Norm conditions for convergence of infinite products

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§0. Introduction

A set $\Sigma = \{A_i : i \in \mathcal{I}\}$ of $n \times n$ complex valued matrices is called an **LCP** *set* if any product $A_{i_k} \cdots A_{i_1}, i_1, ..., i_k \in \mathcal{I}$ converges as $k \to \infty$. Such sets arise in constructing self-similar objects, various interpolation schemes, in constructing wavelets of compact support, and in studying nonhomogeneous Markov chains. See for example [**D-L**] and the references therein. Let $\mathcal{M}(\Sigma)$ denote the set of all finite products formed by matrices in Σ plus the identity matrix. A basic result due to G. Schechtman states that an **LCP** set is product bounded (see Theorem I of [**B-W**]). That is, if Σ is an **LCP** set then $\mathcal{M}(\Sigma)$ is a bounded set. It is well known that $\mathcal{M}(\Sigma)$ is bounded iff there exist a norm $||\cdot|| : \mathbf{C}^n \to \mathbf{R}_+$ such that all matrices in Σ are contractions with respect to this norm. Using this result and Theorem 5.1 in the paper of Daubechies-Lagarias (rephrased for **LCP**) one has a set of necessary and sufficient conditions for **LCP** sets. However this theorem is given in terms of an infinite number of conditions. The object of this paper is to find necessary and sufficient conditions for an **LCP** set in terms of the existence of a distinguished norm on $\mathbf{C}^n$. A sufficient condition of this type for a finite set Σ was given in [**E-K-N**]. It was proved that a paracontracting finite set Σ is **LCP**. Here Σ is called paracontracting (with respect to a norm $||\cdot||$) if

$$Ax \neq x \Rightarrow ||Ax|| < ||x||, x \in \mathbf{C}^n, A \in \Sigma.$$

Discussing necessary conditions, in [**Els**] it was shown that if a finite set Σ is an **LCP** set with continuous limit function then Σ is paracontracting with respect to some norm. Consult with [**D-L**] for the natural definition of a continuous limit function. Note also Theorem 4.2 of [**D-L**] which characterizes finite **LCP** sets with a continuous limit function. In [**Els**] it is conjectured that any finite **LCP** set is paracontracting. In this paper we give a necessary and sufficient norm condition on a set $\Sigma = \{A_1, A_2\}$ to be an **LCP** set.

Main Theorem. *The following conditions are equivalent.*
(i) The set $\Sigma = \{A_1, A_2\} \subset M_n(\mathbf{C})$ is an **LCP** set.
(ii) There exists a norm $||\cdot||$ on $\mathbf{C}^n$ so that
(a) $||A_i|| \le 1, i = 1, 2;$
(b) $spec(A_i) \cap S^1 \subset \{1\}$, i=1,2;
(c) $||A_1 A_2 x|| = ||x|| \iff A_1 x = A_2 x = x.$

Here S^1 stands for the unit circle in the complex plane and $spec(A)$ denotes the spectrum of a matrix A.

§1. Preliminary results

As usual, denote by $M_{mn}(\mathbf{C})$ the vector space of all $m \times n$ matrices with complex valued entries. We let $M_n(\mathbf{C}) = M_{nn}(\mathbf{C})$. Let $\Sigma = \{A_1, ..., A_m\} \subset M_n(\mathbf{C})$. For each $B \in \mathcal{M}(\Sigma)$ we denote by $\Sigma(B) \subset \Sigma$ the subset of all matrices in Σ which appear in the product B. (If $B = I$ and $I \notin \Sigma$ then $\Sigma(B) = \emptyset$.) We wish to determine if Σ is an **LCP** set. Clearly any nontrivial strict subset $\Sigma' \subset \Sigma$ has to be an **LCP** set. Note that $card(\Sigma') < m$. Suppose we already verified that each strict subset $\Sigma' \subset \Sigma$ is an **LCP** set. Thus, to show that Σ is an **LCP** set it is left to show that any left infinite product containing each $A_i, i = 1, ..., m$, an infinite number of times converges. Set

$$\mathcal{B}(\Sigma) = \{B : B \in \mathcal{M}(\Sigma), \Sigma(B) = \Sigma\}.$$

The following lemma follows straightforward from [**B-W**].

Lemma. *Let $\Sigma \subset M_n(\mathbf{C})$ be a finite set. Assume that $\mathcal{B}(\Sigma)$ is bounded. Then any left infinite product containing each element of Σ an infinite number of times converges iff $\mathcal{B}(\Sigma)$ is an* **LCP** *set.*

Theorem 1. *Let $\Sigma = \{A_1, ..., A_m\} \subset M_n(\mathbf{C})$ be an* **LCP** *set. Assume*

$$\cap_{i=1}^m Ker(I - A_i) = \{0\}. \tag{1}$$

Then there exists a norm $||\cdot||$ on $\mathbf{C}^n$ and $0 < q < 1$ such that

$$||A_i|| \le 1, i = 1, ..., m, \tag{2}$$

$$||B|| \le q, B \in \mathcal{B}(\Sigma). \tag{3}$$

In particular, any infinite product of matrices in Σ containing each A_i an infinite number of times converges to zero.

Proof. Clearly, $\mathcal{B}(\Sigma)$ is an **LCP** set. Consider some left infinite product from the elements of $\mathcal{B}(\Sigma)$ which converges to a matrix denoted by C. As each A_i appears in this product an infinite number of times it follows that $A_iC = C, i = 1, ..., m$. Hence, each column of C is in $Ker(I - A_i), i = 1, ..., m$. The assumption (1) yields that $C = 0$. By Theorem I(b) in [**B-W**]

$$\hat{\rho}(\mathcal{B}(\Sigma)) = \limsup_{l \to \infty} \sigma(\mathcal{B}(\Sigma)^l)^{\frac{1}{l}} < 1.$$

Here $\sigma(\cdot)$ is the spectral norm on $M_n(\mathbf{C})$. Hence, by Lemma 2 in [**Els**] for any $q \in (\hat{\rho}(\mathcal{B}(\Sigma)), 1)$ there exists a norm $|||\cdot|||$ on $\mathbf{C}^n$ such that for the induced operator norm $|||\cdot|||$ one has

$$|||B||| \le q, B \in \mathcal{B}(\Sigma).$$

Set

$$||x|| = \sup\{|||Cx|||, C \in \mathcal{M}(\Sigma)\}.$$

Clearly, (2) holds. For any $B \in \mathcal{B}(\Sigma)$ and $C \in \mathcal{M}(\Sigma)$ we have that $CB \in \mathcal{B}(\Sigma)$. Hence

$$|||CBx||| \le q|||x||| \le q||x||.$$

In particular, $||Bx|| \le q||x||$ and (3) follows. $\diamond$

The above theorem can be extended as follows.

Theorem 2. *Assume the conditions of Theorem 1. Set*

$$\Omega = \{\Sigma' : \Sigma' \subset \Sigma, \cap_{X \in \Sigma'} Ker(I - X) = \{0\}\},$$
$$\mathcal{B} = \cup_{\Sigma' \in \Omega} \mathcal{B}(\Sigma').$$

Then there exists a norm $||\cdot||$ *on* $\mathbf{C}^n$ *and* $0 < q < 1$ *such that* (2) *and the following condition hold.*

$$||B|| \le q, B \in \mathcal{B}. \tag{4}$$

In particular, for any $\Sigma' \in \Omega$ *any infinite product of matrices in* Σ *containing each* $A_i \in \Sigma'$ *infinite number of times converges to zero.*

Proof. Consider a left infinite product of matrices in $\mathcal{B}$ which converges to C. Since Ω is a finite set it follows that such a product must contain an infinite number of matrices in some set $\Sigma' \in \Omega$. The definition of Ω yields that $C = 0$. Now the arguments of the proof of Theorem 1 yield this theorem. $\diamond$

Theorem 3. *Let* $\Sigma = \{A_1, ..., A_m\} \subset M_n(\mathbf{C})$ *be an* **LCP** *set. Set*

$$K(\Sigma) = \cap_{i=1}^m Ker(I - A_i),$$
$$\Omega = \{\Sigma' : \Sigma' \subset \Sigma, \cap_{X \in \Sigma'} Ker(I - X) = K(\Sigma)\},$$
$$\mathcal{B} = \cup_{\Sigma' \in \Omega} \mathcal{B}(\Sigma') = \{B : \cap_{A_i \in \Sigma(B)} Ker(I - A_i) = K(\Sigma)\}.$$

Then there exists a norm $||\cdot||$ *on* $\mathbf{C}^n$ *and* $0 < \gamma$ *such that* (2) *and the following condition hold.*

$$||Bx|| \le ||x|| - \gamma||x - Bx||, B \in \mathcal{B}. \tag{5}$$

Proof. Assume first that $K(\Sigma) = \{0\}$. Then Theorem 2 yields (5) with $\gamma = \frac{1-q}{2}$ by observing that $||x - Bx|| \leq ||x|| + ||Bx|| \leq 2||x||$. Assume now that $k = dim(K(\Sigma)) \geq 1$. If $k = n$, i.e. $A_i = I, i = 1, ..., m$, then the theorem is trivial. Assume that $1 \leq k < n$. By considering $\bar{\Sigma} = T\Sigma T^{-1}$ for some $T \in GL_n(\mathbf{C})$ without loss of generality we may assume that $K(\Sigma) = span(e_1, ..., e_k)$ where $e_i = (\delta_{i1}, ..., \delta_{in})^T, i = 1, ..., n$, is the standard basis of $\mathbf{C}^n$. Hence, each $A \in \mathcal{M}(\Sigma)$ has the following upper block diagonal form

$$A = (A_{ij})_1^2,$$
$$A_{11} = I \in M_k(\mathbf{C}), A_{12} \in M_{k(n-k)}(\mathbf{C}), A_{21} = 0 \in M_{(n-k)k}(\mathbf{C}), A_{22} \in M_{(n-k)(n-k)}(\mathbf{C}).$$

Define

$$\Sigma_{22} = \{A_{22} : (A_{ij})_1^2 \in \Sigma\}.$$

Clearly, Σ_{22} is an **LCP** set. Furthermore, Σ_{22} satisfies the assumptions of Theorem 1. Apply Theorem 2 to the set Σ_{22}. We then have a norm $||\cdot||_2$ on $\mathbf{C}^{n-k}$ and $0 < q < 1$ such that

$$\forall A = (A_{ij})_1^2 \in \mathcal{M}(\Sigma) \ \ ||A_{22}|| \leq 1,$$
$$\forall B = (B_{ij})_1^2 \in \mathcal{B} \ \ ||B_{22}|| \leq q.$$

Choose some norms $||\cdot||_1, ||\cdot||_2$ on $\mathbf{C}^k, \mathbf{C}^{n-k}$ respectively. For $t > 0$ let

$$||(x_1^T, x_2^T)^T||(t) = ||x_1||_1 + t||x_2||_2, x_1 \in \mathbf{C}^k, x_2 \in \mathbf{C}^{n-k}.$$

As $\mathcal{B}$ is a bounded set it follows that there exists a constant $\kappa > 0$ so that

$$||B_{12}x_2||_1 \leq \kappa ||x_2||_2, B = (B_{ij})_1^2 \in \mathcal{B}, x_2 \in \mathbf{C}^{n-k}.$$

Choose $t > 0$ so that $\gamma' = t(1-q) - \kappa > 0$. Then

$$||Bx||(t) = ||x_1 + B_{12}x_2||_1 + t||B_{22}x_2||_2 \leq ||x_1||_1 + (\kappa + tq)||x_2||_2 = ||x||(t) - \gamma'||x_2||_2,$$
$$x = (x_1^T, x_2^T)^T, x_1 \in \mathbf{C}^k, x_2 \in \mathbf{C}^{n-k}, B = (B_{ij})_1^2 \in \mathcal{B}.$$

Observe next that

$$||x - Dx||(t) = ||D_{12}x_2||_1 + t||(I - D_{22})x_2||_2 \leq (\kappa + t(1+q))||x_2||_2, D = (D_{ij})_1^2 \in \mathcal{B}.$$

We thus deduce that

$$||Bx||(t) \leq ||x||(t) - \gamma''||x - Dx||(t), B, D \in \mathcal{B}, \gamma'' = \frac{\gamma'}{\kappa + t(1+q)}.$$

Define

$$||x|| = \sup_{C \in \mathcal{M}(\Sigma)} ||Cx||(t), x \in \mathbf{C}^n.$$

Then (2) holds. Observe next that if $\Sigma' \in \Omega$ then any set Σ'' such that $\Sigma' \subset \Sigma'' \subset \Sigma$ is in Ω. Hence, $\forall C \in \mathcal{M}(\Sigma), \ \ C\mathcal{B} \subset \mathcal{B}$. Thus

$$||CBx||(t) \leq ||x||(t) - \gamma''||x - Bx||(t), C \in \mathcal{M}(\Sigma), B \in \mathcal{B}.$$

Therefore

$$||Bx|| \le ||x||(t) - \gamma''||x - Bx||(t) \le ||x|| - \gamma''||x - Bx||(t), B \in \mathcal{B}.$$

As all norms on $\mathbf{C}^n$ are equivalent it follows that there exists $\epsilon > 0$ so that $||x||(t) \ge \epsilon||x||, x \in \mathbf{C}^n$. Let $\gamma = \gamma''\epsilon$ and the theorem follows. $\diamond$

Note that Theorem 3 gives a construction of a norm so that each matrix in Σ is a contraction and each matrix in $\mathcal{B}$ is paracontracting with respect to this norm. It is plausible to conjecture that the conditions (2) and (5) will yield that any left infinite product which contains an infinite number of each A_i will converge. However, the condition (5) involves an infinite number of matrices while we are looking for a condition which is stated in terms of a finite number of matrices.

§2. Proof of Main Theorem.

(i) $\Rightarrow$ (ii). Let $\Sigma = \{A_1, A_2\}$ be an **LCP** set. We claim that (ii) follows from Theorem 3. Consider the norm satisfying the assumptions of Theorem 3. Clearly, the condition (a) is satisfied. As the powers of each A_i are convergent we deduce that either $spec(A_i) \cap S^1 = \emptyset$ or $spec(A_i) \cap S^1 = \{1\}$, i.e. the sequence $A_i^k, k = 1, ...,$ either converges to the zero matrix or a nontrivial projection. That is (b) holds. We now prove (c). Clearly,

$$A_1 x = A_2 x = x \Rightarrow ||A_1 A_2 x|| = ||x||.$$

On the other hand as $A_1 A_2 \in \mathcal{B}$ by (5) it follows from $||A_1 A_2 x|| = ||x||$ that $A_1 A_2 x = x$. By [**D-L**] we have for **LCP** sets

$$Ker(I - A_1 A_2) = Ker(I - A_1) \cap Ker(I - A_2).$$

Hence, $A_1 x = A_2 x = x$.

(ii) $\Rightarrow$ (i). The conditions (a) and (b) yield that if $1 \in spec(A_i)$ then 1 is a geometrically simple eigenvalue, i.e. all the Jordan blocks corresponding to an eigenvalue 1 are 1×1 blocks. Furthermore, $spec(A_i)\backslash\{1\}, i = 1, 2,$ is contained in the open unit disk of the complex plane. It then follows that $\{A_1\}, \{A_2\}$ are **LCP** sets. Fix $d = (d_i)_1^\infty \subset \{1, 2\}^{\mathbf{N}}$. Set

$$d^k = (d_1, ..., d_k), A_{d^k} = A_{d_k} \cdots A_{d_1}, k = 1, ...,.$$

It is left to show that the sequence $(A_{d^k})_1^\infty$ converges for d having infinite numbers of 1 and 2. This is equivalent to the following observation. There exists a unique subsequence $d^{m_i}, i = 1, ...,$ so that the subsequence $A_{d^{m_i}}, i = 1, ...,$ is a (standard) sequence of left products of matrices from the set

$$\mathcal{C} = \mathcal{B}(A_1) \cup \mathcal{B}(A_2) \cup \{A_1 A_2\}$$

such that the matrix A_1A_2 appears an infinite number of times.

Assume first that $K(\Sigma)=\{0\}$. Let

$$B=\{x: x\in \mathbf{C}^n, ||x||\le 1\},\ B_0=\{x: x\in \mathbf{C}^n, ||x||<1\}.$$

Our assumption yields that $A_1A_2B\subset B_0$. Hence, there exists $q<1$ so that $||A_1A_2||\le q$. Consider the product $A_{d^{m_i}}$ as a (unique) product in the elements of $\mathcal{C}$. Let t_i be the number of times that the matrix A_1A_2 appears in this product. Note that $\{t_i\}_1^\infty$ is a nondecreasing sequence which converges to infinity. The condition (2) for $m=2$ implies that

$$||A_{d^k}||\le q^{t_i}, m_i\le k<m_{i+1}.$$

Hence $\lim_{k\to\infty} A_{d^k}=0$ and we deduce that Σ is a **LCP** set.

Assume that $1\le k=dim(K(\Sigma))<n$. (The case $K(\Sigma)=\mathbf{C}^n$ is trivial.) By considering $T\Sigma T^{-1}$ we may assume that each $A\in\mathcal{M}(\Sigma)$ has an upper block triangular form as $A=(A_{ij})_1^2$ in the proof of Theorem 3. We first claim that

$$\lim_{k\to\infty}(A_{d^k})_{22}=0. \tag{6}$$

On $\mathbf{C}^{n-k}$ introduce the norm

$$||x_2||_2=\min_{x_1\in\mathbf{C}^k}||(x_1^T,x_2^T)^T||, x_2\in\mathbf{C}^{n-k}.$$

We claim that the set $\Sigma_{22}=\{(A_1)_{22},(A_2)_{22}\}$ satisfies the conditions (ii) of Main Theorem with $||\cdot||_2$ as above. Furthermore,

$$||(A_1)_{22}(A_2)_{22}x_2||_2=||x||_2\Rightarrow x_2=0. \tag{7}$$

Indeed, fix $x_2\in\mathbf{C}^{n-k}$. Choose $y_1\in\mathbf{C}^k$ so that $||x_2||_2=||(y_1^T,x_2^T)^T||$. Let $A=(A_{ij})_1^2\in\mathcal{M}(\Sigma)$. From the definition of $||\cdot||_2$ it follows

$$||A_{22}x_2||_2\le||((A_{11}y_1+A_{12}x_2)^T,(A_{22}x_2)^T)^T||\le||(y_1^T,x_2^T)^T||=||x_2||_2,$$
$$||A_{22}x_2||=||x_2||_2\Rightarrow||A(y_1^T,x_2^T)^T||=||(y_1^T,x_2^T)^T||.$$

Hence Σ_{22} satisfies the assumptions (ii) of the main theorem. Furthermore, as $K(\Sigma_{22})=\{0\}$ we deduce (7). Hence, by the first part of this proof (6) holds. We now show that the products $A_{d^k}, k=1,...,$ converge for any d which have an infinite number of 1 and 2. It is enough to show that for any $x\in\mathbf{C}^n$ the sequence $A_{d^k}x, k=1,...,$ converge. The condition (6) yields $\lim_{k\to\infty} dist(K(\Sigma),A_{d^k}x)=0$. For a given $\epsilon>0$ choose $k=k(\epsilon)$ so that $dist(K(\Sigma),A_{d^k}x)<\epsilon$. That is

$$\exists z^k\in K(\Sigma)\ ||A_{d^k}x-z^k||<\epsilon.$$

Let $p>k$. Then $A_{d^p}=BA_{d^k}$ for some $B\in\mathcal{M}(\Sigma)$. Hence

$$||A_{d^p}x-A_{d^k}x||=||(B-I)A_{d^k}x||=||(B-I)(z^k+A_{d^k}x-z^k)||=$$
$$||(B-I)(A_{d^k}x-z^k)||\le 2||A_{d^k}x-z^k||<2\epsilon.$$

Therefore, the sequence $A_{d^k}x, k=1,...,$ is a Cauchy sequence which is convergent. The proof of the theorem is completed. ⋄

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