PARTITIONS OF UNITY IN SOBOLEV SPACES OVER INFINITE DIMENSIONAL STATE SPACES

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Abstract

We prove some general results on the existence of partitions of unity in Sobolev type spaces on various infinite dimensional manifolds. As special cases we obtain in particular, (continuous) partitions of unity a) in the Malliavin test functions on an abstract Wiener space; b) in the first order Sobolev space on pinned and free loop spaces; c) in the first order Sobolev spaces associated with reversible Fleming-Viot processes.

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0 Introduction

The purpose of this paper is to prove existence of (continuous) partitions of unity in Sobolev type spaces over infinite dimensional (not necessarily linear) state spaces. In the case of the Sobolev spaces D_p^r over an abstract Wiener space we even prove the existence of partitions of unity which are simultaneously in all of them, i.e., in the Malliavin test function space D^∞. This was, however, essentially known (see [Go71], [Pi77] and Remark 2.2 (ii)), but our general techniques developed in Section 1 below, leads to a quite simple new proof of this fact, which is presented in Section 2. Subsequently, we concentrate on partitions of unity in Sobolev type spaces of order one, more precisely Dirichlet spaces. If the Dirichlet space is regular, in particular its state space a locally compact separable metric space, the existence of partitions of unity is almost trivial (see Section 3 below). This is entirely different for more general Dirichlet forms on merely Polish state spaces. In Section 4 we consider a certain class of Dirichlet spaces $(\mathcal{E}, D(\mathcal{E}))$ on Polish state spaces studied in [RS93], and in Theorem 4.3 we give quite general conditions which ensure the existence of partitions of unity in $D(\mathcal{E})$. In Section 5 we present applications of Theorem 4.3. We prove that it applies, if e.g. the intrinsic metric associated with $(\mathcal{E}, D(\mathcal{E}))$ is uniformly equivalent to the initial metric (cf. Theorem 5.1.1). Furthermore, we show that partitions of unity exist for the following Dirichlet spaces which were constructed and analyzed recently (cf. [DR92], [ALR93], [MR92], [ORS93]); a) Dirichlet spaces on pinned and free loop spaces (cf. Theorem **??** and Remark 5.2.4); b) Dirichlet spaces on Banach spaces of gradient type (cf. Theorem 5.3.2); c) Dirichlet spaces on spaces of measures (cf. Theorem 5.4.1).

1 Preliminaries and some abstract results

For the convenience of the reader we first recall some basic concepts which will be used throughout this paper. Let E be a topological space. A covering of

E is *locally finite* if every point of E has a neighborhood which intersects only finitely many elements of the covering. Let $\{U_\alpha\}$ and $\{V_i\}$ be two coverings. We say that $\{V_i\}$ is a *refinement of* $\{U_\alpha\}$, or $\{V_i\}$ is *subordinated to* $\{U_\alpha\}$ if each V_i is contained in some U_α. A topological space E is *paracompact* if it is Hausdorff, and every open covering has a locally finite open refinement (*open* here means, it consists of open sets). Typical examples of paracompact spaces are metric spaces (cf. e.g. [B74]). We shall make use of the following lemma for the proof of which we refer also to [B74]. For any subset $A \subset E$ let $\bar{A}$ be the closure of A in E and set $A^c := E \setminus A$.

Lemma 1.1 *Let E be a paracompact space and $\{U_i\}$ be a locally finite open covering of E. Then there exists a locally finite open covering $\{V_i\}$ such that $\bar{V}_i \subset U_i$ for each i.*

Definition 1.2 *Let W be a linear space of real valued functions on E, and let $\{U_\alpha\}$ be an open covering of E. We say that $\{\Psi_i\}$ is a* partition of unity of class W subordinated to $\{U_\alpha\}$ *if the following conditions are fullfilled:*

(i). $\Psi_i \in W$ *for each* i.

(ii). There exists a locally finite open covering $\{V_i\}$ subordinated to $\{U_\alpha\}$ such that supp $\Psi_i \left(:= \overline{\{\Psi_i \neq 0\}}\right)$ is contained in V_i for each i.

(iii). $\Psi_i(x) \geq 0$ *and* $\sum_i \Psi_i(x) = 1$ *for each* $x \in E$.

We shall say that E *admits partitions of unity of class W* if E is paracompact, and if, given an open covering $\{U_\alpha\}$, there always exists a partition of unity of class W subordinated to $\{U_\alpha\}$.

Note that if E is a manifold and $W = C^k$ or $W = C^\infty$, then the above definition of partition of unity coincides with the classical definition given in any text book on differentiable manifolds.

Partition of unity provides a means of getting from local to global results. It is therefore important to determine conditions under which a space admits partitions of unity. In the case of infinite dimensional manifolds, certain difficulties arise with the construction of partitions of unity of class C^∞ or class C^k. However, as will be seen in this paper, if we consider certain weak notions of smooth functions, e.g. smooth functions in the sense of the Malliavin calculus, or functions being in the domain of Dirichlet spaces (which can be regarded as a weak notion of C^1), it is still possible to construct partitions of unity. This is the main topic of this paper.

Below there are three abstract results concerning partitions of unity, which identify the underlying technical conditions that are needed. The proofs are quite simple.

Proposition 1.3 *Let E be a paracompact space, and let W be a linear space of real– valued functions on E. Suppose that the following assumptions are fulfilled:*

(1.1). If $\phi \in C_b^\infty(\mathbb{R})$ with $\phi(0) = 0$ and $u \in W$, then $\phi \circ u \in W$

(1.2). For any locally finite open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of E, there exists a locally finite open refinement $\{V_i\}_{i\in\mathbb{N}}$ of $\{U_\alpha\}$ and a family of functions $\{u_i\}_{i\in\mathbb{N}}$ in W such that for all $i \in \mathbb{N}$, supp$[u_i] \subset V_i$, $u_i \geq 0$, and for every $x \in E$ there exists $i \in \mathbb{N}$ such that $u_i(x) \geq 1$.

Then E admits partitions of unity of class W.

Proof: Let $\{U_\alpha\}$ be an open covering of E. Since E is paracompact, we may assume that $\{U_\alpha\}$ is locally finite. Let $\{V_i\}_{i\in\mathbb{N}}$ and $\{u_i\}_{i\in\mathbb{N}}$ be as in (1.2). We take a function $\phi \in C_b^\infty(\mathbb{R})$ such that ϕ is increasing, $\phi(t) = 0$ for $t \leq 0$, and $\phi(t) = 1$ for $t \geq 1$. We define $\Psi_1 := \phi \circ u_1$ and

$$\Psi_i := \phi\left(\sum_{j=1}^{i} u_j\right) - \phi\left(\sum_{j=1}^{i-1} u_j\right), \text{ for } i \geq 2.$$

Then $\Psi_i \in W$ for all $i \in \mathbb{N}$. One can easily check that

$$\operatorname{supp}[\Psi_i] \subset \operatorname{supp}[u_i]\,(\subset V_i) \text{ for all } i \in \mathbb{N},$$

and $\sum_i \Psi_i(x) = 1$ for all $x \in E$. Therefore, $\{\Psi_i\}_{i\in\mathbb{N}}$ is a partition of unity of class W subordinated to $\{U_\alpha\}$. □

Corollary 1.4 *Let E be a separable metric space with metric d and W be a linear space of real–valued functions on E. Suppose that (1.1) and in addition (1.3) below is fulfilled.*

(1.3). For any two closed subsets F_1, F_2 of E with

$$d(F_1, F_2) = \inf\{d(x_1, x_2) | x_1 \in F_1, x_2 \in F_2\} > 0,$$

there exists a non-negative function $u \in W$ such that $u(x) \geq 1$ for all $x \in F_1$ and $u(x) = 0$ for all $x \in F_2$.

Then E admits partitions of unity of class W.

Proof: Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a locally finite open covering of E. Since E has the Lindelöf property, we may assume that Λ is a countable index set. Let $\alpha \in \Lambda$ and $y \in U_\alpha$ be fixed. We set

$$c := d\left(y, U_\alpha^c\right) := \inf\{d(y, w) \mid w \in E \setminus U_\alpha\},$$

$$V_{\alpha,1} := \left\{x \in E \mid d\left(x, U_\alpha^c\right) > c\, 2^{-3}\right\},$$

and

$$V_{\alpha,n} := \left\{x \in E \mid c\, 2^{-n-2} < d\left(x, U_\alpha^c\right) < c\, 2^{-n+2}\right\}$$

for $n \in I\!N$, $n \geq 2$.

Clearly $V_{\alpha,n} \subset U_\alpha$ for each $n \in I\!N$ and $\{V_{\alpha,n}\}_{n\in I\!N}$ is a locally finite open covering of U_α. By (1.3) we can find a non-negative function $u_{\alpha,1} \in W$ such that $u_{\alpha,1}(x) \geq 1$ if $d\left(x, U_\alpha^c\right) \geq c\, 2^{-1}$ and $u_{\alpha,1}(x) = 0$ if $d\left(x, U_\alpha^c\right) \leq c\, 2^{-2}$. Moreover, for each $n \in I\!N$, $n \geq 2$ we can find non-negative functions $u_{\alpha,n} \in W$ such that $u_{\alpha,n}(x) \geq 1$ if $c\, 2^{-n} \leq d\left(x, U_\alpha^c\right) \leq c 2^{-n+1}$ and $u_{\alpha,n}(x) = 0$ if $d\left(x, U_\alpha^c\right) \geq c 2^{-n+2}$ or $d\left(x, U_\alpha^c\right) \leq c 2^{-n-1}$.

Thus one can check that $\operatorname{supp}[u_{\alpha,n}] \subset V_{\alpha,n}$ for all $n \geq 1$, and for each $x \in U_\alpha$ there exists $n \in I\!N$ such that $u_{\alpha,n}(x) \geq 1$. Repeating the above procedure for each $\alpha \in \Lambda$, and then rewriting the countable index set $\{(\alpha, n) \mid \alpha \in \Lambda, n \in I\!N\}$ by $\{i \mid i \in I\!N\}$, we obtain $\{V_i\}_{i\in I\!N}$ and $\{u_i\}_{i\in I\!N}$ satisfying (1.2). Therefore, it follows from Proposition 1.3 that E admits partitions of unity of class W. $\square$

Remark 1.5 *In the situation of Corollary 1.4, suppose that condition (1.3) is replaced by the weaker condition:*

(1.4). For only locally finite open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of E, there exists a locally finite open refinement $\{G_\beta\}_{\beta\in\Gamma}$ such that if F_1, F_2 are closed subsets of E satisfying $d(F_1, F_2) > 0$ and $F_1 \subset G_\beta$ for some $\beta \in \Gamma$, then there exists a non-negative function $u \in W$ with $u(x) \geq 1$ on F_1 and $u(x) = 0$ on F_2.

Then E admits partitions of unity of class W.

Proof: Let $\{U_\alpha\}$ be a locally finite open covering of E. Let $\{G_\beta\}$ be as in (1.4). Then the argument of the proof of Corollary 1.4 goes through if we replace $\{U_\alpha\}$ by $\{G_\beta\}$. Therefore, there exists a partition of unity of class W subordinated to $\{G_\beta\}$ and consequently subordinated to $\{U_\alpha\}$. □

2 Abstract Wiener spaces

Let (E, H, μ) be an abstract Wiener space in the sense of Gross [Gr65]. That is, E is a separable Banach space and μ a zero mean Gaussian measure on its Borel σ-algebra $\mathcal{B}(E)$ such that the support of μ is the whole space E. H is the reproducing kernel Hilbert space of μ, i.e., H consists of all elements $h \in E$ such that the two measures μ and $\mu \circ \tau_h^{-1}$ are equivalent, where $\tau_h : E \to E$ is the h-shift operator defined by $\tau_h(x) = x + h$ for all $x \in E$.

H is continuously and densely imbedded into E and hence if we identify H with its dual space, then the topological dual E^* of E is densely and continuously imbedded into H.

Let D_p^r $r > 0, p > 1$, be the Sobolev spaces of Wiener functionals on E and let $D^\infty := \bigcap_{r>0} D_p^r$ be the Malliavin test functionals. There are several ways to define the Sobolev spaces D_p^r which are all equivalent. We refer to [Wa84] or [Su88] for details. Here we mainly follow [Su88].

An infinite dimensional Banach space in general does not admit partitions of unity of class C^∞. But in the abstract Wiener space case we have the space D^∞ of Malliavin test functionals which is an infinite dimensional analogue of $S(\mathbb{R}^d)$, i.e., the space of classical Schwartz test functions on $\mathbb{R}^d$. Therefore, D^∞ is a good alternative class of functions to be used for partitions of unity on an infinite dimensional abstract Wiener space.

Note that an element of D^∞ may fail to have a continuous modification (see [Wa84, Example 1.3]). Nevertheless, there are enough elements in D^∞ which have continuous modifications. In particular, we have the following result.

Theorem 2.1 *Let W be the family of all continuous functions which are in D^∞. Then the abstract Wiener space E admits partitions of unity of class W.*

Proof: In this case by [Wa84, Proposition 1.11] (1.2) is fulfilled, and by [Su88, Lemma 2.5] (1.3) is fulfilled. Therefore, by Corollary 1.4 E admits partitions of unity of class W. $\square$

Remark 2.2 *(i) Let E be a separable Banach space. In general E does not admit partitions of unity of class C^1. But E always admits partitions of unity of class C^{1-0} where C^{1-0} denotes the class of all locally Lipschitz continuous functions(cf. [Ch93, I. Lemma 2.1]).*

(ii) It was known for a long time that an abstract Wiener space (E, H, μ) admits partitions of unity of class H- C^∞ (i.e., the class of functions which are infinitely differetiable along H, cf. [Go71][Pi77]). But it was surprising for us that nobody realized that the functions used in [Go71] can be used to construct partitions of unity of class W (with W as in Theorem 2.1). For example, F.Getzler erronously stated in [Ge86] that "Unfortunately, since Wiener spaces are not locally compact, D^∞ partitions of unity do not exist in general".

3 Regular Dirichlet forms

Let $(\mathcal{E}, D(\mathcal{E}))$ be a (non-symmetric) Dirichlet form on $L^2(E, m)$, where E is a Hausdorff topological space and m is a σ-finite Borel measure on $\mathcal{B}(E)$. $(\mathcal{E}, D(\mathcal{E}))$ is also called *Dirichlet space*. For the definition of Dirichlet forms and the corresponding notations used here, we refer to [MR92] (see also [BoH91] and [FuOT94]. Throughout this section we denote by W the family of all continuous functions which are m-versions of elements in $D(\mathcal{E})$. In this and the next section we shall see that E admits partitions of unity of class W in the regular case resp. in the case where $(\mathcal{E}, D(\mathcal{E}))$ is generated by a square field operator and the state space is polish. Recall that a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E, m)$ is called *regular* if (and only if) its symmetric part $\left(\tilde{\mathcal{E}}, D(\mathcal{E})\right)$ (defined by $\tilde{\mathcal{E}}(u,v) = \frac{1}{2}[\mathcal{E}(u,v)+\mathcal{E}(v,u)]$ for $u, v \in D(\mathcal{E})$) is regular in the sense of [Fu80], i.e., E is a locally compact separable metric space, $C_0(E) \cap D(\mathcal{E})$ is dense both in $D(\mathcal{E})$ (w.r.t. the norm $\tilde{\mathcal{E}}_1^{\frac{1}{2}}$) and in $C_0(E)$ (w.r.t. the uniform norm), where $C_0(E)$ denotes the set of all continuous functions on E with compact support. Furthermore, as usual we set $\mathcal{E}_1 := \mathcal{E} + (\ ,\ )_{L^2(E;m)}$.

Theorem 3.1 *Let $(\mathcal{E}, D(\mathcal{E}))$ be a regular Dirichlet form on $L^2(E,m)$. Then E admits partitions of unity of class W.*

Proof: Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a locally finite open covering of E. Since E is Lindelöf we can always find a locally finite open refinement $\{V_i\}_{i\in\mathbb{N}}$ of $\{U_\alpha\}$. Since E is locally compact we can achieve that each V_i is relatively compact. Applying Lemma 1.1 we may find open sets $V_{i,1}, V_{i,2}$ such that $\bar{V}_{i,1} \subset V_i$, $\bar{V}_{i,2} \subset V_{i,1}$, $i \in \mathbb{N}$ and both $\{V_{i,1}\}_{i\in\mathbb{N}}$, $\{V_{i,2}\}_{i\in\mathbb{N}}$ are locally finite open coverings subordinated to $\{V_i\}_{i\in\mathbb{N}}$. Let $i \in \mathbb{N}$. Since $\bar{V}_{i,2}$ is compact and $\bar{V}_{i,2} \cap V_{i,1}^c = \emptyset$ there is a non–negative function $g_i \in C_o(E)$ such that $g_i(x) = 2$ for $x \in \bar{V}_{i,2}$ and $g_i(x) = 0$ for $x \in V_{i,1}^c$. Applying [Fu80, Lemma 1.4.2] we can find $u_i \in W$ such that $\mathrm{supp}[u_i] \subset \mathrm{supp}[g_i] \subset V_i$ and $u_i(x) \geq 1$ for all $x \in \bar{V}_{i,1}$. In this way we have constructed $\{V_i\}_{i\in\mathbb{N}}$ and $\{U_i\}_{i\in\mathbb{N}}$ satisfying (1.2). But for a Dirichlet space (1.1) is always fulfilled and E is paracompact (since E is a metric space). Hence by Proposition 1.3 E admits partitions of unity of class W. $\square$

Remark 3.2 *We would like to emphasize that in contrast to the results in the following section the above Theorem 3.1 is basically known to experts, but we could not find an appropriate explicit reference.*

4 Dirichlet forms of generalized square field operator type

It is known that if a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is not regular but still quasi-regular, it may happen that there is no non-vanishing continuous function in the domain $D(\mathcal{E})$ (cf. [MR92, Chap. IV, Sect.4]). Hence in order to construct partitions of unity of class W, where W is as in Section 3, we need to assume that sufficently many continuous functions are in $D(\mathcal{E})$. Under some mild conditions we can then still construct partitions of unity of class W, provided $(\mathcal{E}, D(\mathcal{E}))$ is a kind of perturbation of a Dirichlet form of generalized square field operator type. This type of Dirichlet forms has been introduced and studied in [RS93]. Let us recall the precise definitions.

Let (E, ρ) be a complete, separable metric space equipped with its Borel σ-algebra $\mathcal{B}(E)$. Let m, μ and k be positive σ-finite measures on $(E, \mathcal{B}(E))$ and J a symmetric finite positive measure on $E \times E$. We start with a core D of functions. Suppose D is an algebra of bounded, continuous, real-valued functions on E, so that D separates points in E and D is closed under composition with smooth functions which vanish at the origin. We assume that each member u of D is square integrable with respect to the measures m, μ and k. We also assume that for $u, v \in D$, if $u = v$ m-a.e., then $u = v$ (which is the case, for example, if $\operatorname{supp}[m] = E$). Thus we may regard D as a subspace of $L^2(E; m)$. Finally, we assume that D does not vanish identically at any point in E, and combined with the fact that D is a point separating algebra, this implies that D is in fact dense in $L^2(E; m)$.

Next we assume that we are given a (generalized) *square field operator* Γ. This means that $\Gamma : D \times D \to L^1(E; \mu)$ is a positive, symmetric, bilinear mapping, where positivity means that for each $u \in D$ we have $\Gamma(u) := \Gamma(u, u) \geq 0$ μ-a.e.

We now define a bilinear form $\mathcal{E}$ on the core D by setting, for $u, v \in D$,

$$\mathcal{E}(u, v) = \int_E \Gamma(u, v) d\mu + \int_{E \times E} (u(z_1) - u(z_2))(v(z_1) - v(z_2)) J(dz_1 dz_2) + \int_E uv dk. \tag{4.1}$$

We assume that $(\mathcal{E}, D)$ is closable in $L^2(E; m)$ and that its closure $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form (see Section 5 below for examples). Then the map $\Gamma : D \times D \to L^1(E; \mu)$ is continuous in the $\tilde{\mathcal{E}}_1^{1/2}$-norm and so Γ extends to a continuous bilinear map on $D(\mathcal{E})$. For notational convenience we will continue to denote this map by Γ. Since any $\tilde{\mathcal{E}}_1$-Cauchy sequence is a Cauchy-sequence in $L^2(E;k)$, there is a unique bounded linear map $\Phi : D(\mathcal{E}) \to L^2(E;k)$ such that $\Phi u = u$ k-a.e. for $u \in D$ and $\|\Phi u\|^2_{L^2(E;k)} \leq \mathcal{E}_1(u,u)$ for all $u \in D(\mathcal{E})$.

Lemma 4.1 *Let $u, v \in D(\mathcal{E})$. If $u \leq v$ m-a.e., then $\Phi u \leq \Phi v$ k-a.e. .*

Proof: We show first that if $f, g \in D$ and $u = f + |g|$ then $\Phi u = u$ k-a.e. . To this end let $\phi_n, n \geq 1$, be smooth functions on $\mathbb{R}$ satisfying $\phi_n(0) = 0$ and $|\phi_n'| \leq 1$ and such that $\phi_n \to |\cdot|$ pointwise as $n \to \infty$. Let $u_n = f + \phi_n \circ g$ then $u_n \in D$ and $u_n \underset{n\to\infty}{\to} u$ in $D(\mathcal{E})$ (cf. [MR92, Chap. I, Proposition 4.17]). Hence $u_n \underset{n\to\infty}{\to} \Phi u$ in $L^2(E;k)$. But obviously we have that $u_n(z) \underset{n\to\infty}{\to} u(z)$ for all $z \in E$. Hence $\Phi u = u$ k-a.e. .

This fact together with the linearity of Φ implies that $\Phi(f \wedge g) = f \wedge g$ k-a.e. for all $f, g \in D$. Let now $u, v \in D(\mathcal{E})$ be such that $u \leq v$ m-a.e. . Take $u_n \in D, v_n \in D$ such that $u_n \underset{n\to\infty}{\to} u$ and $v_n \underset{n\to\infty}{\to} v$ in $D(\mathcal{E})$. Let $w_n := u_n \wedge v_n$, $n \in \mathbb{N}$, then $w_n \underset{n\to\infty}{\to} u \wedge v := u$ in $D(\mathcal{E})$, hence $\Phi(w_n) \underset{n\to\infty}{\to} \Phi(u)$. But also $v_n \underset{n\to\infty}{\to} \Phi v$ in $L^2(E;k)$ and, as just shown, $\Phi(w_n) = w_n \leq v_n$ k-a.e. for all $n \in \mathbb{N}$. Thus $\Phi(u) \leq \Phi(v)$ k-a.e. $\square$

Lemma 4.2 *For $u \in D(\mathcal{E})$, we have*

$$\mathcal{E}(u,u) \leq \int \Gamma(u) d\mu + 4J(E \times E)\|u\|^2_{L^\infty(E;m)} + \int (\Phi u)^2 dk \ . \tag{4.2}$$

Proof: See [RS93, Lemma 3.1]. $\square$

Theorem 4.3 *Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form as above and let W be the family of all continuous functions which are m-versions of elements in $D(\mathcal{E})$. Suppose that the following two conditions are fullfilled:*

(i). $\Gamma(u \vee v) \leq \Gamma(u) \vee \Gamma(v)$ μ-a.e. for all $u, v \in D(\mathcal{E})$

(ii). For some countable dense subset $\{x_i | i \in \mathbb{N}\}$ in (E, ρ) there exists a countable collection $\{f_{ij} | i \geq 1, j \geq 1\}$ of functions in $D(\mathcal{E})$ satisfying:

$$\sup_{i,j} \Gamma(f_{ij}) \ =\ : \phi \in L^1(E;\mu) \ , \tag{4.3}$$

$$\sup_j f_{ij}(z) \ =\ h(z)\rho_1(z, x_i) \ \ \text{m-a.e. } z \in E \text{ for all } i \in \mathbb{N} \ , \tag{4.4}$$

where h is some strictly positive bounded continuous function in $D(\mathcal{E})$) and ρ_1 is some bounded metric on E which is uniformly equivalent to ρ, (i.e., the identity from (E,ρ) to (E,ρ_1) and its inverse are uniformly continuous).

Then E admits partitions of unity of class W.

Proof: Fix $i \in I\!N$ and for each $n \geq 1$ define the function

$$u_n(z) := \left(\sup_{j=1}^{n} f_{ij}(z)\right) \vee 0 \ . \tag{4.5}$$

Then $u_n \in D$ for all $n \geq 1$. We are going to show that

$$\sup_n \mathcal{E}_1(u_n, u_n) < \infty. \tag{4.6}$$

To this end let $C := \sup_{z \in E} \rho_1(z, x_i)$. It follows from (4.4) that $0 \leq u_n \leq Ch$ m-a.e. for all n. Hence

$$\sup_n \|u_n\|_{L^2(E;m)} \leq C\|h\|_{L^2(E;m)} < \infty \tag{4.7}$$

Moreover, applying Lemmas 4.1 and 4.2 we have for all $n \geq 1$,

$$\mathcal{E}(u_n, u_n) \leq \int \Gamma(u_n) d\mu + 4J(E \times E)C\|h\|_{L^\infty(E;m)} + \int |C\Phi h|^2 dk \tag{4.8}$$

Employing the assumption (i) and (4.3) we conclude that

$$\sup_n \mathcal{E}(u_n, u_n) \leq \int \phi d\mu + 4J(E \times E)C\|h\|_{L^\infty(E;m)} + \int |C\Phi h|^2 dk < \infty \tag{4.9}$$

Combining (4.7) and (4.9) we obtain (4.6). Note that by (4.4) $u_n \underset{n\to\infty}{\rightarrow} h\rho_1(\cdot, x_i)$ in $L^2(E;m)$. Hence applying [MR92, I.2.12] we conclude from (84.6) that

$$h\rho_1(\cdot, x_i) \in D(\mathcal{E}) \text{for all } i \in I\!N. \tag{4.10}$$

Applying 4.3(i) to $(-u)$ and $(-v)$ we have that

$$\Gamma(u \wedge v) \leq \Gamma(u) \vee \Gamma(v) \ \mu\text{–a.e. for all } u, v \in D(\mathcal{E}). \tag{4.11}$$

For an arbitrary closed subset $F \subset E$ and $z \in E$ we define

$$\phi_F(z) := h(z)\rho_1(z, F) := h(z) \inf\{\rho_1(z, x) | x \in F\}. \tag{4.12}$$

and set, for $m, n \geq 1$

$$v_{n,m}(z) := inf\{h(z)\rho_1(z, x_i) | 1 \leq i \leq n, x_i \in F^{\frac{1}{m}}\} \tag{4.13}$$

where $F^{\frac{1}{m}} := \{d(\cdot, F) < \frac{1}{m}\}$. Then $\lim_{m\to\infty}\lim_{n\to\infty} v_{n,m}(z) = \phi_F(z)$, since ρ_1 is uniformly equivalent to ρ and $\{x_i | i \in I\!N\}$ is dense in (E, ρ). Similarly to the above argument but using both (4.11) and condition (i) one can check that $\phi_F \in D(\mathcal{E})$ and hence $\phi_F \in W$, since it is continuous.

We now define

$$\begin{aligned} V_1 &:= \{z \in E | h(z) > 2^{-1}\} \\ V_n &:= \{z \in E | 2^{-n-2} < h(z) < 2^{-n}\}, \text{for } n \geq 2. \end{aligned}$$

For an arbitrary locally finite open covering of $\{U_\alpha\}_{\alpha\in\Lambda}$ of E, let

$$\{G_\beta\}_{\beta\in\Lambda\times I\!N} := \{U_\alpha \cap V_i | \alpha \in \Lambda, i \in I\!N\}$$

Then $\{G_\beta\}_{\beta\in\Lambda\times I\!N}$ is a locally finite open refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$. Letting F_1, F_2 be two closed subsets of E such that $\rho_1(F_1, F_2) > 0$ and $F_1 \subset G_\beta$ for some $\beta \in \Lambda \times I\!N$, we define

$$u(z) = \frac{\phi_{F_2}}{inf\{\phi_{F_2}(z) | z \in F_1\}}$$

Then $u \in W$ with $u(x) \geq 1$ on F_1 and $u(x) = 0$ on F_2. Hence by Remark 1.5 E admits partitions of unity of class W. □

The following lemma is useful for applying Theorem 4.3

Lemma 4.4 *Suppose that for $u, v \in D$ we have*

$$|\Gamma(u, \varphi(v))| \leq |\Gamma(u, v)| \ \mu\text{-}a.e. \tag{4.14}$$

whenever φ is a smooth function on $I\!R$ with $\varphi(0) = 0$ and $|\varphi'(x)| \leq 1$. Then condition (i) in Theorem 4.3 is fulfilled.

Proof: See [RS93, Lemma 3.2] □

5 Applications

5.1 The intrinsic metric for Dirichlet forms.

Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form on $L^2(E; m)$ of the type considered in Section 4.The intrinsic metric (relative to $(\mathcal{E}, D(\mathcal{E}))$) on E is defined by

$$d(x, y) := \sup_{f\in\mathcal{C}}(f(x) - f(y)) \tag{5.1.1}$$

where

$$\mathcal{C} := \{u \in D(\mathcal{E}) \cap C(E) | \Gamma(u,u) \leq 1\} \tag{5.1.2}$$

(cf. [BiMo91], [Da89], and [St93]).

Theorem 5.1.1 *Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form of the type considered in Section 4 and assume for simplicity that $1 \in D(\mathcal{E})$ and $\Gamma(1,1) = 0$. Suppose that condition (i) in Theorem 4.3 is fulfilled and, in addition, that the intrinsic metric d defined by (5.1.1) is uniformly equivalent to ρ. Then condition (ii) in Theorem 4.3 is fulfilled and hence E admits partitions of unity of class W.*

Proof: Let $\epsilon > 0$. Then there exists $\delta > 0$ such that

$$\rho(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon \text{ for all } f \in \mathcal{C} \tag{5.1.3}$$

where $\mathcal{C}$ is as specified by (5.1.2). Let $\{x_i |\ i \in \mathbb{N}\}$ be a dense subset in (E, ρ). We can find $f_{k,i}^{(n)} \in \mathcal{C}$ such that

$$d(x_k, x_i) = \sup_n \left(f_{k,i}^{(n)}(x_k) - f_{k,i}^{(n)}(x_i) \right). \tag{5.1.4}$$

Let $z \in E$. We may find x_{k_0} such that $\rho(z, x_{k_0}) < \delta$, hence by (5.1.3) and (5.1.4),

$$\begin{aligned} d(z, x_i) &= \sup_{f \in \mathcal{C}} (f(z) - f(x_i)) \\ &\leq \epsilon + \sup_n \left(f_{k_0,i}^{(n)}(x_{k_0}) - f_{k_0,i}^{(n)}(x_i) \right) \\ &\leq 2\,\epsilon + \sup_{n,k} \left(f_{k,i}^{(n)}(z) - f_{k,i}^{(n)}(x_i) \right) \\ &\leq 2\,\epsilon + d(z, x_i) \end{aligned}$$

Therefore,

$$d(z, x_i) = \sup_{n,k} \left(f_{k,i}^{(n)}(z) - f_{k,i}^{(n)}(x_i) \right)$$

for all $i \in \mathbb{N}$. We now define

$$f_{i,(k,n)} := \left(f_{k,i}^{(n)}(z) - f_{k,i}^{(n)}(x_i) \right)$$

and rearrange $\{f_{i,(k,n)} |\, i,k,n \in \mathbb{N}\}$ by $\{f_{ij} |\, i \in \mathbb{N}, j \in \mathbb{N}\}$.

Then condition (ii) of Theorem 4.3 is fulfilled with $\phi \equiv 1, h \equiv 1, \rho_1 = d$ and $\{f_{ij} |\, i \in \mathbb{N}, j \in \mathbb{N}\}$ as above. □

Remark 5.1.2 *Under the condition of Theorem 5.1.1 one can check that* $(\mathcal{E}, D(\mathcal{E}))$ *is quasi-regular in the sense of [MR92] (cf. [RS93, Theorem 3.4]). We would, however, like to emphasize that the condition that the intrinsic metric is uniformly equivalent to the initial metric, is not necessary for neither the existence of partitions of unity nore quasi-regularity. This can be seen in the following examples (see e.g. Remark 5.3.2).*

5.2 Partitions of unity on free loop spaces.

Let us recall the framework of [ALR93].

Let $g := (g_{ij})$ be a uniformly elliptic Riemanian metric with bounded derivatives over $\mathbb{R}^d$ and $\Delta_g := (\det g)^{-1/2} \sum \frac{\partial}{\partial x_i}\left[(\det g)^{1/2} g^{ij} \frac{\partial}{\partial x_j}\right]$ the corresponding Laplacian. Let $p_t(x,y), x, y \in \mathbb{R}^d, t \geq 0$, be the associated heat kernel with respect to the Riemannian volume element. Let $W(\mathbb{R}^d)$ denote the set of all continuous paths $\omega : [0,1] \to \mathbb{R}^d$ and let $\mathcal{L}(\mathbb{R}^d) := \{\omega \in W(\mathbb{R}^d) | \omega(0) = \omega(1)\}$, i.e., $\mathcal{L}(\mathbb{R}^d)$ is the free loop space over $\mathbb{R}^d$. Let P_1^x be the law of the bridge defined on $\{\omega \in \mathcal{L}(\mathbb{R}^d) | \omega(0) = \omega(1) = x\}$ coming from the diffusion on $\mathbb{R}^d$ generated by Δ_g and let

$$\mu := \int P_1^x \, p_1(x,x) dx \tag{5.2.1}$$

be the *Bismut / Høegh–Krohn measure* on $\mathcal{L}(\mathbb{R}^d)$ which is σ-finite, but not finite. We consider $\mathcal{L}(\mathbb{R}^d)$ equipped with the Borel σ-algebra coming from the uniform norm $\| \, \|_\infty$ on $\mathcal{L}(\mathbb{R}^d)$ which makes it a Banach space. The tangent space $T_\omega \mathcal{L}(\mathbb{R}^d)$ at a loop $\omega \in \mathcal{L}(\mathbb{R}^d)$ was introduced in [JL91] as the space of *periodical* vector fields $X_t(\omega) = \tau_t(\omega) h(t), t \in [0,1]$, along ω. Here τ denotes the stochastic parallel transport associated with the Levi-Civita connection of $(\mathbb{R}^d, g)$ and h belongs to the linear space H_0 consisting of all absolutely continuous maps $h : [0,1] \to T_{\omega(0)}\mathbb{R}^d \equiv \mathbb{R}^d$ such that

$$(h,h)_{H_0} := \int_0^1 |h'(s)|^2 ds + \int_0^1 |h(s)|^2 ds < \infty \tag{5.2.2}$$

(where $|v|^2 := g_{\omega(0)}(v,v)$) and $\tau_1(\omega) h(1) = h(0)$ and $\tau_1(\omega)$ =holonomy along ω (cf. [JL91] for details). Note that if we consider $\mathcal{L}(\mathbb{R}^d)$ as continuous maps from S^1 to $\mathbb{R}^d$ this notion is invariant by rotations of S^1 and (5.2.2) induces an inner product on $T_\omega \mathcal{L}(\mathbb{R}^d)$ which turns it into a Hilbert space. Below we shall also need the Hilbert space $H_\omega\mathcal{L}(\mathbb{R}^d)(\supset T_\omega\mathcal{L}(\mathbb{R}^d))$ with inner product $(\ ,\ )_H$ which is constructed analogously but without the holonomy condition, i.e., H_0 is replaced by H which denotes the linear space of all absolutely continuous maps $h : [0,1] \to T_{\omega(0)}\mathbb{R}^d \equiv \mathbb{R}^d$ satisfying (5.2.2). Let $\mathcal{F}C_0^\infty$ denote the linear span of the set of all functions $u : \mathcal{L}(\mathbb{R}^d) \to \mathbb{R}$ such that there exist $k \in \mathbb{N}, f \in C_0^\infty((\mathbb{R}^d)^k), t_1, \dots, t_k \in [0,1]$ with

$$u(\omega) = f(\omega(t_1), \dots, \omega(t_k)), \omega \in \mathcal{L}(\mathbb{R}^d)\ . \tag{5.2.3}$$

Note that $\mathcal{F}C_0^\infty$ is dense in $L^2(\mu) :=$ (real) $L^2(\mathcal{L}(\mathbb{R}^d);\mu)$. Let $\mathcal{F}C^\infty$, $\mathcal{F}C_b^\infty$ be defined correspondingly with $C^\infty((\mathbb{R}^d)^k)$ resp. $C_b^\infty((\mathbb{R}^d)^k)$ replacing $C_0^\infty((\mathbb{R}^d)^k)$. We define the directional derivative of $u \in \mathcal{F}C^\infty$, u as in (5.2.3), at $\omega \in \mathcal{L}(\mathbb{R}^d)$ with respect to $X(\omega) \in H_\omega\mathcal{L}(\mathbb{R}^d)$ by

$$\begin{aligned} \partial_h u(\omega) := \partial_X u(\omega) &:= \sum_{i=1}^k d_i f(\omega(t_1), \dots, \omega(t_k)) X_{t_i}(\omega) \\ &= \sum_{i=1}^k g_{\omega(t_i)}(\nabla_i f(\omega(t_1), \dots, \omega(t_k)), \tau_{t_i}(\omega) h(t_i)) \end{aligned} \tag{5.2.4}$$

where $h \in H$ with $X(\omega) = (\tau_t(\omega)h(t))_{t\in[0,1]}$ and ∇_i resp. d_i denotes the gradient (with respect to g) resp. the differential relative to the i-th coordinate of f. We extend ∂_h to all of $\mathcal{F}C^\infty$ by linearity. Note that if we consider u as a function on $W(\mathbb{R}^d)$ then

$$\partial_X u(\omega) = \frac{d}{ds} u(\omega + sX(\omega))_{|s=0},\ \omega \in \mathcal{L}(\mathbb{R}^d)\ . \tag{5.2.5}$$

Hence $\partial_X u$ is well-defined by (5.2.4) (i.e., independent of the special representation of u).

Let for $u \in \mathcal{F}C^\infty$ and $\omega \in \mathcal{L}(\mathbb{R}^d)$, $\tilde{D}u(\omega)$ be the unique element in H such that $(\tilde{D}u(\omega), h)_H = \partial_h u(\omega)$ for all $h \in H$ and let $Du(\omega)$ be its projection onto H_0. Define for $u, v \in \mathcal{F}C_0^\infty$

$$\mathcal{E}(u,v) = \int_{\mathcal{L}(\mathbb{R}^d)} (Du, Dv)_{H_0}\, d\mu. \tag{5.2.6}$$

By our assumptions on g and (5.2.9) below it follows that $\mathcal{E}(u,u) < \infty$ for all $u \in \mathcal{F}C_0^\infty$. By [L92a, L92b], [ALR93] the densely defined quadratic form

$(\mathcal{E}, \mathcal{F}C_0^\infty)$ is closable on $L^2(\mu)$. Clearly, the closure $(\mathcal{E}, \mathcal{F}C_0^\infty)$ is of the type discussed in the preceding section (see (4.1)) with core $\mathcal{F}C_0^\infty$ and

$$\Gamma(u,v) = (Du, Dv)_{H_0},\, u, v \in D(\mathcal{E}) \tag{5.2.7}$$

where we denote the closure of D with domain $D(\mathcal{E})$ also by D. We note that D satisfies the chain rule, in particular if ϕ is a smooth function on $I\!R$ with $\phi(0) = 0$, and bounded first derivative, then

$$D\phi(u) = \phi'(u)D(u) \text{for all } u \in D(\mathcal{E}). \tag{5.2.8}$$

For $u \in \mathcal{F}C^\infty$ we have that

$$\|Du\|_{H_0} \leq \|\tilde{D}u\|_H \tag{5.2.9}$$

and if $u(\omega) = f(\omega(s_1), \ldots, \omega(s_k))$ then

$$\tilde{D}u(\omega)(s) = \sum_{i=1}^{k} G(s, s_i)\, \tau_{s_i}(\omega)^{-1}\, \nabla_i\, f(\omega(s_1), \ldots, \omega(s_k)) \tag{5.2.10}$$

where G is the Green function of $-\frac{d^2}{dt^2} + 1$ with Neumann boundary conditions on $[0,1]$, i.e.,

$$G(s,u) = \frac{e}{2(e^2-1)}(e^{u+s-1} + e^{1-(u+s)} + e^{|u-s|-1} + e^{1-|u-s|}). \tag{5.2.11}$$

Lemma 5.2.1 *There exists a strictly positive bounded continuous function $h \in D(\mathcal{E})$.*

Proof: For $k \in I\!N$ take a function $f_k \in C_0^\infty(I\!R^d)$ such that $0 \leq f_k \leq 1$ and $f_k(x) = 1$ if $|x| \leq k$. Let $g_k(\omega) = f_k(\omega(0))$ for $\omega \in \mathcal{L}(I\!R^d)$. Then $g_k \in \mathcal{F}C_0^\infty \subset D(\mathcal{E})$. Set for $n \in I\!N$,

$$h_n(\omega) = \sum_{k=1}^{n} \frac{g_k(\omega)}{2^n(\mathcal{E}_1(g_k, g_k)^{1/2} + 1)}$$

and

$$h(\omega) = \sum_{k=1}^{\infty} \frac{g_k(\omega)}{2^n(\mathcal{E}_1(g_k, g_k)^{1/2} + 1)}\ .$$

Then $h_n \underset{n\to\infty}{\longrightarrow} h$ uniformly on $\mathcal{L}(I\!R^d)$ and in $D(\mathcal{E})$. Hence h is a strictly positive bounded continuous function in $D(\mathcal{E})$. $\square$

Lemma 5.2.2 *Let $u \in D(\mathcal{E})$ and $\phi \in \mathcal{F}C_b^\infty$. Then $u\,\phi \in D(\mathcal{E})$ and $D(u\,\phi) = u\,D\,\phi + \phi\,D\,u$ μ-a.e. .*

Proof: First assume that $\phi \in \mathcal{F}C_0^\infty$. By an easy calculation it follows by (5.2.10) that $\omega \mapsto \tilde{D}\,\phi(\omega)$ is bounded from $\mathcal{L}(I\!R^d)$ to H. Thus by (5.2.9) $\omega \mapsto D\,\phi(\omega)$ is bounded from $\mathcal{L}(I\!R^d)$ to H_0. Approximating u by $u_n \in \mathcal{F}C_b^\infty, n \in I\!N$, w.r.t.$\mathcal{E}_1^{1/2}$, the assertion follows, if $\phi \in \mathcal{F}C_0^\infty$. Suppose now that $\phi \in \mathcal{F}C_b^\infty$, say $\phi(\omega) = f(\omega(t_1), \ldots, \omega(t_k))$. For $n \in I\!N$ we take a function $\zeta_n \in C_0^\infty\left((I\!R^d)^k\right)$ such that $\zeta_n(x) = 1$ whenever $|x| \leq n$ and first order derivatives bounded by 1. Set $f_n := \zeta_n f$ and $\phi_n(\omega) := f_n(\omega(t_1), \ldots, \omega(t_k)), \omega \in \mathcal{L}(I\!R^d)$. Then $\phi_n \to \phi$ pointwise. Moreover, since $\nabla_i\, f_n \overrightarrow{n} \to \infty \nabla_i f$, it follows from (5.2.10), (5.2.9), and our assumptions on g that $D\,\phi_n(\omega) \to D\,\phi(\omega)$ in H_0 for all $\omega \in \mathcal{L}(I\!R^d)$, and that $\|D\,\phi_n(\omega)\|_{H_0}$, $n \in I\!N$, are uniformly bounded in n and ω. Now by the product rule for $u\,\phi_n$, one can easily check that $u\,\phi_n$, $n \in I\!N$, is a Cauchy sequence in $D(\mathcal{E})$ and hence the assertion follows. □

Recall that W is the family of all continuous functions which are μ-versions of elements in $D(\mathcal{E})$.

Theorem 5.2.3 *$\mathcal{L}(I\!R^d)$ admits partitions of unity of class W.*

Proof: By (5.2.8) and Lemma 4.4 we see that condition (i) in Theorem 4.3 is fulfilled. Hence we only need to show that condition (ii) in 4.3 is fulfilled. This was essentially already done in [RS93, Subsection 4(b)] resp. [ALR93]. Since it is short and a minor modification is needed, we repeat it here.

Let $\phi \in C_b^\infty(I\!R)$ be an odd and increasing function such that $|\phi| \leq 2$, $\phi' \leq 1$, $\phi'' \leq 0$ on $[0, \infty)$, and $\phi(x) = x$ for $x \in [-1, 1]$. Let $\{s_k | k \in I\!N\}$ be a dense set of $[0, 1]$ and fix $\omega_0 \in \mathcal{L}(I\!R^d)$. Let $x^l : I\!R^d \to I\!R$, $1 \leq l \leq d$, be the standard linear coordinates and define for $j \in I\!N$

$$(5.2.12) \qquad u_j(\omega) := h(\omega) \sup_{k \leq j} \sup_{l \leq d} \left|\phi(x^l(\omega(s_k) - \omega_0(s_k)))\right|, \omega \in \mathcal{L}(I\!R^d).$$

where h is as specified in Lemma 5.2.1.

Applying first condition (i) of Theorem 4.3 and then Lemma 5.2.2 as well as the chain rule for D we obtain that for μ-a.e. $\omega \in \mathcal{L}(I\!R^d)$

$$(5.2.13) \qquad \Gamma(u_j)(\omega) \leq 4 \sup_{k \leq j} \sup_{l \leq d} (h^2(\omega) \|D(x^l(\omega(s_k) - \omega_0(s_k)))\|_{H_0}^2 + \|D\,h(\omega)\|_{H_0}^2).$$

Hence by our assumptions on g and by (5.2.10) there exists $c \in]0, \infty)$ such that for all $\omega_0 \in \mathcal{L}(\mathbb{R}^d)$

$$\Gamma(u_j)(\omega) \le c(h^2(\omega) + \|D\,h(\omega)\|^2_{H_0})\mu\text{-a.e..} \tag{5.2.14}$$

But the function on the right hand side of (5.2.14) is in $L^1(E;\mu)$. Let now $\{\omega_i | i \in \mathbb{N}\}$ be a dense subset of $\mathcal{L}(\mathbb{R}^d)$ and define $f_{ij} := u_{ij} \wedge 1$ where u_{ij} is equal to u_j with ω_0 replaced by ω_i, $i, j \in \mathbb{N}$. Then condition (ii) in Theorem 4.3 is fulfilled with the metric $\rho(\omega, \omega') := 1 \wedge \|\omega - \omega'\|_\infty$, $\omega, \omega' \in \mathcal{L}(\mathbb{R}^d)$. Hence $\mathcal{L}(\mathbb{R}^d)$ admits partitions of unity of class W by Theorem 4.3.

□

Remark 5.2.4 *We would like to emphasize that we also have partitions of unity within the continuous functions in $D(\mathcal{E})$ when $(\mathcal{E}, D(\mathcal{E})$ is the Dirichlet form on the pinned loop space over a compact oriented Riemannian manifold (without boundary) which was constructed in [DR92]. Also in that case the corresponding square field operator Γ satisfies the chain and hence again by Lemma 4.4 condition (i) in Theorem 4.3 is fulfilled. Condition (ii) in Theorem 4.3 follows directly from the calculations in [DR92].*

5.3 Quasi-regular gradient-type Dirichlet forms on Banach space.

Let E be a (real) separable Banach space with dual E', and μ a finite measure on $\mathcal{B}(E)$ which charges every weakly open set. Define a linear space of functions on E by

$$\mathcal{F}C_b^\infty = \{f(l_1, \ldots, l_m) | m \in \mathbb{N}\,, f \in C_b^\infty(\mathbb{R}^m),\, l_1, \ldots, l_m \in E'\}. \tag{5.3.1}$$

By the Hahn-Banach theorem, $\mathcal{F}C_b^\infty$ separates the points of E. The support condition of μ means that we can regard $\mathcal{F}C_b^\infty$ as a subspace of $L^2(E;\mu)$, and a monotone class argument shows that it is dense in $L^2(E;\mu)$. Define for $u \in \mathcal{F}C_b^\infty$ and $k \in E$,

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds} u(z + s\,k)|_{s=0},\ z \in E. \tag{5.3.2}$$

Observe that if $u = f(l_1, \ldots, l_m)$, then

$$\frac{\partial u}{\partial k} = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(l_1, \ldots, l_m)\,{}_{E'}\langle l_i, k\rangle_E, \tag{5.3.3}$$

which shows that $\partial u/\partial k$ is again a member of $\mathcal{F}C_b^\infty$. Also let us assume that there is a separable real Hilbert space $(H, \langle\, , \rangle_H)$ densely and continuously embedded into E. Identifying H with its dual H' we have that

$$E' \subset H \subset E \text{ densely and continuously,} \tag{5.3.4}$$

and ${}_{E'}\langle\, , \rangle_E$ restricted to $E' \times H$ coincides with $\langle\, , \rangle_H$. Observe that by (5.3.3) and (5.3.4), for $u \in \mathcal{F}C_b^\infty$ and fixed $z \in E$, the map $k \to (\partial u/\partial k)(z)$ is a continuous linear functional on H. Define $\nabla u(z) \in H$ by

$$\langle \nabla u(z)\, , k \rangle_H = \frac{\partial u}{\partial k}(z), \quad k \in H. \tag{5.3.5}$$

Define a bilinear form on $\mathcal{F}C_b^\infty$ by

$$\mathcal{E}_\mu(u,v) = \int \langle \nabla u(z), \nabla v(z) \rangle_H \, \mu(dz). \tag{5.3.6}$$

Assumption 5.3.1 *We assume that the form $\mathcal{E}_\mu$ in (5.3.6) is closable in $L^2(E;\mu)$.*

Let $\mathcal{L}_\infty(H)$ denote the set of all bounded linear operators on H with operator norm $\| \, \|_\infty$. Suppose $z \mapsto A(z)$, $z \in E$, is a map from E to $\mathcal{L}_\infty(H)$ such that $z \mapsto \langle A(z)\, h_1\, , h_2 \rangle_H$ is $\mathcal{B}(E)$-measurable for all $h_1, h_2 \in H$. Furthermore, assume that

(5.3.7) there exists $\alpha \in (0,\infty)$ such that $\langle A(z)\, h, h \rangle_H \geq \alpha \|h\|_H^2$ for all $h \in H$,

and that $\|\tilde{A}\|_\infty \in L^1(E;\mu)$ and $\|\check{A}\|_\infty \in L^\infty(E;\mu)$, where $\tilde{A} := \frac{1}{2}(A + \hat{A}), \check{A} := \frac{1}{2}(A - \hat{A})$ and $\hat{A}(z)$ denotes the adjoint of $A(z), z \in E$. Let $c \in L^\infty(E;\mu)$ and $b, d \in L^\infty(E \to H;\mu)$ such that for all $u \in \mathcal{F}C_b^\infty$ with $u \geq 0$,

$$\int (\langle d, \nabla u \rangle_H + c\, u) d\mu \geq 0 \text{ and } \int (\langle b, \nabla u \rangle_H + c\, u) d\mu \geq 0. \tag{5.3.8}$$

For $u, v \in \mathcal{F}C_b^\infty$ let

$$\mathcal{E}_A(u,v) = \int \langle A(z) \nabla u(z), \nabla v(z) \rangle_H \mu(dz), \tag{5.3.9}$$

and

$$\begin{aligned} \mathcal{E}(u,v) &= \mathcal{E}_A(u,v) + \int u \langle d, \nabla v \rangle_H \, d\mu \\ &+ \int \langle b, \nabla u \rangle_H v \, d\mu + \int u\, v\, c\, d\mu. \end{aligned} \tag{5.3.10}$$

Then Example 3e of [MR92, Chapter II] shows that the forms $(\mathcal{E}_A, \mathcal{F}C_b^\infty)$ and $(\mathcal{E}, \mathcal{F}C_b^\infty)$ are closable and that their closures $(\mathcal{E}_A, D(\mathcal{E}_A))$ and $(\mathcal{E}, D(\mathcal{E}))$ are Dirichlet forms.

One can check that $\mathcal{E}$ and $\mathcal{E}_A$ are equivalent. More precisely, there exists a constant K, such that for all $u \in \mathcal{F}C_b^\infty$

$$\mathcal{E}_A(u,u) \leq \mathcal{E}(u,u) \leq K\,[\mathcal{E}_A(u,u) + (u,u)] \tag{5.3.11}$$

(cf. [RS93, 4.13]). Obviously (5.3.11) implies that $D(\mathcal{E}_A) = D(\mathcal{E})$ Hence in order to prove the existence of partitions of unity in the continuous functions in $D(\mathcal{E})$, we only need to consider the continuous functions in $D(\mathcal{E}_A)$. Again let W be the family of continuous functions which are μ-versions of elements in $D(\mathcal{E}) = D(\mathcal{E}_A)$

Theorem 5.3.2 *E admits partitions of unity of class W.*

Proof: In this case the square field operator is given on $D := \mathcal{F}C_b^\infty$ by $\Gamma(u,v)(z)$ $:= \langle A(z)\,\nabla u(z), \nabla v(z)\rangle_H$, which clearly satisfies (4.14) and hence satisfies condition 4.3(i) by Lemma 4.4. The fact that condition 4.3(ii) is also fulfilled with $h \equiv 1$ follows by [RS93, Subsection 4(a)]. □

Remark 5.3.3 *Let d be the intrinsic metric associated to $(\mathcal{E}_A, D(\mathcal{E}_A))$ defined by (5.1.1). Assume for simplicity that $A(z) = \mathrm{id}_H$ for all $z \in E$, and suppose $H \neq E$. Then d is not uniformly equivalent to the metric given by $\|\ \|_E$. It is not even finite. Indeed, since $\|\ \|_H$ and $\|\ \|_{E'}$, are not equivalent on E', we can find $l_n \in E'$ with $\|l_n\|_H = 1$, $n \in I\!N$, and $\sup_n \|l_n\|_{E'} = +\infty$. By the uniform boundedness principle there exists $z_0 \in E$ such that $l_n(z_0) \nearrow \infty$ as $n \to \infty$, hence we may assume that $l_n(z_0) \geq n$ for all $n \in I\!N$. Let $f \in C_b^\infty(I\!R)$ with $f(0) = 0$, $f' \leq 1$, and*

$$u_n(z) := n\,f(\frac{1}{n}\,l_n(z)),\ z \in E.$$

Then $\|\nabla u_n(z)\|_H = f'(l_n(z))\ \|l_n\|_H \leq 1$, hence

$$d(z_0, 0) \geq \sup_n n\,f(\frac{1}{n}\,l_n(z_0)) = +\infty.$$

5.4 Dirichlet forms with state spaces consisting of probability measures

Let us recall the framework in [ORS93].

Let $E := \mathcal{M}(S)$ be the space of probability measure on a Polish space S with Borel σ-algebra $\mathcal{B}(S)$. Let E be equipped with the topology of weak convergence. There exist uniformly continuous functions ϕ_i, $i \in \mathbb{N}$, on S such that $\|\phi_i\|_\infty \leq 1$ and the topology on E is generated by the metric

$$\rho(\mu, \nu) = \sup_i \left(\int \phi_i d\mu - \int \phi_i d\nu \right), \ \mu, \nu \in E. \tag{5.4.1}$$

Set $\langle \mu, \phi \rangle := \int \phi d\mu$ for $\phi \in C_b(S)$ and μ a finite positive measure on $\mathcal{B}(S)$, and let

$$\mathcal{F}C_b^\infty := \{ f(\langle \cdot\,, \psi_1 \rangle, \ldots, \langle \cdot\,, \psi_m \rangle) | m \in \mathbb{N}, \psi_i \in C_b(S), 1 \leq i \leq m, f \in C_b^\infty(\mathbb{R}^m) \}. \tag{5.4.2}$$

For $u = f(\langle \cdot\,, \psi_1 \rangle, \ldots, \langle \cdot\,, \psi_m \rangle) \in \mathcal{F}C_b^\infty$ and $x \in S$ define

$$\frac{\partial u}{\partial \epsilon_x}(\mu) := \frac{d}{ds} u(\mu + s\epsilon_x)|_{s=0}, \ \mu \in \mathcal{M}(S), \tag{5.4.3}$$

and

$$\nabla u(\mu) := \left(\frac{\partial u}{\partial \epsilon_x}(\mu) \right)_{x \in S}. \tag{5.4.4}$$

Here ϵ_x denotes Dirac measures in $x \in S$.For $\mu \in \mathcal{M}(S)$ and $f, g \in L^2(S; \mu)$ we also set

$$\langle f, g \rangle_\mu := \int fg \, d\mu - \int f \, d\mu \int g \, d\mu. \tag{5.4.5}$$

Note that for $u \in \mathcal{F}C_b^\infty$, $\mu \in \mathcal{M}(S)$ the map $x \mapsto \partial u / \partial \epsilon_x(\mu)$ is bounded, hence belongs to the space $L^2(S; \mu)$, i.e., $\nabla u(\mu) \in L^2(S; \mu)$. Consequently, if m is a finite positive measure on the Borel sets $\mathcal{B}(E)$ of E we can define

$$\mathcal{E}(u, v) := \int \langle \nabla u(\mu), \nabla v(\mu) \rangle_\mu \ m(d\mu), \ u, v \in \mathcal{F}C_b^\infty. \tag{5.4.6}$$

Clearly, $\mathcal{F}C_b^\infty$ separates the points of E and therefore if $\mathrm{supp}[m] = S$, then $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is a densely defined positive symmetric bilinear form on $L^2(E; m)$. If it is closable, then its closure $(\mathcal{E}, D(\mathcal{E}))$ is clearly of the type studied in Section 4 with core $\mathcal{F}C_b^\infty$, and

$$\Gamma(u, v)(\mu) := \langle \nabla u(\mu), \nabla v(\mu) \rangle_\mu, \ \mu \in E. \tag{5.4.7}$$

It is easy to check that $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form and by Lemma 4.4, Γ satisfies condition 4.3 (i). Since $(\mathcal{M}(S), \rho)$ is separable we can find $\nu_j \in \mathcal{M}(S)$, $j \in \mathbb{N}$, which are dense in $\mathcal{M}(S)$. Let $\Phi \in C_b^\infty[0,\infty)$, strictly increasing, with $\Phi(0) = 0$, $\Phi' \le 1$, and $\Phi'' \le 0$. Defining for $i, j \in \mathbb{N}$

$$f_{ij}(\mu) := \Phi\Big(\int \phi_j \, d\mu - \int \phi_j d\nu_i\Big), \ \mu \in E, \tag{5.4.8}$$

we see that $f_{ij} \in \mathcal{F}C_b^\infty$. Moreover, for $\mu \in E$ we have

$$\begin{aligned} \Gamma(f_{ij})(\mu) &= \Big(\Phi'\Big(\int \phi_j \, d\mu - \int \phi_j d\nu_i\Big)\Big)^2 \langle \phi_j, \phi_j \rangle_\mu \\ &\le \int \phi_j^2 \, d\mu \le 1 \in L^1(E; m), \end{aligned} \tag{5.4.9}$$

and

$$\Phi(\rho(\mu, \nu_i)) = \sup_j f_{ij}(\mu) \text{ for all } \mu \in E, \ i \in \mathbb{N} \tag{5.4.10}$$

(cf. [RS93, (4.38)–(4.41)]). Since $\rho_1 = \Phi \circ \rho$ is a metric on E which is uniformly equivalent to ρ, Theorem 4.3 with $h \equiv 1$ applies (cf. [RS93, (4.38 - (4.41)]) and hence we conclude that the following theorem holds.

Theorem 5.4.1 *The space $E := \mathcal{M}(S)$ admits partition of unity of class W where W is the family of continuous functions which are m-versions of elements in $D(\mathcal{E})$.*

For concrete examples of measures m for which (5.4.6) is closable and hence Theorem 5.4.1 holds, we refer to [ORS93]. These in particular include the reversible invariant measure of the Fleming-Viot process on $\mathcal{M}(S)$.

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