have

$$|\mathbf{E}\,(1-\tau)u(\tau U)b(U, Z_{i_1})b(U, Z_{i_2})| \ll \mathbf{E}\,|b(U, Z_{i_1})b(U, Z_{i_2})||\mathbf{E}_{\,Z_1} u(\tau U)|. \tag{4.8}$$

Notice that $Z_1 = X_1 + \dots + X_{2k+2}$. Thus

$$\mathbf{E}_{\,Z_1} u(\tau U) = \mathbf{E}_{\,Z_1} \exp\{itQ(X_1 + \dots + X_{2k+2} + Z_2 + Z_3 + W)\}.$$

Applying the symmetrization Lemma 7.1 with $L = 0$ and $C = 0$, with $Y_1 = X_1 + \dots + X_{2k}$, $Y_2 = X_{2k+1}$, $Y_3 = X_{2k+2}$, as well as with $Z_2 + Z_3 + W$ instead of W, we obtain

$$|\mathbf{E}_{\,Z_1} u(\tau U)| \ll \sqrt{\varphi(t)}. \tag{4.9}$$

The estimates (4.7)–(4.9) together imply

$$|\mathbf{E}\,(1-\tau)b^2(U)v(\tau U)| \ll \sqrt{\varphi(t)}\mathbf{E}\,\left(\sum_{j=1}^{3} |b(U, Z_j)|\right)^2. \tag{4.10}$$

Using

$$|\tau| \ll 1, \quad |U| \ll 1, \quad |a| \ll 1, \quad |X_*| \ll A, \quad |X_j| \ll A$$

we derive $|b(U, Z_j)| \ll q_k^4 |t| A$. Thus (4.10) and $A^2 \ll s$ yield (4.5).

5. The basic inequality for characteristic functions

In this section we shall prove the following basic inequality.

Lemma 5.1. *For all $t \ge 0$ and $\varepsilon \ge 0$ we have*

$$\varphi(t)\varphi(t+\varepsilon) \ll \Lambda^2 \mu(\varepsilon). \tag{5.1}$$

Recall that the functions φ and μ are defined by (2.8) and (2.9), and that $\Lambda = q_k^k$.

Before proving Lemma 5.1 – the main result of the section, we introduce some notation and prove several technical Lemmas.

Instead of

$$\sum_{m\in\mathbb{Z}^k,\ |m|_\infty \le D} f(m)$$

we shall write $(1/p_D)\mathbf{E}\,f(M)$, where $p_D \stackrel{\text{def}}{=} \mathbf{P}\,\{M = 0\} = (2[D]+1)^{-k}$, and where M denotes a r. v. uniformly distributed on the lattice points in the box $B\,(D)$.

Lemma 5.2 *(symmetrization–desymmetrization)*. *Let M, N and J denote lattice r.v. uniformly distributed in boxes $B(D)$, $B(2D)$ and $B(4D)$ respectively, for some integer $D \geq 0$. Then*

$$\mathbf{E}\, f(M) \leq 4^k \mathbf{E}\, f(\tilde{N}) \leq 8^k \mathbf{E}\, f(J),$$

for any nonnegative function $f : \mathbb{Z}^k \to [0, \infty)$.

Proof. Notice that

$$p \overset{\text{def}}{=} \mathbf{P}\{M = 0\} = (2D+1)^{-k}, \qquad q \overset{\text{def}}{=} \mathbf{P}\{N = 0\} = (4D+1)^{-k}.$$

For $s \in \mathbb{Z}^k$ such that $|s|_\infty \leq D$, we have

$$\mathbf{P}\{\tilde{N} = s\} = \mathbf{E}_{\bar{N}} \mathbf{P}_N\{N = s + \bar{N}\} \geq q \mathbf{E}_{\bar{N}} \mathbf{I}\{|\bar{N}|_\infty \leq D\} = q^2 (2D+1)^k.$$

Therefore

$$\mathbf{E}\, f(M) = \sum_{|s|_\infty \leq D} p f(s) \leq p q^{-2} (2D+1)^{-k} \sum_{|s|_\infty \leq D} \mathbf{P}\{\tilde{N} = s\} f(s) \leq 4^k \mathbf{E}\, f(\tilde{N}).$$

Obviously $\mathbf{P}\{\tilde{N} = s\} \leq q$, and

$$\mathbf{E}\, f(\tilde{N}) \leq q \sum_{|s|_\infty \leq 4D} f(s) \leq 4^k \mathbf{E}\, f(J)$$

since $q / \mathbf{P}\{J = 0\} \leq 4^k$, which concludes the proof of the Lemma.

Recall that $X = \tilde{X}_2 + \cdots + \tilde{X}_{2k+1}$ and $Y = \tilde{X}_1$, where $X_1, X_2, \ldots$ is a sequence of i.i.d. r.v. uniformly distributed on the lattice points in the box $B(r)$.

Lemma 5.3. *Let $\tilde{N} = N - \bar{N}$ be a symmetrization of a lattice r.v. N uniformly distributed in the box $B(4A)$. Write $\psi_t(z) \overset{\text{def}}{=} \mathbf{E} \exp\{it(Qz, X)\}$, for $z \in \mathbf{R}^k$. Then*

$$\varphi(t)\varphi(t+\varepsilon) \leq 16^k \mathbf{E}\, \psi_t(\tilde{N} + Y) \psi_{t+\varepsilon}(Y). \tag{5.2}$$

Proof. Fix a $z \in \mathbb{Z}^k$. Let us prove that

$$\mathbf{E}\, \psi_t(Y) \leq 16^k \mathbf{E}\, \psi_t(\tilde{N} + z), \qquad \text{whenever} \quad |z|_\infty \leq 2A. \tag{5.3}$$

Let J denote a lattice r.v. uniformly distributed in the box $B(2A)$. Notice that $Y = X_1 - \bar{X}_1$ is a symmetrization of lattice r.v. X_1 uniformly distributed in the box $B(A)$. Desymmetrizing and symmetrizing by Lemma 5.2 we have

$$\mathbf{E}\, \psi_t(Y) \leq 4^k \mathbf{E}\, \psi_t(J) = 4^k (4A+1)^{-k} \sum_{|j|_\infty \leq 2A} \psi_t(j)$$

$$\le 4^k(4A+1)^{-k} \sum_{|n|_\infty \le 4A} \psi_t(n+z) \le 16^k \mathbf{E}\, \psi_t(\tilde N + z),$$

which implies (5.3).

Let us prove (5.2). Using (5.3) we have ($\bar Y$ denotes an independent copy of Y)

$$\varphi(t)\varphi(t+\varepsilon) = \mathbf{E}\left(\mathbf{E}_{\bar Y}\psi_t(\bar Y)\right)\psi_{t+\varepsilon}(Y)$$

(apply (5.3) with $\bar Y$ instead of Y and with $z = Y$)

$$\le 16^k \mathbf{E}\left(\mathbf{E}_{\tilde N}\psi_t(\tilde N + Y)\right)\psi_{t+\varepsilon}(Y) = 16^k \mathbf{E}\, \psi_t(\tilde N + Y)\psi_{t+\varepsilon}(Y),$$

which concludes the proof of the Lemma.

Lemma 5.4. *For $z = (z_1, \dots, z_k) \in \mathbf{R}^k$,*

$$\mathbf{E}\, \exp\{i(z, X)\} \ll \sum_{m \in \mathbb{Z}^k} \prod_{j=1}^{k} \left(\frac{1}{1 + A^2(z_j - 2\pi m_j)^2}\right)^{2k}.$$

Proof. A coordinate (say ξ) of the r. v. X_1 is a random variable uniformly distributed on integers in the interval $[-A, A]$. Thus its characteristic function is given by $\mathbf{E}\, \exp\{it\xi\} = (2A+1)^{-1} D_A(t)$, for $t \in \mathbf{R}$ with the Dirichlet kernel

$$D_A(t) = \sum_{n=-A}^{A} \exp\{itn\} = \frac{\sin\bigl(t(A+1/2)\bigr)}{\sin(t/2)}.$$

The random variable X has independent identically distributed coordinates and is a symmetrization of $X_2 + \cdots + X_{2k+1}$. Therefore

$$\mathbf{E}\, \exp\{i(z, X)\} = \prod_{j=1}^{k} \{u(z_j)\}^{2k} \qquad \text{with} \quad u(x) = (2A+1)^{-2} D_A^2(x). \tag{5.4}$$

The function $u(x)$ is even and 2π-periodic. Since $u(x) \le 1$, $\sin y \le 1$ and $\sin(x/2) \ge x/\pi$, for $0 \le x \le \pi$, it follows that

$$u(x) \le \frac{5}{1 + A^2 x^2}, \qquad \text{for} \quad 0 \le |x| \le \pi.$$

By the periodicity of u,

$$\{u(x)\}^{2k} \ll \sum_{m \in \mathbb{Z}} \frac{\mathbf{I}\{|x - 2\pi m| \le \pi\}}{\left(1 + A^2(x - 2\pi m)^2\right)^{2k}} \ll \sum_{m \in \mathbb{Z}} \frac{1}{\left(1 + A^2(x - 2\pi m)^2\right)^{2k}},$$

which together with (5.4) imply the result of the Lemma.

Lemma 5.5. *Let $B > 0$. Assume that $\theta \leq 10k$. Then*

$$S \overset{\text{def}}{=} \sup_{z \in \mathbf{R}^k} \sum_{n \in \mathbb{Z}^k} \Big(\frac{1}{1 + B^2 |z-n|^2} \Big)^{\theta} \ll \Big(1 + \frac{1}{B} \Big)^k, \qquad \textit{for} \quad 2\theta \geq k+1, \tag{5.5}$$

and

$$S_0 \overset{\text{def}}{=} \sum_{n \neq 0,\ n \in \mathbb{Z}^k} \Big(\frac{1}{1 + A^2 |n|^2} \Big)^{\theta} \ll A^{-\theta}, \qquad \textit{for} \quad \theta \geq k. \tag{5.6}$$

Proof. Let us prove (5.5). Notice that $|z-n|^2 \geq (z_j - n_j)^2$, for $1 \leq j \leq k$. Therefore

$$\begin{aligned} S &\leq \sup_{z \in \mathbf{R}^k} \sum_{n \in \mathbb{Z}^k} \prod_{j=1}^{k} \Big(\frac{1}{1 + B^2 (z_j - n_j)^2} \Big)^{\theta/k} \\ &= \sup_{z \in \mathbf{R}^k} \prod_{j=1}^{k} \sum_{n=-\infty}^{\infty} \Big(\frac{1}{1 + B^2 (z_j - n)^2} \Big)^{\theta/k} \\ &\leq \Big(\sup_{z \in \mathbf{R}} F(z) \Big)^k, \end{aligned}$$

where

$$F(z) = \sum_{n=-\infty}^{\infty} \Big(\frac{1}{1 + B^2 (z-n)^2} \Big)^{\theta/k}.$$

Since the function $F(z)$ has period 1, we may assume that $|z| \leq 1/2$. We have (notice that $|z-n| \geq |n| - |z| \geq |n|/2$, for $n \neq 0$)

$$\begin{aligned} F(z) &\leq 1 + \sum_{n \neq 0} \Big(\frac{1}{1 + B^2 (z-n)^2} \Big)^{\theta/k} \ll 1 + \sum_{n=1}^{\infty} \Big(\frac{1}{1 + B^2 n^2} \Big)^{\theta/k} \\ &\ll 1 + \int_0^{\infty} \frac{du}{(1 + B^2 u^2)^{\theta/k}} = 1 + \frac{1}{B} \int_0^{\infty} \frac{du}{(1 + u^2)^{\theta/k}} \ll 1 + \frac{1}{B}. \end{aligned}$$

Collecting the estimates, we obtain (5.5).

Let us prove (5.6). Obviously

$$S_0 \leq k \sum_{n_1 \neq 0} \sum_{(n_2, \dots, n_k) \in \mathbb{Z}^{k-1}} \Big(\frac{1}{1 + A^2 |n|^2} \Big)^{\theta}.$$

Notice that $1 + A^2 |n|^2 \geq A^2 n_1^2$ and $1 + A^2 |n|^2 \geq 1 + A^2 (n_2^2 + \cdots + n_k^2)$. Therefore

$$S_0 \ll A^{-\theta} \sum_{n=1}^{\infty} n^{-\theta} \sum_{m \in \mathbb{Z}^{k-1}} \Big(\frac{1}{1 + A^2 |m|^2} \Big)^{\theta/2} \ll A^{-\theta}$$

by (5.5) and $A \geq 1$.

Proof of Lemma 5.1. In order to prove (5.1) it suffices to show that

$$\varphi(t)\varphi(t+\varepsilon) \ll \Lambda^2 A^{-k} + \Lambda^2 \mu(\varepsilon). \tag{5.7}$$

Indeed, the minimal value of $\mu(\varepsilon)$ is equal to $A^{-k} = \mu(A^{-1})$.

Let us apply first Lemma 5.3 followed by Lemma 5.4. We get

$$\varphi(t)\varphi(t+\varepsilon) \ll \mathbf{E}\,\psi_t(\tilde{N}+Y)\psi_{t+\varepsilon}(Y) \ll \mathbf{E} \sum_{m\in\mathbb{Z}^k} \sum_{q\in\mathbb{Z}^k} J(m,q),$$

where

$$J(m,q) \stackrel{\text{def}}{=} \prod_{j=1}^{k} \left(\frac{1}{(1+A^2(u_j-2\pi m_j)^2)(1+A^2(v_j-2\pi q_j)^2)} \right)^{2k},$$

and where $u = tQ(\tilde{N}+Y)$ and $v = (t+\varepsilon)QY$.

Several times applying the inequality $(1+a)(1+b) \ge 1+a+b$, for $a \ge 0$, $b \ge 0$, we obtain

$$\prod_{j=1}^{k} (1+x_j^2) \ge 1+|x|^2, \qquad \text{for} \quad x \in \mathbf{R}^k,$$

and

$$J(m,q) \le (1+A^2|u-2\pi m|^2 + A^2|v-2\pi q|^2)^{-2k}.$$

Using the identity $2a^2+2b^2 = (a+b)^2+(a-b)^2$ we have

$$J(m,q) \ll I(m+q, m-q),$$

where

$$I(m+q,m-q) \stackrel{\text{def}}{=} \big(1+A^2|u+v-2\pi m-2\pi q|^2 + A^2|u-v-2\pi m+2\pi q|^2\big)^{-2k}.$$

Note that

$$\sum_{m\in\mathbb{Z}^k} \sum_{q\in\mathbb{Z}^k} I(m+q,m-q) \le \sum_{m\in\mathbb{Z}^k} \sum_{q\in\mathbb{Z}^k} I(m,q).$$

Indeed, the map $(m,q) \longmapsto (m+q,m-q) : \mathbb{Z}^k \times \mathbb{Z}^k \to \mathbb{Z}^k \times \mathbb{Z}^k$ is injective, and the last sum has more summands than the previous one.

Collecting the bounds of J and I we get

$$\varphi(t)\varphi(t+\varepsilon) \ll \mathbf{E} \sum_{m\in\mathbb{Z}^k} \sum_{q\in\mathbb{Z}^k} I(m,q) \tag{5.8}$$

with $I(m,q) = \big(1+W^2+A^2|u+v-2\pi q|^2\big)^{-2k}$, where $W^2 = A^2|u-v-2\pi m|^2$.

Denote $B^2 \overset{\text{def}}{=} A^2/(1+W^2) \geq 1/(1+W^2)$. Applying the inequality (5.5) of Lemma 5.5 we obtain

$$\begin{aligned}\sum_{q\in\mathbb{Z}^k} I(m,q) &\ll (1+W^2)^{-2k} \sup_{z\in\mathbf{R}^k} \sum_{q\in\mathbb{Z}^k} \Big(\frac{1}{1+B^2|z-q|^2}\Big)^{2k} \\ &\ll (1+W^2)^{-2k}\Big(1+\frac{1}{B}\Big)^k \ll \frac{1}{(1+W^2)^k}. \end{aligned} \tag{5.9}$$

Inequalities (5.8) and (5.9) together imply

$$\varphi(t)\varphi(t+\varepsilon) \ll \mathbf{E} \sum_{m\in\mathbb{Z}^k} \frac{1}{(1+A^2|tQ\tilde N - \varepsilon QY - 2\pi m|^2)^k}. \tag{5.10}$$

The random variables $\tilde N$ and Y are bounded, $|\tilde N|_\infty \leq 8A$ and $|Y|_\infty \leq 2A$. Furthermore, $|Qx| \leq q_k^2|x|$ and $|x| \leq \sqrt{k}|x|_\infty$. Therefore $|tQ\tilde N| \leq D$, and $|\varepsilon QY| \leq D$,where $D \overset{\text{def}}{=} 8(1+t+\varepsilon)\sqrt{k}q_k^2 A$. Consequently,

$$|m|_\infty > D \Longrightarrow |tQ\tilde N - \varepsilon QY - 2\pi m| \geq 2\pi|m| - 2D \geq (2\pi-2)|m|$$

since $|m| \geq |m|_\infty > D$. Applying inequality (5.6) of Lemma 5.5 we have

$$\begin{aligned} &\mathbf{E} \sum_{|m|_\infty > D} \frac{1}{(1+A^2|tQ\tilde N - \varepsilon QY - 2\pi m|^2)^k} \\ &\ll \sum_{m\neq 0} \frac{1}{(1+A^2|m|^2)^k} \\ &\ll A^{-k}. \end{aligned} \tag{5.11}$$

Thus the estimates (5.10) and (5.11) together imply $\varphi(t)\varphi(t+\varepsilon) \ll A^{-k} + I_1$, where

$$I_1 \overset{\text{def}}{=} \mathbf{E} \sum_{|m|_\infty \leq D} \frac{1}{(1+A^2|tQ\tilde N - \varepsilon QY - 2\pi m|^2)^k}.$$

Writing the sum in the formula for I_1 as expectation and applying Lemma 5.2 to symmetrize the result we get

$$I_1 \ll I_2 \overset{\text{def}}{=} (1/p_D)\mathbf{E} \frac{1}{(1+A^2|tQ\tilde N - \varepsilon QY - 2\pi \tilde M|^2)^k}, \tag{5.12}$$

where M denotes a lattice r.v. uniformly distributed in the box $B(D)$.

Notice that $\mathbf{E}\exp\{-W\eta^2/2\} = 1/\sqrt{1+W}$, for $W \geq 0$, where η denotes a standard normal $N(0,1)$ random variable. For a sequence $\eta_1, \eta_2, \dots$ of independent copies of η, let $\xi \stackrel{\mathrm{def}}{=} \eta_1^2 + \cdots + \eta_{2k}^2$. Then

$$\mathbf{E}\exp\{-W^2\xi/2\} = 1/(1+W^2)^k. \tag{5.13}$$

Applying this formula to I_2 in (5.12) we get

$$I_2 = (1/p_D)\mathbf{E}\exp\{-\frac{\xi A^2}{2}|tQ\tilde{N} - \varepsilon QY - 2\pi\tilde{M}|^2\}.$$

Let G denote a centered Gaussian r.v. assuming values in $\mathbf{R}^k$ with identity covariance operator. Then

$$\mathbf{E}\exp\{i(x,G)\} = \exp\{-|x|^2/2\}, \qquad \text{for} \quad x \in \mathbf{R}^k. \tag{5.14}$$

Thus

$$\begin{aligned} I_2 &= (1/p_D)\mathbf{E}\exp\{iA\sqrt{\xi}(G, tQ\tilde{N} - \varepsilon QY - 2\pi\tilde{M})\} \\ &= (1/p_D)\mathbf{E}\, f_1(tA\sqrt{\xi}QG) f_2(\varepsilon A\sqrt{\xi}QG) f_3(2\pi A\sqrt{\xi}G), \end{aligned}$$

where

$$\begin{aligned} f_1(z) &= \mathbf{E}_{\tilde{N}}\exp\{i(z,\tilde{N})\}, \\ f_2(z) &= \mathbf{E}_{Y}\exp\{-i(z,Y)\}, \\ f_3(z) &= \mathbf{E}_{\tilde{M}}\exp\{-i(z,\tilde{M})\} \end{aligned}$$

denote the characteristic functions of $\tilde{N}$, Y and $\tilde{M}$ respectively. These characteristic functions are nonnegative since $\tilde{N}$, Y and $\tilde{M}$ are symmetrizations. Using $f_1 \leq 1$ and applying (5.14) again, we obtain

$$I_2 \ll (1/p_D)\mathbf{E}\exp\{-\frac{\xi A^2}{2}|\varepsilon QY - 2\pi\tilde{M}|^2\}$$

(apply (5.13))

$$= (1/p_D)\mathbf{E}\,\frac{1}{(1 + A^2|\varepsilon QY - 2\pi\tilde{M}|^2)^k}$$

(desymmetrize $\tilde{M}$ by Lemma 5.2)

$$\ll I_3 \stackrel{\mathrm{def}}{=} \mathbf{E}\sum_{m\in\mathbb{Z}^k}\frac{1}{(1 + A^2|\varepsilon QY - 2\pi m|^2)^k}.$$

Notice that $|\varepsilon QY| \le H$, where $H \overset{\text{def}}{=} 2q_k^2\sqrt{k}\varepsilon A$. Similar arguments as for the proof of (5.11) show that

$$\mathbf{E} \sum_{|m|_\infty > H} \frac{1}{(1 + A^2|\varepsilon QY - 2\pi m|^2)^k} \ll A^{-k}.$$

Therefore

$$I_3 \ll A^{-k} + I_4, \quad \text{where} \quad I_4 \overset{\text{def}}{=} \mathbf{E} \sum_{|m|_\infty \le H} \frac{1}{(1 + A^2|\varepsilon QY - 2\pi m|^2)^k}.$$

The number of summands (say n) in the last sum is bounded from above by $n \ll \max\{1; H^k\} \ll \Lambda^2 \max\{1; (\varepsilon A)^k\}$. Therefore

$$I_4 \ll n I_5, \quad \text{where} \quad I_5 \overset{\text{def}}{=} \sup_{z \in \mathbf{R}^k} \mathbf{E} \frac{1}{(1 + A^2|\varepsilon QY - z|^2)^k}.$$

Since Q is invertible with the smallest eigenvalue $q_1^2 = 1$, we have $|Qx| \ge |x|$, for all $x \in \mathbf{R}^k$. Hence

$$I_5 \ll I_6 \overset{\text{def}}{=} \sup_{z \in \mathbf{R}^k} \mathbf{E} \frac{1}{(1 + A^2\varepsilon^2|Y - z|^2)^k}.$$

Recall that $Y = X_1 - \bar{X}_1$ is a symmetrization of the lattice r.v. X_1 which is uniformly distributed in the box $B(A)$. Let Z denote a lattice random variable which is uniformly distributed in the box $B(2A)$. Applying desymmetrization Lemma 5.2 again and estimate (5.5) of Lemma 5.5, we have

$$\begin{aligned} I_6 &\ll \sup_{z \in \mathbf{R}^k} \mathbf{E} \frac{1}{(1 + A^2\varepsilon^2|Z - z|^2)^k} \\ &\ll A^{-k} \sup_{z \in \mathbf{R}^k} \sum_{m \in \mathbb{Z}^k} \frac{1}{(1 + A^2\varepsilon^2|m - z|^2)^k} \\ &\ll A^{-k} \Big(1 + \frac{1}{A\varepsilon}\Big)^k. \end{aligned}$$

Collecting the estimates of I_1 to I_6 we obtain

$$\varphi(t)\varphi(t+\varepsilon) \ll A^{-k} + \Lambda^2 I_7 \tag{5.15}$$

with $I_7 \overset{\text{def}}{=} A^{-k}\max\{1;\ (\varepsilon A)^k\}\max\big\{1;\ (\varepsilon A)^{-k}\big\}$. If $\varepsilon A \le 1$ we get $I_7 = (\varepsilon A^2)^{-k}$. If $\varepsilon A \ge 1$ we have $I_7 = \varepsilon^k$. Thus (5.15) yields the inequality (5.7), which completes the proof of Lemma 5.1.

6. The integration procedure for $A^{-4/k} \le |t| \le 1$

Theorem 6.1. *Let $\varphi(t)$, $t \ge 0$ denote a continuous nonnegative function such that $\varphi(0) = 1$. Assume that*

$$\varphi(t)\varphi(t+\varepsilon) \le \Lambda^2 \mu(\varepsilon), \qquad \text{for all} \quad t \ge 0 \quad \text{and} \quad \varepsilon \ge 0, \tag{6.1}$$

with some $\Lambda \ge 1$ independent of t and ε, where $\mu(\varepsilon)$ is defined in (2.9). Let $L = A^{-4/k}$. Then

$$\int_L^1 \sqrt{\varphi(t)}\,\frac{dt}{t} \ll \frac{\Lambda}{A^2}, \qquad \text{for} \quad k \ge 9.$$

Proof. The inequality (6.1) implies that (put $t = 0$ and note that $\Lambda \ge 1$)

$$\varphi(t) \le \Lambda^2 \mu(t) \qquad \text{and} \qquad \varphi(t)\varphi(t+\varepsilon) \le \Lambda^4 \mu(\varepsilon). \tag{6.2}$$

In order to derive the result starting with (6.2) we may assume that

$$\varphi(t) \le \mu(t) \qquad \text{and} \qquad \varphi(t)\varphi(t+\varepsilon) \le \mu(\varepsilon). \tag{6.3}$$

Indeed, we may replace φ in (6.2) by φ/Λ^2, and we may integrate over $\sqrt{\varphi/\Lambda^2}$ instead of $\sqrt{\varphi}$.

Thus assuming (6.3) we have to prove that

$$\int_L^1 \sqrt{\varphi(t)}\,\frac{dt}{t} \ll \frac{1}{A^2}, \qquad \text{for} \quad k \ge 9.$$

Let us introduce the sets

$$B_l = [L,1] \cap \{t: \ 2^{-l-1} \le \varphi(t) \le 2^{-l}\} \quad \text{and} \quad D_l = [L,1] \cap \{t: \ \varphi(t) \le 2^{-l-1}\},$$

for $l = 0, 1, 2, \ldots$ Notice that the sets B_l and D_l are closed and that $\cup_{l=0}^m B_l \cup D_m = [L,1]$ since (6.3) yields $\varphi(t) \le 1$, for $t \in [L,1]$, provided $k \ge 4$. Furthermore, (6.3) implies that $\varphi(t) \le t^k$, for $t \ge A^{-1}$, and $B_l \subset [L_l, 1]$, where $L_l = 2^{-(l+1)/k}$, for $k \ge 4$.

We have

$$\int_L^1 \sqrt{\varphi(t)}\,\frac{dt}{t} \le \sum_{l=0}^m \int_{B_l} \sqrt{\varphi(t)}\,\frac{dt}{t} + \int_{D_m} \sqrt{\varphi(t)}\,\frac{dt}{t}$$

$$\le \sum_{l=0}^m 2^{-l/2} \int_{B_l} \frac{dt}{t} + 2^{-m/2} \ln L^{-1}$$

since $\varphi(t) \le 2^{-l}$, for $t \in B_l$, $\varphi(t) \le 2^{-m}$, for $t \in D_m$, and $D_m \subset [L, 1]$.

We have $L = A^{-4/k}$ and

$$2^{-m/2} \ln L^{-1} = \frac{2}{k} 2^{-m/2} \ln A^2 \le A^{-2}$$

provided that we choose

$$m = \frac{2}{\ln 2} \ln(A^2 \ln A^2).$$

Therefore it remains to show that

$$\sum_{l=0}^{m} I_l \ll \frac{1}{A^2}, \qquad \text{where} \quad I_l = 2^{-l/2} \int_{B_l} \frac{dt}{t}. \tag{6.4}$$

For an estimation of I_l we need a description of the structure of the sets B_l with $l \le m$. Let $t, t' \in B_l$ denote points such that $t' > t$. The inequality (6.3) and the definition of B_l imply

$$4^{-l-1} \le \mu(t' - t). \tag{6.5}$$

If $t - t' \le A^{-1}$ then by (6.5) and the definition of $\mu(\varepsilon)$ we get

$$t' - t \le \delta, \qquad \text{where} \quad \delta = A^{-2} 4^{(l+1)/k}. \tag{6.6}$$

If $t - t' \ge A^{-1}$ then by (6.5) and the definition of $\mu(\varepsilon)$ we have as well

$$t' - t \ge \rho, \qquad \text{where} \quad \rho = 4^{-(l+1)/k}.$$

For $k > 8$ and sufficiently large $A \ge C$ we note that

$$\delta < \rho, \qquad \text{provided} \quad l \le m. \tag{6.7}$$

The verification of (6.7) is elementary and is based on the fact that $l \le m$.

The estimate (6.7) implies that either $t - t' \le \delta$ or $t - t' \ge \rho$. Therefore it follows from (6.6) and (6.7) that

$$t \in B_l \Longrightarrow B_l \cap (t + \delta, t + \rho) = \emptyset. \tag{6.8}$$

Assuming that the inequalities

$$I_l \ll A^{-2} l 2^{-l/2 + 4l/k}, \qquad \text{hold for} \quad l \le m, \tag{6.9}$$

we obtain (6.4) since the series $\sum_{l=0}^{\infty} l 2^{-l/2+4l/k}$ is convergent for $k > 8$.

Thus it remains to prove (6.9). If the set B_l is empty then (6.9) is obviously fulfilled. If B_l is nonempty then define $e_1 \overset{\text{def}}{=} \min\{t : \ t \in B_l\}$. Choosing $t = e_1$

and using (6.8) we see that the interval $(e_1+\delta, e_1+\rho)$ does not intersect B_l. Similarly, let e_2 denote the smallest $t \geq e_1+\rho$ such that $t \in B_l$. Then the interval $(e_2+\delta, e_2+\rho)$ does not intersect B_l. Repeating this procedure we construct a sequence $L_l \leq e_1 < e_2 < \cdots < e_s \leq 1$ such that

$$B_l \subset \cup_{j=1}^s [e_j, e_j+\delta] \qquad \text{and} \qquad e_{j+1} \geq e_j + \rho. \tag{6.11}$$

The sequence $e_1 < \cdots < e_s$ can not be infinite. Indeed, due to (6.11),

$$1 \geq e_s \geq e_1 + (s-1)\rho \geq L_l + (s-1)\rho \geq s\rho,$$

and therefore $s \leq \rho^{-1}$.

From (6.11) we can finally prove (6.9). Indeed,

$$I_l \leq 2^{-l/2} \sum_{j=1}^s \int_{e_j}^{e_j+\delta} \frac{dt}{t} = 2^{-l/2} \sum_{j=1}^s \ln\left\{1 + \frac{\delta}{e_j}\right\}$$

$$\leq 2^{-l/2} \sum_{j=1}^s \frac{\delta}{e_j} \ll l 2^{-l/2+4l/k} A^{-2}$$

since $\ln(1+x) \leq x$, for $x \geq 0$, $e_1 \geq L_l \geq \rho$, $e_1 \geq L_l \geq \rho$, $s \leq 1/\rho$, and

$$\sum_{j=1}^s \frac{1}{e_j} \leq \sum_{j=1}^s \frac{1}{e_1 + (j-1)\rho} \leq \frac{1}{\rho} \sum_{j=1}^s \frac{1}{j} \ll \frac{\ln \rho}{\rho} \ll l 4^{l/k}.$$

7. A symmetrization inequality

The following symmetrization inequality slightly improves an inequality due to Götze (1979). This inequality is in fact a generalization of a classical inequality due to Weyl (1915/16) and van der Corput, see, e.g. Graham and Kolesnik (1991).

Lemma 7.1. *Let $L \in \mathbf{R}^k$ and $C \in \mathbf{R}$. Let Y_1, Y_2, Y_3, W denote arbitrary independent random variables taking values in $\mathbf{R}^k$. Define a real-valued polynomial of the second order by $P(x) = (Qx, x) + (L, x) + C$, for $x \in \mathbf{R}^k$. Then*

$$\begin{aligned} &2|\mathbf{E}\exp\{itP(Y_1+Y_2+Y_3+W)\}|^2 \leq \\ &\mathbf{E}\exp\{2it(Q\tilde{Y}_1, \tilde{Y}_2)\} + \mathbf{E}\exp\{2it(Q\tilde{Y}_1, \tilde{Y}_3)\}. \end{aligned}$$

Furthermore, if Y_2, Y_3 are i.i.d. then

$$|\mathbf{E}\exp\{itP(Y_1+Y_2+Y_3+W)\}|^2 \leq \mathbf{E}\exp\{2it(Q\tilde{Y}_1, \tilde{Y}_2)\}.$$

Proof. Write $A = (QY_1, Y_1) - (Q\bar{Y}_1, \bar{Y}_1)$ and $B = (L, Y_1) - (L, \bar{Y}_1)$. Using conditional expectations and Hölder's inequality we obtain

$$
\begin{aligned}
&|\mathbf{E}\,\exp\{itP(Y_1+Y_2+Y_3+W)\}|^2\\
&\le \mathbf{E}\,|\mathbf{E}_{Y_1}\exp\{it(QY_1,Y_1)+2it(QY_1,Y_2+Y_3+W)+it(L,Y_1)\}|^2\\
&= \mathbf{E}\,\exp\{itA+itB+2it(QY_1,Y_2+Y_3+W)-2it(Q\bar{Y}_1,Y_2+Y_3+W)\}\\
&\le \mathbf{E}\,|\mathbf{E}_{Y_2,Y_3}\exp\{2it(Q\tilde{Y}_1,Y_2+Y_3)\}|\\
&= \mathbf{E}\,|\mathbf{E}_{Y_2}\exp\{2it(Q\tilde{Y}_1,Y_2)\}||\mathbf{E}_{Y_3}\exp\{2it(Q\tilde{Y}_1,Y_3)\}|\\
&\le \frac{1}{2}\mathbf{E}\,\exp\{2it(Q\tilde{Y}_1,\tilde{Y}_2)\}+\frac{1}{2}\mathbf{E}\,\exp\{2it(Q\tilde{Y}_1,\tilde{Y}_3)\},
\end{aligned}
$$

where the last inequality follows from the elementary inequality $2ab \le a^2 + b^2$.

8. A smoothing Lemma

Lemma 8.1. *Let F and G be arbitrary distribution functions with the characteristic functions f and g. Then*

$$
\sup_x |F(x) - G(x)| \le \frac{1}{2\pi}\int_{-H}^{H} |f(t) - g(t)|\,\frac{dt}{|t|} + R, \tag{8.1}
$$

for any $H > 0$, where

$$
|R| \le \frac{1}{H}\int_{-H}^{H}\bigl|f(t)\bigr|dt + \frac{1}{H}\int_{-H}^{H}\bigl|g(t)\bigr|dt.
$$

Introduce the function $2K(s) = K_1(s) + iK_2(s)/(\pi s)$, where $K_1(s) = K_2(s) = 0$, for $s \ge 1$, and

$$
K_1(s) = 1 - |s|, \qquad K_2(s) = \pi s\bigl(1-|s|\bigr)\cot \pi s + |s|, \qquad \text{for} \quad |s| \le 1.
$$

It is known $\bigl($Prawitz (1972)$\bigr)$ that for all $x \in \mathbf{R}$ and any $H > 0$

$$
F(x+) \le \frac{1}{2} + \text{V.P.}\int_{\mathbf{R}} \exp(-ixt)\frac{1}{H}K\Bigl(\frac{t}{H}\Bigr)f(t)dt, \tag{8.2}
$$

$$
F(x-) \ge \frac{1}{2} - \text{V.P.}\int_{\mathbf{R}} \exp(-ixt)\frac{1}{H}K\Bigl(-\frac{t}{H}\Bigr)f(t)dt, \tag{8.3}
$$

where $F(x+) = \lim_{z\downarrow x} F(z)$, $F(x-) = \lim_{z\uparrow x} F(z)$, and V.P. denotes Cauchy's Principal Value, V.P. $\int_{\mathbf{R}} \dots dt = \lim_{h\downarrow 0}\int_{|t|\ge h} \dots dt$. Note that all integrals are real and that the expressions under the signs of integrals vanish unless $|t| \le H$.

The following lemma is elementary.

Lemma 8.2. *For $0 \le s \le 1$ we have*

$$K_2(0) = 1, \quad K_2(1) = 0, \quad K_2(\tfrac{1}{2}) = \frac{1}{2}, \quad K_2'(s) \le 0, \quad K_2(s) + K_2(1-s) = 1.$$

Furthermore,

$$\begin{aligned} &1 - 2(1-s)\sin^2\frac{\pi s}{2} \le K_2(s) \le 1, && \text{for} \quad 0 \le s \le \frac{1}{2}, \\ &0 \le K_2(s) \le 2s\sin^2\frac{\pi(1-s)}{2}, && \text{for} \quad \frac{1}{2} \le s \le 1. \end{aligned}$$

It follows from Lemma 8.1 that $|1 - K_2(s)| \le 2|s|$, for all $s \in \mathbf{R}$. Therefore (8.2), (8.3) and the definition of the function K imply

$$F(x) = \frac{1}{2} + \frac{i}{2\pi}\text{V.P.}\int_{-H}^{H} \exp(-ixt) f(t)\,\frac{dt}{t} + R, \tag{8.4}$$

for any $H > 0$, where

$$|R| \le \frac{1}{H}\int_{-H}^{H} \big|f(t)\big|\,dt.$$

As a consequence of (8.4) we derive Lemma 8.1.

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