GEOMETRY OF 1-TORI IN GL_n

Nikolai Vavilov

Department of Mathematics and Mechanics
University of Sanct-Petersburg
Petrodvorets, 198904, Russia

and

Fakultät für Mathematik
Universität Bielefeld
33615, Bielefeld, Deutschland

and

Departement Wiskunde en Informatica
Universitaire Instelling Antwerpen
B-2610 Wilrijk (Antwerpen), Belgium

Introduction

Geometry of unipotent (long) root subgroups in Chevalley groups and related geometries have been extensively studied and are very well understood, see, for example, [AS], [Co1] – [Co4], [Cu], [Ka1], [LS], [Mc], [Ne], [Ti1] – [Ti4], [W1], [ZS1] and references in [Ka2], [Kn], [Z1]. At the same time very little is known about the geometry of tori. Anisotropic groups do

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not have unipotents and in this case the study of tori is a necessity. Here one could mention, for example, very interesting results of B.N.Weisfeiler and D.G.James [We1] – [We3], [JW]. However even for Chevalley groups it might be interesting to describe orbits on pairs, triples, etc. of tori of various classes, identify corresponding spans, classify irreducible subgroups generated by tori from this class, etc. This may lead to geometries similar to, but different from buildings and their residues (compare [We3], p.365). Results in this spirit might shed a new light on the known results concerning description of subgroups (see the surveys [Kn], [V7] – [V9], [Z1] – [Z3]). They would be also the first step towards generalization of results on subgroups generated by pseudoreflections, two-dimensional elements or the like [C1], [C2], [Cox], [Hu], [HW1] – [HW3], [Ko1] – [Ko3], [ST], [W2], [W3], [Wa], [ZS2], [ZS3] to other types of groups and to other types of semisimple elements.

In this paper we study in some detail the model case of *1-tori* in the general linear group $G = \mathrm{GL}(n,T)$ over a skew-field T. The 1-tori are one parameter groups of pseudoreflections with the same axis and centre. First we study the orbits of G acting by conjugation on pairs (X,Y) of 1-tori (referred to as *orbitals* in the sequel). It turns out, that when the skew-field T contains at least 7 elements, there is a close analogy with the root subgroups. Namely, as for the root subgroups one can define the angle ϕ between two 1-tori X and Y, taking values 0, $\pi/3$, $\pi/2$, $2\pi/3$ and π. When $\phi < \pi$ the above orbitals are almost completely defined by the angle between X and Y. (As for the root subgroups the angle $\pi/3$ leads to two possible configurations, fused by the external automorphism of G.) Unlike root subgroups the angle π leads to $|T|-1$ distinct orbitals, falling into two essentially different cases, in one of which the span $\langle X,Y\rangle$ is isomorphic to $\mathrm{GL}(2,T)$, in another one it is triangularizable. There is another important aspect in which this result differs from the case of root subgroups. Namely, whereas for pairs (X,Y) of root subgroups the orbitals and the corresponding spans $\langle X,Y\rangle$ do not depend on the skew-field, in our case the fields K with $|K| \leq 7$ are true exceptions. In these cases the shape of the span $\langle X,Y\rangle$ for a given orbital differs from the generic case.

We show how the above result immediately implies description of the irreducible subgroups in G generated by 1-tori (the analogue of McLaughlin's theorem for pseudoreflections, see [V6]) and description of overgroups of the diagonal subgroup $D = \mathrm{D}(n,T)$ (see [B], [BV], [V1] for the first proof of this result; [Se] for a proof for finite fields; [V2], [V4] for different proofs; [BK], [K1] – [K4], for the description in the exceptional cases $|K| \leq 5$; [Fr], [L1] for proofs of a weeker statement about the maximality of the monomial subgroup; [V3], [Ki] for the case of the special linear group; and surveys [Z1] – [Z3], [V7] – [V9] for the general background and further references).

Of course for a (commutative) field $T = K$ the results of the present paper are a very special case of the forthcoming papers by V.V.Nesterov and the author, where the same problems are studied for the microweight tori (see [V8], [V10]) and long root tori in arbitrary Chevalley groups. However there are at least three reasons to publish this elementary case separately. Firstly, this is the only case (among the groups of normal type) which makes sense over skew-fields, so that the results of this paper are slightly more general in this respect. Secondly, it is very easy to completely describe the spans also in the exceptional cases of very small fields. In the general case we treat only the *generic* case of *large* fields. Thirdly, in the present paper we can afford to use only very rudimentary matrix calculations. Thus it may serve as an elementary introduction and an invitation to the forthcoming much more sophisticated and technical papers.

The paper is organized as follows. In § 1 we recall the necessary notation and background. In § 2 we introduce the obvious invariants of the orbits and reduce description of the orbits of $\mathrm{GL}(n, T)$ on pairs of 1-tori to the cases of $\mathrm{B}(3, T)$ and $\mathrm{GL}(2, T)$, which are treated in § 3 and § 4 respectively. In § 5 we summarize the results and discuss the exceptional cases. In § 6 we show how the description of subgroups of $\mathrm{GL}(n, T)$ containing $\mathrm{D}(n, T)$ follows from the result of the present paper. Finally, in § 7 we prove an analogue of McLaughlin's theorem for irreducible linear groups, generated by 1-tori.

§ 1. Notation

The following standard notation will be used throughout the paper. Let T be a skew-field, $T^* = T \backslash \{0\}$ be its multiplicative group, $G = \mathrm{GL}(n, T)$ be the general linear group of degree n over T. We will use the following subgroups of G: the group $B = \mathrm{B}(n, T)$ of upper triangular matrices, the group $B^- = \mathrm{B}^-(n, T)$ of lower triangular matrices, the group $D = \mathrm{D}(n, T)$ of diagonal matrices, the monomial subgroup $N = \mathrm{N}(n, T)$. By $U = \mathrm{U}(n, T)$ and $U^- = \mathrm{U}^-(n, T)$ we denote the unipotent radicals of B and B^- respectively, i.e. the groups of upper and lower unitriangular matrices.

The quotient group N/D is isomorphic to S_n, the symmetric group on n letters. Denote by $W = \mathrm{W}_n$ the group of permutation matrices in G. We identify S_n and W_n via the isomorphism $\pi \mapsto w_\pi$, where w_π is the matrix whose entry in the position (i, j) is $\delta_{i,\pi j}$. As usual we denote by w_{ij} the permutation matrix corresponding to the transposition (i, j).

Denote by $V = T^n$ the *right* vector space of columns of height n over T. Sometimes we identify a matrix $g \in G$ with the corresponding linear transformation of the space T^n. This transformation multiplies a column by g on the *left*. When we want to stress this geometric viewpoint we refer to the elements of G as *transformations*. By $v_1, \dots, v_n$ we denote the standard base of T^n, i.e. v_i is the column, which has 1 as its i-th coordinate, while all other coordinates are zeros. The dual space $V^* = {}^nT$ is the *left* vector space of rows of length n. By $u_1, \dots, u_n$ we denote the standard base of ${}^nT$, dual to $v_1, \dots, v_n$ with respect to the standard pairing.

For a matrix $g \in G$ we denote by g_{ij} its entry in the position (i,j), so that $g = (g_{ij})$, $1 \le i,j \le n$. As usual $g^{-1} = (g'_{ij})$ denotes the inverse of g, e denotes the identity matrix and e_{ij} is a standard matrix unit, i.e. the matrix whose entry in the position (i,j) is 1 and all the remaining entries are zeros. Thus $g = \sum g_{ij}e_{ij}$. By g^t we denote the *formal* transpose of g, whose entry in the position (i,j) equals g_{ji} considered as an element of T. (In the correct definition of a transpose g_{ji} should be considered an element of the opposite skew-field T^0).

By $t_{ij}(\xi) = e + \xi e_{ij}$ for $\xi \in T$ and $1 \le i \ne j \le n$ we denote an *elementary transvection*. For given $i \ne j$ we consider the corresponding unipotent *root subgroup* $X_{ij} = \{t_{ij}(\xi), \xi \in T\}$. The subgroup $\mathrm{E}(n,T)$ of G, generated by all X_{ij}, $1 \le i \ne j \le n$, is called the *elementary* subgroup of G. When $T = K$ is commutative, it coincides with the special linear group $\mathrm{SL}(n,K)$. Similarly, by $d_i(\varepsilon) = e + (\varepsilon - 1)e_{ii}$ we denote an 'elementary' *pseudoreflection*. For a given i we consider the corresponding '1-torus' $Q_i = \{d_i(\varepsilon),\ \varepsilon \in T^*\}$. We write $Q = Q_1$. Clearly, $\mathrm{GL}(n,T)$ is generated by $\mathrm{E}(n,T)$ and Q.

Let us recall that the *residual space* of a transformation $g \in G$ is the space $S(g) = \{ga - a \mid a \in T^n\}$ and the *residue* of g is the dimension of $S(g)$ or, in other words, the rank of the transformation $g - e$. We denote the residue of g by $\mathrm{res}(g)$. It follows immediately from the definition that $\mathrm{res}(gh) \le \mathrm{res}(g) + \mathrm{res}(h)$ for all $g, h \in G$, and one may find many further properties of $S(g)$ and $\mathrm{res}(g)$ in [HO]. Sometimes an element g such that $\mathrm{res}(g) = m$ is called *m-dimensional*.

The most important individual elements of $\mathrm{GL}(n,T)$ are of course the 1-*dimensional* elements, which play the crucial role in the study of linear groups. The general form of an 1-dimensional transformation is $x_{ab}(\xi) = e + a\xi b$, where $a = (\alpha_1, \dots, \alpha_n)^t$ is a column of height n, $\xi \in T$, and $b = (\beta_1, \dots, \beta_n)$ is a row of length n.

The subspace generated by the column a is called the *centre* of the transformation $x_{ab}(\xi)$. The hyperplane orthogonal to the row b is called the *axis* of $x_{ab}(\xi)$. Sometimes we will loosely apply the terms 'centre' and 'axis' to a and b themselves. The centre and axis of an 1-dimensional transformation x will be denoted by $a(x)$ and $b(x)$ respectively. Note that they are defined only projectively, i.e. only up to an invertible scalar factor. For any matrix $g \in \mathrm{GL}(n,T)$ one has $gx_{ab}(\xi)g^{-1} = x_{ga,bg^{-1}}(\xi)$. In particular, $a(gxg^{-1}) = ga(x)$ and $b(gxg^{-1}) = b(x)g^{-1}$.

Let $ba = \beta_1\alpha_1 + \dots + \beta_n\alpha_n = \delta$. If $\delta = 0$ the transformation $x_{ab}(\xi)$ is a *transvection* for every $\xi \in$T. If $\delta \ne 0$ then changing ξ if necessary we can always assume that $ba=\delta=1$. In that case to guarantee the invertibility of $x_{ab}(\xi)$ one has to require $\xi\neq -1$. Set $T^{\#}=T\backslash\{-1\}$. In this case the transformations $x_{ab}(\xi)$, $\xi\in T^{\#}$, are called *pseudoreflections*. Usually we will write a pseudoreflection in the form $d_{ab}(\varepsilon)=x_{ab}(\varepsilon-1)$, where $\varepsilon=\xi+1\in T^*$. Clearly, $d_{ab}(\varepsilon)d_{ab}(\eta)=d_{ab}(\varepsilon\eta)$ for two $\varepsilon,\eta\in T^*$.

Fix some r and consider the conjugate $y(\varepsilon)=xd_r(\varepsilon)x^{-1}$ of $d_r(\varepsilon)$ by x:

$$y(\varepsilon)=\begin{pmatrix} 1+\alpha_1(\varepsilon-1)\beta_1 & \alpha_1(\varepsilon-1)\beta_2 & \dots & \alpha_1(\varepsilon-1)\beta_n \\ \alpha_2(\varepsilon-1)\beta_1 & 1+\alpha_2(\varepsilon-1)\beta_2 & \dots & \alpha_2(\varepsilon-1)\beta_n \\ \vdots & \vdots & & \vdots \\ \alpha_n(\varepsilon-1)\beta_1 & \alpha_n(\varepsilon-1)\beta_2 & \dots & 1+\alpha_n(\varepsilon-1)\beta_n \end{pmatrix},$$

where $\alpha_i=x_{ir}$ are the entries of the r-th column of x, while $\beta_j=x'_{rj}$ are the entries of the r-th row of the inverse matrix $x^{-1}=(x'_{ij})$. Thus $xd_r(\varepsilon)x^{-1}=d_{ab}(\varepsilon)$

In this paper we consider *one-parameter groups of pseudoreflections*

$$Q_{ab}=\{d_{ab}(\varepsilon),\ \varepsilon\in T^*\}.$$

For brevity sake such one-parameter groups will be called 1-*tori*. Over a skew-field every row and column whose product equals 1, may be interpreted as the first row of an invertible matrix and the first column of the inverse of this matrix respectively. It follows that over a skew-field every 1-torus is conjugate to the torus $Q=Q_1$.

§ 2. Initial reduction

Our purpose is to describe orbits of $G=\mathrm{GL}(n,T)$ acting by simultaneous conjugation on pairs of 1-tori

$$(X,Y)\mapsto(gXg^{-1},gYg^{-1}),\qquad g\in G,$$

and to identify the corresponding spans. In this section we discuss the obvious invariants of these orbits and reduce their description to two special cases in degrees 2 and 3 respectively.

We start with some obvious general observations. Let X and Y be two 1-tori with the corresponding centres a_1 and a_2 and axes b_1 and b_2. Let $l=l(X,Y)=\dim\langle a_1,a_2\rangle$ and $m=m(X,Y)=\dim\langle b_1,b_2\rangle$. Then, clearly, l and m can take values 1 or 2 and are not changed by simultaneous conjugation. The products b_1a_2 and b_2a_1 also remain invariant in the projective sence. This means that if b_1a_2 was equal to zero, respectively distinct from zero, for a given pair (X,Y), then the same holds for any conjugate pair (gXg^{-1},gYg^{-1}). Denote by $p=p(X,Y)$ and $q=q(X,Y)$ the invariants of a pair (X, Y), taking value 0 if $b_1 a_2 = 0$ and $b_2 a_1 = 0$, respectively, and 1 otherwise. It turns out that generically these obvious discrete invariants almost suffice to distinguish all 'small' orbits.

However to distinguish 'typical' orbits we have to introduce one more continuous parameter. Namely, assume that both products $b_1 a_2$ and $b_2 a_1$ are distinct from zero (which is, of course, the 'typical' case) and consider the matrix

$$\begin{pmatrix} b_1 a_1 & b_1 a_2 \\ b_2 a_1 & b_2 a_2 \end{pmatrix}.$$

This matrix is defined only up to multiplication of its columns and rows by invertible scalar factors, but, clearly, the ratio

$$c = c(X, Y) = \frac{(b_1 a_2)(b_2 a_1)}{(b_1 a_1)(b_2 a_2)} \in T^*.$$

is an actual invariant of the orbit defined by the pair (X, Y). Then an orbit is completely determined by the invariants l, m, p, q and c and in each occurring case the corresponding span can be easily identified. Generically (when $l = m = 2$, $p = q = 1$ and $c \neq 1$) it is $\mathrm{GL}(2, T)$.

As a first step we reduce the problem to degree $n \leq 3$.

Lemma 1. *Let X and Y be two 1-tori in $\mathrm{GL}(n, T)$, $n \geq 4$. Then there exists $g \in \mathrm{GL}(n, T)$ such that $gXg^{-1}, gYg^{-1} \leq H$, where H is the subgroup of degree 3 generated by Q, X_{12}, X_{21}, X_{13} and X_{23}.*

Proof. Let $(X, Y) = (xQx^{-1}, yQy^{-1})$ be two 1-tori. Without loss of generality we may from the very start assume that $x = 1$ and thus $X = Q$. Let $Y = Q_{ab}$, as above, where $a = (\alpha_1, \dots, \alpha_n)^t$ is the first column of y and $b = (\beta_1, \dots, \beta_n)$ is the first row of y^{-1}.

Either $\alpha_2 = \ldots = \alpha_n = 0$, or at least one of α_i, $2 \leq i \leq n$, is distinct from zero. If not all α_i, $2 \leq i \leq n$, are zeros, then, possibly after a conjugation by a permutation matrix, we may assume that $\alpha_2 \neq 0$. Conjugating by $d_2(\alpha_2^{-1})$ we may even assume that $\alpha_2 = 1$. Now, conjugating by a transvection $t_{32}(-\alpha_3) \ldots t_{n2}(-\alpha_n)$ we may assume that $\alpha_3 = \ldots = \alpha_n = 0$. These conjugations preserve $X = Q$. In any case after conjugation we may assume that the centres of X and Y are both contained in the subspace $v_1 T \oplus v_2 T \leq T^n$.

It remains to repeat the same argument for β_i. Either $\beta_3 = \ldots = \beta_n = 0$ and we are done, or at least one of β_i, $3 \leq i \leq n$, is distinct from zero. In the second case, possibly after a conjugation by a permutation matrix, we may assume that $\beta_3 \neq 0$. Conjugating by $d_3(b_3)$ we may even assume that $\beta_3 = 1$. Now, conjugating by a transvection $t_{34}(\beta_4) \ldots t_{3n}(\beta_n)$ we may assume that $\beta_4 = \ldots = \beta_n = 0$. These conjugations preserve X and the subspace $v_1T + v_2T \le T^n$. Now the axes of X and Y are both contained in the subspace $Tu_1 \oplus Tu_2 \oplus Tu_3 \le {}^nT$, which, together with the above claim about the centres, proves the lemma.

It follows from the above lemma that – possibly after a simultaneous conjugation – we may assume that

$$X, Y \le \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Repeating the same argument, as above, but with columns and rows interchanged, we see that – again up to simultaneous conjugation – one has

$$X, Y \le \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix}.$$

Obviously, any pair of parabolic subgroups is simultaneously conjugate to a pair P_1, wP_2w^{-1}, where P_1 and P_2 are standard parabolic subgroups and w is an element of the Weyl group. In particular, only one of the two following possibilities may occur for the intersection of two non-conjugate maximal parabolic subgroups in $\mathrm{GL}(3, T)$: either their intersection is $\mathrm{GL}(2, T)$ (if they are opposite), or it is $\mathrm{B}(3, T)$ (otherwise). Thus the previous lemma immediately implies.

Lemma 2. *Let X and Y be two 1-tori in $\mathrm{GL}(n, T)$, $n \ge 4$. Then there exists $g \in \mathrm{GL}(n, T)$ such that $gXg^{-1}, gYg^{-1} \le H$, where H is either the subgroup of degree 2 generated by Q, X_{12} and X_{21}, isomorphic to $\mathrm{GL}(2, T)$, or the subgroup of degree 3 generated by Q_2, X_{12}, X_{13} and X_{23}.*

Thus from now on we may assume that X and Y are both contained in one of the following subgroups:

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & 0 & 1 \end{pmatrix}.$$

In Sections 3 and 4 we treat these cases under assumption $|T| \ge 7$ and then in Section 5 summarize the results. In the sequel we refer to the case $|T| \ge 7$ as *generic*. The cases $|T| \le 5$ are real exceptions.

§ 3. The case of $\mathrm{B}(3, T)$

In this section we describe the orbits of $\mathrm{GL}(3, T)$ (or, in fact, $\mathrm{B}(3, T)$) acting by conjugation on pairs of 1-tori contained in the following subgroup

$$P = \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & 0 & 1 \end{pmatrix}.$$

It turns out that this case furnishes exactly one new orbit as compared with the orbits occuring in $\mathrm{GL}(2,T)$. The span of X and Y, where (X,Y) is a representative of this orbit, coincides with P.

The following lemma is a variation of Lemma 1.

Lemma 3. *Any pair (X,Y) of 1-tori in $\mathrm{GL}(n,T)$ such that $l(X,Y)=1$ or $m(X,Y)=1$ is conjugate to a pair of subgroups in $\mathrm{B}(2,T)$.*

Proof. Keep the setting from the proof of Lemma 1. Let, moreover a_1, a_2 be the centres and b_1 and b_2 be the axes of the tori X and Y respectively. As in lemma 1 we may assume that $X=Q$.

If $a_1=a_2$, then $a_1=a_2=v_1$. Either $\beta_2=\ldots=\beta_n=0$, or at least one of β_i, $2\le i\le n$, is distinct from zero. In the first case we are done, in the second case we may assume that $\beta_2\neq 0$ and $\beta_i=0$ for all $3\le i\le n$. Thus $X,Y\le QX_{12}\le B(2,T)$.

If $b_1=b_2$, we get in the same way, that $X,Y\le QX_{21}\le B^-(2,T)$. But $B^-(2,T)$ is conjugate to $B(2,T)$ by a permutation matrix.

Lemma 4. *There is one $\mathrm{B}(3,T)$-orbit of pairs of 1-tori in P such that $l(X,Y)=m(X,Y)=2$.*

Proof. Denote by a_1 and a_2, respectively by b_1 and b_2, the centres, respectively the axes, of X and Y. Then, clearly, $a_1,a_2\in v_1T\oplus v_2T$ and $b_1,b_2\in Tu_2\oplus Tu_3$. This means, that conjugation by X_{12} preserves the axes, whereas conjugation by X_{23} preserves the centres. Conjugating by a transvection from X_{12} we may assume that $a_1=v_2$ and conjugating by a transvection from X_{23} (which does not change a_1) we may assume that $b_2=u_2$. Now $l=2$ implies that $a_2\notin v_2T$ and $m=2$ implies that $b_1\notin Tu_2$. This means that conjugating by an element from Q_3 and by an element from Q_1 respectively, we may assume that $a_2=v_1+v_2$ and $b_1=u_2+u_3$. This shows that *any* pair of 1-tori in P such that $l=m=2$ is conjugate to the pair

$$X=\left\{x(\varepsilon)=\begin{pmatrix}1&0&0\\0&\varepsilon&\varepsilon-1\\0&0&1\end{pmatrix},\ \varepsilon\in T^*\right\},$$

$$Y=\left\{y(\varepsilon)=\begin{pmatrix}1&\varepsilon-1&0\\0&\varepsilon&0\\0&0&1\end{pmatrix},\ \varepsilon\in T^*\right\}.$$

Clearly for this pair one has $p=q=1$ and $c=1$. Now we identify the span of this pair in the generic case.

Lemma 5. *Let T be a skew-field having at least 5 elements. If X and Y are two 1-tori in P such that $l(X,Y)=m(X,Y)=2$, then $\langle X,Y\rangle=P$*

Proof. By the preceding lemma we may from the very start assume that X and Y have the above form. Denote the subgroup $\langle X, Y\rangle$ by Z. We want to prove that $Z = P$.

Take any three commuting elements $\varepsilon, \eta, \theta \neq 0, 1$ and form the product

$$z = x(\varepsilon)y(\eta)x(\theta) = \begin{pmatrix} 1 & (\eta-1)\theta & (\eta-1)(\theta-1) \\ 0 & \varepsilon\eta\theta & \varepsilon\eta(\theta-1)+(\varepsilon-1) \\ 0 & 0 & 1 \end{pmatrix} \in Z.$$

Now we set $\theta = \eta^{-1}\varepsilon^{-1}(\varepsilon\eta - \varepsilon + 1)$ so that $\varepsilon\eta(\theta-1)+(\varepsilon-1) = 0$. Then $\theta \neq 0$ imposes one additional restriction on η, namely $\eta \neq 1-\varepsilon^{-1}$, whereas $\theta \neq 1$ is automatically fulfilled since $\eta \neq 0$ and $\varepsilon \neq 1$.

Consider the element $u = zy(\varepsilon\eta\theta)^{-1} \in Z$. Clearly $u \in X_{12}X_{13}$ and we want to prove that we may choose such ε and η that $u \notin X_{13}$. Indeed, $u = t_{12}(\lambda)t_{13}(*)$, where

$$\lambda = (\eta-1)\eta^{-1}e^{-1}(\varepsilon\eta-\varepsilon+1) - (\varepsilon\eta-\varepsilon+1)+1,$$

and an easy calculation shows that under the above restrictions on ε, η one has $\lambda \neq 0$ provided that $\varepsilon\eta + 1 \neq 0$. For a given $\varepsilon \neq 0, 1$ this imposes on η four linear restrictions altogether, so that the desired ε and η may be found whenever T contains at least 5 elements (as is well known [Co] every non-central element of an infinite skew-field is contained in an infinite subfield).

Now the proof is finished as follows. Since X_{13} coincides with the centre of P, one has $[g, u] = [g, t_{12}(\lambda)]$ for any $g \in P$. Taking $g = y(\xi^{-1})$, where $\xi \in T^*$, we get $t_{12}(\lambda(\xi-1)) \in Z$. Since ξ is arbitrary, this shows that $X_{12} \leq Z$. Since moreover $Y \leq Z$, one has $Q_2 \leq Z$, and now since $X \leq Z$, one has $X_{23} \leq Z$. But then $X_{13} = [X_{12}, X_{23}] \leq Z$ and thus, finally, $Z = P$.

The excluded cases of fields K containing less than 5 elements are true exceptions. The case of $\mathbb{F}_2$ is vacuous. On the other hand for the cases $\mathbb{F}_3$ and $\mathbb{F}_4$ a straightforward computation shows that X and Y generate a proper subgroup of P of order 6 or 24, respectively (in these cases P itself is of order $2\cdot 3^3$ or $2^6\cdot 3$, respectively).

§ 4. The case of $\mathrm{GL}(2, T)$

Now we start describing the orbits of $\mathrm{GL}(2, T)$ acting by conjugation on pairs of 1-tori in the generic case. In this section we identify the corresponding spans. Under a slightly different disguise this problem has been treated in [B], [BV], [V1] and in terms even closer to our present terms in [V2], [V3]. These papers gave (among other things) a description of subgroups of $\mathrm{GL}(n, T)$ over a skew-field T, $|T| \geq 7$, containing the group $\mathrm{D}(n, T)$ of diagonal matrices and, more generally, normalized by it. In the special case $n = 2$ one of the main results of these papers may be summarized as follows.

Lemma 6. *Let T be a skew-field having at least 7 elements. Then there are only three proper intermediate subgroups between* $\mathrm{D}(2,T)$ *and* $\mathrm{GL}(2,T)$*, namely,* $\mathrm{N}(2,T)$*,* $\mathrm{B}(2,T)$ *and* $\mathrm{B}^-(2,T)$*.*

It is an elementary exercise in matrix calculus to give a direct proof of this result independent of the general case (see [V2], for example). To the contrary, an analogue of this result for $\mathrm{SL}(2,K)$ is extremely deep [Ki] (see also [V8], [V9]).

This result (and its higher dimensional analogues) is in general no longer valid when $|K|\le 5$. A very easy argument (see [BV] or [V1]) shows that this result implies the following slightly more general one.

Lemma 7. *Assume as in Lemma 3. Let H be a subgroup of* $\mathrm{GL}(2,T)$*, normalized by* $\mathrm{D}(2,T)$*. Then one of the following holds:*

(1) $\mathrm{E}(2,T)\le H\le \mathrm{GL}(2,T)$,

(2) $\mathrm{U}^{\pm}(2,T)\le H\le \mathrm{B}^{\pm}(2,T)$,

(3) $H\le \mathrm{N}(2,T)$.

This result already almost answers our question. Indeed for the case $n=2$ the group $\mathrm{D}(2,T)$ is the product of Q and the centre of $\mathrm{GL}(2,T)$. It follows that any subgroup containing Q is normalized by $\mathrm{D}(2,T)$ and thus satisfies the conclusion of Lemma 4. In particular, if H is generated by Q and another 1-torus Q_{ab}, then it must either coincide with $\mathrm{GL}(2,T)$ or to be contained in $\mathrm{B}(2,T)$ or $\mathrm{N}(2,T)$ up to conjugacy.

The following 'connectivity theorem' shows that the case $\mathrm{N}(2,T)$ may be discarded. This is essentially a special case of [V5], 7^0 or of [V4], Proposition 1. In these papers we assumed that $xDx^{-1}\le \mathrm{N}$, but we may repeat essentially the same argument. Below we follow [V5] since it is more elementary, the proof from [V4] (see also [V8]) is more natural but it uses some facts about Bruhat decomposition.

Lemma 8. *Let T be a skew-field having at least 4 elements. If $xQx^{-1}\le$* $\mathrm{N}(n,T)$ *for some $x\in G$, then in fact $xQx^{-1}\le$* $\mathrm{D}(n,T)$*.*

Proof. Suppose x has at least two non-zero entries x_{p1} and x_{q1}, $p\ne q$, in the first column. The inverse $x^{-1}=(x'_{ij})$ has at least one non-zero entry in the first row, say $x'_{1s}\ne 0$. For all $\varepsilon\in T^*$ one has

$$y(\varepsilon)=xd_1(\varepsilon)x^{-1}=e+\sum_{i,j}x_{i1}(\varepsilon-1)x'_{1j}e_{ij}.$$

At least one of the indices p,q is distinct from s. If $p\ne s$, then clearly $y(\varepsilon)_{ps}=x_{p1}(\varepsilon-1)x'_{1s}\ne 0$ for all $\varepsilon\ne 0,1$. If q is also distinct from s we are done, since in this case $y(\varepsilon)_{qs}$ is also distinct from 0 for all $\varepsilon\ne 0,1$ and thus $y(\varepsilon)$ cannot be monomial. Thus we are left with the case, when $q = s$. In this case $y(\varepsilon)_{ss} = 1 + x_{s1}(\varepsilon - 1)x'_{1s}$ equals zero at most for one value of $\varepsilon \neq 0$. Altogether this prohibits three values of ε. Since $|T| \geq 4$ we can find an ε such that $y(\varepsilon)$ is not monomial.

The above contradiction shows that x has exactly one non-zero entry in the first column, say $x_{p1} \neq 0$. The same argument with the roles of x and x^{-1} interchanged shows that x^{-1} has exactly one non-zero entry in the first row, say $x'_{1s} \neq 0$. Since x^{-1} and x are mutually inverse, $p = s$. Thus, finally, $xQx^{-1} = Q_p$, which proves the lemma.

The two above lemmas immediately imply the following result.

Lemma 9. *Let X and Y be two 1-tori in $\mathrm{GL}(2,T)$, $|T| \geq 7$. Then either $X, Y \leq \mathrm{B}(2,T)$ up to simultaneous conjugacy, or $\langle X, Y\rangle = \mathrm{GL}(2,T)$.*

§ 5. Pairs of 1-tori

In this section we prove the main result of this paper. In analogy with the root subgroups (see [AS], [Ka1], [Co1], [Co2], [DV]), we introduce the 'angle' between two 1-tori which almost completely describes if not the orbits themselves, then, at least, there spans. Unlike the root subgroups this angle is purely conventional and expresses what the angle between two root subgroups with the same values of l, m, p, q would have been. For example, from a different viewpoint one could say that in the case 7 below the angle is $2\pi/3$.

Theorem 1. *Assume that $|T| \geq 7$. Then for any $n \geq 3$ there are $|T| + 5$ orbits of $\mathrm{GL}(n,T)$ acting by simultaneous conjugation on pairs (X, Y) of 1-tori. These orbits can be distinguished by the values of l, m, p, q and c. The values of these invariants on the orbits and the corresponding spans are identified in the following table.*

NN.	ϕ	l	m	p	q	c	$\langle X, Y\rangle$
1.	0	1	1	1	1	1	Q_1
2.	$\pi/3$	1	2	1	1	1	Q_1X_{12}
3.	$\pi/3$	2	1	1	1	1	Q_1X_{21}
4.	$\pi/2$	2	2	0	0	–	Q_1Q_2
5.	$2\pi/3$	2	2	0	1	–	$Q_1Q_2X_{12}$
6.	$2\pi/3$	2	2	1	0	–	$Q_1Q_2X_{12}$
7.	π	2	2	1	1	1	$Q_2X_{12}X_{13}X_{23}$
$8*$.	π	2	2	1	1	$\neq 1$	$\mathrm{GL}(2,T)$

Proof. Lemmas 3, 4 and 9 imply that, with exactly one exception, for all orbits (X, Y) one has the following alternative: either $X, Y \leq \mathrm{B}(2, T)$, or $\langle X, Y \rangle = \mathrm{GL}(2, T)$. The exceptional orbit is represented by

$$(Q_{v_1, u_1+u_2}, Q_{v_1+v_3, u_1}) \sim (Q, Q_{v_1+v_3, u_1+u_2}),$$

this is exactly the case when $\phi = \pi$ and $c = 1$. Now Lemma 5 implies that under the assumption $|T| \geq 5$ the span of this pair has the required shape. On the other hand the following lemma settles the case when $X, Y \in \mathrm{B}(2, T)$.

Lemma 10. *Assume that $|T| \geq 3$. Then all pairs $X, Y \leq \mathrm{B}(2, T)$ belong to one of the orbits 1 – 6 listed above and their spans have the required shape.*

Proof. Without loss of generality we may assume that $X = Q_i$, $i = 1, 2$. In turn, for $Y = Q_{ab}$ we have an alternative: either $a = v_1$, or $b = u_2$.

If $X = Q_1$ and $a = v_1$, then either $b = u_1$, in which case $X = Y$, or $b = u_1 + \lambda u_2$, $\lambda \neq 0$. Conjugating by an element of Q_2, we may assume that $\lambda = 1$, which leads to one orbit, represented by $(X, Y) = (Q_1, Q_{v_1, u_1+u_2})$. Clearly, $\langle X, Y \rangle \geq X_{12}$. This gives us the orbits corresponding to $\phi = 0$ and the first case for $\phi = \pi/3$.

The case of $X = Q_2$ and $b = u_2$ is similar, but it leads to the second case for $\phi = \pi/3$, represented by $(Q_2, Q_{v_1+v_2, u_2})$ (or, what is the same, by $(Q_1, Q_{v_1+v_2, u_1})$. Clearly, this case is fused with the first one by the contragredient map.

On the other hand, if $X = Q_1$ and $b = u_2$, then either $a = v_2$, in which case $Y = Q_2$, or $a = \lambda u_1 + u_2$, $\lambda \neq 0$. Conjugating by an element of Q_1 we may assume that $\lambda = 1$, which leads to one orbit, represented by $(X, Y) = (Q_1, Q_{v_1+v_2, u_2})$. Since $|T| \geq 3$, one has $[X, Y] = X_{12}$. This gives us the orbit, corresponding to $\phi = \pi/2$ and the first case for $\phi = 2\pi/3$.

The case of $X = Q_2$ and $a = v_1$ is similar, but it leads to the second case for $\phi = 2\pi/3$, represented by (Q_2, Q_{v_1, u_1+u_2}). Clearly, this case is fused with the first one, if we are allowed to permute X and Y. This finishes the proof of the lemma.

Finally, when $\langle Q, Q_{ab} \rangle = \mathrm{GL}(2, T)$ one must have $a = v_1 + \lambda v_2$, $\lambda \neq 0$, and $b = u_1 + \mu u_2$, $\mu \neq 0$. Conjugating by an element of Q_2 one may even assume that $a = v_1 + v_2$. Now for this orbit one has $\phi = \pi$ and $c = 1 + \mu \neq 1$. Since c is left invariant by simultaneous conjugation, for every $c \neq 1$ we get one orbit, represented by $(Q, Q_{v_1+v_2, u_1+(c-1)u_2})$.

It remains only to describe the orbits and the corresponding spans in the exceptional cases of fields containing less, than 7 elements. Only the case of $\phi = \pi$ is problematic. The case of $K = \mathbb{F}_2$ is vacuous. In the cases $\mathbb{F}_3$, $\mathbb{F}_4$ and $\mathbb{F}_5$ the 1-torus is generated by a pseudoreflection of order 2 (i.e. a reflection), 3, or 4, respectively. Thus, our problem amounts to the description of orbits of $\mathrm{GL}(2,K)$ on pairs of such conjugate pseudoreflections.

In the case $\mathbb{F}_3$ Lemma 8 breaks down. Out of 12 reflections 10 lie in $\mathrm{B}(2,3)$ or $\mathrm{B}^-(2,3)$, but each of the remaining two $\pm(e_{12}+e_{21})$ generates $\mathrm{N}(2,3)$ together with $d_1(-1)$. Thus in this case the spans in 7 and 8(-1) differ from the generic case.

As observed in § 3, for the case $\mathbb{F}_4$ the span for the orbit 7 differs from the generic case. Otherwise no alterations are necessary. It follows from the results of [K2] that Lemma 6 remains valid for $\mathbb{F}_4$. Indeed, let ω be a generator of $\mathbb{F}_4$, $\omega^3=1$. Then 14 out of 20 pseudoreflections conjugate to $d_1(\omega)$ lie in $\mathrm{B}(2,4)$ or $\mathrm{B}^-(2,4)$. Up to conjugation by a monomial matrix the remaining 6 have the form

$$\begin{pmatrix} 0 & \omega^2 \\ \omega^2 & \omega^2 \end{pmatrix}$$

It is immediate that each of these matrices together with $d_1(\omega)$ generates $\mathrm{GL}(2,4)$.

Finally, let $K=\mathbb{F}_5$. In this case 18 out of 30 pseudoreflections conjugate to $d_1(2)$ lie in $\mathrm{B}(2,5)$ or $\mathrm{B}^-(2,5)$. Up to conjugation by a monomial matrix 8 of the remaining 12 have the form

$$\begin{pmatrix} 0 & 1 \\ 3 & 3 \end{pmatrix}$$

and thus generate the whole group $\mathrm{GL}(2,5)$ together with $d_1(2)$. The remaining 4 are conjugate by a monomial matrix to

$$\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

and a straightforward computation (see [BK], for example) shows that each of these matrices together with $d_1(2)$ generates a subgroup of order 96.

§ 6. Overgroups of the diagonal subgroup

In this section we show that Theorem 1 immediately implies results on overgroups of the overgroups of the diagonal subgroup $D=\mathrm{D}(n,T)$ in the general linear group $G=\mathrm{GL}(n,T)$ over a skew-field T, see [B], [BV], [V1]. These overgroups are described in terms of the closed sets of roots in the root system Φ of type A_{n-1}.

Recall that Φ may be realized as follows:

$$\Phi=\big\{(i,j)\mid 1\le i\ne j\le n\big\}.$$

The Weyl group $W(\mathrm{A}_{n-1}) = S_n$ acts in the obvious way and it remains to define the sum of two roots. One sets $(i,j)+(j,h)=(i,h)$ and says that $(i,j)+(h,k)$ is not a root if $j\neq h$ and $i\neq k$.

A subset $S\subseteq\Phi$ is called *closed* if with any two roots $\alpha,\beta\in S$ such that there sum is a root, also $\alpha+\beta\in S$. One may associate with S several subgroups in $G=\mathrm{GL}(n,T)$. Let $E(S)$ be the subgroup of G spanned by all X_{ij}, such that $(i,j)\in S$. Then, clearly, $E(S)$ is normalized by D and the subgroup $G(S)=DE(S)$ contains D. One denotes by $N(S)$ the normalizer of $G(S)$ in G. If T contains at least 3 elements $N(S)$ is spanned over $G(S)$ by all permutation matrices $w\in S_n$ such that $wS=S$ (see [B] or [BV] for a direct proof, but in fact this is a very special case of Tits conjugacy theorem).

Then the standard description of subgroups in G containing D asserts that for any such subgroup H, $D\leq H\leq G$, there exists a unique closed set of roots $S\subseteq\Phi$ such that $G(S)\leq H\leq N(S)$. In [B], [BV], [V1] it is proven that the standard description holds whenever T contains at least 7 elements, i.e. that the following result holds.

Theorem 2. *Let T be a skew-field which contains at least 7 elements. Then for any subgroup H of $\mathrm{GL}(n,T)$ containing $\mathrm{D}(n,T)$ there exists a unique closed set of roots S such that*

$$G(S)\leq H\leq N(S).$$

When $T=K$ is a (commutative) field, this theorem asserts precisely, that all intermediate subgroups are *algebraic*, i.e. groups of K-rational points of the overgroups of the the corresponding algebraic torus $\overline{D}=\mathrm{D}(n,\overline{K})$ in the general linear group $\overline{G}=\mathrm{GL}(n,\overline{K})$ over the algebraic closure $\overline{K}$ of the field K (see [BT]). Nothing like that holds for non-split tori, even for the field $\mathbb{R}$ of real numbers (see references in [V8], [V9]).

In fact in the papers quoted above a slightly different language has been used. The closed sets of roots were identified with *D-nets of ideals* in T, i.e. square tables $\sigma=(\sigma_{ij})$, $1\leq i,j\leq n$, of ideals $\sigma_{ij}\trianglelefteq T$, such that

(1) $\sigma_{ij}\sigma_{jh}\subseteq\sigma_{ih}$ for all i,j,h,

(2) $\sigma_{ii}=T$ for all i.

Namely for any closed set of roots S one may define the corresponding D-net σ of ideals as follows: $\sigma_{ij}=T$ if either $i=j$ or $(i,j)\in S$ and $\sigma_{ij}=0$ otherwise. Now the group $G(S)$ may be defined as follows:

$$G(S)=\big\{a=(a_{ij})\in G\mid a_{ij}\in\sigma_{ij},\ 1\leq i,j\leq n\big\}.$$

This language is more suitable for generalizations to rings, [BV], [V1], [V7].

Standard arguments from the theory of linear groups (see [Z3]) reduce Theorem 1 to the question of non-existence of proper primitive irreducible subgroups of $\mathrm{GL}(n,T)$, containing $\mathrm{D}(n,T)$ (compare [Ko4] or [V3], where this is done for the case of the special linear group, the case of GL_n is simpler). Using McLaughlin's theorem [Mc] and its non-commutative version [L2], [V6] (Lemmas 11 and 12 below) one readily sees (compare Proposition 2 of [V2–II]) that this is equivalent to the following statement.

Lemma 11. *Assume that T is a skew-field containing at least 7 elements and H is a non-monomial subroup of the general linear group $G = \mathrm{GL}(n,T)$, containing the group $D = \mathrm{D}(n,T)$ of diagonal matrices. Then H contains a unipotent root subgroup.*

Clearly this lemma – and thus also Theorem 2 – immediately follows from our Theorem 1. Indeed, if $x \in H$ is a non-monomial matrix, then it follows from Lemma 8 that for some i the 1-torus $X = xQ_ix \leq H$ is non-monomial (otherwise one would have $xDx^{-1} \leq D$, which contradicts the assumption that x is non-monomial). Being non-monomial X cannot coincide with Q_i. Since the centres $v_1,\dots,v_n$ of the tori $Q_1,\dots,Q_n$ span the whole space T^n and the pairing ${}^nT \times T^n \longrightarrow T$ is non-degenerate, it follows that X cannot be orthogonal to all Q_i, $1 \leq i \leq n$. Thus there exists such an i that X is distinct from Q_i and not orthogonal to it. Then by Theorem 1 already the span $\langle X, Q_i\rangle \leq H$ contains a unipotent root subgroup.

Of course, for the case of GL_n this proof is not much simpler than the direct proof from the papers [B], [BV], [V1]. But already for other classical groups, not to say for exceptional ones, it is usually an enormous advantage to work with two one-dimensional tori lying in something very small, like D_4, instead of calculating in the group itself.

§ 7. Irreducible groups generated by 1-tori

In this section we prove an analogue of McLaughlin's theorem for irreducible subgroups generated by 1-tori. We recall the theorem first.

Lemma 12. *Let K be a field containing at least 3 elements, H be an irreducible subgroup of $G = \mathrm{GL}(n,K)$ generated by unipotent root subgroups. Then either $H = \mathrm{SL}(n,K)$, or $n = 2l$ is even and $H \sim \mathrm{Sp}(n,K)$.*

Let now T be a skew-field. Recall that $\mathrm{E}(n,T)$ is the elementary subgroup of degree n over T, generated by all elementary transvections of degree n. In the non-commutative case symplectic group does not exist. The following analogue of McLaughlin's theorem has been established in [L2], [V6].

Lemma 13. *Let T be a non-commutative skew-field, H be an irreducible subgroup of $G = \mathrm{GL}(n,T)$ generated by unipotent root subgroups. Then $H = \mathrm{E}(n,T)$.*

Now we prove an analogue of these results for pseudoreflections. Like Theorem 2 the following result may be deduced from Theorem 1. However we prefer to give a direct proof in the style of [B], [BV], [V1].

Theorem 3. *Let T be a skew-field containing at least 7 elements. Then any irreducible subgroup H of $G = \mathrm{GL}(n, T)$, generated by 1-tori, coincides with G.*

Proof. To reduce to the preceding lemmas we have to prove that H contains a unipotent root subgroup. Passing from H to its conjugate, we may from the very start assume that $Q_1 \leq H$. Let

$$Q_{ab} = \{x_{ab}(\xi),\ \xi \in T^{\#}\} \leq H,$$

be another 1-torus, contained in H. Here we keep notation from Section 1, in particular $a = (\alpha_1, \dots, \alpha_n)^t$ is a column of height n and $b = (\beta_1, \dots, \beta_n)$ is a row of length n subject to the condition $\beta_1\alpha_1 + \dots + \beta_n\alpha_n = 1$.

Since H is irreducible, there exists an 1-torus $Q_{ab} \leq H$ such that $\beta_1 \neq 0$ and not all $\alpha_2, \dots, \alpha_n$ are equal to 0. We fix such a and b and denote $x_{ab}(\xi)$, $\xi \in T^{\#}$, simply by $x(\xi)$. One has $x(\xi)^{-1} = x(\overline{\xi})$, where $\overline{\xi} = -\xi(1 + \xi)^{-1}$.

If $\alpha_1 \neq 0$ we pick up commuting $\xi, \eta \neq 0, -1$ such that $\xi \neq \eta$, $\xi \neq \overline{\eta}$, $\alpha_1\xi\beta_1 \neq -1$, $\alpha_1\eta\beta_1 \neq -1$. Such ξ and η exist since T contains at least 7 elements. When T is non-commutative $\beta_1\alpha_1$ is contained in an infinite subfield, thus we may always assume that ξ and η both commute with $\beta_1\alpha_1$. Now consider the following matrix

$$y = x(\xi)d_1(\varepsilon)x(\overline{\eta}) \in H,$$

where

$$\varepsilon = (1 + \alpha_1\xi\beta_1)^{-1}\alpha_1\xi\eta^{-1}{\alpha_1}^{-1}(1 + \alpha_1\eta\beta_1).$$

A straightforward calculation shows that $y_{1j} = 0$ for all $2 \leq j \leq n$, and that y_{j1} differs from α_j by an invertible factor. In particular, not all b_{j1}, $2 \leq j \leq n$, are equal to 0. Take $\theta \neq 0, 1$, and set $z = [y, d_1(\theta)] \in H$. Thus $z \neq e$ is a nontrivial transvection in H. But then H contains the whole unipotent root subgroup

$$X = \{d_1(\varepsilon)zd_1(\varepsilon)^{-1},\ \varepsilon \in T^*\} \cup \{e\}.$$

If $\alpha_1 = 0$, then we may set $y = x(\xi)$, $\xi \neq 0, -1$, in the preceding argument.

Let now L be the normal subgroup of H, generated by X. If H were imprimitive, it must be monomial ([ZS1], Lemma 1.8), and then it cannot contain a root subgroup. Thus we may assume that H is primitive and then by Clifford's theorem (which remains valid for skew-fields, see [Z3]),

L is irreducible (see [ZS1], Lemma 1.10). Then the two preceding lemmas imply that either $H = \mathrm{E}(n, T)$, or $T = K$ is commutative, $n = 2l$ is even and $L \sim \mathrm{Sp}(n, K)$. But the second possibility is excluded, since the normalizer of $\mathrm{Sp}(n, K)$ in $\mathrm{GL}(n, K)$ coincides with $\mathrm{GSp}(n, K)$ (see [KL]) and does not contain pseudoreflections. Since, moreover, $Q_1 \leq H$, one has $H = \mathrm{GL}(n, T)$.

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