# A Characterization of M–matrices

Reinhard Nabben
Fakultät für Mathematik
Universität Bielefeld
Postfach 10 01 31
33 501 Bielefeld
Germany

**Abstract**

Since Ostrowski introduced the class of $M$–matrices more than fifty equivalent definitions of these matrices were established. Here we give another characterization using Schur complements.

*Mathematics Subject Classification* (1991): 15A57, 15A48.

One of the most beautiful classes of structured matrices is the class of $M$–matrices introduced by Ostrowski in 1937 [O]:

**Definition 1.1** *A matrix $A \in \mathbb{R}^{n,n}$ is called a Z–matrix if $A$ can be expressed in the form*

$$A = sI - B,$$

*where $B$ is a nonnegative matrix, i.e. $B \geq 0$, $s \in \mathbb{R}$ (here $I$ denotes the identity matrix). If in addition, $s$ is greater than the spectral radius of $B$ ($s > \rho(B)$), $A$ is called M–matrix.*

$M$–matrices arise in many problems in the mathematical, physical, chemical, biological and social sciences. Since 1937 many properties and equivalent definitions of $M$–matrices were established. The first systematic effort was made by Fiedler and Pták in 1962 [FP] followed by the work of Varga [V]. The book of Berman and Plemmons gives 50 equivalent definitions for a $Z$–matrix to be an $M$–matrix. Here we just mention the following ones

**Theorem 1.2** *Let $A$ be a $Z$–matrix. Then the following properties are equivalent:*

*i) $A$ is an $M$–matrix;*

*ii) all principal minors of $A$ are positive;*

*iii) all the leading principal minors of $A$ are positive;*

*iv) $A^{-1}$ is a nonnegative matrix.*

Before we state our new characterization of $M$–matrices we need well–known definitions and notations.

**Notation 1.3** *For $n \in \mathbb{N}$ we write $N := \{1, \ldots, n\}$. If $\alpha, \beta \subseteq N$ we denote by $A_{\alpha\beta}$ the submatrix of $A \in \mathbb{R}^{n,n}$ whose rows are indexed by the elements of $\alpha$ and whose columns are indexed by the elements of $\beta$. Moreover, $\bar{\alpha} := N \backslash \alpha$.*
*If $\alpha = \{1, \ldots, r\}$ for a $r \in N$, then $\alpha$ and $\bar{\alpha}$ constitute a partition of $A$. Then we write*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{1.1}$$

*where $A_{11} = A_{\alpha\alpha}$, $A_{12} = A_{\alpha\bar{\alpha}}$ $A_{21} = A_{\bar{\alpha}\alpha}$, $A_{22} = A_{\bar{\alpha}\bar{\alpha}}$. If $A_{11}$ is nonsingular then*

$$(A/A_{11}) := A_{22} - A_{21}A_{11}^{-1}A_{12} \tag{1.2}$$

*denotes the Schur complement of $A$ with respect to $A_{11}$.*

**Theorem 1.4** *Let $A$ be a $Z$–matrix. Then the following conditions are equivalent:*

*1) $A$ is an $M$–matrix;*

*2) there exists a partition of $A$, i.e.*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

*such that all leading principal minors of $A_{11}$ and $(A/A_{11})$ are positive;*

*3) condition 2) holds for all partitions of $A$.*

*Proof.* If there exists a partition (1.1) of $A$ with $A_{11}$ nonsingular, one obtains the factorization of $A$

$$A = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & (A/A_{11}) \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}.$$

Hence the inverse of $A$ is given by ([HJ], p. 472)

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}(A/A_{11})^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{1,2}(A/A_{11})^{-1} \\ -(A/A_{11})^{-1}A_{2,1}A_{11}^{-1} & (A/A_{11})^{-1} \end{bmatrix}. \tag{1.3}$$

If $A$ satisfies 1) then with Theorem 1.2 each principal submatrix $A_{11}$ of $A$ is also an $M$–matrix. Hence, $A_{11}$ is nonsingular. Since $A$ is a $Z$–matrix, the Schur complement $(A/A_{11})$ is also a $Z$–matrix. Moreover, the inverse of $(A/A_{11})$ is a principal submatrix of $A^{-1}$ (1.3). Thus $(A/A_{11})$ is an $M$–matrix. Therefore, 3) holds.

Obviously, 2) holds if 3) holds. Now assume 2). Since $det A = det A_{11} det(A/A_{11})$, $A$ is nonsingular. Moreover we obtain with Theorem 1.2 that $A_{11}$ and $(A/A_{11})$ are $M$–matrices, thus $A_{11}^{-1} \geq 0$ and $(A/A_{11})^{-1} \geq 0$. Next consider the explicit form of $A^{-1}$ given in (1.3). Since $A$ is a $Z$–matrix we obtain

$$\begin{aligned} A_{11}^{-1} + A_{11}^{-1}A_{12}(A/A_{11})^{-1}A_{21}A_{11}^{-1} &\geq 0 \\ -A_{11}^{-1}A_{1,2}(A/A_{11})^{-1} &\geq 0 \\ -(A/A_{11})^{-1}A_{2,1}A_{11}^{-1} &\geq 0 \\ (A/A_{11})^{-1} &\geq 0. \end{aligned}$$

Thus, $A$ is an $M$–matrix. □

In fact we proved that if $A$ is a $Z$–matrix then $A$ is an $M$–matrix if and only if there exists a partition of $A$ such that $A_{11}$ and $(A/A_{11})$ are $M$–matrices. Since $A_{11}$ and $(A/A_{11})$ are $Z$–matrices one can use each of the equivalent conditions to describe that $A_{11}$ and $(A/A_{11})$ are $M$–matrices. Hence, Theorem 1.4 gives a number of new characterizations of $M$–matrices. That the Schur complement of an $M$–matrix is again an $M$–matrix is first proved by Crabtree [C]. However, the converse seems to be new.

The fifty equivalent definitions of an $M$–matrix given in [BP] are ordered in groups. The conditions belonging to each group are equivalent without assuming that the matrix itself is a $Z$–matrix. In order to arrange condition 2) of Theorem 1.4 in one of these groups we need the following Theorem proved by Crabtree and Haynsworth [CH]:

**Theorem 1.5** *Let $A$ be partitioned as in (1.1) with $\alpha = \{1, \ldots, r\}$, $1 \leq r < n$. Consider another partition of $A$ given by $\tilde{\alpha} = \{1, \ldots, s\}$ with $r < s \leq n$. If $A_{\alpha\alpha}$ and $A_{\tilde{\alpha}\tilde{\alpha}}$ are nonsingular, then $(A_{\tilde{\alpha}\tilde{\alpha}}/A_{\alpha\alpha})$ is a leading principal submatrix of $(A/A_{\alpha\alpha})$.*

Thus, if 2) of Theorem 1.4 holds, then the determinant of $(A_{\tilde{\alpha}\tilde{\alpha}}/A_{\alpha\alpha})$ is positive for all $\tilde{\alpha} = \{1, \ldots, s\}$ with $r < s \leq n$. But $(A_{\tilde{\alpha}\tilde{\alpha}}/A_{\alpha\alpha}) = det(A_{\tilde{\alpha}\tilde{\alpha}})/det(A_{\alpha\alpha})$. Therefore, all principal minors of $A$ are positive. Conversely, if iii) of Theorem 1.2 holds we obtain 2) of Theorem 1.4 similarly. Hence, iii) of Theorem 1.2, which is condition (E17) in [BP], and 2) of Theorem 1.4 are equivalent for arbitrary matrices.

We have seen that Theorem 1.4 can be proved in several ways. However, the first proof we mentioned, seems to be the most transparent one.

# References

[BP] A. Berman, R.J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences,* Academic Press, New York (1979).

[C] D.E. Crabtree, *Characteristic roots of $M$–matrices*, Proc. Amer. Math. Soc. 17(1966), 1435-1439.

[CH] D.E. Crabtree, E.V. Haynsworth, *An identity for thr Schur complement of a matrix,* Proc. Amer. Math. Soc. 22(1969), 364-366.

[FP] M. Fiedler and V. Pták, *On matrices with nonpositive off-diagonal elements and positive principal minors,* Czechoslovak Math. J. 12 (1962), 382-400.

[HJ] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, (1985).

[O] A.M. Ostrowski, *Über die Determinanten mit überwiegender Hauptdiagonale*, Comment. Math. Helv. 10(1937), 69-96.

[V] R.S. Varga, *On recurring theorems on diagonal dominance*, Lin. Alg. Appl. 13(1976), 1-9.