Resultants and the Continuity of the Join Pairing on Chow Varieties

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Introduction

In [13] Lawson showed the equivalence of the stable Chow variety $\mathcal{C}$ and the product $\prod_{i>0} K(\mathbb{Z}, 2i) =: K(\mathbb{Z}, even)$ of Eilenberg MacLane spaces $K(\mathbb{Z}, 2i)$. The invested complex suspension theorem is fundamental for the definition of Lawson homology [7]. The original proof of this theorem uses geometric measure theory. Here an algebraic and as we think more elementary proof (Thm. 3.1) is given going back to the construction of Chow varieties via resultants [5]. Once the basic continuity statements are proved, we can stay close to the geometric intuition [13], which is behind the measure theoretic arguments, by the Chow/Van der Waerden approach to Chow varieties. The observation is that one can calculate the Chow form of a suspended cycle (Prop. 2.2) from the original Chow form. Thus suspension is an algebraic map.

To prove that the join pairing is algebraic (Thm. 2.3), hence continuous, one just has to combine the continuity of the complex suspension with classical results (Barlet [1]). (Independently Barlet [2] obtained a proof of Thm. 2.3 using different methods.)

In [4] the join pairing was used to define an infinite loop space structure on $K(\mathbb{Z}, even)$ which is compatible with the infinite loop space structure on BU induced by Whitney sum. The required continuity checks (Lemma 4.5) for the operation of [4] also follow easily in the Chow/Van der Waerden picture.

Section 1 describes Chow forms in the terms of Chow and Van der Waerden and some properties of them. In section 2 the complex suspension and join of cycles and several stabilizations are defined. The Chow form of a suspended cycle is determined from a given Chow form and continuity of the suspension and the join pairing are shown. Section 3 contains a proof of the complex suspension theorem. Finally the last section recollects the continuity statements which make the total Chern class map an infinite loop map.

Throughout this paper we work in the category kTop of compactly generated, weak Hausdorff spaces [17]. All proofs are given for algebraic cycles with support in $\mathbb{CP}^n$ but they directly work for cycles with support in an algebraic subset $X \subset \mathbb{CP}^n$ (see also [13]), except that one has to take care of π_0–phenomena (compare [13, p. 265]).

[1] This paper is an outgrowth of my diploma thesis [19]. I am indebted to R. Schwänzl for suggesting the problem to me, to P. Lima–Filho for calling my attention to [2], and to the DFG for support.

1 Review of the Chow – Van der Waerden approach to Chow varieties

There are many different ways to define the Chow form of a variety $V \subset \mathbb{CP}^n$ ([5],[11],[8]). The definition of Chow and Van der Waerden is constructive and therefore very useful for proofs of continuity. First some notations and basic definitions:

1.1 Notation: Let $\mathbb{C} \subset K \subset \Omega$ be fields.

i. A projective algebraic set $V_K \subset \Omega\mathbb{P}^n$ *over* K is the set of common zeros of a homogeneous ideal $I \subset K[x_0, \ldots, x_n]$.

ii. Let $V_K \subset \Omega\mathbb{P}^n$ be a projective algebraic set. The homogeneous ideal $I_K(V)$ which is generated by

$$\{f \in K\left[x_0, \ldots, x_n\right] \mid f \text{ homogeneous and } f(\underline{v}) = 0 \; \forall \underline{v} \in V\}$$

in $K\left[x_0, \ldots, x_n\right]$ is the *vanishing ideal* of V_K.

An algebraic set $V_K \subset \Omega\mathbb{P}^n$ is called *irreducible* or *a variety* iff $I_K(V) \subset K[x_0, \ldots, x_n]$ is prime. The dimension of a variety V_K is its dimension as a topological subspace of $\Omega\mathbb{P}^n$ in the Zariski topology. It follows from the Noetherian Normalization Theorem [12, Chap. II, Thm. 3.1] that the dimension of $V_K \subset \Omega\mathbb{P}^n$ is independent of the extension field Ω of K, that means

$$\dim V_K = \dim V_K \cap K\mathbb{P}^n \, .$$

From now on Ω will denote a fixed *universal field* in the sense of Van der Waerden [21], given as the algebraic closure of $\mathbb{C}(U)$, where U is a countably infinite set of algebraically independent elements over $\mathbb{C}$. K is always a subfield of Ω with finite transcendence degree over $\mathbb{C}$.

Suppose $\underline{\xi} = (\xi_0, \ldots, \xi_n) \in \Omega^{n+1}$. Then one can define $K[\underline{\xi}] := K[\xi_0, \ldots, \xi_n]$. If $\underline{\xi} \in \Omega\mathbb{P}^n$ then $K[\underline{\xi}]$ depends on the choice of a representative of $\underline{\xi}$ in Ω^{n+1}. The definition of a special representative of points in $\Omega\mathbb{P}^n$ makes this ring adjunction unique up to canonical isomorphisms:

1.2 Definition: Let $(\xi_0, \ldots, \xi_n) = \underline{\xi} \in \Omega\mathbb{P}^n$ and i minimal such that $\xi_i \neq 0$. Then

$$\tau\underline{\xi} := \left(\tau\frac{\xi_0}{\xi_i}, \ldots, \tau\frac{\xi_i}{\xi_i}, \ldots, \tau\frac{\xi_n}{\xi_i}\right) \in \Omega^{n+1}, \;\; \tau \in \Omega - \overline{K(\xi_0, \ldots, \xi_n)}.$$

For every point $\underline{\xi} \in \Omega\mathbb{P}^n$ the ring $K[\tau\underline{\xi}]$ depends on choices up to canonical isomorphisms, which extend to K–automorphisms of Ω. From now on the condition $f(\underline{\xi}) = 0$, $f \in K[x_0, \ldots, x_n]$, is understood as $f(\tau\underline{\xi}) = 0$ for $\underline{\xi} \in \Omega\mathbb{P}^n$. Also one defines $K(\tau\underline{\xi})$ to be the quotient field of $K[\tau\underline{\xi}]$.

The advantage of viewing varieties in $\mathbb{CP}^n$ as varieties in $\Omega\mathbb{P}^n$ *over* $\mathbb{C}$ is, that varieties in $\Omega\mathbb{P}^n$ over K are completely described by single points in $\Omega\mathbb{P}^n$.

1.3 Definition: Let $V_K \subset \Omega\mathbb{P}^n$ be an algebraic set over K. A point $\underline{\xi}$ is a *general point* of V_K if it satisfies the following conditions:

i. $\underline{\xi} \in V_K$
ii. for all $f \in K[x_0, \ldots, x_n]$, $f(\underline{\xi}) = 0$, it follows that $f \in I_K(V) \subset K[x_0, \ldots, x_n]$.

As with the definition of $K[\tau\underline{\xi}]$ general points of a variety V_K are only determined up to K–automorphisms of Ω, that is, if $\underline{\xi}$ is a general point of V_K and σ a K–automorphisms of Ω, then $(\sigma(\xi_0), \ldots, \sigma(\xi_n))$ is a general point of V_K, too. We also say the general points are conjugate to each other. It is easy to show that an algebraic set in $\Omega\mathbb{P}^n$ has a general point iff it is irreducible. Also one can prove that the projective coordinate ring of a variety V_K is naturally isomorphic to $K[\tau\underline{\xi}]$ for a general point $\underline{\xi}$ of V_K [21]. The isomorphism is given by

$$E_{\tau\underline{\xi}} : \begin{array}{ccll} K[x_0, \ldots, x_n]\big/ I_K(V) & \longrightarrow & K[\tau\underline{\xi}] & \\ z & \longmapsto & z & , z \in K \\ x_i & \longmapsto & \tau\xi_i & , i = 0, \ldots, n\ . \end{array}$$

Therefore a variety V has infinitely many general points iff $\dim V > 0$. For the definition of the Chow form of a variety over $\mathbb{C}$ one looks at the intersections of varieties over K with K–general hyperplanes.

1.4 Definition and Notation: Let $u_i^j \in \Omega$, $i = 0, \ldots, n$, $j = 1, \ldots, r$ be algebraically independent over K. Denote

$$\mathcal{U}(n, r) := \begin{pmatrix} u_0^1 & \cdots & u_n^1 \\ \vdots & & \vdots \\ u_u^r & \cdots & u_n^r \end{pmatrix} .$$

The $(n-r)$*–dimensional linear K–general subspace* H_{n-r} in $\Omega\mathbb{P}^n$ is the zero set of $\mathcal{U}(n, r) \cdot \underline{x}$ viewed as r forms in $\Omega[x_0, \ldots, x_n]$. H_{n-1} is also called a K–general hyperplane.

Observe that algebraic independence of the u_i^j implies linear independence of the $\underline{u}^j = (u_0^j, \ldots, u_n^j)$, because linear dependence can be expressed in terms of determinants of submatrices of $\mathcal{U}(n, r)$.

1.5 Lemma: Let $V_K \subset \Omega\mathbb{P}^n$ be a variety of dimension r and H the K–general hyperplane given by $\langle \underline{u}, \underline{x} \rangle = 0$. Then the intersection of V_K and H is a variety over $K(\underline{u})$ of dimension $r - 1$.

Proof: [22, p. 140].

Lemma 1.5 proves the following theorem:

1.6 Theorem: Let $V_{\mathbb{C}} \subset \Omega\mathbb{P}^n$ be a variety of dimension r over $\mathbb{C}$ and H_{n-r} the $\mathbb{C}$–general $(n-r)$–dimensional linear subspace of $\Omega\mathbb{P}^n$. The intersection $V_{\mathbb{C}} \cap H_{n-r}$ is a 0–dimensional variety over $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$.

0–dimensional varieties over a field, which is not algebraically closed, consist of finitely many points. All of them are general points [21, §129].

1.7 Construction and Definition: Let $V_{\mathbb{C}}$ be as in Theorem 1.6. Further let $V_{\mathbb{C}} \cap H_{n-r} = \{\underline{q}^1, \ldots, \underline{q}^d\}$. As mentioned above $\underline{q}^k$, $k = 1, \ldots, d$, are general points of $V_{\mathbb{C}} \cap H_{n-r}$ viewed as a variety over $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$. Therefore they are conjugated to each other over $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$. On the other hand it is obvious that $\underline{q}^k$, $k = 1, \ldots, d$ are general points of $V_{\mathbb{C}}$ viewed as a variety over $\mathbb{C}$.

Without loss of generality let $q^k_m \neq 0$, $k = 1, \ldots, d$, $m \in \{0, \ldots, n\}$. Now choose

$$\underline{p}^k := \left(\frac{q^k_0}{q^k_m}, \frac{q^k_1}{q^k_m}, \ldots, \frac{q^k_m}{q^k_m}, \ldots, \frac{q^k_n}{q^k_m} \right) \in \Omega^{n+1}$$

as a representation of $\underline{q}^k$, $k = 1, \ldots, d$. The p^k_i are now algebraic over $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$, because $V_{\mathbb{C}} \cap H_{n-r}$ has dimension 0.

Let $u^0_0, \ldots, u^0_n$ be algebraically independent over $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$ and define

$$\begin{aligned} L_k &:= \langle \underline{u}^0, \underline{p}^k \rangle && , k = 1, \ldots, d \\ G(\underline{u}^0) &:= \prod_{k=1}^{d} L_k = \prod_{k=1}^{d} \sum_{i=0}^{n} u^0_i p^k_i && \in \overline{\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)}[\underline{u}^0] \,. \end{aligned}$$

Now let Z be the splitting field of the minimal polynomials of the p^k_i, $i = 0, \ldots, n$, $k = 1, \ldots, d$. Z is a Galois extension of $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$, because the characteristic of $\mathbb{C}$ is 0. Now let φ be an element of the Galois group of Z over $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$. The general points of a variety are conjugated to each other, therefore $\varphi(\underline{p}^k)$ must be equal to $\underline{p}^l$ for some $l \in \{1, \ldots, d\}$. Hence one obtains $\varphi(G(\underline{u}^0)) = G(\underline{u}^0)$, that is the coefficients of G remain invariant under the Galois operations and are therefore elements of $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$.

Now let $c \in \mathbb{C}[\underline{u}^1, \ldots, \underline{u}^r]$ be the common denominator of the coefficients of $G(\underline{u}^0)$. Then

$$F(\underline{u}^0, \ldots, \underline{u}^r) := cG(\underline{u}^0)$$

is an element of $\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$.

The form $F(\underline{u}^0, \ldots, \underline{u}^r)$ is a *Chow form* of $V_{\mathbb{C}}$. The *degree* of $V_{\mathbb{C}}$ is defined to be the degree d of F as polynomial in $\mathbb{C}[\underline{u}^1, \ldots, \underline{u}^r][\underline{u}^0]$. In fact F is *multihomogeneous* of *multidegree* d, which means it is homogeneous of degree d as form in $\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^{j-1}, \underline{u}^{j+1}, \ldots, \underline{u}^r][\underline{u}^j]$, $j = 0, \ldots, r$ [5, p. 694].
It follows directly from the construction, that the degree of a variety is the number of points in which a general linear $(n-r)$–dimensional linear hyperplane meets the variety. This definition of the degree of a variety coincides with the definition of the degree by the Hilbert polynomial [10, p. 48].

The Chow form of a variety is uniquely determined up to multiplication by elements in $\mathbb{C}^*$. Viewed as an element of $\overline{\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)}[\underline{u}^0]$ the Chow form F of a variety V splits into linear factors $\langle \underline{u}^0, \underline{p}^k \rangle$, where every $\underline{p}^k$ is a general point of V. Therefore two varieties are equal iff their Chow forms coincide up to a factor in $\mathbb{C}^*$.

In the following some properties of Chow forms are described, especially their relations to resultants.

1.8 Lemma: The Chow form $F(\underline{u}^0, \ldots, \underline{u}^r)$ of a r–dimensional variety $V_{\mathbb{C}}$ is irreducible in $\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$.

Proof: [5, p. 693].

1.9 Reminder: Let $S \subset \Omega$ be a ring and $f_1, \ldots, f_k \in S[x_0, \ldots, x_n]$ be homogeneous polynomials. The *resultant of the homogeneous equations*

$$f_1 = \ldots = f_k = 0$$

is an ideal $J(f) \subset S$, which is generated by determinants of matrices, where one fixed row of such a matrix consists of coefficients of one f_j [21, § 130]. Therefore there is a set of generators of $J(f)$ consisting of elements of the polynomial ring in the coefficients of $f_1, \ldots, f_k$ over $\mathbb{Z}$. And the ideal $J(f)$ is generated by the image of those elements in S. Because the ideal $J(f)$ is generated by determinants, it is homogeneous in the coefficients of each f_j, $j = 1, \ldots, k$.
We also say $J(f)$ is the resultant of the homogeneous polynomials $f_1, \ldots, f_k$. The ideal $J(f)$ has the following property [21, §130]:

$$\begin{array}{rl} & f_1 = \ldots = f_k = 0 \text{ has a solution in } \Omega\mathbb{P}^n \\ \Longleftrightarrow & \text{J(f) is the zero ideal in } S\ . \end{array}$$

Let $I(f) \subset S[x_0, \ldots, x_n]$ denote the ideal which is generated by $f_1, \ldots, f_k$. The zero set of $I(f)$ in $\Omega\mathbb{P}^n$ is empty iff $J(f) \neq 0$, but the ideal $J(f)$ depends on the choice of generators of $I(f)$. The radical of $J(f)$ only depends on the radical of $I(f)$:

To see this let $g_1, \ldots, g_l \in S[x_0, \ldots, x_n]$ be homogeneous polynomials and $I(g)$ be the ideal which is generated by $g_1, \ldots, g_l$. Further let $J(g)$ be the resultant of $g_1 = \ldots = g_l = 0$ Suppose that $\sqrt{I(g)} = \sqrt{I(f)}$. Then $g_1 = \ldots = g_l = 0$ has a solution in $\Omega\mathbb{P}^n$ iff $f_1 = \ldots = f_k = 0$ has a solution in $\Omega\mathbb{P}^n$. Therefore $J(g) = 0 \Leftrightarrow J(f) = 0$. But this implies $\sqrt{J(g)} = \sqrt{J(f)}$.

Therefore we denote by $\sqrt{J(f)}$ the *resultant of the ideal* $I(f)$.

The Chow form of a variety $V_{\mathbb{C}}$ arises from the intersection with H_{n-r}. If one cuts this 0–dimensional set with another sufficiently general hyperplane, the intersection is empty. Therefore one gets a set of homogeneous polynomials, whose resultant is not the zero ideal. In fact the radical of this resultant is generated by the Chow form:

1.10 Theorem: Let $V_{\mathbb{C}} \subset \Omega\mathbb{P}^n$ be a r–dimensional variety and $f_1, \ldots, f_k$ homogeneous generators of $I_{\mathbb{C}}(V) \subset \mathbb{C}[x_0, \ldots, x_n]$. The Chow form $F(\underline{u}^0, \ldots, \underline{u}^r)$ generates the radical $\sqrt{R} \subset \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$ of the resultant R of the following equations:

$$\begin{array}{l} f_1(x_0, \ldots, x_n) = \ldots = f_k(x_0, \ldots, x_n) = 0 \\ u_0^0 x_0 + \ldots + u_n^0 x_n = 0 \\ \mathcal{U}(n, r) \cdot \underline{x} = 0 \end{array} \tag{1.i}$$

viewed as polynomials in $\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r][x_0, \ldots, x_n]$.

Proof: The first step is to show that $\sqrt{R}$ is a prime ideal in $\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$:
The proof of this statement uses the same techniques as the proof of Lemma 1.5 in [22, p. 140]. Let

$$A := \{(\underline{p}, \underline{y}^0, \ldots, \underline{y}^r) \in \Omega\mathbb{P}^n \times (\Omega\mathbb{P}^n)^{r+1} \mid \underline{p} \in V_{\mathbb{C}} \text{ and } \langle \underline{p}, \underline{y}^0 \rangle = \ldots = \langle \underline{p}, \underline{y}^r \rangle = 0\} .$$

The vanishing ideal of A is generated by the polynomials in (1.i). The projection pr_1 onto the first factor is again the set $V = pr_1(A)$ and the projection pr_2 of A onto the second factor $(\Omega\mathbb{P}^n)^{r+1}$ gives us the zero set of R.
We have to prove that $pr_2(A)$ is irreducible. Suppose $pr_2(A)$ is reducible, then pr_2^{-1} would define a partition of A, because pr_1, pr_2 are continuous in the Zariski topology. Therefore it suffices to prove that A is irreducible over $\mathbb{C}$:
Suppose $A = U_1 \cup U_2$. Let $\underline{\xi} \in \Omega\mathbb{P}^n$ be a general point of $V_{\mathbb{C}}$ and look at

$$A_{\underline{\xi}} = \{\underline{\xi}\} \times (\Omega\mathbb{P}^n)^{r+1} \cap A$$

this is canonically isomorphic to the zero set of $\langle \underline{\xi}, \underline{u}^0 \rangle, \ldots, \langle \underline{\xi}, \underline{u}^r \rangle$ hence a linear subspace which is irreducible. Because $A_{\underline{\xi}}$ is irreducible, it is contained in U_1 of in U_2. Suppose $A_{\underline{\xi}} \subset U_1$ and $\xi_i \neq 0$.
The vanishing ideal of $A_{\underline{\xi}}$ is generated by $\langle \underline{x}, \underline{u}^0 \rangle, \ldots, \langle \underline{x}, \underline{u}^r \rangle, \tau\xi_0 x_i - \tau\xi_i x_0, \ldots, \tau\xi_n x_i - \tau\xi_i x_n$ in $\mathbb{C}(\tau\underline{\xi})[\underline{x}, \underline{u}^0, \ldots, \underline{u}^r]$ (1.2).
Because $A_{\underline{\xi}} \subset U_1$ we have $I_{\mathbb{C}}(U_1) \subset I_{\mathbb{C}(\tau\underline{\xi})}(A_{\underline{\xi}}) \cap \mathbb{C}[\underline{x}, \underline{u}^0, \ldots, \underline{u}^r]$.
It follows directly from the definition of a general point, that the intersection of $\mathbb{C}[\underline{x}]$ and the ideal which is generated by $\tau\xi_0 x_i - \tau\xi_i x_0, \ldots, \tau\xi_n x_i - \tau\xi_i x_n$ in $\mathbb{C}(\tau\underline{\xi})[\underline{x}]$ is again the vanishing ideal of $V_{\mathbb{C}}$. Hence $I_{\mathbb{C}(\tau\underline{\xi})}(A_{\underline{\xi}}) \cap \mathbb{C}[\underline{x}, \underline{u}^0, \ldots, \underline{u}^r] = I_{\mathbb{C}}(A)$.
Now we obtain $I_{\mathbb{C}}(U_1) \subset I_{\mathbb{C}}(A)$, which means $A \subset U_1$ and A is irreducible. That proves that $\sqrt{R}$ is prime.

Now we can prove the theorem:
The Chow form F of $V_{\mathbb{C}}$ is $F(\underline{u}^0, \ldots, \underline{u}^r) = c \prod_{i=1}^{d} \langle \underline{p}^i, \underline{u}^0 \rangle$, where $\{\underline{p}^1, \ldots, \underline{p}^d\}$ is the intersection of the zero set of $\mathcal{U}(n, r)\underline{x}$ and $V_{\mathbb{C}}$, $c \in \mathbb{C}[\underline{u}^1, \ldots, \underline{u}^r]$. Denote by $Z \subset \overline{\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)}$ the splitting field of the p^i_j, $i = 1, \ldots, d$, $j = 0, \ldots, n$.
Now view the polynomials in (1.i) as polynomials in $Z[\underline{u}^0][x_0, \ldots, x_n]$ and determine the resultant $R \cdot Z[\underline{u}^0]$. The zero set of $R \cdot Z[\underline{u}^0]$ equals the zero set of $\prod_{i=1}^{d} \langle \underline{p}^i, \underline{u}^0 \rangle$. But that means the radical of the ideal which is generated by R in $Z[\underline{u}^0]$ is the ideal which is generated by F:

$$(F)_{Z[\underline{u}^0]} = \sqrt{R \cdot Z[\underline{u}^0]} \quad \text{in } Z[\underline{u}^0] .$$

Cutting down with $\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$ we first obtain $(F)_{\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]} \subset (F)_{Z[\underline{u}^0]} \cap \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$.
Now let $F \cdot h \in (F)_{Z[\underline{u}^0]} \cap \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$. And let σ be an element of the Galois group of Z over $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$. Then it is $F \cdot h = \sigma(F \cdot h) = F \cdot \sigma(h)$ and that means $h \in \mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$, because Z is a Galois extension. Therefore $h = \frac{h_1}{h_2}$, $h_1 \in \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r], h_2 \in \mathbb{C}[\underline{u}^1, \ldots, \underline{u}^r]$, and the common divisor of h_1 and h_2 is 1. Now h_2 must be a divisor of F. But F is

irreducible and $\underline{u}^0$ does not appear in h_2, hence $h_2 \in \mathbb{C}$. That means

$$(F)_{Z[\underline{u}^0]} \cap \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r] = (F)_{\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]} .$$

Now it remains to show that $\sqrt{R} \subset \sqrt{R \cdot Z[\underline{u}^0]} \cap \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$ equals $\sqrt{R}$. For the inverse inclusion we have to show that $F \in \sqrt{R}$. We have already seen that $F \in \sqrt{R \cdot Z[\underline{u}^0]} \cap \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$. That means

$$F^k = \sum_{i=1}^{l} g_i z_i \quad , g_i \in R \, , z_i \in Z[\underline{u}^0] \, , k \in \mathbb{N}^* .$$

Z is a Galois extension of $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$ and the Galois group $G(Z : \mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r))$ is finite. F and g_i, $i = 1, \ldots, l$ are elements in $\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$ and therefore they stay invariant under the action of the group $G(Z : \mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r))$. Now one obtains:

$$\mid G(Z : \mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)) \mid \cdot F^k = \sum_{\sigma \in G(Z:\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r))} \sigma(\sum_{i=1}^{l} g_i z_i) = \sum_{i=1}^{l} g_i \underbrace{\sum_{\sigma \in G(Z:\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r))} \sigma(z_i)}_{:= \tilde{z}_i}$$

and the $\tilde{z}_i$ are invariant under Galois action, hence elements in $\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)[\underline{u}^0]$. But then there are $s_1, \ldots, s_l \in \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$ and $t \in \mathbb{C}[\underline{u}^1, \ldots, \underline{u}^r]$ such that

$$t \cdot F^k = \sum_{i=1}^{l} g_i \cdot s_i .$$

Because g_i, $i = 1, \ldots, l$ are elements of R we have $tF^k \in R$. But that means $tF \in \sqrt{R}$. We have proved that $\sqrt{R}$ is a prime ideal, therefore $t \in \sqrt{R}$ or $F \in \sqrt{R}$.
Suppose $t \in \sqrt{R}$. We know that the polynomials (1.i) get common zeros $\underline{p}^1, \ldots, \underline{p}^d$ if we set $\underline{u}^0 = (0, \ldots, 0)$. That means the resultant R is mapped onto the zero ideal if $\underline{u}^0$ is mapped to $(0, \ldots, 0)$. But $t \in \mathbb{C}[\underline{u}^1, \ldots, \underline{u}^r]$ and therefore it is not mapped onto zero and hence not an element of $\sqrt{R}$. That finishes the proof of the theorem. □

If the multihomogeneous monomials in $\mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$ are lexicographically ordered, then the coefficients of a Chow form F of degree d define a point in $\mathbb{CP}^N$, $N = \binom{n+d}{d}^{r+1} - 1$, because Chow forms are only determined up to a factor in $\mathbb{C}^*$. In the following the coefficients of a Chow form are also denoted *Chow coordinates.*
Therefore the space of r–dimensional varieties in $\mathbb{CP}^n$ of degree d constitutes a subspace of $\mathbb{CP}^N$ and hence it is canonically topologized.

1.11 Definition: The *Chow variety* $\mathcal{C}_{r,\bullet}(\mathbb{CP}^n)$ as a set is the free abelian semi group generated by the r–dimensional varieties $V \subset \Omega\mathbb{P}^n$ over $\mathbb{C}$.
Let V_i be r–dimensional varieties and $F_i(\underline{u}^0, \ldots, \underline{u}^r)$ the Chow forms of V_i. The *Chow form* of the cycle $\sum_{i=1}^{m} n_i V_i$ is defined to be

$$F(\underline{u}^0, \ldots, \underline{u}^r) = \prod_{i=1}^{m} F_i(\underline{u}^0, \ldots, \underline{u}^r)^{n_i} .$$

The *degree* of $\sum_{i=1}^{m} n_i V_i$ is again the multidegree of the multihomogeneous form $F(\underline{u}^0, \ldots, \underline{u}^r)$ and hence equals $\sum_{i=1}^{m} n_i \cdot \deg V_i$.

The degree of cycles defines a natural filtration of the Chow variety $\mathcal{C}_{r,\bullet}(\mathbb{CP}^n)$.

1.12 Definition: The *Chow variety* $\mathcal{C}_{r,d}(\mathbb{CP}^n)$ is defined as follows:

$$\mathcal{C}_{r,d}(\mathbb{CP}^n) := \{V \in \mathcal{C}_{r,\bullet}(\mathbb{CP}^n) \mid \deg V = d\} .$$

The Chow variety $\mathcal{C}_{r,d}(\mathbb{CP}^n)$ again can be topologized as a subset of $\mathbb{CP}^N$, by the definition of Chow forms.

1.13 Example: There are three natural homeomorphisms:

i. $\mathcal{C}_{0,d}(\mathbb{CP}^n)$ is homeomorphic to the d–fold symmetric product $SP^d(\mathbb{CP}^n)$ of $\mathbb{CP}^n$.
ii. $\mathcal{C}_{r,1}(\mathbb{CP}^n)$ coincides with $\mathcal{G}_{r+1}^{n-r}$ the Grassmannian of $(r+1)$–dimensional linear subspaces of $\mathbb{C}^{n+1}$. The Chow coordinates of a linear subspace coincide with its Plücker coordinates.
iii. $\mathcal{C}_{n-1,d}(\mathbb{CP}^n) \cong \mathbb{CP}^N$, $N = \binom{n+d}{d} - 1$, because the Chow forms of cycles of codimension 1 and degree d coincide naturally with homogeneous forms of degree d in $\mathbb{C}[x_0, \ldots, x_n]$.

$\mathcal{C}_{r,d}(\mathbb{CP}^n)$ is an algebraic set [5, Thm. 2] and hence compact. It consists of algebraic cycles whose Chow forms are multihomogeneous forms of multidegree d. The following theorem of Chow and Van der Waerden demonstrates which multihomogeneous forms of multidegree d are Chow forms of algebraic cycles.

1.14 Theorem: Let $F \in \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$ be multihomogeneous of multidegree d. F is a Chow form of an algebraic cycle in $\mathcal{C}_{r,d}(\mathbb{CP}^n)$ iff the following conditions hold:

There are d points $\underline{p}^i \in \overline{\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)}^{n+1} - \{0\}$ and $c \in \mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)$ satisfying:

i. $F(\underline{u}^0, \ldots, \underline{u}^r) = c \prod_{i=1}^{d} \langle \underline{p}^i, \underline{u}^0 \rangle$, in $\overline{\mathbb{C}(\underline{u}^1, \ldots, \underline{u}^r)}[\underline{u}^0]$.
ii. $L_i(\underline{u}^j) := \langle \underline{p}^i, \underline{u}^j \rangle = 0$ for $i = 1, \ldots, d$, $j = 1, \ldots, r$.
iii. For every $i \in \{1, \ldots, d\}$ one has:
Let S^j be skew symmetric $(n+1) \times (n+1)$–matrices, $j = 1, \ldots, r$, whose entries $s_{k,l}^j \in \Omega$ are algebraically independent over $\mathbb{C}(\underline{p}^i)$, $k, l = 0, \ldots, n$. In this situation $L_i(\underline{u}^0)$ divides the form $F(\underline{u}^0, S^1\underline{p}^i, \ldots, S^r\underline{p}^i)$ in $\mathbb{C}(\underline{p}^i)[s_{0,0}^1, \ldots, s_{n,n}^r][\underline{u}^0]$.

Proof: [5, Thm. 1].

1.15 Remark: Let $\underline{p} \in \Omega^{n+1} - \{0\}$ and $L(\underline{x}) := \langle \underline{p}, \underline{x} \rangle$. Then it is easy to see that $\underline{v} \in \Omega^{n+1}$ is an element of the zero set of L iff there is a skew symmetric matrix S satisfying $\underline{v} = S\underline{p}$. Therefore the third condition can be reformulated as:
Let $v_k^j \in \Omega$, $k = 0, \ldots, n$, $j = 1, \ldots, r$ and $i \in \{1, \ldots, d\}$, such that

$$L_i(\underline{v}^j) = 0 \;\forall j = 1, \ldots, r .$$

Then $L_i(\underline{u}^0)$ divides $F(\underline{u}^0, \underline{v}^1, \ldots, \underline{v}^r)$ in $\mathbb{C}(\underline{p}^i)[\underline{v}^1, \ldots, \underline{v}^r][\underline{u}^0]$.

Let $V_{\mathbb{C}} = \sum_{i=1}^{m} n_i V_i \in \mathcal{C}_{r,d}(\mathbb{CP}^n)$. Then the algebraic set $supp(V_{\mathbb{C}}) := \bigcup_{i=1}^{m} V_i$ is called the *support* of $V_{\mathbb{C}}$. From a given Chow form F the support of the cycle, whose Chow form is F, can be determined:

1.16 Lemma: Let $V_{\mathbb{C}} \in \mathcal{C}_{r,d}(\mathbb{CP}^n)$ and $F(\underline{u}^0, \ldots, \underline{u}^r)$ be the Chow form of $V_{\mathbb{C}}$. Further let S^i be skew symmetric $(n+1) \times (n+1)$–matrices whose entries $s^i_{j,k}$ are algebraically independent over $\mathbb{C}$. The coefficients of

$$F(S^0\underline{x}, \ldots, S^r\underline{x}) \in \mathbb{C}[x_0, \ldots, x_n][s^0_{0,1}, \ldots, s^r_{n-1,n}] \ ,$$

are denoted $f_l(x_0, \ldots, x_n) \in \mathbb{C}[x_0, \ldots, x_n]$. Then the zero set of the f_l is $supp(V_{\mathbb{C}})$.

Proof: [5, Thm. 3].

From Theorem 1.10 one concludes directly the following corollary, which makes the connection to Ŝafarevic̑'s [20, p. 65] definition of Chow varieties transparent. The equivalence of the definitions of Chow / Van der Waerden and Ŝafarevic̑ is implicitly proved in the proof of 1.10.

1.17 Corollary: Let $F(\underline{u}^0, \ldots, \underline{u}^r)$ be the Chow form of $V \in \mathcal{C}_{r,d}(\mathbb{CP}^n)$, then the following statements are equivalent.

i. $\underline{p}$ is contained in $supp(V)$
ii. $F(\underline{v}^0, \ldots, \underline{v}^r) = 0$ for all $\underline{v}^0, \ldots, \underline{v}^r \in \Omega^{n+1}$ such that $\langle \underline{v}^0, \underline{p} \rangle = \ldots = \langle \underline{v}^r, \underline{p} \rangle = 0$.

2 Stabilizations and pairings of Chow varieties

The additive structure on the free abelian semi group $\mathcal{C}_{r,\bullet}(\mathbb{CP}^n)$ is given by formal addition of cycles, i. e. multiplication of the Chow forms of cycles. Therefore $\mathcal{C}_{r,\bullet}(\mathbb{CP}^n)$ is a topological semi group. In particular there is a continuous pairing

$$+ : \mathcal{C}_{r,d_1}(\mathbb{CP}^n) \times \mathcal{C}_{r,d_2}(\mathbb{CP}^n) \longrightarrow \mathcal{C}_{r,d_1+d_2}(\mathbb{CP}^n) \ .$$

There is a second pairing on Chow varieties: Let $V \subset \Omega\mathbb{P}^n$ and $W \subset \Omega\mathbb{P}^m$ be varieties and $I_{\mathbb{C}}(V) \subset \mathbb{C}[x_0, \ldots, x_n]$ and $I_{\mathbb{C}}(W) \subset \mathbb{C}[x_{n+1}, \ldots, x_{n+m+1}]$ be the vanishing ideals of V and W. Define $V \# W \subset \Omega\mathbb{P}^{n+m+1}$ to be the zero set of the ideal $J \subset \mathbb{C}[x_0, \ldots, x_{n+m+1}]$ which is generated by the elements of $I_{\mathbb{C}}(V)$ and $I_{\mathbb{C}}(W)$ viewed as elements in $\mathbb{C}[x_0, \ldots, x_{n+m+1}]$.

2.1 Definition: The algebraic set $V \# W$ is called the *complex join* of V and W. Further we call $\Sigma V := \mathbb{CP}^0 \# V$ the *complex suspension* of V.

There are other equivalent ways to define the join of varieties. For example let $C(V)$ and $C(W)$ be the affine cones of V and W, i. e. $C(V)$ is the zero set of $I_{\mathbb{C}}(V)$ in $\mathbb{C}^{n+1}$. Then the join of V and W is uniquely determined by

$$C(V \# W) := C(V) \times C(W) \ .$$

From the second definition we deduce that the dimension of $V \divideontimes W$ is $\dim V + \dim W + 1$. And one obtains $V \divideontimes W = (V \divideontimes \mathbb{CP}^m) \cap (\mathbb{CP}^n \divideontimes W)$.
The join of two varieties is again a variety. But one wants to calculate the degree of the join of two varieties:
The varieties $V \divideontimes \mathbb{CP}^m$ and $\mathbb{CP}^n \divideontimes W$ intersect transversally, hence by Bezout [18, Chap. 5, Thm. 5.16]:

$$\begin{aligned} \deg V \divideontimes W &= \deg (V \divideontimes \mathbb{CP}^m) \cdot \deg (\mathbb{CP}^n \divideontimes W) \\ &= \deg V \cdot \deg W . \end{aligned}$$

By linear extension to algebraic cycles the join defines an associative pairing on Chow varieties:

$$\mathcal{C}_{r_1,d_1}(\mathbb{CP}^{n_1}) \times \mathcal{C}_{r_2,d_2}(\mathbb{CP}^{n_2}) \longrightarrow \mathcal{C}_{r_1+r_2+1,d_1 d_2}(\mathbb{CP}^{n_1+n_2+1}) .$$

2.2 Proposition: The complex suspension $\not\Sigma : \mathcal{C}_{r-1,d}(\mathbb{CP}^{n-1}) \to \mathcal{C}_{r,d}(\mathbb{CP}^n)$ is algebraic and hence continuous.

Proof: Let $V = \sum\limits_{m=1}^{k} n_i V_i \in \mathcal{C}_{r-1,d}(\mathbb{CP}^{n-1})$ and $F(\underline{u}^1, \dots, \underline{u}^r)$ be the Chow form of V. Let $\underline{u}^i = (u_1^i, \dots, u_n^i)$, $i = 1, \dots, r$ and let $v_k^j \in \Omega$ be algebraically independent over $\mathbb{C}(\underline{u}^1, \dots, \underline{u}^r)$, $k = 0, \dots, n$, $j = 0, \dots, r$.
For suitable $\underline{p}^l \in \left(\overline{\mathbb{C}(\underline{u}^2, \dots, \underline{u}^r)}\right)^n$, $l = 1, \dots, d$, the Chow form of F splits:

$$\begin{aligned} F_V(\underline{u}^1, \dots, \underline{u}^r) &= \prod_{l=1}^{d} \sum_{k=1}^{n} p_k^l u_k^1 \qquad\qquad \in \overline{\mathbb{C}(\underline{u}^2, \dots, \underline{u}^r)}[\underline{u}^1] \\ &= \sum_{1 \le i_1 \le \dots \le i_d \le n} u_{i_1}^1 \cdot \ldots \cdot u_{i_d}^1 \underbrace{\sum_{\sigma \in \Sigma(i_1, \dots, i_d)} p_{\sigma(i_1)}^1 \cdot \ldots \cdot p_{\sigma(i_d)}^d}_{=:\, G_{i_1,\dots,i_d}(\underline{u}^2, \dots, \underline{u}^r)} , \end{aligned}$$

such that $G_{i_1,\dots,i_d} \in \mathbb{C}[\underline{u}^2, \dots, \underline{u}^r]$. The group $\Sigma(i_1, \dots, i_d)$ is that subgroup of Σ_n which only permutes the numbers $i_1, \dots, i_d \in \{1, \dots, n\}$. Each $\underline{p}^l$ is contained in one of the V_j and a solution of the following equations:

$$\langle \underline{u}^i, \underline{x} \rangle = 0 \ , i = 2, \dots, r .$$

On the other hand the Chow form of $\not\Sigma V$ is obtained from points $\underline{q}^l$ of one of the $\not\Sigma V_j$ which are solutions of the equation:

$$\begin{pmatrix} v_0^1 & v_1^1 & \dots & v_n^1 \\ \vdots & \vdots & & \vdots \\ v_0^r & v_1^r & \dots & v_n^r \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = 0 \quad .$$

This equation is equivalent to

$$\underbrace{\begin{pmatrix} v_0^1 & v_1^1 & \dots & v_n^1 \\ 0 & v_1^2 v_0^1 - v_1^1 v_0^2 & \dots & v_n^2 v_0^1 - v_n^1 v_0^2 \\ \vdots & \vdots & & \vdots \\ 0 & v_1^r v_0^1 - v_1^1 v_0^r & \dots & v_n^r v_0^1 - v_n^1 v_0^r \end{pmatrix}}_{=:\, A} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = 0 \quad ,$$

and the entries of the matrix A are algebraically independent over $\mathbb{C}(\underline{u}^1, \dots, \underline{u}^r)$.

Step 1: Observe the following injective ring homomorphism:

$$\begin{array}{rccll} \tilde{\varphi}: & \mathbb{C}[\underline{u}^2, \dots, \underline{u}^r, \underline{v}^0, \underline{v}^1] & \longrightarrow & \mathbb{C}[\underline{v}^0, \dots, \underline{v}^r] & \\ & u_k^i & \longmapsto & v_k^i v_0^1 - v_k^1 v_0^i & ,i = 2, \dots, r \;, k = 1, \dots, n \\ & c & \longmapsto & c & \forall c \in \mathbb{C}[\underline{v}^0, \underline{v}^1] \;\; . \end{array}$$

It is extendable to an automorphism φ of Ω. Then one obtains:

$$\varphi^{-1}(A) = \begin{pmatrix} v_0^1 & v_1^1 & \dots & v_n^1 \\ 0 & u_1^2 & \dots & u_n^2 \\ \vdots & \vdots & & \vdots \\ 0 & u_1^r & \dots & u_n^r \end{pmatrix} \quad .$$

Now one can calculate the points $\underline{q}^i$ from which the Chow form of $\not\Sigma V$ arises:

$$\begin{aligned} \underline{q}^i &= (t_i, \varphi(p_1^i), \dots, \varphi(p_n^i)) \qquad\qquad ,i = 1, \dots, d \\ t_i &= -\frac{v_1^1 \varphi(p_1^i) + \dots + v_n^1 \varphi(p_n^i)}{v_0^1} \;. \end{aligned}$$

Now one can determine the Chow form as in Construction 1.7. We start with the form G using the notations of 1.7 multiplied by $(v_0^1)^d$. Let I_k be the index set $I_k := \{(i_1, \dots, i_d) \in \mathbb{N}^d \mid 1 \le i_1 \le \dots \le i_k \le n, 1 \le i_{k+1} \le \dots \le i_d \le n\}$ for $0 \le k \le d$.

$$\begin{aligned} & G(\underline{v}^0, \dots, \underline{v}^r) := \;\; (v_0^1)^d \prod_{i=1}^d \sum_{j=0}^n q_j^i v_j^0 \;\; = \;\; \prod_{i=1}^d \left(v_0^1 t_i v_0^0 + \sum_{j=1}^n v_0^1 \varphi(p_j^i) v_j^0 \right) \;\; = \\ & \prod_{i=1}^d \left(-(v_1^1 \varphi(p_1^i) + \dots + v_n^1 \varphi(p_n^i)) v_0^0 + \sum_{j=1}^n v_0^1 \varphi(p_j^i) v_j^0 \right) \;\; = \\ & \prod_{i=1}^d \left(-v_0^0 \sum_{j=1}^n v_j^1 \varphi(p_j^i) + v_0^1 \sum_{j=1}^n \varphi(p_j^i) v_j^0 \right) \;\; = \\ & \sum_{k=0}^d \sum_{I_k} (-v_0^0)^k v_{i_1}^1 \cdot \ldots \cdot v_{i_k}^1 (v_0^1)^{d-k} v_{i_{k+1}}^0 \cdot \ldots \cdot v_{i_d}^0 \sum_{\sigma \in \Sigma(i_1, \dots, i_d)} \varphi(p_{\sigma(i_1)}^1) \cdot \ldots \cdot \varphi(p_{\sigma(i_d)}^d) \;\; = \\ & \sum_{k=0}^d \sum_{I_k} (-v_0^0)^k v_{i_1}^1 \cdot \ldots \cdot v_{i_k}^1 (v_0^1)^{d-k} v_{i_{k+1}}^0 \cdot \ldots \cdot v_{i_d}^0 \cdot \varphi \left(\sum_{\sigma \in \Sigma(i_1, \dots, i_d)} p_{\sigma(i_1)}^1 \cdot \ldots \cdot p_{\sigma(i_d)}^d \right) \;\; = \\ & \sum_{k=0}^d \sum_{I_k} (-v_0^0)^k v_{i_1}^1 \cdot \ldots \cdot v_{i_k}^1 (v_0^1)^{d-k} v_{i_{k+1}}^0 \cdot \ldots \cdot v_{i_d}^0 \cdot \varphi(G_{i_1, \dots, i_d}(\underline{u}^2, \dots, \underline{u}^r)) \;\; . \end{aligned}$$

This calculation shows that the coefficients of the form G are polynomials in the coefficients of F_V. These polynomials depend only on the dimension, the codimension and the degree of V.

Step 2: The form $G \in \mathbb{C}[\underline{v}^0, \ldots, \underline{v}^r]$ is homogeneous of degree d in $\underline{v}^0, \underline{v}^2, \ldots, \underline{v}^r$ and homogeneous of degree rd in $\underline{v}^1$. Therefore G is not the Chow form of $\not\Sigma V$, but as in Construction 1.7 there is a $T \in \mathbb{C}(\underline{v}^1, \ldots, \underline{v}^r)$ such that

$$G(\underline{v}^0, \ldots, \underline{v}^r) = F_{\not\Sigma V}(\underline{v}^0, \ldots, \underline{v}^r) T ,$$

where $F_{\not\Sigma V}$ denotes the Chow form of $\not\Sigma V$. Let $T_1 \in \mathbb{C}[\underline{v}^1, \ldots, \underline{v}^r]$ be the common denominator of T. Then T_1 has to be a divisor of F_V in $\mathbb{C}[\underline{v}^0, \ldots, \underline{v}^r]$. But this can only happen if $T_1 \in \mathbb{C}^*$, because the irreducible factors of F_V are polynomials of positive degree in $\underline{v}^0$. This proves that $T \in \mathbb{C}[\underline{v}^1]$ and T is homogeneous of degree $(r-1)d$.
For the case $r = 1$ the theorem is proved. For the case $r > 1$ one has to show that $T(\underline{v}^1) = (v_0^1)^{(r-1)d}$. Without loss of generality let V be from now on a variety:

Step 3: Determine the zero set of $T(\underline{v}^1)$ as a polynomial in $\mathbb{C}[\underline{v}^1]$:
Let $\underline{w} = (0, v_1^1, \ldots, v_n^1)$ and $\underline{\tilde{w}} = (v_1^1, \ldots, v_n^1)$. Then one obtains:

$$\begin{aligned}
&G(\underline{v}^0, \underline{w}, \underline{v}^2, \ldots, \underline{v}^r) = \\
&\sum_{k=0}^{d} \sum_{\substack{1 \le i_1 \le \ldots \le i_k \le n \\ 1 \le i_{k+1} \le \ldots \le i_d \le n}} (-v_0^0)^k v_{i_1}^1 \cdot \ldots \cdot v_{i_k}^1 0^{d-k} v_{i_{k+1}}^0 \cdot \ldots \cdot v_{i_d}^0 \cdot \varphi(G_{i_1,\ldots,i_d}(\underline{u}^2, \ldots, \underline{u}^r)) = \\
&\sum_{1 \le i_1 \le \ldots \le i_d \le n} (-v_0^0)^d v_{i_1}^1 \cdot \ldots \cdot v_{i_d}^1 \cdot G_{i_1,\ldots,i_d}(v_0^2 \cdot \underline{\tilde{w}}, v_0^3 \cdot \underline{\tilde{w}}, \ldots, v_0^r \cdot \underline{\tilde{w}}) = \\
&\quad (-v_0^0)^d \sum_{1 \le i_1 \le \ldots \le i_d \le n} v_{i_1}^1 \cdot \ldots \cdot v_{i_d}^1 \cdot (v_0^2)^d \cdot \ldots \cdot (v_0^r)^d \cdot G_{i_1,\ldots,i_d}(\underline{\tilde{w}}, \ldots, \underline{\tilde{w}}) = \\
&\quad (-v_0^0)^d \cdot (v_0^2)^d \cdot \ldots \cdot (v_0^r)^d \sum_{1 \le i_1 \le \ldots \le i_d \le n} v_{i_1}^1 \cdot \ldots \cdot v_{i_d}^1 \cdot G_{i_1,\ldots,i_d}(\underline{\tilde{w}}, \ldots, \underline{\tilde{w}}) = \\
&\quad (-v_0^0)^d \cdot (v_0^2)^d \cdot \ldots \cdot (v_0^r)^d \cdot F_V(\underline{\tilde{w}}, \ldots, \underline{\tilde{w}}) .
\end{aligned}$$

But $F_V(\underline{\tilde{w}}, \ldots, \underline{\tilde{w}})$ must be zero, because the dimension of V is greater than 0, and $F_V(\underline{\tilde{w}}, \ldots, \underline{\tilde{w}})$ is a divisor of a resultant which has to be 0. It is easy to show that the Chow form $F_{\not\Sigma V}(\underline{v}^0, \underline{w}, \underline{v}^2, \ldots, \underline{v}^r) \neq 0$ using the property that $F_{\not\Sigma V}$ generates the radical of a resultant. Therefore $T(0, v_1^1, \ldots, v_n^1) = 0$ for all $v_1^1, \ldots, v_n^1 \in \Omega$ and hence v_0^1 divides T.
Now we claim that v_0^1 is the only irreducible factor of T: Suppose $T(\underline{w}) = 0$ such that $w_0 \neq 0$, $\underline{w} \in \mathbb{CP}^n$. We may assume $w_0 = 1$. For the Chow form of $\not\Sigma V$ one has

$$F_{\not\Sigma V}(\underline{v}^0, \underline{w}, \underline{v}^2, \ldots, \underline{v}^r) = 0 \text{ iff } \langle \underline{p}, \underline{w} \rangle = 0 \text{ for all } \underline{p} \in \not\Sigma V \cap H_{n-(r-1)} .$$

But this would mean $\langle \underline{x}, \underline{w} \rangle \in \mathbb{C}[\underline{x}]$ is an element of $I_{\mathbb{C}}(\not\Sigma V)$ because $\not\Sigma V \cap H_{n-(r-1)}$ contains general points of $\not\Sigma V$. This implies that $w_0 = 0$, because $\langle \underline{x}, \underline{w} \rangle$ is irreducible and $I_{\mathbb{C}}(\not\Sigma V)$ is generated by elements in $\mathbb{C}[x_1, \ldots, x_n]$. This is a contradiction to the assumption $w_0 = 1$.

On the other hand we have $G(\underline{v}^0, \underline{w}, \underline{v}^2, \ldots, \underline{v}^r) = 0$, because $T(\underline{w}) = 0$. That means there is a point $\underline{p} \in \not\Sigma V$ satisfying the following equation:

$$\begin{pmatrix} v_0^0 & v_1^0 & \ldots & v_n^0 \\ 1 & w_1^1 & \ldots & w_n^1 \\ 0 & v_1^2 - w_1^1 v_0^2 & \ldots & v_n^2 - w_n^1 v_0^2 \\ \vdots & \vdots & & \vdots \\ 0 & v_1^r - w_1^1 v_0^r & \ldots & v_n^r - w_n^1 v_0^r \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = 0 \ ,$$

which is equivalent to

$$\begin{pmatrix} v_0^0 & v_1^0 & \ldots & v_n^0 \\ 1 & w_1^1 & \ldots & w_n^1 \\ v_0^2 & v_1^2 & \ldots & v_n^2 \\ \vdots & \vdots & & \vdots \\ v_0^r & v_1^r & \ldots & v_n^r \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = 0 \quad ,$$

because in the case where V is irreducible, the form G equals $\varphi(F_{\not\Sigma V})$ up to a factor in $\mathbb{C}[v_0^1]$. But this means that $F_{\not\Sigma V}(\underline{v}^0, \underline{w}, \underline{v}^2, \ldots, \underline{v}^r) = 0$, which leads to a contradiction as above. □

As a corollary we obtain the continuity of the complex join:

2.3 Theorem: The map $\mathcal{C}_{r_1,d_1}(\mathbb{CP}^{n_1}) \times \mathcal{C}_{r_2,d_2}(\mathbb{CP}^{n_2}) \longrightarrow \mathcal{C}_{r_1+r_2+1,d_1d_2}(\mathbb{CP}^{n_1+n_2+1})$ is algebraic and hence continuous.

Proof: The image of the continuous map

$$\mathcal{C}_{r_1,d_1}(\mathbb{CP}^{n_1}) \times \mathcal{C}_{r_2,d_2}(\mathbb{CP}^{n_2}) \xrightarrow{(\divideontimes \mathbb{CP}^{n_2}, \mathbb{CP}^{n_1} \divideontimes)} \mathcal{C}_{r_1+n_2+1,d_1}(\mathbb{CP}^{n_1+n_2+1}) \times \mathcal{C}_{r_2+n_1+1,d_2}(\mathbb{CP}^{n_1+n_2+1})$$

consists only on pairs of cycles whose irreducible components intersect transversally. Therefore the intersection pairing is well defined and analytic on the image of $(\divideontimes \mathbb{CP}^{n_2}, \mathbb{CP}^{n_1} \divideontimes)$ [1, Thm. VI.2.10 and p.142]. The Chow varieties are compact and hence the join is algebraic by GAGA (e. g. see [9, Chap. IV.1]). □

Now we want to define stabilizations of Chow varieties. For this purpose let $L \in \mathcal{C}_{r,1}(\mathbb{CP}^n)$ be a fixed linear subspace. Further let *inc* be the map of Chow varieties, which is induced by the inclusion

$$\begin{array}{rccc} inc: & \Omega\mathbb{P}^n & \longrightarrow & \Omega\mathbb{P}^{n+1} \\ & (p_0, \ldots, p_n) & \longmapsto & (p_0, \ldots, p_n, 0) \end{array} \quad .$$

The following diagram is commutative:

$$\begin{tikzcd}
& \mathcal{C}_{r,d+1}(\mathbb{CP}^n) \arrow[rr, "inc"] \arrow[dd, "\Sigma\!\!\!/"'] & & \mathcal{C}_{r,d+1}(\mathbb{CP}^{n+1}) \arrow[dd, "\Sigma\!\!\!/"'] \\
\mathcal{C}_{r,d}(\mathbb{CP}^n) \arrow[ur, "+L"] \arrow[rr, "inc"] \arrow[dd, "\Sigma\!\!\!/"'] & & \mathcal{C}_{r,d}(\mathbb{CP}^{n+1}) \arrow[ur, "+inc(L)"] \arrow[dd, "\Sigma\!\!\!/"'] & \\
& \mathcal{C}_{r+1,d+1}(\mathbb{CP}^{n+1}) \arrow[rr, "inc"] & & \mathcal{C}_{r+1,d+1}(\mathbb{CP}^{n+2}) \\
\mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1}) \arrow[ur, "+\Sigma\!\!\!/ L"] \arrow[rr, "inc"] & & \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+2}) \arrow[ur, "+\Sigma\!\!\!/ inc(L)"'] &
\end{tikzcd}$$

The map $+L$ is obviously continuous. The continuity of the map inc follows directly from the definition of Chow forms. Hence we can define several stabilizations of Chow varieties:

2.4 Notation:
$\mathcal{C}_r(\mathbb{CP}^n) := \lim_{d} \mathcal{C}_{r,d}(\mathbb{CP}^n)$, $\mathcal{D}(d) := \lim_{r,n} \mathcal{C}_{r,d}(\mathbb{CP}^n)$, $\mathcal{C} := \lim_{r,d,n} \mathcal{C}_{r,d}(\mathbb{CP}^n)$, $\mathfrak{C} := \coprod_{d} \mathcal{D}(d)$.

From the examples 1.13 one deduces $\mathcal{C}_0(\mathbb{CP}^n)$ to be homeomorphic to the reduced infinite symmetric product $\widetilde{\mathrm{SP}}(\mathbb{CP}^n)$. And $\mathcal{D}(1)$ is homeomorphic to BU the classifying space of U–bundles.
$\mathcal{C}_r(\mathbb{CP}^n)$ and $\mathcal{C}$ are commutative topological monoids by formal addition of cycles with unit L. $\mathfrak{C}$ is a commutative topological monoid by formal addition of cycles with unit $\emptyset$ the empty cycle.
Intertwining coordinates the join pairing extends to a multiplicative pairing on $\mathcal{C}$, $\mathfrak{C}$, and $\mathcal{D}(1)$. This multiplicative pairing on $\mathcal{D}(1)$ coincides with the Whitney sum on BU. In the next section the homotopy type of $\mathcal{C}$ will be determined.

3 The Complex Suspension Theorem

3.1 The Complex Suspension Theorem: The complex suspension

$$\Sigma\!\!\!/ : \mathcal{C}_r(\mathbb{CP}^n) \longrightarrow \mathcal{C}_{r+1}(\mathbb{CP}^{n+1})$$

is a homotopy equivalence.

As in [13] the theorem will be proved in two steps. First one can show that $\Sigma\!\!\!/ \mathcal{C}_{r,d}(\mathbb{CP}^n)$ is a strong deformation retract of a space $\mathcal{T}_d \subset \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1})$. The second step is to show that the inclusion $\mathcal{T} \hookrightarrow \mathcal{C}_{r+1}(\mathbb{CP}^{n+1})$ induces isomorphisms on homotopy groups.

3.2 Definition: Let $\{0\} \times \mathbb{CP}^n \subset \mathbb{CP}^{n+1}$ denote the hyperplane given by the equation

$x_0 = 0$. Then define

$$\mathcal{T}_d := \left\{ \sum_{i=1}^{m} n_i V_i \in \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1}) \mid V_i \not\subset \{0\} \times \mathbb{CP}^n \, , i = 1, \ldots, m \right\} .$$

Fixing an element of $\mathcal{T}_1$ one defines $\mathcal{T} := \lim\limits_{d \to \infty} \mathcal{T}_d$.

Let $V \in \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1})$ and F_V be the Chow form of V. V is an element of $\mathcal{T}_d$ iff $F_V(\underline{u}^0, \ldots, \underline{u}^r, \underline{e}) \neq 0$, $\underline{e} = (1, 0, \ldots, 0)$. This is obviously true for varieties for F_V generates the radical of a resultant by 1.10. Hence it is true for cycles. This condition is polynomial in the coefficients of F_V, hence $\mathcal{T}_d$ is a Zariski open subset of $\mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1})$.

We first construct a map induced on Chow varieties by

$$\begin{array}{rccc} \varphi : & \mathbb{CP}^{n+1} \times \mathbb{R}_+^* & \longrightarrow & \mathbb{CP}^{n+1} \\ & ((p_0, \ldots, p_{n+1}), t) & \longmapsto & (tp_0, p_1, \ldots, p_{n+1}) \end{array} .$$

This map induces a retraction ϕ of $\mathcal{T}_d$. But first one has to check how this induced map is defined on Chow forms.

3.3 Lemma: Let $V \in \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1})$ and $F_V(\underline{u}^0, \ldots, \underline{u}^{r+1})$ be the Chow form of V. Then $\phi(V, t)$ has the Chow form $F_V({}^t\underline{u}^0, \ldots, {}^t\underline{u}^{r+1})$, where ${}^t\underline{u}^i = (tu_0^i, u_1^i, \ldots, u_{n+1}^i)$.

Proof: Let V be a variety and $f_1, \ldots, f_k$ homogeneous generators of $I_{\mathbb{C}}(V)$. Then F_V generates the radical of the resultant of

$$f_1(x_0, \ldots, x_{n+1}) \, , \ldots , \; f_k(x_0, \ldots, x_{n+1}) \, , \; \begin{pmatrix} u_0^0 & \cdots & u_{n+1}^0 \\ \vdots & & \vdots \\ u_0^{r+1} & \cdots & u_{n+1}^{r+1} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ \vdots \\ x_{n+1} \end{pmatrix} .$$

The Chow form of $\phi(V, t)$ is the generator of the radical of the resultant of

$$f_1(\frac{1}{t}x_0, \ldots, x_{n+1}) \, , \ldots , \; f_k(\frac{1}{t}x_0, \ldots, x_{n+1}) \, , \; \begin{pmatrix} u_0^0 & \cdots & u_{n+1}^0 \\ \vdots & & \vdots \\ u_0^{r+1} & \cdots & u_{n+1}^{r+1} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ \vdots \\ x_{n+1} \end{pmatrix}$$

which is obviously the same as the radical of the resultant of

$$f_1(x_0, \ldots, x_{n+1}) \, , \ldots , \; f_k(x_0, \ldots, x_{n+1}) \, , \; \begin{pmatrix} tu_0^0 & u_1^0 & \cdots & u_{n+1}^0 \\ \vdots & \vdots & & \vdots \\ tu_0^{r+1} & u_1^{t+1} & \cdots & u_{n+1}^{r+1} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ \vdots \\ x_{n+1} \end{pmatrix} .$$

The algebraic independence of the u_i^j remains invariant under multiplication with t, hence the lemma is proved. □

Suppose V lies in the image of $\Sigma\!\!\!/$. Then the vanishing ideal of each irreducible component is generated by polynomials in $\mathbb{C}[x_1, \ldots, x_{n+1}]$, that means x_0 does not appear in these generators. Hence one obtains directly from the proof

$$\phi(\quad , t) \mid_{\Sigma\!\!\!/\mathcal{C}_{r,d}(\mathbb{CP}^n)} = id_{\Sigma\!\!\!/\mathcal{C}_{r,d}(\mathbb{CP}^n)} \;\; \forall t \in \mathbb{R}_+^* .$$

On the other hand the algebraic independence of the u_i^j remains invariant under the multiplication with t which implies

$$F_V(\underline{u}^0,\ldots,\underline{u}^r,\underline{e}) \neq 0 \Longleftrightarrow F_V({}^t\underline{u}^0,\ldots,{}^t\underline{u}^r,t\underline{e}) \neq 0 \ .$$

Hence $\phi(\mathcal{T}_d,t) \subset \mathcal{T}_d$. We claim that ϕ extends continuously to $\mathcal{T}_d \times]0,\infty]$. The following lemma is very useful for this purpose:

3.4 Lemma: Let $V \in \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1})$ be a variety and F_V be a fixed representative of the Chow form of V. Suppose

$$F_V({}^t\underline{u}^0,\ldots,{}^t\underline{u}^{r+1}) = t^k F_V(\underline{u}^0,\ldots,\underline{u}^{r+1}) \ \ \forall t \in \mathbb{C}^*,$$

then V lies in the image of $\not\Sigma$ and $k=d$.

Proof: Let $\{\underline{p}^1,\ldots,\underline{p}^d\}$ be the intersection of V with a $\mathbb{C}$–general $(n-r)$–dimensional linear subspace H_{n-r}. Then the Chow form F_V splits in $\overline{\mathbb{C}(\underline{u}^1,\ldots,\underline{u}^{r+1})}[\underline{u}^0]$:

$$\begin{aligned}
F_V({}^t\underline{u}^0,\ldots,{}^t\underline{u}^{r+1}) &= t^k F_V(\underline{u}^0,\ldots,\underline{u}^{r+1}) \\
&= \frac{t^k}{t^d}\prod_{i=1}^d t\langle \underline{p}^i,\underline{u}^0\rangle \\
&= \frac{t^k}{t^d}\prod_{i=1}^d \left(p_0^i(tu_0^0) + \sum_{j=1}^{n+1}(tp_j^i)u_j^0 \right) \\
&= \left(\frac{t^k}{t^d}p_0^1(tu_0^0) + \sum_{j=1}^{n+1}(\frac{t^k}{t^d}tp_j^1)u_j^0 \right) \prod_{i=2}^d \left(p_0^i(tu_0^0) + \sum_{j=1}^{n+1}(tp_j^i)u_j^0 \right) \quad .
\end{aligned}$$

$(p_0^i,tp_1^i,\ldots,tp_{n+1}^i)$ is a general point of V for every $t \in \mathbb{C}^*$. Hence $(tp_0^i,p_2^i,\ldots,p_{n+1}^i)$ is a general point for V, too. One has $p_0^i \neq 0$ unless $k=0$. Now let $f \in I_{\mathbb{C}}(V)$. Then tp_0^i is a zero of $f(x_0,p_1^i,\ldots,p_{n+1}^i)$ for every $t \in \mathbb{C}^*$. But this means that the polynomial has infinitely many zeros and hence there must be a set of generators of $I_{\mathbb{C}}(V)$ which are constant as polynomials in x_0. Therefore V lies in the image of $\not\Sigma$. Hence $k=d$ because $(u_0^0)^d t^d$ is a monomial of the Chow form of V whose coefficient is constant in t. □

Now we want to extend ϕ: Let $V \in \mathcal{T}_d$ and F_V be a fixed representative of the Chow form of V. Let k be the degree of $F_V({}^t\underline{u}^0,\ldots,{}^t\underline{u}^{r+1})$ as a polynomial in t. For $t \in \mathbb{N}$ one obtains a sequence $(\phi(V,t))_{t\in\mathbb{N}}$ in the Chow variety. If one divides each element of this sequence by t^k, the sequence remains invariant, because Chow forms are only determined up to a factor in $\mathbb{C}^*$. If t approaches ∞ we obtain that the original coefficients of t^k remain invariant and the original coefficients of t^l, $l<k$ tend to zero. That means the sequence is convergent. Therefore $\phi(V,\infty) := \lim_{t\to\infty}\phi(V,t)$ is an element of the Chow variety $\mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1})$. Furthermore we have already seen that the Chow form $F_{\phi(V,\infty)}({}^t\underline{u}^0,\ldots,{}^t\underline{u}^{r+1})$ is homogeneous of degree k as a polynomial in t.
If one could check that this degree k in t equals d, then by 3.4 the map ϕ retracts $\mathcal{T}_d$ to $\not\Sigma\mathcal{C}_{r,d}(\mathbb{CP}^n)$:

The degree k in t cannot be lower than the degree d of V, because the monomial $(u_0^0)^d t^d$ appears in the Chow form of $V \in \mathcal{T}_d$. Hence $k \geq d$.
On the other hand from Lemma 3.4 for each irreducible component V_i one knows that $F_{\phi(V_i,\infty)}({}^t\underline{u}^0, \ldots, {}^t\underline{u}^{r+1})$ is homogeneous of degree deg V_i or of degree zero as polynomial in t. But in the case where the degree is zero for one irreducible component one obtains $k < d$. This contradicts $k \geq d$. Hence k has to be d. This proves the following lemma:

3.5 Lemma: The map $\phi : \mathcal{T}_d \times [1, \infty] \to \mathcal{T}_d$ is a homotopy, which retracts $\mathcal{T}_d$ to $\Sigma\!\!\!/\,\mathcal{C}_{r,d}(\mathbb{CP}^n)$.

The first part of the complex suspension theorem is proved. Now some preparations for the second part of the proof, where we have to check, that the inclusion $\mathcal{T} \hookrightarrow \mathcal{C}_{r+1}(\mathbb{CP}^{n+1})$ induces isomorphisms on homotopy groups:

3.6 Notation: In the following let $\underline{z}_1 := (1, 1, 0, \ldots, 0) \in \mathbb{CP}^{n+2}$ and

$$\mathcal{C}'_{r+1,d}(\mathbb{CP}^{n+2}) := \{V \in \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+2}) \mid \underline{z}_1 \notin supp(V)\} .$$

Further let $pr_1 : \mathbb{CP}^{n+2} - \{\underline{z}_1\} \to \mathbb{CP}^{n+1}$ be the projection from $\underline{z}_1$ onto the hyperplane given by the equation $x_0 = 0$.

Suppose $V \in \mathcal{C}'_{r+1,d}(\mathbb{CP}^{n+2})$ then the algebraic set $pr_1(supp(V))$ has dimension $r + 1$ because the projection is an algebraic map. Now we want to define a map π_1 for cycles $V \in \mathcal{C}'_{r+1,d}(\mathbb{CP}^{n+2})$ such that $supp(\pi_1(V)) = pr_1(supp(V))$.
Let θ be the following ring homomorphism:

$$\begin{array}{rcccl} \theta : & \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r] & \longrightarrow & \mathbb{C}[\underline{\tilde{u}}^0, \ldots, \underline{\tilde{u}}^r] & \\ & u_0^j & \longmapsto & -u_1^j & , j = 0, \ldots, r \\ & z & \longmapsto & z & \forall z \in \mathbb{C}[\underline{\tilde{u}}^0, \ldots, \underline{\tilde{u}}^r], \end{array}$$

where $\underline{\tilde{u}}^j := (u_1^j, \ldots, u_{n+2}^j)$ and $\underline{u}^j = (u_0^j, \ldots, u_{n+2}^j)$.

3.7 Lemma: θ maps the Chow form of a cycle $V \in \mathcal{C}'_{r+1,d}(\mathbb{CP}^{n+2})$ to the Chow form of a cycle, which is denoted $\pi_1(V)$, in $\mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1})$, such that $supp(\pi_1(V)) = pr_1(supp(V))$.

For the proof of this lemma we need some properties of the Chow form of a pure dimensional algebraic set:

3.8 Remark: Let $V \subset \mathbb{C}^n$ be an algebraic set whose irreducible components have fixed dimension r. Denote the irreducible components by $V_1, \ldots, V_k$, $V_i \neq V_j$ for $i \neq j$. Further let $f_1, \ldots, f_l$ be homogeneous generators of the vanishing ideal of V and $g_1^j, \ldots, g_{l_j}^j$ homogeneous generators of V_j, $j = 1, \ldots, k$.
The Chow form $F_V \in \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^r]$ of V is the product of the Chow forms F_j of the V_j, $j = 1, \ldots, k$. By Theorem 1.10 the form F_j is the generator of the radical of the resultant of

$$g_1^j(\underline{x}) = \ldots = g_{l_j}^j(\underline{x}) = \langle \underline{u}^0, \underline{x} \rangle = \ldots = \langle \underline{u}^r, \underline{x} \rangle = 0 .$$

For $\underline{v}^0, \ldots, \underline{v}^r \in \Omega^{n+1}$ this implies $F_V(\underline{v}^0, \ldots, \underline{v}^r) = 0$ iff there is a $j \in \{1, \ldots, k\}$ such that the zero set of $\langle \underline{v}^0, \underline{x} \rangle, \ldots, \langle \underline{v}^r, \underline{x} \rangle$ meets V_j.

Hence $F_V(\underline{v}^0, \ldots, \underline{v}^r) = 0$ iff the zero set of $\langle \underline{v}^0, \underline{x} \rangle, \ldots, \langle \underline{v}^r, \underline{x} \rangle$ meets V, which says the polynomials

$$f_1(\underline{x}), \ldots, f_l(\underline{x}), \langle \underline{v}^0, \underline{x} \rangle, \ldots, \langle \underline{v}^r, \underline{x} \rangle$$

have a common zero in $\Omega\mathbb{P}^n$. Therefore F_V generates the radical of the resultant of

$$f_1(\underline{x}) = \ldots = f_l(\underline{x}) = \langle \underline{u}^0, \underline{x} \rangle = \ldots = \langle \underline{u}^r, \underline{x} \rangle = 0 \ .$$

Proof of 3.7: Let $V \in \mathcal{C}'_{r+1,d}(\mathbb{CP}^{n+2})$ be a variety and F_V be the Chow form of V. Further let $f_1, \ldots, f_k \in \mathbb{C}[x_0, \ldots, x_{n+2}]$ be homogeneous generators of the vanishing ideal of V. And let $g_1, \ldots, g_l \in \mathbb{C}[x_1, \ldots, x_{n+2}]$ be homogeneous generators of the vanishing ideal of the algebraic set $pr_1(V)$. F_V is the generator of the radical of the resultant of the following equations:

$$f_1(\underline{x}) = \ldots = f_k(\underline{x}) = \begin{pmatrix} u_0^0 & u_1^0 & \cdots & u_{n+2}^0 \\ \vdots & \vdots & & \vdots \\ u_0^r & u_1^r & \cdots & u_{n+2}^r \end{pmatrix} \cdot \underline{x} = 0 \ . \tag{3.i}$$

But the resultant of (3.i) is surjectively mapped under θ onto the resultant of

$$f_1(\underline{x}) = \ldots = f_k(\underline{x}) = \begin{pmatrix} -u_1^0 & u_1^0 & \cdots & u_{n+2}^0 \\ \vdots & \vdots & & \vdots \\ -u_1^r & u_1^r & \cdots & u_{n+2}^r \end{pmatrix} \cdot \underline{x} = 0 \ . \tag{3.ii}$$

Therefore the radical of the ideal which is generated by $\theta(F_V)$in $\mathbb{C}[\underline{\tilde{u}}^0, \ldots, \underline{\tilde{u}}^r]$ is the radical of the resultant of (3.ii). But the resultant of (3.ii) is the same as the resultant of

$$g_1(\underline{x}) = \ldots = g_l(\underline{x}) = \begin{pmatrix} u_1^0 & \cdots & u_{n+2}^r \\ \vdots & & \vdots \\ u_1^r & \cdots & u_{n+2}^r \end{pmatrix} \cdot \underline{x} = 0 \ . \tag{3.iii}$$

To check this let $\underline{v}^0, \ldots, \underline{v}^r \in \Omega^{n+2}$ be a zero of the resultant of (3.ii). That means the equations

$$f_1(\underline{x}) = \ldots = f_k(\underline{x}) = \begin{pmatrix} -v_1^0 & v_1^0 & \cdots & v_{n+2}^0 \\ \vdots & \vdots & & \vdots \\ -v_1^r & v_1^r & \cdots & v_{n+2}^r \end{pmatrix} \cdot \underline{x} = 0 \ .$$

have a solution $(q_0, \ldots, q_{n+2}) \in V$. But $(q_1 - q_0, q_2, \ldots, q_{n+2})$ is contained in $pr_1(V)$ and therefore a solution of the equations

$$g_1(\underline{x}) = \ldots = g_l(\underline{x}) = \begin{pmatrix} v_1^0 & \cdots & v_{n+2}^r \\ \vdots & & \vdots \\ v_1^r & \cdots & v_{n+2}^r \end{pmatrix} \cdot \underline{x} = 0 \ .$$

Therefore the radical of the resultant of (3.iii) is a subideal of the radical of the resultant of (3.ii). In the analogous way one checks the inverse inclusion.
By Remark 3.8 we know that the radical of the resultant of (3.iii) is generated by the

Chow form $F_{pr_1(V)}$ of $pr_1(V)$. That means that the radical of the ideal which is generated by $\theta(F_V)$ is the ideal which is generated by $F_{pr_1(V)}$. From this we obtain

$$\theta(F_V) = F_1^{n_1} \cdot \ldots \cdot F_s^{n_s}$$

where $F_1, \ldots, F_s$ are the Chow forms of all irreducible components of $pr_1(V)$ and $n_j \in \mathbb{N} - \{0\}$, $j = 1, \ldots, s$.
This proves that $\theta(F_V)$ is a Chow form of multidegree d and the support of the cycle $\pi_1(V)$, whose Chow form is $\theta(F_V)$, is $pr_1(V)$. $\square$

3.9 Notation: $\mathcal{C}_{n+1,d}(\mathbb{CP}^{n+2})$ can be identified with the equivalence classes of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_{n+2}]$ of degree d (Lemma 1.13), where $f \sim g$ iff $f = zg$ for one $z \in \mathbb{C}^*$. Let $Div'_d \subset \mathcal{C}_{n+1,d}(\mathbb{CP}^{n+2})$ denote the set of cycles

$$\sum_{i_0,\ldots,i_{n+2}=d} a_{i_0,\ldots,i_{n+2}} x_0^{i_0} \cdot \ldots \cdot x_{n+2}^{i_{n+2}} \quad ,$$

such that $a_{d,0,\ldots,0} \neq 0$ and $\sum\limits_{i_0+i_1=d} a_{i_0,i_1,0,\ldots,0} = 0$.
Div'_d is a quasi projective algebraic subset of $\mathcal{C}_{n+1,d}(\mathbb{CP}^{n+2})$ of codimension 1. Hence it has dimension $\binom{n+2+d}{d} - 2$.
Because elements in Div'_d are only determined up to a factor in $\mathbb{C}^*$, we can suppose $a_{d,0,\ldots,0} = 1$. The space Div'_d is contractible by the following homotopy:
Let I be the index set $I := \{(i_0, \ldots, i_{n+2}) \in \mathbb{N}_0 \mid i_0 + \ldots + i_{n+2} = d, i_0 \neq d\}$.

$$\begin{array}{llcl} \Phi : & Div'_d \times [0,1] & \longrightarrow & Div'_d \\ & \left(x_0^d + \sum\limits_I a_{i_0,\ldots,i_{n+2}} x_0^{i_0} \cdot \ldots \cdot x_{n+2}^{i_{n+2}}, t \right) & \longmapsto & x_0^d + \sum\limits_I t \cdot a_{i_0,\ldots,i_{n+2}} x_0^{i_0} \cdot \ldots \cdot x_{n+2}^{i_{n+2}} \quad . \end{array}$$

$\Phi(Div'_d, 0)$ is the hyperplane $x_0 = 0$ with multiplicity d. Simple calculations show that $\underline{z}_0 := (1, 0, \ldots, 0)$ and $\underline{z}_1 := (1, 1, 0, \ldots, 0)$ are not contained in the support of any cycle in Div'_d.

Now we are able to define a map which will prove the second part of the complex suspension theorem:

3.10 Definition:

$$\Psi := \pi_1 \circ \cap \circ (\not\Sigma, \Phi) : \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1}) \times Div'_k \times [0,1] \longrightarrow \mathcal{C}_{r+1,dk}(\mathbb{CP}^{n+1}) \quad ,$$

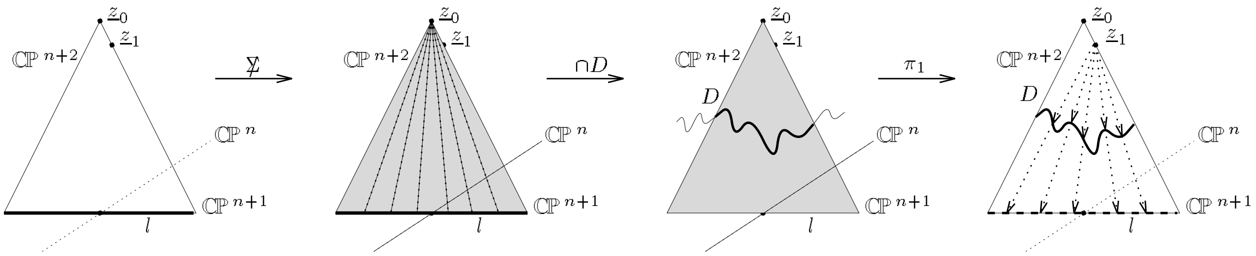
where $\cap$ denotes the intersection of cycles in the sense of Barlet [1].
We want to check that Ψ is well defined: First we have $\Phi(Div'_k, [0,1]) \subset Div'_k$.
Each irreducible component of a cycle in $\not\Sigma\mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1})$ meets the point $\underline{z}_0$, and $\underline{z}_0$ is not contained in the support of any element in Div'_k. Therefore a pair of cycles in $\not\Sigma\mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1}) \times Div'_k$ has general position and the intersection is well defined. The support of the resulting cycle is just the geometric intersection of the supports of the original cycles. Hence the support of the resulting cycle cannot contain the point $\underline{z}_1$,

because $\underline{z}_1$ is not contained in the support of any cycle in Div'_k. That means the map π_1 is well defined in this situation.

The maps $\not\Sigma$, Φ, and π_1 are algebraic and $\cap$ is analytic, hence Ψ is analytic. Ψ has nice geometric properties. First the map $\Psi(\_\_,\_\_,0)$ is multiplication by k. Moreover it moves subsets of $\mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1}) - \mathcal{T}_d$ to $\mathcal{T}_{dk}$ for fixed $D \in Div'_k$. On the other hand Ψ maps cycles in $\not\Sigma\mathcal{C}_{r,d}(\mathbb{CP}^n)$ to cycles in $\not\Sigma\mathcal{C}_{r,dk}(\mathbb{CP}^n)$.

3.11 Lemma: The map $\Psi\mid_{\not\Sigma\mathcal{C}_{r,d}(\mathbb{CP}^n)\times Div'_k\times[0,1]}$ is the map which sends a cycle $V \in \not\Sigma\mathcal{C}_{r,d}(\mathbb{CP}^n)$ to the cycle $k\cdot V$.

Proof: An element in $\not\Sigma\mathcal{C}_{r,d}(\mathbb{CP}^n)$ consists of lines through a point $(0,p_1,\dots,p_{n+1})$ and the point $(1,0,\dots,0)$ in $\mathbb{CP}^{n+1}$. Therefore it suffices to check the case $r=0$ and $d=1$. Ψ first takes the complex suspension of this line l. The complex suspension is cut with a cycle $D \in Div'_d$, and the common points of D and l are projected from $\underline{z}_1$ to the hyperplane $x_0=0$ in $\mathbb{CP}^{n+2}$.



Therefore $\Psi(l,D,t)$ is a closed subset of l which has dimension 1. But a closed 1–dimensional subset of a line is the line itself. Hence $\Psi(l,D,t)=l$. $\square$

For a fixed $D \in Div'_k$ the map Ψ is a homotopy. Now one has to determine for which $D \in Div'_d$ a fixed cycle $V \in \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1})$ is moved to $\mathcal{T}_{dk}$.

3.12 Lemma: Let π_0 denote the projection from $\underline{z}_0$ onto the hyperplane $x_0=0$. There is an isomorphism $F := (\pi_0\mid_{\overline{\pi_1^{-1}(\mathbb{CP}^n)}})^{-1} : \mathbb{CP}^{n+1} \to \overline{\pi_1^{-1}(\mathbb{CP}^n)}$ such that

$$\Psi(V,D,1) \notin \mathcal{T}_{dk} \Longrightarrow F(supp(V)) \subset supp(D)\ .$$

A proof of this lemma is given in [13, p.286,287], where no arguments of continuity are needed, one only has to check this pointwise.

For a fixed cycle $V \in \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1})$ let $\mathcal{B}(V)$ denote the set of those cycles $D \in Div'_d$ such that $\Psi(V,D,1) \notin \mathcal{T}_{dk}$. As one has already seen $\mathcal{B}(V) = \{D \in Div'_k \mid F(V) \subset supp(D)\}$. In general $\mathcal{B}(V)$ is a proper subset of Div'_k.

3.13 Lemma: Let $V = \sum_{i=1}^{m} n_i V_i \in \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1})$. For the complex codimension of $\mathcal{B}(V) \subset Div'_k$ one obtains:

$$\mathrm{codim}_{\mathbb{C}}\mathcal{B}(V) = \max_i\ \mathrm{codim}_{\mathbb{C}}\mathcal{B}(V_i) \geq \binom{r+k+1}{k} - 1\ .$$

Proof: We have to expand the ideas of [13, p.288] (see also [7, p. 83]). It suffices to check the case $m = 1$, therefore let V be a variety. V can be viewed as an affine variety in $\mathbb{C}^{n+2} \cong \mathbb{CP}^{n+2} - \{\underline{z}_0\}$. In this situation the set Div'_k can be viewed as a subset of the polynomials of degree $\leq d$ in $x_1, \ldots, x_{n+2}$. And the set $\mathcal{B}(V) \subset Div'_k$ is the set of those polynomials, which vanish on $F(V) \cap \mathbb{C}^{n+2}$.
$F(V)$ is $r+1$–dimensional. If one projects $F(V) \cap \mathbb{C}^{n+2}$ on a linear subspace of dimension $r+1$, then without loss of generality the image of $F(V) \cap \mathbb{C}^{n+2}$ has dimension $r+1$, too, because $F(V) \cap \mathbb{C}^{n+2}$ contains regular points. Up to linear isomorphisms one can suppose the linear subspace to be the zero set of $x_{r+2} = \ldots = x_{n+2} = 0$.
In this situation a polynomial $f \in \mathbb{C}[x_1, \ldots, x_{r+1}]$ can only vanish on the image of $F(V) \cap \mathbb{C}^{n+2}$ under the projection, if $f = 0$. This is obvious, because the zero set of f is a r–dimensional subset of the linear subspace.
Therefore the complex codimension of $\mathcal{B}(V)$ in $\mathcal{C}_{n+1,k}(\mathbb{CP}^{n+1})$ is greater or equal to the dimension of the space of polynomials in $\mathbb{C}[x_1, \ldots, x_{r+1}]$ of degree $\leq k$. This is $\binom{r+k+1}{k}$. Because $Div'_k \subset \mathcal{C}_{n+1,k}(\mathbb{CP}^{n+2})$ has codimension 1 the lemma is proved. □

Now let $\mathcal{B}$ denote the following set:

$$\begin{aligned} \mathcal{B} &:= (\Psi\, |_{\mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1}) \times Div'_k \times \{1\}})^{-1}(\mathcal{C}_{r+1,dk}(\mathbb{CP}^{n+1}) - \mathcal{T}_{dk}) \\ &= \{(V, D) \in \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1}) \times Div'_k \mid \Psi(V, D, 1) \notin \mathcal{T}_{dk}\}\ . \end{aligned}$$

The set $\mathcal{C}_{r+1,dk}(\mathbb{CP}^{n+1}) - \mathcal{T}_{dk}$ is an algebraic set and Ψ is an analytic map, hence $\mathcal{B}$ is an analytic set. Let $V \in \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1})$ then $V \times Div'_k \cap \mathcal{B} \subset \mathcal{B}(V)$. Therefore the codimension of $\mathcal{B}$ is greater or equal $\binom{r+k+1}{k} - 1$. Now choose $E(r,d) \in \mathbb{N}$ such that $\binom{r+k+1}{k} - 1 > \dim \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1})$, $\forall k \geq E(r,d)$. Further denote by pr_2 the projection from $\mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1}) \times Div'_k$ onto the second factor. Then one obtains

$$\begin{aligned} \dim pr_2(\mathcal{B}) &\leq \dim \mathcal{B} \\ &\leq \dim \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1}) + \dim Div'_k - \tbinom{r+k+1}{k} + 1 \\ &< \dim Div'_k\ . \end{aligned}$$

Hence the projection from $\mathcal{B}$ on Div'_k cannot be surjective $\forall k > E(r,d)$. Therefore one obtains a cycle $\mathcal{D} \in Div'_k$ for each $k \geq E(r,d)$ such that the map $\Psi : \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1}) \times \{\mathcal{D}\} \times [0,1] \to \mathcal{C}_{r+1,dk}(\mathbb{CP}^{n+1})$ has the following properties:

$$\Psi(V, \mathcal{D}, 0) = k \cdot V \quad \text{and} \quad \Psi(\mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1}), \mathcal{D}, 1) \subset \mathcal{T}_{dk}\ .$$

To finish the proof of the complex suspension theorem one has to check that the inclusion $i : \mathcal{T} \hookrightarrow \mathcal{C}_{r+1}(\mathbb{CP}^{n+1})$ induces isomorphisms on homotopy groups. For this purpose let $\Sigma\!\!\!/^{r+1}\mathbb{CP}^0$ be the basepoint $*$ of $\mathcal{T}$ and $\mathcal{C}_{r+1}(\mathbb{CP}^{n+1})$. First of all the monoid $\mathcal{C}_{r+1}(\mathbb{CP}^{n+1})$ is connected. This is obvious from the properties of Ψ. Further the addition on homotopy groups, which is induced by the H–cogroup structure of the spheres, coincides with the addition which is induced by the topological monoid structure of $\mathcal{T}$ and $\mathcal{C}_{r+1}(\mathbb{CP}^{n+1})$.

Surjectivity: Let $\alpha \in \pi_j(\mathcal{C}_{r+1}(\mathbb{CP}^{n+1}))$. There is $d \in \mathbb{N}$ such that $\alpha : S^j \to \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1})$. For $k \geq E(d,r)$ there exists $\alpha_k \in \pi_j(\mathcal{T}_{dk})$ such that $i_*[\alpha_k] \stackrel{\Psi}{=} [k \cdot \alpha]$. Therefore $\alpha_{k+1} - \alpha_k \in \pi_j(\mathcal{T})$ and $i_*[\alpha_{k+1} - \alpha_k] = \alpha$.

Injectivity: Let $\beta \in \pi_j(\mathcal{T})$ such that $i_*[\beta] = *$. Without loss of generality suppose $\beta : S^j \to \Sigma\!\!\!\!/\,\mathcal{C}_r(\mathbb{CP}^n)$, because $\Sigma\!\!\!\!/\,\mathcal{C}_{r,d}(\mathbb{CP}^n)$ is a deformation retract of $\mathcal{T}$.
Now there is a map $h : S^j \times [0,1] \to \mathcal{C}_{r+1}(\mathbb{CP}^{n+1})$ such that $h(\ ,0) = i_*(\beta)$ and $h(\ ,1) = *$. There exists $d \in \mathbb{N}$ such that $h(S^j \times [0,1]) \subset \mathcal{C}_{r+1,d}(\mathbb{CP}^{n+1})$. For each $k \geq E(d,r)$ there exists $h_k : S^j \times [0,1] \to \mathcal{T}_{dk}$ such that $i_*[h_k] \stackrel{\Psi}{=} [k \cdot h]$. Because Ψ restricted to $\Sigma\!\!\!\!/\,\mathcal{C}_{r,d}(\mathbb{CP}^n)$ is multiplication by d one obtains $h_k(\ ,1) = *$ and $h_k(\ ,0) = k \cdot \beta$. In this situation one obtains $(h_{k+1} - h_k) : S^i \times [0,1] \to \mathcal{T}$ such that $(h_{k+1} - h_k)(\ ,1) = *$ and $(h_{k+1} - h_k)(\ ,0) = \beta$.

The complex suspension theorem enables one to determine the homotopy type of the monoids $\mathcal{C}_r(\mathbb{CP}^n)$.

3.14 Corollary: The topological monoid $\mathcal{C}_r(\mathbb{CP}^n)$ is homotopy equivalent to a product of Eilenberg–MacLane spaces:

$$\mathcal{C}_r(\mathbb{CP}^n) \simeq \prod_{i=1}^{n-r} K(\mathbb{Z}, 2i) \ .$$

Proof: Let H be the unreduced singular homology theory. By Dold/Thom [6, Satz 7.1] we have

$$\widetilde{\mathrm{SP}}(\mathbb{CP}^{n-r}) \ \simeq \ \prod_{i>0} K(H_k(\mathbb{CP}^{n-r}, \mathbb{Z}), i)$$

Hence $\mathcal{C}_r(\mathbb{CP}^n) \stackrel{\Sigma\!\!\!\!/}{\simeq} \dots \stackrel{\Sigma\!\!\!\!/}{\simeq} \mathcal{C}_0(\mathbb{CP}^{n-r}) \cong \widetilde{\mathrm{SP}}(\mathbb{CP}^{n-r}) \simeq \prod_{i=1}^{n-r} K(\mathbb{Z}, 2i)$. □

So $\mathcal{C}$ is homotopy equivalent to the weak product $\prod_{i>0} K(\mathbb{Z}, 2i)$. The naive group completion of $\mathfrak{C} = \coprod_d \mathcal{D}(d)$ is homotopy equivalent to $\mathcal{C} \times \mathbb{Z}$ [14, Thm. 5.4] using the complex suspension theorem and Dold/Thom [6, Thm. 6.10.III].
The naive group completion of $\mathfrak{C}$ is homotopy equivalent to the homotopy theoretic group completion $\Omega B(\mathfrak{C})$ by [15, Thm. 4.4']. Hence we obtain $\Omega B(\mathfrak{C}) \simeq \prod_{i \geq 0} K(\mathbb{Z}, 2i)$.
The inclusion $BU \cong \mathcal{D}(1) \hookrightarrow \mathcal{C}$ represents the total Chern class [14, Thm. 3.6]. Assuming continuity Boyer, Lawson, Lima–Filho, Mann and Michelsohn [4] proved that there is an $\mathcal{I}_*$–functor T [16] such that the induced infinite loop space structure on $\mathfrak{C}$ is compatible to the infinite loop space structure on $\mathcal{D}(1) \cong BU$ which is induced by the Whitney sum of vector bundles. Hence the total Chern class map is an infinite loop map.

4 An $\mathcal{I}_*$–functor defined by Chow varieties

4.1 Definition[16, Def. I.1.8]: Let $\mathcal{I}_*$ denote the category of finite dimensional hermitian inner product spaces with linear isometries. The morphisms are topologized by the (compactly generated) compact–open topology. An $\mathcal{I}_*$–functor T is a continuous functor $T : \mathcal{I}_* \to \mathrm{kTop}$ together with a commutative, associative, and continuous natural transformation $\omega : T \times T \to T \circ \oplus$ such that

i. if $x \in TV$ and $1 \in T\{0\}$ is the basepoint, then

$$\omega(x,1) = x \in T(V \oplus \{0\}) = TV \ .$$

ii. if $V = V' \oplus V''$ the map $TV' \to TV$ given by $x \mapsto \omega(x,1)$ is a homeomorphism onto a closed subset.

If T is an $\mathcal{I}_*$–functor, then $\lim\limits_{V \subset \mathbb{C}^\infty} TV$ is an infinite loop space [3],[16].

4.2 Example[4, Ex. 4.3]: For each hermitian inner product space V of dimension n, let $T_0V := \mathcal{G}(V \oplus V)$ denote the Grassmannian of n–dimensional linear subspaces of $V \oplus V$ with basepoint $1 = V \oplus \{0\}$. Given a linear isometric embedding $f : V \to W$, define $T_0 f$ by $T_0 f(U) = ((fV)^\perp \oplus \{0\}) \oplus (f \oplus f)(U)$.
There is a natural transformation $\omega_0 : T_0 \times T_0 \to T_0 \circ \oplus$ given by the external direct sum, i. e., for $U \in T_0V$ and $U' \in T_0V'$,

$$\omega_0(U,U') = \tau(U \oplus U') \ ,$$

where $\tau : V \oplus V \oplus V' \oplus V' \to V \oplus V' \oplus V \oplus V'$ is the obvious shuffle isometry interchanging the middle two factors.
(T_0, ω_0) is a $\mathcal{I}_*$–functor and $\lim\limits_{V \subset \mathbb{C}^\infty} TV = BU$ with its standard loop space structure.

In an analogous way to Example 4.2 one defines an $\mathcal{I}_*$–functor with values in Chow varieties:

4.3 Example[4]: For each hermitian inner product space V of dimension n, let $TV := \coprod\limits_d \mathcal{C}_{n-1,d}(\mathbb{P}(V \oplus V)$ denote the Chow monoid of $(n-1)$–dimensional or n–codimensional cycles in $\mathbb{P}(V \oplus V)$ with basepoint $1 = \mathbb{P}(V \oplus \{0\})$. In the case $V = \{0\}$ we obtain $T\{0\} = \coprod\limits_d \mathcal{C}_{-1,d}(\mathbb{P}(\{0\}))$. The set $\mathbb{P}(\{0\}) = \emptyset =: \mathbb{CP}^{-1}$ has as codimension 0–subset only $\mathbb{CP}^{-1}$ itself. Therefore $\coprod\limits_d \mathcal{C}_{-1,d}(\mathbb{P}(\{0\})) \cong \mathbb{N}$. Observe that $\mathbb{CP}^{-1}$ is the identity element of the join pairing.
For a linear isometric embedding $f : V \to W$ define

$$Tf(c) := \mathbb{P}(F(V)^\perp \oplus \{0\}) \# (f \oplus f)_*(c) \ ,$$

where $(f \oplus f)_*(c)$ is well defined for varieties and becomes well defined for cycles by linear extension.
Now define $\omega : TV \times TV' \to T(V \oplus V')$ by

$$\omega(c,c') := \tau_*(c \# c') \ ,$$

where $\tau : V \oplus V \oplus V' \oplus V' \to V \oplus V' \oplus V \oplus V'$ interchanges the middle two factor as above.
Straight forward calculation shows that T is a functor and ω is a commutative, associative, natural transformation [4]. One is only left with proving continuity of the structure data.

We illustrate, how the continuity of T and ω can be proved:

4.4 Lemma: Let $V \subset \mathbb{C}^m$ be a n–dimensional vector space and $W \subset \mathbb{C}^m$ a p–dimensional vector space. If $f : V \to W$ is a linear, isometric embedding, then the map

$$\begin{array}{rccc} Tf: & TV & \longrightarrow & TW \\ & c & \longmapsto & \mathbb{P}(f(V)^\perp \oplus \{0_W\}) \circledast (f \oplus f)_*(c) \end{array}$$ is algebraic and therefore continuous.

Proof: The complex join is continuous and building orthogonal complements is a continuous map on the Grassmannian. Hence it suffices to check that

$$(f \oplus f)_* : TV = \coprod_d \mathcal{C}_{n-1,d}(\mathbb{P}(V \oplus V)) \longrightarrow \coprod_d \mathcal{C}_{n-1,d}(\mathbb{P}(W \oplus W))$$

is continuous. We shall calculate the Chow form $F_{(f\oplus f)_*(c)}$ of the cycle $(f \oplus f)_*(c)$.
We can extend the map $(f \oplus f)$ to an automorphism of the vector space $\mathbb{C}^{2m}$ by choice of the orthogonal complement of $V \oplus V$ in $\mathbb{C}^{2m}$. This automorphism is given by an invertible $(2m \times 2m)$–matrix over $\mathbb{C}$.
Without loss of generality suppose c to consist only on one irreducible component. Let $g_1, \ldots, g_k \in \mathbb{C}[x_0, \ldots, x_{2m-1}]$ be homogeneous generators of the vanishing ideal of c. Then $(f \oplus f)_*(c)$ is the irreducible zero set of

$$g_1(A^{-1}\underline{x}), \ldots, g_k(A^{-1}\underline{x}) \quad .$$

The Chow form F_c of c generates the radical of the resultant of

$$g_1(\underline{x}) = \ldots = g_k(\underline{x}) = \begin{pmatrix} u_0^0 & \cdots & u_{2m-1}^0 \\ \vdots & & \vdots \\ u_0^n & \cdots & u_{2m-1}^n \end{pmatrix} \cdot \underline{x} = 0 \quad .$$

The solutions of this equations are the same as the solutions of

$$g_1(A^{-1}\underline{x}) = \ldots = g_k(A^{-1}\underline{x}) = \begin{pmatrix} u_0^0 & \cdots & u_{2m-1}^0 \\ \vdots & & \vdots \\ u_{2m-1}^n & \cdots & u_{2m-1}^n \end{pmatrix} A^{-1}\underline{x} = 0 \quad . \qquad (4.\mathrm{i})$$

Therefore F_c generates the radical of the resultant of (4.i), too. The following map is a ring automorphism

$$\begin{array}{rccccl} \varphi_f : & \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^{n-1}] & \longmapsto & \mathbb{C}[\underline{u}^0, \ldots, \underline{u}^{n-1}] & \\ & u_j^i & \longmapsto & (\underline{u}^i A)_j & , i = 0, \ldots, n-1 \, , \, j = 0, \ldots, 2m-1 \\ & z & \longmapsto & z & \forall z \in \mathbb{C} \, , \end{array}$$

because the $(\underline{u}^i A)_j$ are algebraically independent over $\mathbb{C}$, $i = 0, \ldots, n-1$, $j = 0, \ldots, 2m-1$, for the entries of A are elements of $\mathbb{C}$. The equations (4.i) are mapped onto

$$g_1(A^{-1}\underline{x}) = \ldots = g_k(A^{-1}\underline{x}) = \begin{pmatrix} u_0^0 & \cdots & u_{2m-1}^0 \\ \vdots & & \vdots \\ u_{2m-1}^{n-1} & \cdots & u_{2m-1}^{n-1} \end{pmatrix} AA^{-1}\underline{x} = 0 \ . \qquad (4.\text{ii})$$

Hence $\varphi_f(F_c) = F_c(\underline{u}^0 A, \ldots, \underline{u}^{n-1}A)$ generates the radical of the resultant of (4.ii). But $F_{(f\oplus f)_*(c)}$ is a generator of the radical of the resultant of (4.ii), too. Hence $\varphi_f(F_c)$ and $F_{(f\oplus f)_*(c)}$ coincide up to a factor in $\mathbb{C}^*$. But this means that $\varphi_f(F_c)$ is the Chow form of $(f\oplus f)_*$. φ_f depends on an extension of the linear isometry $(f \oplus f)$ to a matrix A. But $\varphi_f(F_c)$ coincides with $F_{(f\oplus f)_*}$ and is therefore independent of a chosen extension.
We obtain, that $(f\oplus f)_*$ is given on Chow forms by the map φ_f, which is polynomial in the coefficients of Chow forms. Hence $(f\oplus f)_*$ and Tf are algebraic and therefore continuous maps. □

4.5 Lemma: T is a continuous functor and ω is a continuous natural transformation.

Proof: From the proof of Lemma 4.4 for fixed $V, V' \in \mathcal{I}_*$ we obtain that the map $\omega : TV \times TV' \to F(V \oplus V')$, which was defined by $\omega(c, c') = \tau_*(c \# c')$, is algebraic and hence continuous, for τ is a linear isometry and the join is algebraic.
The continuity of T and ω can now be obtained as follows:
By the choice of an invertible matrix A, which represents an morphism f in $\mathcal{I}_*$, we get a local section from the space of linear isometries on a subspace $V \oplus V \subset \mathbb{C}^{2n}$, into the space of invertible $(2n \times 2n)$–matrices. The entries of such matrices depend locally continuously on f and $V \oplus V$. The map Tf is independent on the choice of such a matrix and hence independent on the choice of a local section. Hence T is continuous. The continuity of ω follows by the same arguments. □

Now we obtain a map

$$BU \cong \lim_{V \subset \mathbb{C}^\infty} T_0(V) \hookrightarrow \lim_{V\subset\mathbb{C}^\infty} T(V) = \lim_{n\to\infty} \mathcal{C}_{n-1,d}(\mathbb{CP}^{2n-1}) \simeq \prod_{i\geq 0} K(\mathbb{Z}, 2i)$$

of infinite loop spaces, which represents the total Chern class [4].

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