Hyperbolic 3-manifolds as 2-fold coverings according to Montesinos

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Abstract

The Fibonacci manifolds and the ten smallest known compact orientable three-dimensional hyperbolic manifolds are obtained as two-fold coverings of the three-dimensional sphere. The corresponding branch sets are described. It is shown that every manifold under consideration admits a Heegaard splitting of genus two.

In this paper we consider compact orientable hyperbolic three-dimensional manifolds. We are interesting in two families of such manifolds. First of them are the Fibonacci manifolds M_n, $n \geq 2$, uniformized by the Fibonacci groups $F(2,2n)$. Manifolds M_n were considered by J. Mennicke, H. Helling and A. C. Kim in [20] and [8]. These manifolds are hyperbolic for $n \geq 4$. It was shown in [11] that M_n can be obtained as the n-fold cyclic covering of the 3-sphere, branched over the figure-eight knot. In the previous preprint [19] we showed that for any $n \geq 2$ the volume of the compact hyperbolic Fibonacci manifold M_{2n} is exactly equal to the volume of the non-compact manifold $S^3 \setminus Th_n$, where the Turk's head link Th_n is a closed 3-strings braid $(\sigma_1\sigma_2^{-1})^n$. In section 1 we will prove that the Fibonacci manifold M_n, $n \geq 2$, can be obtained as a two-fold covering of the 3-sphere, branched over the Turk's head link Th_n.

In section 2 we will discuss the second family of compact 3-manifolds $W(m,n;p,q)$ which can be obtained by (m,n) and (p,q) Dehn surgeries on two components of the Whitehead link. According to the Montesinos theorem [22], any manifold, obtained by Dehn surgeries on a strongly invertible link can be presented as a 2-fold covering of the 3-sphere, branched over some link.

Moreover there is an algorithm for constructing this branch set in [22]. We will apply this algorithm for describing manifolds $W(m,n;p,q)$ as a 2-fold branched coverings of 3-sphere. In particular, by this way we will get the ten smallest known compact orientable hyperbolic 3-manifolds $\mathcal{M}_1,\ldots,\mathcal{M}_{10}$ from [13].

Section 3 is devoted to the conjecture of R. Meyerhoff and W. Neumann [21]. Comparing results of section 1 and section 2 we will prove that the third smallest manifold $\mathcal{M}_3$, constructed in [21], is arithmetic over the field $\mathbb{Q}(\sqrt{-3})$ and its volume is exactly equals to the volume of the regular ideal tetrahedron in Lobachevsky space.

In section 4 we will discuss the Heegaard genus of above manifolds. More exactly, we will prove that the Heegaard genus of the Fibonacci hyperbolic manifolds and of the ten smallest known hyperbolic manifolds $\mathcal{M}_1,\ldots,\mathcal{M}_{10}$ from [13] is equal two.

1. Fibonacci manifolds as two–fold coverings

The Fibonacci groups

$$F(2,m) \;=\; \langle x_1,\, x_2,\, \ldots,\, x_m \;\mid\; x_i\, x_{i+1} \;=\; x_{i+2},\; i \bmod m \rangle.$$

were introduced by J. Conway [6]. It was shown by H. Helling, A. C. Kim and J. Mennicke [8] that the groups $F(2,2n)$, $n \geq 2$, are interesting from the geometrical point of view. The group $F(2,2n)$ is isomorphic to a properly discontinuous cocompact group of isometries which acts without fixed points on a space X_n, where $X_2 = S^3$ – spherical, $X_3 = \mathbb{E}^3$ – Euclidean and $X_n = \mathbb{H}^3$ – hyperbolic for $n \geq 4$.

We will call a three-dimensional manifold $M_n = F(2,2n)$, $n \geq 2$, uniformized by the Fibonacci group to be a *Fibonacci manifold.*

Various properties of hyperbolic Fibonacci manifolds M_n, $n \geq 4$ were investigated in [8], [10], [11] and [18].

For describing geometrical properties of the Fibonacci manifolds we need to define one family of knots and links. Denote by Th_n, $n \geq 2$, the closed 3-strings braid $(\sigma_1\sigma_2^{-1})^n$, where σ_1 and σ_2 are canonical generators [2]. We note that Th_n is a 3-component link if n is divided by three and it is a knot otherwise. This family contains well-known knots and links. In particular, Th_2 is the figure-eight knot, Th_3 are the Borromean rings, Th_4 is the Turk's head knot 8_{18} and Th_5 is the knot 10_{123}.

It was shown by W. Thurston [27] that a complement $S^3 \setminus Th_n$, $n \geq 2$, is a hyperbolic manifold. In [16] the virtual Betti numbers of compact hyperbolic manifolds constructed by Dehn surgeries on knots Th_n were studied. The symmetry groups of knots and links Th_n are described in [25]. In [19] authors proved that the volume of $S^3 \setminus Th_n$ is exactly equal to the volume of the Fibonacci manifold M_{2n}.

The following theorem gives one more relation between the Fibonacci manifolds M_n and the links Th_n.

Theorem 1. *For any $n \geq 2$ the Fibonacci manifold M_n is a two-fold covering of the three-dimensional sphere S^3 branched over the link Th_n.*

Proof. Let $\mathcal{O}(n)$ be an orbifold with the underlying space the 3-sphere and the singular set the figure-eight knot with the branch index n (Figure 1).

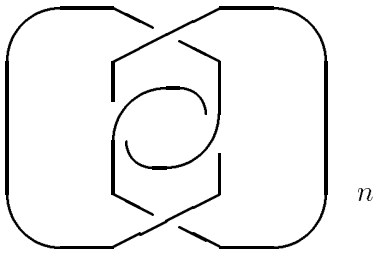

Figure 1. The singular set of the orbifold $\mathcal{O}(n)$.

It was shown by H. Hilden, M. Lozano and J. Montesinos [11] that the Fibonacci manifold M_n is a n-fold cyclic covering of the orbifold $\mathcal{O}(n)$.

It is obvious (see Figure 1) that the orbifold $\mathcal{O}(n)$ has the rotation symmetry τ of the order two, such that the fixed point set of the τ and the singular set of $\mathcal{O}(n)$ are disjoint. The quotient space $\mathcal{O}(n)/\tau$ is an orbifold with the underlying space 3-sphere and the singular set the 2-component link in Figure 2 with branched indices 2 and n.

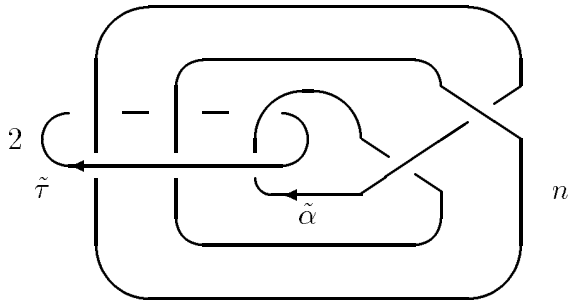

Figure 2. The singular set of the orbifold $6_2^2(2,n)$.

It was shown in [27, ch. 13] that the singular set in Figure 2 is a two-component link 6_2^2 presented in Figure 3 according to [24].

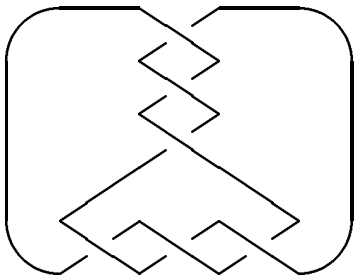

Figure 3. The link 6_2^2.

So we will denote the quotient orbifold $\mathcal{O}(n)/\tau$ by $6_2^2(2,n)$.

We remark that the space X_n, $n \geq 2$, is the universal covering for the Fibonacci manifold M_n, the orbifold $\mathcal{O}(n)$ and the orbifold $6_2^2(2,n)$. Let us denote by $\Gamma_n = F(2,2n)$, Ω_n and Π_n the fundamental groups of the manifold M_n, the orbifold $\mathcal{O}(n)$ and the orbifold $6_2^2(2,n)$ respectively. Therefore groups Γ_n, Ω_n and Π_n are discrete subgroups of the full isometry group of X_n. Moreover there are canonical isomorphisms $M_n = X_n \,/\, \Gamma_n$, $\mathcal{O}(n) = X_n \,/\, \Omega_n$ and $6_2^2(2,n) = X_n \,/\, \Pi_n$. Since from above we have a covering diagram

$$M_n \stackrel{n}{\longrightarrow} \mathcal{O}(n) \stackrel{2}{\longrightarrow} 6_2^2(2,n) \tag{1}$$

which implies an embedding for subgroups

$$\Gamma_n \lhd \Omega_n \lhd \Pi_n, \tag{2}$$

where $\mid \Pi_n : \Omega_n \mid = 2$ and $\mid \Omega_n : \Gamma_n \mid = n$.

For describing the group Π_n we use a representation of the fundamental group $\pi_1(S^3 \setminus 6_2^2)$ from [19]:

$$\langle \tilde{\alpha}, \tilde{\tau} \mid \left(\tilde{\tau}\tilde{\alpha}^{-1}\tilde{\tau}\tilde{\alpha}\tilde{\tau}^{-1}\tilde{\alpha}\tilde{\tau}^{-1}\right)\left(\tilde{\tau}^2\tilde{\alpha}^{-1}\tilde{\tau}\tilde{\alpha}\tilde{\tau}^{-1}\tilde{\alpha}\tilde{\tau}^{-2}\right)\left(\tilde{\tau}\tilde{\alpha}^{-1}\tilde{\tau}\tilde{\alpha}\tilde{\tau}^{-1}\tilde{\alpha}\tilde{\tau}^{-1}\right)^{-1} = \tilde{\alpha}\,\rangle. \tag{3}$$

In this representation generators $\tilde{\alpha}$ and $\tilde{\tau}$ canonical correspond to arcs with the same labels on the link diagram of $6_2^2(2,n)$ in Figure 2.

According to [7] it follows from (3) that the group Π_n of the orbifold $6_2^2(2,n)$ has the following representation:

$$\langle \alpha, \tau \mid \left(\tau\alpha^{-1}\tau\alpha\tau^{-1}\alpha\tau^{-1}\right)\left(\tau^2\alpha^{-1}\tau\alpha\tau^{-1}\alpha\tau^{-2}\right)\left(\tau\alpha^{-1}\tau\alpha\tau^{-1}\alpha\tau^{-1}\right)^{-1} = \alpha, \quad \alpha^n = \tau^2 = 1\rangle, \tag{4}$$

where generators α and τ of Π_n correspond to generators $\tilde{\alpha}$ and $\tilde{\tau}$ of the group $\pi_1(S^3 \setminus 6_2^2)$.

Let us consider a group

$$\mathbb{Z}_n \oplus \mathbb{Z}_2 = \langle a \mid a^n = 1\rangle \oplus \left\langle t \mid t^2 = 1\right\rangle$$

and an epimorphism

$$\theta \ : \Pi_n \ \longrightarrow \mathbb{Z}_n \oplus \mathbb{Z}_2$$

defined by conditions

$$\theta(\alpha) \ = \ a, \quad \theta(\tau) \ = \ t. \tag{5}$$

By the construction of the two-fold cover $\mathcal{O}(n) \ \to \ 6_2^2(2,n)$ the loop τ from the group Π_n lifts to a trivial loop in the group Ω_n. By the same reasons the loop α from the group Π_n lifts to a loop, which generates a cyclic subgroup of the order n in the group Ω_n. Therefore,

$$\Omega_n \ = \ \theta^{-1}(\mathbb{Z}_n) \ = \ \theta^{-1}\left(\langle a \mid a^n \ = \ 1\rangle\right). \tag{6}$$

For the $2n$-fold covering $M_n \ \to \ 6_2^2(2,n)$ both loops α and τ from the group Π_n lift to trivial loops in the group Γ_n. Therefore

$$\Gamma_n \ = \ \theta^{-1}(1) \ = \ \mathrm{Ker}\,\theta. \tag{7}$$

Let T_n be a subgroup of Π_n defined by the following condition:

$$\mathrm{T}_n \ = \ \theta^{-1}(\mathbb{Z}_2) \ = \ \theta^{-1}\left(\left\langle t \mid t^2 \ = \ 1\right\rangle\right). \tag{8}$$

Then we have a sequence of normal subgroups:

$$\Gamma_n \ \lhd \ \mathrm{T}_n \ \lhd \ \Pi_n, \tag{9}$$

where $\mid \Pi_n : \mathrm{T}_n \mid = n$ and $\mid \mathrm{T}_n : \Gamma_n \mid = 2$.

The group T_n is a subgroup of Π_n. Hence it acts by isometries on the universal covering X_n and uniformize an orbifold $X_n \,/\, \mathrm{T}_n$. From (9) we get the following diagram of covers for orbifolds:

$$M_n = X_n \,/\, \Gamma_n \ \stackrel{2}{\longrightarrow} \ X_n \,/\, \mathrm{T}_n \ \stackrel{n}{\longrightarrow} \ 6_2^2(2,n) = X_n \,/\, \Pi_n. \tag{10}$$

Our next step is to describe the orbifold $X_n \,/\, \mathrm{T}_n$. First of all we will prove that the cover

$$p \ : \ X_n \,/\, \mathrm{T}_n \ \stackrel{n}{\longrightarrow} \ 6_2^2(2,n) = X_n \,/\, \Pi_n \tag{11}$$

is cyclic. We will use the following lemma for it.

Lemma. *Let G, K, L be groups and $\theta : G \ \longrightarrow K \oplus L$ be an epimorphism. If $H = \theta^{-1}(K)$ then $H \lhd G$ and $G \,/\, H \cong L$.*

We will apply this Lemma to the epimorphism $\theta : \Pi_n \ \longrightarrow \mathbb{Z}_n \oplus \mathbb{Z}_2$ which was defined by (5). Since $\mathrm{T}_n = \theta^{-1}(\mathbb{Z}_2)$, then $\mathrm{T}_n \lhd \Pi_n$ and $\Pi_n \,/\, \mathrm{T}_n \cong \mathbb{Z}_n$. It means that the cover p is a regular n-fold cyclic cover. Moreover by (8) the cover p is branched over the component with the branch index n of the singular set of the orbifold $6_2^2(2,n)$.

It is known [25], that there is an involution in the symmetry group of 6_2^2 which changes two components (it is evident from Figure 3). Therefore the singular set of the orbifold $6_2^2(2,n)$ is equivalent to the link diagram in Figure 4.

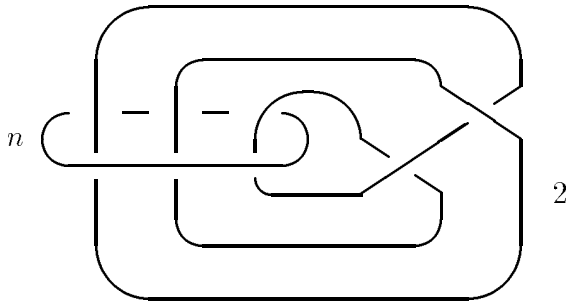

Figure 4.

In Figure 4 one can see that the component with the index n is unknotted. Therefore p is a standard cyclic cover of the 3-sphere which is the underlying space of the orbifold $6_2^2(2,n)$, branched over an unknotted circle. Hence the underlying space of the orbifold $X_n \,/\, \mathrm{T}_n$ is the 3-sphere again. Moreover the component with the index 2 of the singular set of $6_2^2(2,n)$ is the closed 3-string braid $\sigma_1\sigma_2^{-1}$. Therefore this component will lift to the closed 3-string braid $(\sigma_1\sigma_2^{-1})^n$ on the n-fold cyclic covering $X_n \,/\, \mathrm{T}_n$. It is the link Th_n according to our notation.

Summarizing we see that for the orbifold $X_n \,/\, \mathrm{T}_n$ the underlying space is the 3-sphere and the singular set is the link Th_n with the branch index 2. For this orbifold we will be use notation $Th_n(2) = X_n \,/\, \mathrm{T}_n$.

Comparing diagrams (1) and (10) we conclude that the following diagram for covers holds (Figure 5):

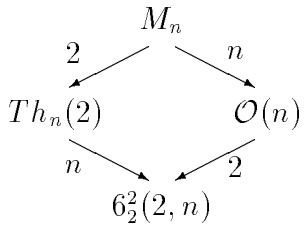

Figure 5. The coverings diagram.

From this diagram we see that the Fibonacci manifold M_n is a 2-fold covering of the orbifold $Th_n(2)$, and the theorem is proven. □

We recall that according to [4] a *π-orbifold* is an orbifold with the 3-sphere as the underlying space and a link with branch indices equal to 2 as the singular set.

Since for $n \geq 4$ the Fibonacci manifold M_n is hyperbolic, from Theorem 1 we get the following

Corollary. *For $n \geq 4$ the π-orbifold $Th_n(2)$ is hyperbolic.*

2. Dehn Surgeries on the Whitehead Link

This section is devoted to the study of compact 3-manifolds connected with the Whitehead link W (see Figure 6). Non-compact 3-manifolds obtained by Dehn surgery on one component of W were investigated in [12].

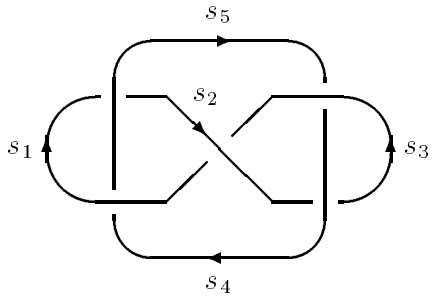

Figure 6. The Whitehead link.

Let $\mathcal{L}$ be a link in S^3. The link $\mathcal{L}$ is called strongly-invertible if there is an orientation-preserving involution of S^3 which induces in each component of $\mathcal{L}$ an involution with two fixed points. Above mentioned involution is called a strongly-invertible involution of the link. The following theorem belongs to J.Montesinos [22]:

Let M be a closed orientable 3-manifold that is obtained by doing surgeries on a strongly-invertible link L of n component. Then M is a 2-fold covering of S^3 branched over a link of at most $n+1$ components. Conversely, every 2-fold cyclic branched covering of S^3 can be obtained in this fashion.

The proof of this theorem given in [22] is constructive. In particular it contains an algorithm for describing of branch set of above 2-fold covering (see also [23]).

It is well-known that the Whitehead link W is a strongly-invertible link. The strongly-invertible involution ρ is shown in Figure 7.

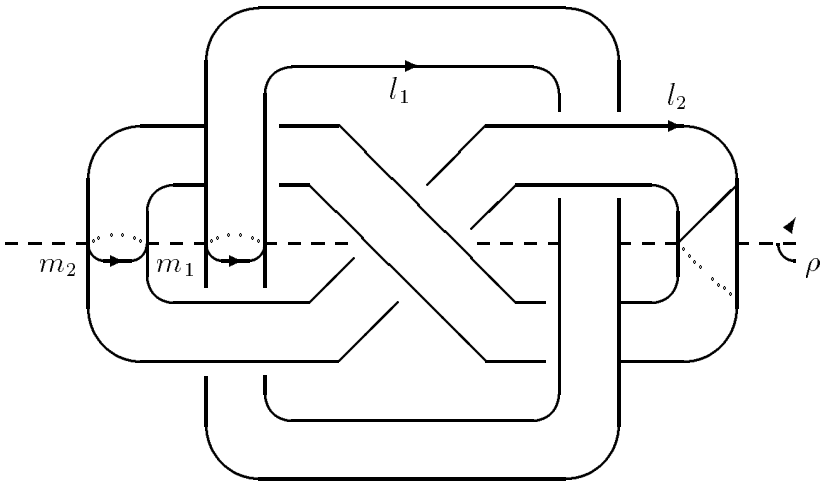

Figure 7.

The aim of this section is to apply the Montesinos algorithm to the Whitehead link W.

Let us denote the compact 3-manifold which can be obtained by $\frac{m}{n}$ and $\frac{p}{q}$ Dehn surgeries on components of W by $W(m,n;p,q)$. By Montesinos theorem the manifolds $W(m,n;p,q)$ is a 2-fold covering of S^3. It follows from [22] that this covering is uniquely determined by choose of strongly-invertible involution. Denote by $\mathcal{L}(m,n;p,q)$ the branched set of 2-fold covering $W(m,n;p,q)$ of S^3 corresponding to the involution ρ shown in Figure 7. Now we will describe the link $\mathcal{L}(m,n;p,q)$.

Let us consider the Wirtinger generators $s_1,\dots s_5$ of the fundamental group $\pi_1(S^3 \setminus W)$ as in Figure 6. According to [5] we choose meridians m_1, m_2 and longitudes l_1, l_2 as following words in generators $s_1,\dots s_5$:

$$\begin{aligned} m_1 &= s_5, & l_1 &= s_3 s_1^{-1}, \\ m_2 &= s_1, & l_2 &= s_5^{-1} s_4 s_2^{-1} s_1. \end{aligned}$$

Let V be a regular tubular neighborhood of the link W in S^3. Without lose of generality one can choose V, meridians m_1, m_2 and longitudes l_1, l_2 on the boundary of V to be invariant under the involution ρ. The factor space of S^3 under ρ is shown in Figure 8.

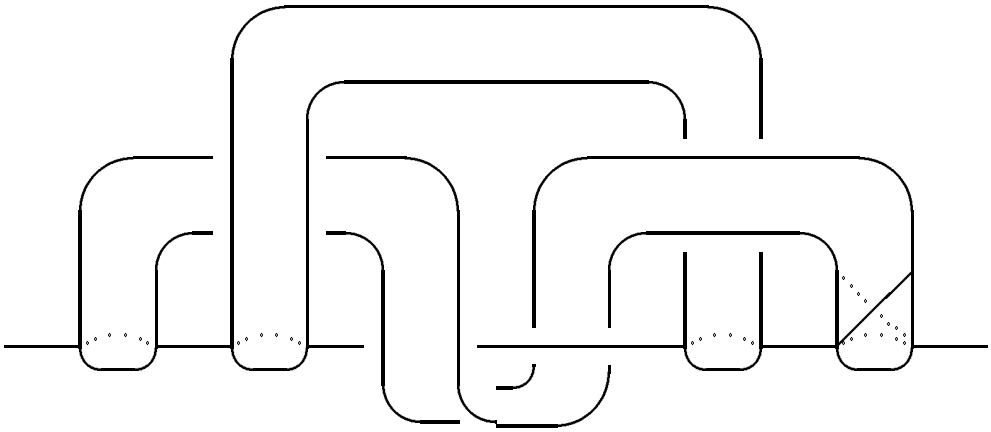

Figure 8. The quotient $(S^3 \setminus V)/\rho$.

The image of the tubular neighborhood V under canonical projection $p : S^3 \to S^3/\rho$ consists of two balls B_1 and B_2. Denote by $Fix(\rho)$ the axis of the involution ρ in S^3. For each ball B_i the intersection $B_i \cup p(Fix(\rho))$ consists of two arcs. By the isotopy of B_i along the image $p(l_i)$ of the longitude l_i $(i = 1,2)$ we get the following figure:

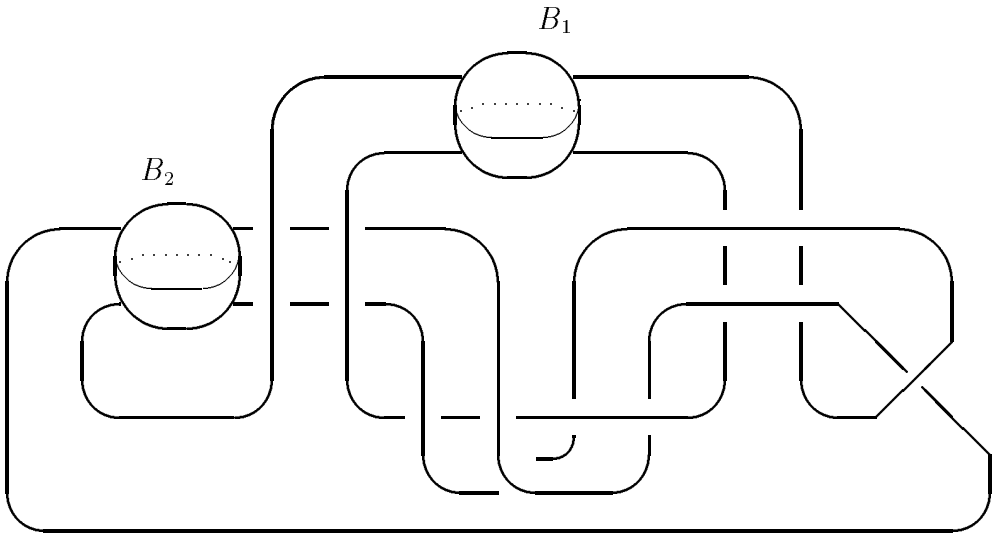

Figure 9.

The B_i with the arcs $B_i \cup p(Fix(\rho))$ is a trivial tangle in a terminology of [23].

By Montesinos algorithm to describe the link $\mathcal{L}(m,n;p,q)$ we need to replace this trivial tangles B_1 and B_2 by $\frac{m}{n}$ and $\frac{p}{q}$ rational tangles, respectively (see Figure 10).

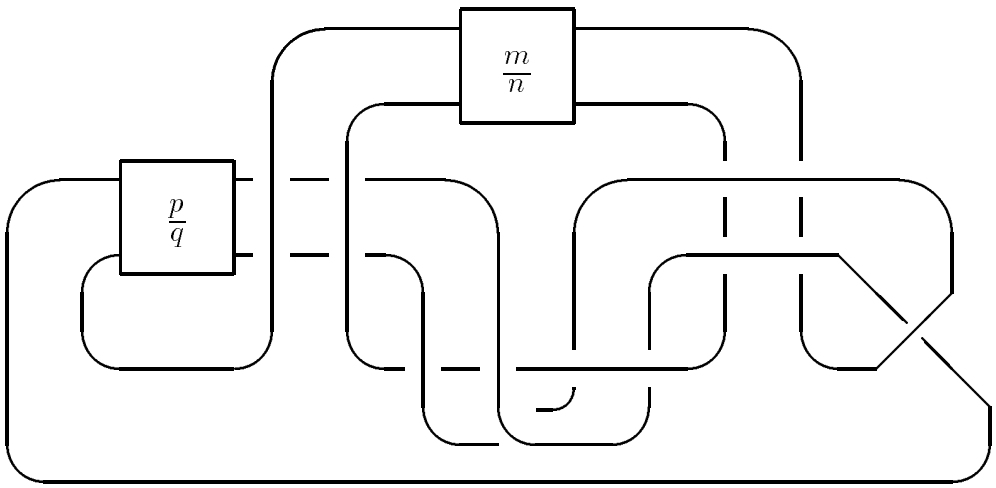

Figure 10. The link $\mathcal{L}(m,n;p,q)$.

In the following figures using Reidemeister's moves we redraw the link $\mathcal{L}(m,n;p,q)$ in more convenient form (Figure 11–13).

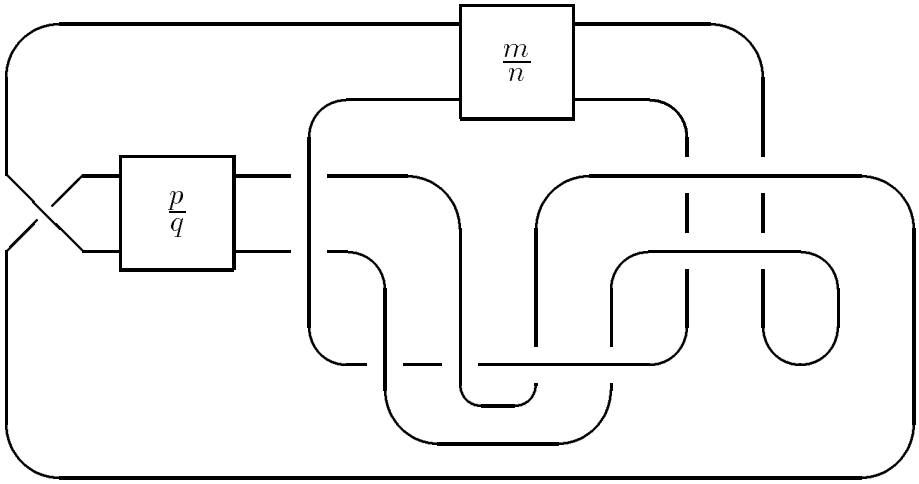

Figure 11. The link $\mathcal{L}(m, n; p, q)$.

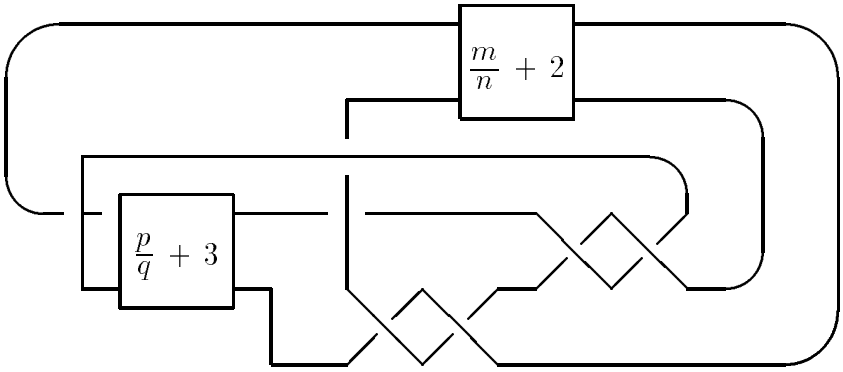

Figure 12. The link $\mathcal{L}(m, n; p, q)$.

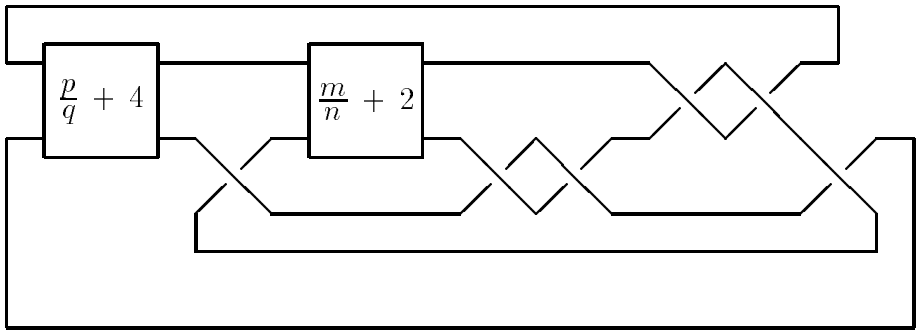

Figure 13. The link $\mathcal{L}(m, n; p, q)$.

As a sequence of above consideration we get the following theorem.

Theorem 2. *Let $M = W(m,n;p,q)$ be the manifold obtained by (m,n), (p,q)-Dehn surgery on the Whitehead link W. Then M is a two-fold covering of S^3 branched over the link $\mathcal{L}(m,n;p,q)$ pictured in Figure 13.*

We remark that the ten smallest known compact orientable hyperbolic 3-manifolds $\mathcal{M}_1,\dots,\mathcal{M}_{10}$ from [13] can be described in the form $W(m,n;p,q)$. Let $\mathcal{L}_1,\dots,\mathcal{L}_{10}$ be corresponding links from Theorem 2. As usual let $S^3 \setminus \mathcal{L}_i$ be a non-compact manifold which is a complement to $\mathcal{L}_i$ in the sphere S^3 and let $\mathcal{L}_i(2)$ be an π-orbifold with the underlying space S^3 and the link $\mathcal{L}_i$ as the singular set. Hyperbolic volumes of these manifolds and orbifolds can be founded using SnapPea program of J. Weeks [1], [12], [29]. In last column there are given notations of links $\mathcal{L}_i$ according to [5] and [24].

	$vol(\mathcal{M}_i)$	$vol(\mathcal{L}_i(2))$	$vol(S^3 \setminus \mathcal{L}_i)$	$\mathcal{L}_i$
$\mathcal{M}_1 = W(5,-2;5,-1)$	$0,9427\dots$	$0,4713\dots$	$9,4270\dots$	9_{49}
$\mathcal{M}_2 = W(1,1;5,-1)$	$0,9813\dots$	$0,4906\dots$	$5,6387\dots$	10_{161}
$\mathcal{M}_3 = W(3,-2;6,-1)$	$1,0149\dots$	$0,5074\dots$	$8,1195\dots$	10^2_{138}
$\mathcal{M}_4 = W(5,-1;5,-1)$	$1,2637\dots$	$0,6318\dots$	$9,2505\dots$	10_{155}
$\mathcal{M}_5 = W(1,1;6,-1)$	$1,2844\dots$	$0,6422\dots$	$5,8430\dots$	$11^2_?$
$\mathcal{M}_6 = W(1,1;1,-2)$	$1,3985\dots$	$0,6992\dots$	$5,8296\dots$	$14_?$
$\mathcal{M}_7 = W(1,-2;6,-1)$	$1,4140\dots$	$0,7070\dots$	$5,9782\dots$	$11^2_?$
$\mathcal{M}_8 = W(2,1;5,-1)$	$1,4140\dots$	$0,7070\dots$	$7,7948\dots$	$11^2_?$
$\mathcal{M}_9 = W(7,-3;5,-1)$	$1,4236\dots$	$0,7118\dots$	$10,6933\dots$	10_{162}
$\mathcal{M}_{10} = W(1,1;3,-2)$	$1,4406\dots$	$0,7203\dots$	$7,1180\dots$	$13_?$

Corollary. *The smallest known closed hyperbolic 3-manifolds $\mathcal{M}_1,\dots,\mathcal{M}_{10}$ are two-fold coverings of S^3 branched over the links $L_1,\dots,L_{10}$ in Figures $A1,\dots,A10$ in the Appendix respectively.*

3. The Meyerhoff–Neumann manifold

This section is devoted to an interesting connection between the Fibonacci manifold M_4 and the third of the ten smallest known manifolds.

We recall that R. Meyerhoff and W. Neumann [21] have obtained the hyperbolic manifold $\mathcal{M}_3 = W(3,-2;6,-1)$ by means of Dehn surgery on the Whitehead link W. It was interesting that $vol(\mathcal{M}_3)$ approximately up to 10^{-50} equals to the volume of the regular ideal tetrahedron in the Lobachevsky space. They asked if these volumes are strictly equal and if the manifold $\mathcal{M}_3$ is arithmetic over the field $\mathbb{Q}(\sqrt{-3})$. The goal of this section is to give affirmative answers on these questions.

Theorem 3. *The Fibonacci manifold M_4 is a two-fold unbranched covering of the Meyerhoff–Neumann manifold $\mathcal{M}_3$.*

Proof. Let us consider the hyperbolic π–orbifold $Th_4(2) = \mathbb{H}^3 / \mathrm{T}_4$. The underlying space of this orbifold is the three-dimensional sphere and the singular set is the Turk's head knot Th_4 in Figure 14.

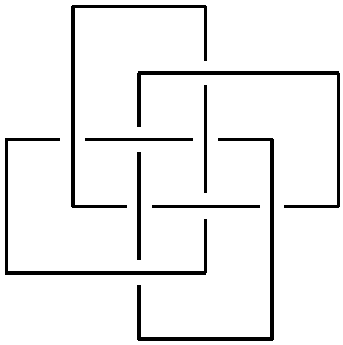

Figure 14. The singular set of the orbifold $Th_4(2)$.

The orbifold $Th_4(2)$ has the rotation symmetry ρ of the order four and the singular set remains invariant under ρ. Let us consider the involution ρ^2. The quotient space $Th_4(2) / \rho^2$ is a π–orbifold $D(2,2)$. The singular set of the orbifold $D(2,2)$ is the two-component link in Figure 15.

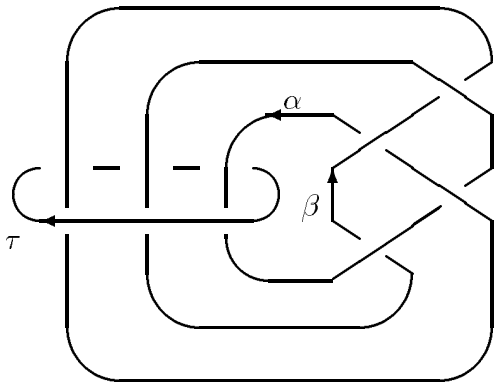

Figure 15. The singular set of the orbifold $D(2,2)$.

Using Wirtinger algorithm and results from [7] for the fundamental group of an orbifold, we can find the fundamental group Δ of the orbifold $D(2,2)$:

$$\Delta = \langle \alpha, \beta, \tau \mid (\tau\alpha\tau\beta)^2 \alpha (\beta\tau\alpha\tau)^2 = \tau\beta\tau, \quad (\beta\alpha)^2 \tau\beta\tau (\alpha\beta)^2 = \tau\alpha\tau, \quad \alpha^2 = \beta^2 = \tau^2 = 1 \rangle. \quad (12)$$

In this representation generators α, β, τ canonical correspond to arcs with the same labels on the link diagram in Figure 15.

By Mostow rigidity theorem the involution ρ^2 isotopic to an isometry of the hyperbolic orbifold $Th_4(2)$. Therefore the group Δ can be realized as a discrete subgroup of the isometry group of the Lobachevsky space $\mathbb{H}^3$. In this case τ is a lifting of the involution ρ^2 on the universal covering. According to Theorem 1, the Fibonacci manifold $M_4 = \mathbb{H}^3 / \Gamma_4$ is the two-fold covering of the π–orbifold $Th_4(2) = \mathbb{H}^3 / \mathrm{T}_4$. Since for M_4, $Th_4(2)$ and $D(2,2) = \mathbb{H}^3 / \Delta$ we have a covering diagram

$$M_4 \stackrel{2}{\longrightarrow} Th_4(2) \stackrel{2}{\longrightarrow} D(2,2), \tag{13}$$

which implies an embedding for subgroups:

$$\Gamma_4 \lhd \mathrm{T}_4 \lhd \Delta, \tag{14}$$

where $| \Delta : \mathrm{T}_4 | = 2$ and $| \mathrm{T}_4 : \Gamma_4 | = 2$.

Let us consider an epimorphism

$$\theta : \Delta \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \left\langle a \mid a^2 = 1 \right\rangle \oplus \left\langle t \mid t^2 = 1 \right\rangle$$

defined by conditions:

$$\theta(\alpha) = \theta(\beta) = a, \quad \theta(\tau) = t. \tag{15}$$

By the construction of the two-fold cover $Th_4(2) \to D(2,2)$ the loop τ from the fundamental group Δ of the orbifold $D(2,2)$ lifts to a trivial loop in the fundamental group T_4 of the orbifold $Th_4(2)$. By the same loops α and β lift to loops which generate cyclic subgroups of the order 2 in the group T_4. Therefore

$$\mathrm{T}_4 = \theta^{-1}(\mathbb{Z}_2) = \theta^{-1}\left(\left\langle a \mid a^2 = 1 \right\rangle\right). \tag{16}$$

Let us consider the 4–fold cover $M_4 \to D(2,2)$. In this case loops α, β and τ lift to a trivial loops in the group Γ_4. Hence

$$\Gamma_4 = \theta^{-1}(1) = \mathrm{Ker}\,\theta. \tag{17}$$

The group

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \left\langle a \mid a^2 = 1 \right\rangle \oplus \left\langle t \mid t^2 = 1 \right\rangle$$

contains a cyclic subgroup of the order two generated by $d = a + t$. Let us define an epimorphism

$$\lambda : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2,$$

by conditions

$$\lambda(a) = \lambda(t) = d. \tag{18}$$

Then for an epimorphism $\varphi = \lambda \circ \theta$ such that

$$\varphi \, : \, \Delta \, \longrightarrow \, \mathbb{Z}_2 \, = \, \Big\langle d \, \mid \, d^2 \, = \, 1 \Big\rangle \tag{19}$$

we have

$$\varphi(\alpha) \, = \, \varphi(\beta) \, = \, \varphi(\tau) \, = \, d. \tag{20}$$

Denote $\Phi = \mathrm{Ker}\, \varphi$ and consider an orbifold $U = \mathbb{H}^3 \, / \Phi$. By the construction of the epimorphism φ the orbifold cover

$$U = \mathbb{H}^3 \, / \Phi \, \stackrel{2}{\longrightarrow} \, D(2,2) = \mathbb{H}^3 \, / \Delta \tag{21}$$

is branched over both components of the singular set of the orbifold $D(2,2)$. In this case loops α, β and τ lift to trivial loops from the group Φ. Therefore U is a hyperbolic orbifold and the singular set of U is empty. Hence U is a hyperbolic manifold.

Our next step is to prove that $U \, = \, \mathcal{M}_3$. By Theorem 2 the manifold $\mathcal{M}_3 = W(3, -2; 6, -1)$ can be obtained as the 2-fold covering of S^3 branched over the 2-component link $\mathcal{L}_3 = \mathcal{L}(3, -2; 6, -1) = 10^2_{138}$ in notation according to SnapPea (Figure 16).

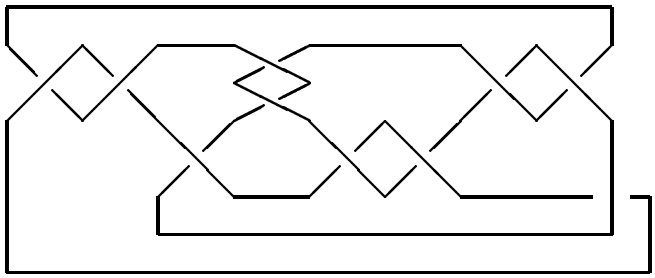

Figure 16. The link $\mathcal{L}_3 = \mathcal{L}(3, -2; 6, -1) = 10^2_{138}$.

By using the Reidemeister moves one can see that two-component links in Figure 15 and Figure 16 are equivalent. Therefore manifolds U and $\mathcal{M}_3$ are obtained as 2-fold coverings of the three-sphere branched over the same link. Hence manifolds U and $\mathcal{M}_3$ are homeomorphic and moreover by Mostow rigidity theorem they are isometric. By this $\mathcal{M}_3 = \mathbb{H}^3 \, / \, \Phi$. Since $\varphi = \lambda \circ \theta$ for fundamental groups we get

$$\Phi \, = \, \mathrm{Ker}\varphi \, > \, \Gamma_4 \, = \, \mathrm{Ker}\theta, \tag{22}$$

and a covering diagram for manifolds

$$M_4 \, = \, \mathbb{H}^3 \, / \, \Gamma_4 \, \stackrel{2}{\longrightarrow} \, N \, = \, \mathbb{H}^3 \, / \, \Phi. \tag{23}$$

Comparing covering diagrams (13), (21) and (23) we get the following diagram in Figure 17 :

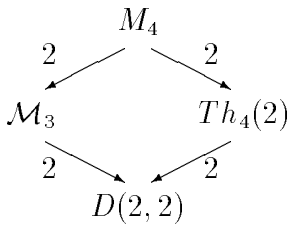

Figure 17. The covering diagram.

Groups Γ_4 and Φ are fundamental groups of hyperbolic manifolds M_4 and $\mathcal{M}_3$ respectively. Hence these groups are torsion-free. Therefore we can conclude that the cover (23) induced by (22) is unbranched. Theorem is proved. □

From [19] and a deep connection between manifolds M_4 and $\mathcal{M}_3$ we have affirmative answers on above questions.

Theorem 4. *The Meyerhoff–Neumann manifold* $\mathcal{M}_3 = W(3,-2;6,-1)$ *is arithmetic over the field* $\mathbb{Q}(\sqrt{-3})$ *and its volume is exactly equal to the volume of the regular ideal tetrahedron in* $\mathbb{H}^3$.

Proof. It was shown in [19], that the volume of the hyperbolic Fibonacci manifold M_4 is strictly equal to the double volume of the regular ideal tetrahedron in $\mathbb{H}^3$. By Theorem 3 $vol(\mathcal{M}_3) = \frac{1}{2}vol(M_4) =$ the volume of the regular ideal tetrahedron $= 1.0149\ldots$ Moreover by (22) fundamental groups Φ and Γ_4 of manifolds $\mathcal{M}_3$ and M_4 are commensurable. It was proved in [8] that the Fibonacci manifold M_4 is arithmetic over the field $\mathbb{Q}(\sqrt{-3})$. Therefore the same is true for the manifold $\mathcal{M}_3$. □

4. The Heegaard genus of the manifold $W(m,n;p,q)$ and of the Fibonacci manifold

Following [31] we recall some known facts from the theory of 3-manifolds.

Let M^3 be a closed orientable 3-manifold. A Pair $(H_g, H_g^{'})$ of handlebodies of genus g is called a *Heegaard decomposition of genus* g of M^3 if $M^3 = H_g \cup H_g^{'}$ and $H_g \cap H_g^{'} = \partial H_g = \partial H_g^{'}$ is a closed orientable surface of genus g. The minimal genus among the genera of all Heegaard decompositions of M^3 is called the *Heegaard genus* of M^3 and is denoted by $h(M^3)$. The three-dimensional sphere S^3 is the alone orientable manifold which Heegaard genus is equal to 0. The Heegaard genus is equal to 1 only for lens spaces and for manifold $S^2 \times S^1$. In particular, if a manifold M^3 admits Euclidean or hyperbolic structure, then $h(M^3) \geq 2$. In general case the problem of determination of the Heegaard genus $h(M^3)$ is open.

The minimal number of elements needed to generate the fundamental group $\pi_1(M^3)$ of a closed 3-manifold M^3 is called the *rank* of $\pi_1(M^3)$. For a 3-

manifold M^3 we denote the rank of $\pi_1(M^3)$ by $r(M^3)$. The following inequality is valued: $r(M^3) \leq h(M^3)$ in obvious way [9]. In particular the Poincare conjecture can be formulate in the following way : $r(M^3) = 0$ if and only if $h(M^3) = 0$. M. Boileau and H. Zieschang [3] have constructed an Seifert fiber space M^3 with : $2 = r(M^3) < h(M^3) = 3$.

We recall that the fundamental groups of hyperbolic manifolds $W(m, n; p, q)$ and of hyperbolic Fibonacci manifolds are two-generated. In will be shown that for these manifolds Heegaard genus is 2.

Let $\mathcal{L}$ be a link in S^3. According to [31] a link $\mathcal{L}$ is said to have a *3-bridge presentation* if there is a genus 0 Heegaard splitting (B_1, B_2) of S^3 such that the link $\mathcal{L}$ intersect B_i, $(i = 1, 2)$, in three unlinked arcs. That is there are three mutually disjoint discs in B_i each of which is bounded by one of the arcs considered and an arc on the boundary of B_i.

The following theorem is due to O. Ja. Viro [28]:

A closed orientable 3-manifold M^3 admits a Heegaard decomposition of genus 2 if and only if M^3 is a two-fold covering of S^3 branched over a link with a 3-bridge presentation.

By Theorem 2 the manifold $M = W(m, n; p, q)$ is a two-fold covering of S^3 branched over the link $\mathcal{L}(m, n; p, q)$ (see Figure 18).

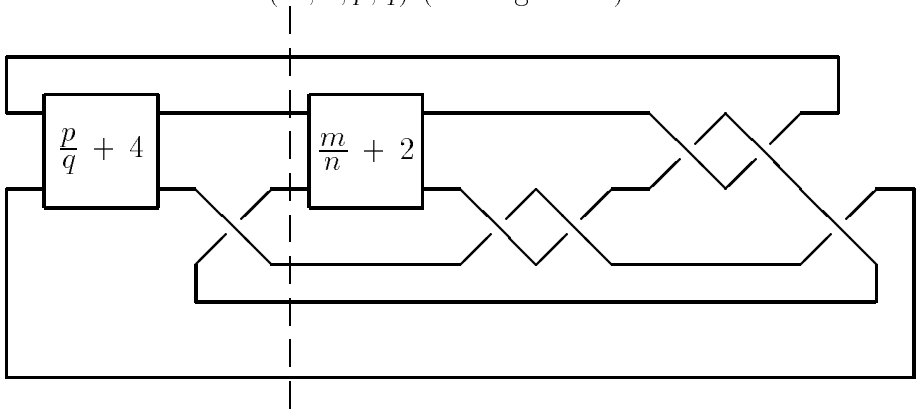

Figure 18. The link $\mathcal{L}(m, n; p, q)$.

Let us consider the Heegaard decomposition of genus 0 (B_1, B_2) of S^3, where boundaries ∂B_1 and ∂B_2 correspond to the dotted line in Figure 18. Then for each $i = 1, 2$ intersection $\mathcal{L}(m, n; p, q) \cap B_i$ consist of three unknotted arcs. Therefore $\mathcal{L}(m, n; p, q)$ admits a 3-bridge presentation. So as a consequence of Viro theorem we have

Proposition 1. *Let $M = W(m, n; p, q)$ be a manifold obtained by (m, n), (p, q)-Dehn surgery on the Whitehead link W. Then $h(M) \leq 2$.*

We recall that $h(M) = 0$ or 1 if and only if M is the 3-sphere, the lens space or $S^2 \times S^1$ [31]. In each of these cases M does not admit a hyperbolic structure [26]. This gives the following improvement of above Proposition.

Proposition 2. *Let $M = W(m,n;p,q)$ be a hyperbolic manifold obtained by (m,n), (p,q)-Dehn surgery on the Whitehead link W. Then $h(M) = 2$.*

The case of a manifold $M = W(1,1;p,q)$ which can be obtained by (p,q)-Dehn surgery on the figure-eight knot, was considered in [23] and [27].

In particular, Heegaard genus on the ten smallest known hyperbolic manifolds is equal to 2.

By the same arguments one can find Heegaard genus of hyperbolic Fibonacci manifolds.

Proposition 3. *For any $n \geq 3$ the Heegaard genus $h(M_n)$ of a Fibonacci manifold M_n is two.*

Proof. By Theorem 1, for any $n \geq 2$ the Fibonacci manifold M_n is a two-fold covering of S^3 branched over the Turk's head link Th_n. The link Th_n is a closed 3-string braid $(\sigma_1\sigma_2^{-1})^n$ and therefore has a 3-bridge presentation. Hence by Viro theorem $h(M_n) \leq 2$. But for $n \geq 4$ the manifold M_n is hyperbolic [8]. The manifold M_3 coincides with the Hantzche–Wendt manifold [32] and admits an Euclidean structure. Therefore $h(M_3) = 2$ for $n \geq 3$. □

The Fibonacci manifold M_2 is the two-fold covering of S^3 branched over the figure-eight knot and coincides with the lens space $L(5,2)$. Therefore $h(M_2) = 1$.

We will make two remarks on above results.

First remark is connected with classical $84(g-1)$–Hurwitz theorem. By this theorem the automorphism group of a compact Riemann surface of genus $g > 1$ is finite and bounded above by $84(g-1)$.

By Mostow rigidity theorem the isometry group of the compact hyperbolic 3-manifold is finite always. Moreover *a priori* it may be isomorphic to an arbitrary finite group [15].

In analogy to Hurwitz theorem for 2-dimensional case one can try to estimate the order of the isometry group of a hyperbolic 3-manifold in terms of Heegaard genus. But the following Proposition shows that it is impossible.

Proposition 4. *There are hyperbolic 3-manifolds of Heegaard genus 2 with an arbitrary large group of isometries.*

Proof. The Fibonacci hyperbolic manifold M_n has an orientation-preserving isometry of the order n. This isometry is induced by an automorphism $x_i \to x_{i+2}$ (i mod 2n) of the fundamental group $\pi_1(M_n) = F(2,2n)$. It was shown in [11] that a quotient space of M_n by this isometry is the orbifold with the underlying space S^3 and the figure-eight knot as the singular set. Therefore by one hand the isometry group contains the cyclic group of the order n and so can be arbitrary large as $n \to \infty$. By the other hand, $h(M_n) = 2$ for $n \geq 3$. □

Proposition 5. *If $n > 10$ then each Heegaard surface of genus 2 in a*

Fibonacci manifold M_n is not invariant under the isometry group.

Proof. Let (H_2, H_2') be a genus two Heegaard decomposition of M_n. Denote by $S_2 = H_2 \cap H_2^{'} = \partial H_2 = \partial H_2^{'}$ the corresponding Heegaard surface. Suppose that S_2 is invariant under the isometry group of M_n. According to [14] by Wiman theorem [30] the order of an automorphism of a Riemann surface of genus $g = 2$ is bounded above by $4 \cdot g + 2 = 10$. Hence $n \leq 10$. Contradiction. $\square$

B. Zimmermann [33] asked if there is some relationship between equivariant Heegaard genus of hyperbolic 3-manifold and its Heegaard genus. Using a technique from [33] it is possible to show that equivariant Heegaard genus of the Fibonacci manifold M_n is equal to $n - 1$ inspite of its Heegaard genus equals two. So there are hyperbolic 3-manifolds with an arbitrary large distance between equivariant Heegaard genus and Heegaard genus.

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References

[1] Adams C., Hildebrand M., Weeks J., *Hyperbolic invariants of knots and links,* Trans. Amer. Math. Soc., **326** (1991), 1–56.

[2] Birman J.S., *Braids, Links and Mapping Class Groups* Annals of Math. Studies, 84, Princeton University Press, Princeton, N.J.,1975.

[3] Boileau M., Zieschang H., *Heegaard genus of closed orientable Seifert 3-manifolds*, Invent. Math. **76** (1984), 455–468.

[4] Boileau M., Zimmermann B., *The π–orbifold group of a link,* Math. Z., **200** (1989), P.187–208.

[5] Burde G., Zieschang H., *Knots,* de Gruyter Studies in Mathematics, Berlin - New-York, 1985.

[6] Conway J., *Advanced problem 5327,* Amer. Math. Monthly, **72** (1965), 915.

[7] Haefliger A., Quach N.D., *Une presentation de groupe fundamental d'une orbifold,* Asterisque, **116** (1984), P.98–107.

[8] Helling H., Kim A.C., Mennicke J., *A geometric study of Fibonacci groups,* SFB 343 Bielefeld, Diskrete Strukturen in der Mathematik, Preprint (1990).

[9] Hempel J., *3-manifolds*, Ann. of Math. Studies 86, Princeton University Press, Princeton, N.J., 1976.

[10] Hempel J., *The lattice of branched covers over the figure–eight knot,* Topology and its Appl., **34** (1990), 183–201.

[11] Hilden H.M., Lozano M.T., Montesinos J.M., *Arithmeticy of the Figure-Eight Knot Orbifolds,* in: Topology'90, de Gruyter, 1992, 169–184.

[12] Hodgson C.D., Meyerhoff R.G., Weeks J., *Surgeries on the Whitehead Link Yield Geometrically Similar Manifolds*, Topology'90, 1992, 195–205.

[13] Hodgson C.D., Weeks J., *Symmetries, isometries and length spectra of closed hyperbolic 3-manifolds*, Preprint 1994.

[14] Kerekjarto B., *Vorlesungen uber Topologie*, Berlin., Springer, 1923.

[15] Kojima S., *Isometry transformations of hyperbolic 3-manifolds,* Topology Appl., **29** (1988), 297–307.

[16] Kojima S., Long D.D., *Virtual Betti numbers of some hyperbolic 3–manifolds*, A Fete of Topology, Academic Press Inc., 1988, pp. 417–442.

[17] Matveev S.V., Fomenko A.T., *Constant energy surfaces of Hamiltonian systems, enumeration of three-dimensional manifolds in increasing order of complexity, and computation of volumes of closed hyperbolic manifolds,*. Russian Mathem. Surveys, **43** (1988), 1, 3–24.

[18] Mednykh A.D., Vesnin A.Ju., *On limit ordinals in the Thurston–Jorgensen theorem on the volumes of three–dimensional hyperbolic manifolds,* Dokl. RAS., **336** (1994), 7–10 (In Russian).

[19] Mednykh A.D., Vesnin A.Ju., *Volumes of the Fibonacci manifolds* SFB 343 Bielefeld, Diskrete Strukturen in der Mathematik, Preprint 94 - 015 (1994).

[20] Mennicke J.L., *On Fibonacci groups and some other groups*, Proceedings of "Groups - Korea 1988", Lect. Note Math. 1062, pp. 117–123.

[21] Meyerhoff R., Neumann W.D., *An asymptotic formula for the eta invariants of hyperbolic 3-manifolds,* Comment. Math. Helv., **67** (1992), 28–46.

[22] Montesinos J.M., *Surgery on links and double branched covers of S^3,* Knots, Groups, and 3-Manifolds, Princeton University Press, Princeton, 1975, 227–259.

[23] Montesinos J.M., Whitten W., *Constructions of two–fold branched covering spaces,* Pacific J. Math., **125** (1986), 415–446.

[24] Rolfsen D., *Knots and Links,* Publish of Perish Inc., Berkely Ca., 1976.

[25] Sakuma M., Weeks J., *Examples of canonical decompositions of hyperbolic link complements*, Preprint, 1993.

[26] Scott P.. *The Geometries of 3-Manifolds*, Bull. London Math. Soc. **15** (1986), 401-487.

[27] Thurston W.P., *The Geometry and Topology of Three-Manifolds,* Lecture Notes, Princeton University 1980.

[28] Viro O.Ja., *Linkings, 2-sheeted branched coverings and braids*, Math. USSR, Sbornik, **16** (1972), 223–236.

[29] Weeks J., *Hyperbolic structures on 3-manifolds,* Princeton Univ. Ph. D. Thesis, 1985.

[30] Wiman A., *Uber die hyperelliptischen Curven und diejenigen vom Geschlechte p=3 welche eindenregen Transformationen in sich zulassen,* Bihang Till. Kongl. Svenska Veienskaps-Akademiens Handlingar, **21** (1895-96), 1–23.

[31] Zieschang H., *On Heegaard Diagrams of 3-manifolds*, Asterisque, 163-164 (1988), 247–280.

[32] Zimmermann B., *On the Hantzche–Wendt manifold*, Monat. Math. **110** (1990), 321–327.

[33] Zimmermann B., *Finite group actions on handelbodies and equivariant Heegaard genus for 3-manifolds*, Topology and Appl. **43** (1992), 263–274.

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Appendix. Links corresponding to the ten smallest known closed hyperbolic 3-manifolds.

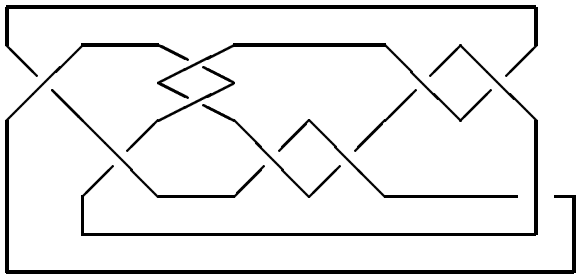

Figure A1. The knot $\mathcal{L}_1 = \mathcal{L}(5,-2;5,-1) = 9_{49}$.

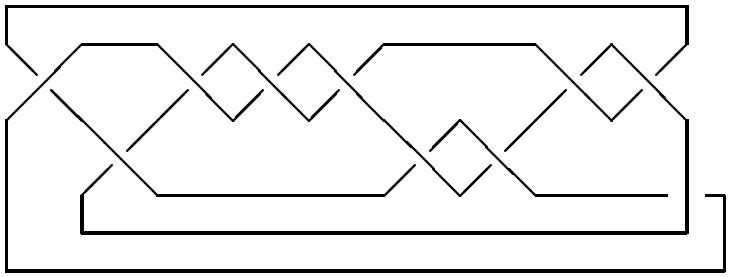

Figure A2. The knot $\mathcal{L}_2 = \mathcal{L}(1,1;5,-1) = 10_{161}$.

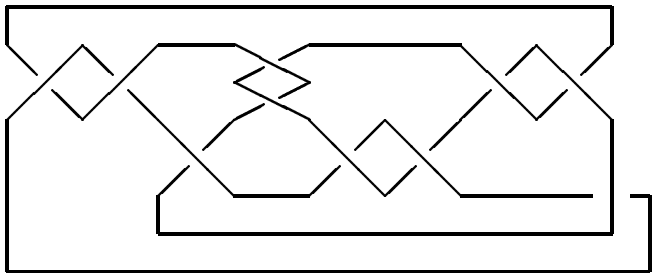

Figure A3. The link $\mathcal{L}_3 = \mathcal{L}(3,-2;6,-1) = 10^2_{138}$.

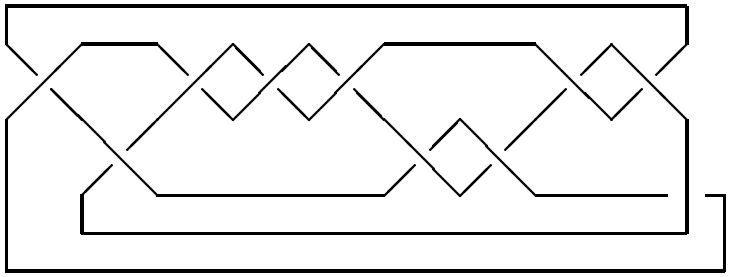

Figure A4. The knot $\mathcal{L}_4 = \mathcal{L}(5,-1;5,-1) = 10_{155}$.

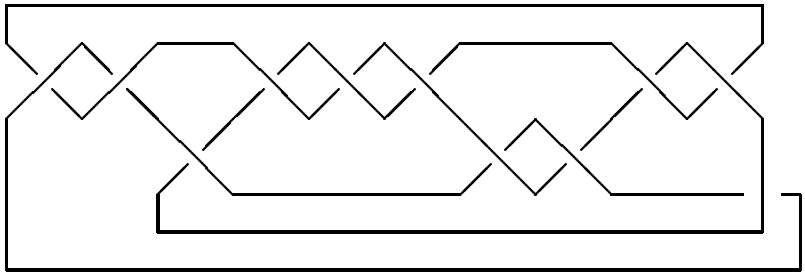

Figure A5. The link $\mathcal{L}_5 = \mathcal{L}(1,1;6,-1) = 11^2_?$.

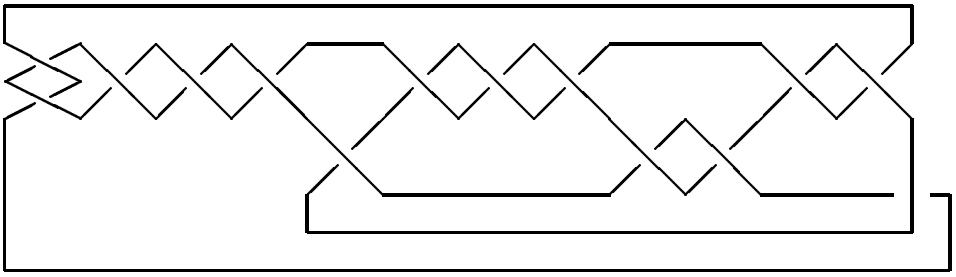

Figure A6. The knot $\mathcal{L}_6 = \mathcal{L}(1,1;1,-2) = 14_?$.

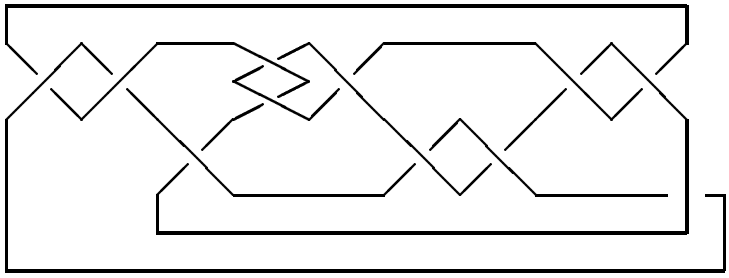

Figure A7. The link $\mathcal{L}_7 = \mathcal{L}(1,-2;6,-1) = 11^2_?$.

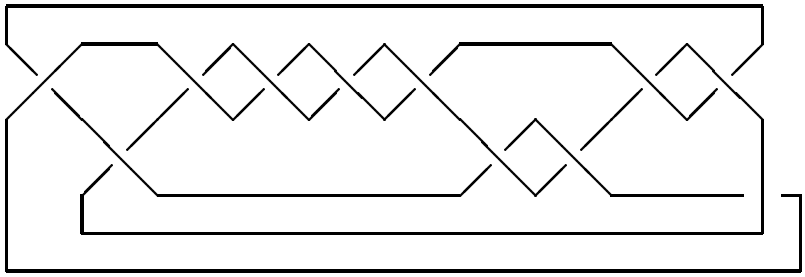

Figure A8. The link $\mathcal{L}_8 = \mathcal{L}(2,1;5,-1) = 11^2_?$.

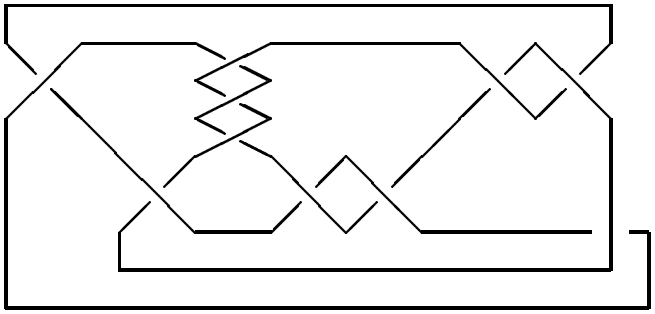

Figure A9. The knot $\mathcal{L}_9 = \mathcal{L}(7,-3;5,-1) = 10_{162}$.

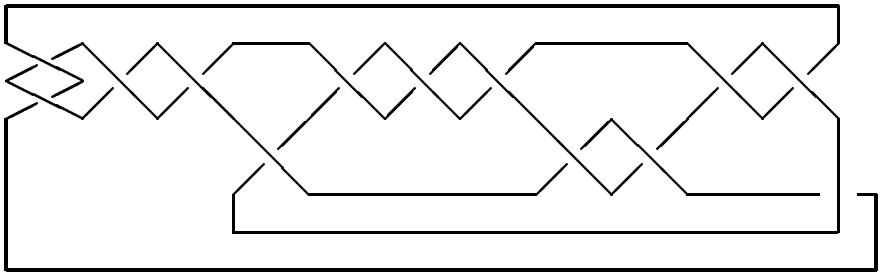

Figure A10. The knot $\mathcal{L}_{10} = \mathcal{L}(1,1;3,-2) = 13_?$.