On the tameness of certain 2-point algebras

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Abstract

In [Ge1] it was proved that all tame distributive 2-point algebras are factors of algebras contained in a small list given in that paper. Also there were given tameness proofs for the algebras of that list with the exception of essentially two cases. It was pointed out already in that paper that in order to complete the tameness proofs one essentially had to study two 2-point algebras with homogenous relations, call them Λ and $\Gamma_{m,A}$, see 1.2 for details. The subject of this paper is to supply these proofs. Indeed we manage in the case of Λ to show that the universal Galois covering with group $\mathbb{Z}$ is locally support-finite and tame, thus so is Λ. In the case of $\Gamma_{m,A}$ we use the generalized 1-point extensions of [Dr3], to prove that the universal Galois covering with group $\mathbb{Z}$ is tame, and the tameness of the algebra $\Gamma_{m,A}$ itself will follow from a result announced by Drozd ([Dd]).

1 Introduction

1.1. Let us first fix some notations. k will be an algebraically closed field. In this paper by a k-algebra we mean a locally bounded spectroid in the sense of [GR]. Usually we present such a category Λ in the form $k[Q]/I$ where $k[Q]$ is the k-linear hull of the path category of a quiver Q and I is an admissable ideal. We denote by Λ-mod the category of contravariant k-linear functors $\Lambda \to k$-mod. We compose k-linear maps from right to left but morphisms in other categories as well as arrows from left to right.

The (Jacobson) radical of a locally bounded spectroid Λ is denoted by $\operatorname{Rad}\Lambda$.

1.2. Consider the quiver $Q\colon \overset{\sigma}{\circlearrowright} a \underset{\gamma}{\overset{\nu}{\rightleftarrows}} b \overset{\rho}{\circlearrowleft}$ and the categories

$$\Lambda 9'_{m,A} := k[Q]/\langle \sigma\nu - \nu\rho, \nu\gamma - \sigma^2, \rho\gamma - \gamma\sigma, \gamma\nu - \rho^m(1_b - A\rho), \sigma^2\nu, \rho^2\gamma, \sigma^4, \rho^{m+2}\rangle$$

$$\Lambda 9''_{m,A} := k[Q]/\langle \sigma\nu - A\nu\rho, \nu\gamma - \sigma^2, \rho\gamma - \gamma\sigma, \gamma\nu - \rho^m, \sigma^2\nu, \rho^2\gamma, \sigma^3, \rho^{m+1}\rangle$$

$$\Gamma_{m,A} := k[Q]/\langle \sigma\nu - A\nu\rho, \nu\gamma - \sigma^2, \rho\gamma - \gamma\sigma, \gamma\nu, \sigma^3, \rho^{m+1}\rangle$$

$$\Lambda 10 := k[Q]/\langle \sigma\nu - \nu\rho, \nu\gamma - \sigma^2, \gamma\nu - \rho^2, \rho\gamma\rangle$$

$$\Lambda 10'_1 := k[Q]/\langle \sigma\nu - \nu\rho, \nu\gamma - \sigma^2, \gamma\nu - \rho^2, \rho\gamma, \sigma^2\nu\rangle$$

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Observe that $\Lambda 9'_{m,A}$ is selfinjective, modulo its socle it is isomorphic to $\Lambda 9''_{m,A}$, and this in turn degenerates into $\Gamma_{m,A}$, thus by [Ge2] the tameness of $\Gamma_{m,A}$ will imply the tameness of $\Lambda 9''_{m,A}$ and $\Lambda 9'_{m,A}$. Similarily $\Lambda 10$ has a projective-injective module, modulo its socle it is isomorphic to $\Lambda 10'_1$; see [Ge1, 4.2, 4.3]. For brevity we write $\Lambda 10'_1 =: \Lambda$. Concerning $\Gamma_{m,A}$ we note that $m \in \mathbb{N}$ with $m \geq 3$ and $0 \neq A \in k$. We point out that $\operatorname{Rad}^3 \Lambda = 0$ and that the elements of $\operatorname{Rad}^3 \Gamma_{m,A}$ are polynomials in ρ.

Now consider the quiver $\tilde{Q}$:

$$\begin{tikzcd}
\cdots & a_{-1} \arrow[r, "\sigma_{-1}"] \arrow[dr, "\nu_{-1}"'] & a_0 \arrow[r, "\sigma_0"] \arrow[dr, "\nu_0"'] & a_1 \arrow[r, "\sigma_1"] \arrow[dr, "\nu_1"'] & a_2 & \cdots \\
\cdots & b_{-1} \arrow[r, "\rho_{-1}"'] \arrow[ur, "\gamma_{-1}"] & b_0 \arrow[r, "\rho_0"'] \arrow[ur, "\gamma_0"] & b_1 \arrow[r, "\rho_1"'] \arrow[ur, "\gamma_1"] & b_2 & \cdots
\end{tikzcd}$$

and the categories

$$\tilde{\Lambda} := k[\tilde{Q}]/\langle \sigma_i \nu_{i+1} - \nu_i \rho_{i+1}, \nu_i \gamma_{i+1} - \sigma_i \sigma_{i+1}, \gamma_i \nu_{i+1} - \rho_i \rho_{i+1}, \rho_i \gamma_{i+1},$$

$$\sigma_i \sigma_{i+1} \nu_{i+2} \mid i \in \mathbb{Z} \rangle$$

$$\tilde{\Gamma}_{m,A} := k[\tilde{Q}]/\langle \sigma_i \nu_{i+1} - A\nu_i \rho_{i+1}, \nu_i \gamma_{i+1} - \sigma_i \sigma_{i+1}, \rho_i \gamma_{i+1} - \gamma_i \sigma_{i+1}, \gamma_i \nu_{i+1},$$

$$\sigma_i \sigma_{i+1} \sigma_{i+2}, \rho_i \cdots \rho_{i+m} \rho_{i+m+1} \mid i \in \mathbb{Z} \rangle$$

There is a natural action of the (additive) group $\mathbb{Z}$ on these categories inducing Galois coverings $\tilde{\Lambda} \to \Lambda$ and $\tilde{\Gamma}_{m,A} \to \Gamma_{m,A}$.

2 Λ is tame

2.1. We will show, that the Galois-covering $\tilde{\Lambda}$ of Λ is locally support-finite and tame, which implies by [Pe, 3.2] the tameness of Λ (the hypothesis k uncountable may be omitted).

We first have to introduce some notation. For $i \leq j$ we set

$$\tilde{\Lambda}^{(i,j)} := \tilde{\Lambda}\{a_i, b_i, a_{i+1}, \ldots, a_{j+1}, b_{j+1}\},$$

the full subcategory of $\tilde{\Lambda}$ that contains the objects $\{a_i, \ldots, b_{j+1}\}$; for brevity write $\tilde{\Lambda}^{(i,i)} =: \tilde{\Lambda}^{(i)}$,

$$\tilde{\Lambda}^{[i]} := \tilde{\Lambda}\{a_{i-1}, a_i, b_i, a_{i+1}, b_{i+1}, b_{i+2}\}.$$

We also give names to some indecomposable regular $\tilde{\Lambda}^{(i)}$-modules. For $C \in P_1(k) := k \cup \{\infty\}$ we set $M_C^{(i)}(x) := k$ for $x \in \tilde{\Lambda}^{(i)}$, and for $C \in k$ we take $M_C^{(i)}(\gamma_i) := C \cdot \mathrm{id}_k$, $M_C^{(i)}(\sigma_i) = M_C^{(i)}(\rho_i) = M_C^{(i)}(\nu_i) = \mathrm{id}_k$, while $M_\infty^{(i)}(\rho_i) = 0$, $M_\infty^{(i)}(\gamma_i) = M_\infty^{(i)}(\nu_i) = M_\infty^{(i)}(\sigma_i) = \mathrm{id}_k$; note that $M_0^{(i)}$ and $M_\infty^{(i)}$ lie in different inhomogenous tubes $\mathcal{R}_0^{(i)}$ and $\mathcal{R}_\infty^{(i)}$ of $\tilde{\Lambda}^{(i)}$-mod, while the modules $M_C^{(i)}$ for $C \in k \setminus \{0,1\}$ lie in the mouths of the homogenous tubes $\mathcal{R}_C^{(i)}$.

2.2. Note that $\tilde{\Lambda}^{[i]} \cong [M_1^{(i)}]\left(\tilde{\Lambda}^{(i)}[M_1^{(i)}]\right)$. Thus it is not hard, to determine $\tilde{\Lambda}^{[i]}$-mod with the methods developed in [Ri1] and [Ri2]. We find $\tilde{\Lambda}^{[i]}$-mod $= \mathcal{P}^{[i]} \vee \mathcal{R}^{[i]} \vee \mathcal{I}^{[i]}$, with $\mathcal{P}^{[i]}$ a preprojective and $\mathcal{I}^{[i]}$ a preinjective component of type $\tilde{D}_3$, while

$$\mathcal{R}^{[i]} = \bigvee_{C \in P_1(k)} \mathcal{R}_C^{[i]} \quad \text{with} \quad \mathcal{R}_C^{[i]} = \mathcal{R}_C^{(i)} \text{ for } C \neq 1.$$

$\mathcal{R}_1^{[i]}$ is an inhomogenous tube of rank 2 with a projective and an injective module in its mouth. Note that the indecomposable sincere modules all lie in this tube. Moreover $\mathcal{R}^{[i]}$ is a separating tubular family. Finally, $M \in \mathcal{P}^{[i]}$ implies $M(b_{i+2}) = 0$ and $M \in \mathcal{P}^{[i]}$ implies $M(a_{i-1}) = 0$.

2.3. We will show by induction on $j \geq i+2$ the following claim which will imply that $\tilde{\Lambda}$ is locally support-finite:
Let $M \in \tilde{\Lambda}^{(i,j)}$-mod *be indecomposable, then*

$$\begin{aligned} M(b_{j+1}) \neq 0 \text{ and } M(a_{j+1}) = 0 &\implies M \in \tilde{\Lambda}^{[j-1]}\text{-mod} \\ M(a_{j+1}) \neq 0 &\implies M \in \tilde{\Lambda}^{(j-1,j)}\text{-mod} \end{aligned}$$

We start with $j = i+2$ and see

$$\tilde{\Lambda}^{(i,i+2)} = [M_0^{(i+1)}]\left(\tilde{\Lambda}^{[i+1]}[M_\infty^{(i+1)}]\right).$$

For the first case we only have to observe that $\mathrm{Hom}_{\tilde{\Lambda}^{[i+1]}}(X, M_0^{(i+1)}) \neq 0$ implies $X(a_{i+2}) = 0$ (thus for $M \in [M_0^{(i+1)}]\tilde{\Lambda}^{[i+1]}$-mod indecomposable, we have $M(a_{i+2}) = 0$ or $M(b_{i-1}) = 0$). In the second case our assertion follows, since

$$\mathrm{Hom}_{\tilde{\Lambda}^{[i+1]}}(M_\infty^{(i+1)}, X) \neq 0 \neq \mathrm{Hom}_{\tilde{\Lambda}^{[i+1]}}(X, M_0^{(i+1)})$$

is impossible for $X \in \tilde{\Lambda}^{[i+1]}$-mod, compare [GP, 1.4,ii)].

Suppose now our assertion to be true for some $j \geq i+2$ and let $M \in \tilde{\Lambda}^{(i,j+1)}$-mod be indecomposable with $M(a_{j+2}) \oplus M(b_{j+2}) \neq 0$. Decompose $M\,|_{\tilde{\Lambda}^{(i,j)}} = \oplus_{t=1}^m M_t$ into indecomposable summands, then $\mathrm{Hom}_{\tilde{\Lambda}^{(i,j)}}(M_\infty^{(j)}, M_t) \neq 0$ or $\mathrm{Hom}_{\tilde{\Lambda}^{(i,j)}}(M_1^{(j)}, M_t) \neq 0$ for every $t \in \{1, \ldots, m\}$. It is easy to see that this is only possible if $M_t(a_{j+1}) \neq 0 \neq M_t(b_{j+1})$, thus by induction hypothesis $M_t \in \tilde{\Lambda}^{(j-1,j)}$-mod and $M \in \tilde{\Lambda}^{(j-1,j+1)}$-mod. Now we can apply the first part of the proof again.

2.4. By the foregoing subsections it remains to show that $\tilde{\Lambda}^{(i,i+1)}$ is tame. This was done already in [Ge1, 4.4]; alternatively we can use generalized 1-point extensions.

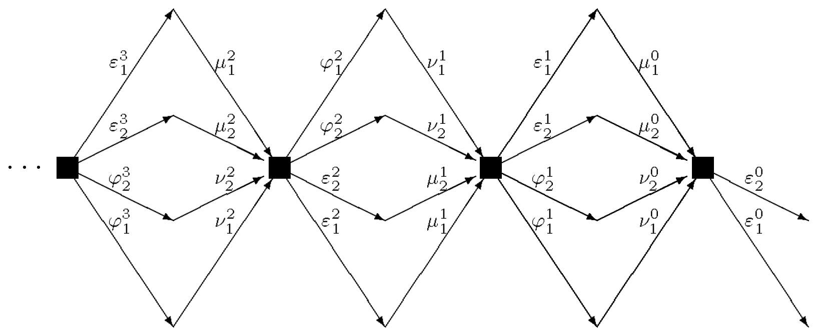
3 Vector space categories

3.1. The proof of the tameness of $\tilde{\Gamma}_{m,A}$ will be done in the nect section by reduction to a vector space category using the generalized 1-point extension technique. In this section we will recall the needed notions about vector space categories and prove some auxiliary results.

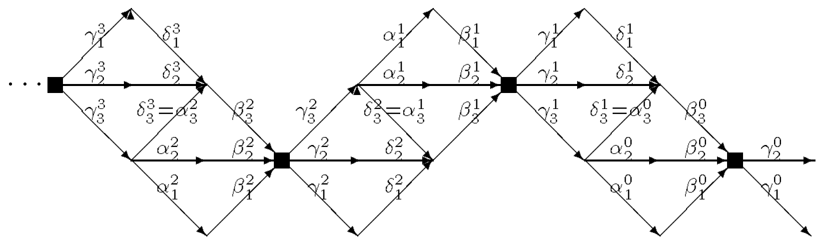
A vector space category over k is a pair $(K, |-|)$ consisting of an aggregate K and a k-linear functor $|-| : K \to k\text{-mod}$. The vector space category is called faithful provided the functor $|-|$ is faithful. The associated subspace category has as objects the triples $X = (X_0, X_\omega, \gamma_X)$ where $X_0 \in K$, $X_\omega \in k\text{-mod}$ and $\gamma_X \in \mathrm{Hom}_K(X_\omega, |X_0|)$. Morphisms are just the pairs (f_0, f_ω) where f_0 is a morphism in K and f_ω is a k-linear map satisfying the obvious commutativity conditions. The subspace category is again an aggregate. For basic results about this category and in particular for the proper definition of tameness for a vector space category we refer to [Si].

3.2. An aggregate K is frequently decribed by the quiver Δ of its spectroid $\mathrm{ind}\, K$. One obtains a display functor $k[\Delta] \to \mathrm{ind}\, K$ whose kernel will be an admissible ideal I of $k[\Delta]$. If we dispose of a full display functor then we even obtain $k[\Delta]/I \cong \mathrm{ind}\, K$. Conversely it is possible to define an aggregate by setting its spectroid as $k[\Delta]/I$.

In the the study of a vector space category attached to $\tilde{\Gamma}_{m,A}$ in the next section we will encounter the following quiver Δ.



In [Dr2] vector space categories whose quivers resemble the patterns from [Ri2] were sudied. For this reason the concept of a pattern-like quiver (Γ, T) was introduced. In general Γ is a quiver with particular properties and T is a subset of the vertex set of Γ satisfying certain axioms. We only will need the following pattern-like quiver Γ which will turn out to be closely related to Δ. Note that in this example $T = T_\Gamma$ is just the set of vertices having at least 5 direct neighbours in Γ.



Let us denote by R_Γ the set of all relations $\sum_{j=1}^3 \alpha_j^i \beta_j^i$, $\sum_{j=1}^3 \gamma_j^i \delta_j^i$ and $\beta_j^i \gamma_j^i$. In [Dr2, 2.3] an admissible ideal J_Γ of $k[\Gamma]$ and a faithful functor $|-|_\Gamma : k[\Gamma]/J_\Gamma \to k\text{-mod}$ were constructed such that $R_\Gamma \subseteq J_\Gamma$ and furthermore $|x|$ is 2-dimensional for $x \in T_\Gamma$ and 1-dimensional for all other vertices.

It is shown in [Dr3, 3.5] that this construction is unique in the following sense: If J is another admissible ideal of $k[\Gamma]$ containing R_Γ, $|-|: k[\Gamma]/J \to k\text{-mod}$ is a faithful functor taking values 2 on all elements of T whereas 1 on all other vertices and finally the images of the k-linear maps $|\bar{\beta}^i_j|$ are pairwise different subspaces for $j = 1, 2, 3$, then up to an automorphism of $k[\Gamma]$ which fixes the vertices the ideal J coincides with J_Γ and up to a natural isomorphism the functor $|-|$ coincides with $|-|_\Gamma$.

3.3. We want to transfer the construction of J_Γ and $|-|_\Gamma$ to Δ. For this purpose we define a functor $\Phi : k[\Delta] \to k[\Gamma]$ by sending ε^i_j to γ^i_j, φ^i_j to $-\gamma^i_3\alpha^{i-1}_j$, μ^i_j to $\delta^{i+1}_j\beta^i_3$ and ν^i_j to β^i_j for $j = 1, 2$. It is easy to see that this functor happens to be fully faithful.

By R_Δ we denote the set of relations in $k[\Delta]$ given by $\varepsilon^i_1\mu^{i-1}_1 + \varepsilon^i_2\mu^{i-1}_2 + \varphi^i_1\nu^{i-1}_1 + \varphi^i_2\nu^{i-1}_2$, $\mu^i_j\varphi^i_j$ and $\nu^i_j\varepsilon^i_j$. By T_Δ we denote the set of all vertices of Δ which have at least 6 direct neighbours in Δ. Defining as J_Δ the preimage of I_Γ under Φ we obtain an admissible ideal of $k[\Delta]$ which contains R_Δ. Let us denote the induced faithful functor $k[\Delta]/J_\Delta \to k[\Gamma]/J_\Gamma$ by $\bar{\Phi}$. The composition $|-|_\Delta := |-|_\Gamma\bar{\Phi}$ will be a faithful functor again which maps all vertices in T_Δ to 2-dimensional and all other vertices to 1-dimensional spaces. Moreover, the construction of $|-|_\Gamma$ shows that $\operatorname{Im}|\bar{\mu}^i_1|_\Gamma = \operatorname{Im}|\bar{\mu}^i_2|_\Gamma$, $\operatorname{Ker}|\bar{\varphi}^i_1|_\Gamma = \operatorname{Ker}|\bar{\varphi}^i_2|_\Gamma$ and $\operatorname{Im}|\bar{\mu}^i_1|_\Gamma$, $\operatorname{Im}|\bar{\nu}^i_2|_\Gamma$, $\operatorname{Im}|\bar{\nu}^i_1|_\Gamma$ are three pairwise different subspaces for all $i \in \mathbb{N}_0$.

Lemma. *Let I be an admissible ideal of $k[\Delta]$ containing R_Δ and let $|-| : k[\Delta] \to$ k-mod be a faithful functor which maps all vertices in T_Δ to 2-dimensional and all other vertices to 1-dimensional spaces. Moreover, suppose that $\operatorname{Im}|\bar{\mu}^i_1| = \operatorname{Im}|\bar{\mu}^i_2|$, $\operatorname{Ker}|\bar{\varphi}^i_1| = \operatorname{Ker}|\bar{\varphi}^i_2|$ and $\operatorname{Im}|\bar{\mu}^i_1|$, $\operatorname{Im}|\bar{\nu}^i_1|$, $\operatorname{Im}|\bar{\nu}^i_2|$ are three pairwise different subspaces for all $i \in \mathbb{N}_0$. Then up to an automorphism of $k[\Delta]$ fixing the vertices I coincides with J_Δ and up to a natural isomorphism $|-|$ coincides with $|-|_\Delta$.*

Proof: We consider the ideal J of $k[\Gamma]$ generated by $\Phi(I)$ and consider the induced faithful functor $\Psi : k[\Delta]/I \to k[\Gamma]/J$. One checks that $R_\Gamma \subseteq J$ and constructs in the obvious way a faithful functor $|-|' : k[\Delta] \to k$-mod satisfying $|-|'\Psi = |-|$. Observing that $|-|'$ has the properties needed to apply the above cited uniquenss result from [Dr3], we obtain that up to trivial modifications $J = J_\Gamma$ and $|-|' = |-|_\Gamma$. But this shows that $I = J_\Delta$ and $|-| = |-|_\Delta$.

3.4. The category $k[\Delta]/I_\Delta$ above has an obvious fully faithful endofunctor given by shifting i to $i + 2$. In the next section we will have to apply the following lemma to the restriction of the inverse τ of this shift functor to a certain subcategory M.

Lemma. *Let $(K, |-|)$ is be faithful vector space category and L, M, N be full additive subcategories of K such that $L \cap M = L \cap N = 0$. Suppose that there exists an isomorphism $\tau = (\tau_0, \tau_\omega) : (M, |-|) \to (N, |-|)$ and moreover there exist objects $x_1, \ldots, x_n$ in a spectroid of M and non-zero maps $u_i : x_i \to y_i := \tau_0 x_i$ such that the following properties are satisfied:*

(a) $\dim_k|x_i| = 1$ *and* $|u_i| = (\tau_\omega)_{x_i}$ *for all* $i = 1, \ldots, n$.
(b) $K(M, L) = K(N, L) = 0$.
(c) $K(z, x) = \sum_{i=1}^n K(z, x_i)K(x_i, x)$ *for all* z *in a spectroid of* L *and* x *in a spectroid of* M.
(d) $K(z, y) = \sum_{i=1}^n K(z, y_i)K(y_i, y)$ *for all* z *in a spectroid of* L *and* y *in a spectroid of* N.
(e) $K(z, \oplus_{i=1}^n y_i) = K(z, \oplus_{i=1}^n x_i)u$ *for all* z *in a spectroid of* L *where* $u : \oplus_{i=1}^n x_i \to \oplus_{i=1}^n y_i$ *is the map whose components are* $u_{ii} = u_i$ *and* $u_{ij} = 0$ *for* $i \neq j$.
If all these conditions are satisfied, then $(L \vee M, |-|) \cong (L \vee N, |-|)$.

Proof: We only sketch the proof and skip the technical details. We have to define an isomorphism $\varphi = (\varphi_0, \varphi_\omega) : (L \vee M, |-|) \to (L \vee N, |-|)$ of vector space categories.

The equivalence $\varphi_0 : L \vee M \to L \vee N$ is defined to send indecomposables z in L to themselves and indecomposables x in M to $\tau_0(x)$. Morphisms inside L resp. M are mapped in the obvious way. Thus it only remains to define φ_0 on maps $f : z \to x$ where z is an indecomposable object in L and x is an indecomposable object in M. Using condition (c) we write f as $f = (f_1, \ldots, f_n)(g_1, \ldots, g_n)^T$ where $f_i \in K(z, x_i)$ and $g_i \in K(x_i, x)$. We define $\varphi_0(f) := f = (f_1, \ldots, f_n)u(\tau_0(g_1), \ldots, \tau_0(g_n))^T$.

In additon the natural isomorphism $\varphi_\omega : |-| \to |\varphi_0(-)|$ is given by $(\varphi_\omega)z := \mathrm{id}_z$ for all indecomposables z in L and $(\varphi_\omega)x := (\tau_\omega)_x$ for all indecomposables x in M.

4 $\Gamma_{m,A}$ is tame

4.1. To prove the tameness of $\tilde{\Gamma}_{m,A}$ we will consider this algebra as a generalized 1-point extension, a concept which was introduced in [Dr3]. As background we also need some results on fiber sum functors and tameness which are taken from [Dr1].

Using the notation for full subcategories introduced in section 2, we put

$$\tilde{\Gamma}^n_{m,A} := \tilde{\Gamma}_{m,A}\{a_0, b_0, \ldots, a_n, b_n\}.$$

To prove the tameness of $\tilde{\Gamma}_{m,A}$ it suffices to prove the tameness of $\tilde{\Gamma}^n_{m,A}$ for all $n \in \mathbb{N}$ which will be achieved by induction on n. It is clear that $\tilde{\Gamma}^2_{m,A}$ is still tame since it is a tubular algebra in the sense of [Ri2]. This means that we still have to show $\tilde{\Gamma}^{n+1}_{m,A}$ is tame provided $\tilde{\Gamma}^n_{m,A}$ is tame and $n \geq 3$.

If not mentioned otherwise the Hom and Ext spaces considered in this section are taken with respect to $\tilde{\Gamma}^{n+1}_{m,A}$. Let us denote by P the indecomposable projective $\tilde{\Gamma}^{n+1}_{m,A}$-module induced by the vertex b_n. Let K be the full subcategory supported by all $V \in \tilde{\Gamma}^{n+1}_{m,A}$-mod satisfying $\mathrm{Ext}^1(V, \mathrm{fac}\, P) = 0$. We are interested in the vector space category $(K_{red}, \mathrm{Hom}(P, -))$ where $K_{red} := K/\mathrm{Ker}\,\mathrm{Hom}(P, -)$. Hence by construction this vector space category is faithful. By $\mathrm{ind}\, K_{red}$ we mean a spectroid of the full subcategory of K supported by the indecomposable objects V such that $\mathrm{Hom}(P, V) \neq 0$. Of course this gives also rise to a spectroid of K_{red}.

Using the fiber sum functor with respect to P it was shown in [Dr1, 3.3] that $\tilde{\Gamma}^{n+1}_{m,A}$ is tame if and only if $\tilde{\Gamma}^{n+1}_{m,A}/P$ is tame, the vector space category $(K_{red}, \mathrm{Hom}(P, -))$ is tame and moreover for any natural number n_0 there are only finitly many objects V in ind K_{red} satisfying $\dim_k V \leq n_0$. Here we denote by $\tilde{\Gamma}^{n+1}_{m,A}/P$ the factor algebra of $\tilde{\Gamma}^{n+1}_{m,A}$ by the trace ideal attached to P.

4.2. To verfiy the above properties we will convince ourselves that $\tilde{\Gamma}^{n+1}_{m,A}$ is a generalized 1-point extension with respect to the vertex $s := b_n$. In order to explain this notion we consider the categories $\tilde{\Gamma}^s := \tilde{\Gamma}_{m,A}\{a_0, b_0, \ldots, a_{n-1}, b_{n-1}, a_n\}$ and $\tilde{\Gamma}_s := \tilde{\Gamma}_{m,A}\{a_n, a_{n+1}, b_{n+1}\}$ which may be considered as factor algebras of $\tilde{\Gamma}^{n+1}_{m,A}/P$. Hence $\tilde{\Gamma}^{n+1}_{m,A}/P$-mod contains $\tilde{\Gamma}_s$-mod and $\tilde{\Gamma}^s$-mod. In fact in our case $\tilde{\Gamma}^{n+1}_{m,A}/P$-mod is just the union of these two subcategories which means by definition (see [Dr3, 3]) that $\tilde{\Gamma}^{n+1}_{m,A}$ is a generalized 1-point extension with respect to s.

Note that as a factor algebra of $\tilde{\Gamma}^{n}_{m,A}$ the algebra $\tilde{\Gamma}^s$ is tame and obviously $\tilde{\Gamma}_s$ is of finite representation type. Thus we also obtain that $\tilde{\Gamma}^{n+1}_{m,A}/P$ is tame which is already the first property to be checked.

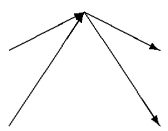
4.3. We call R^+ the radical of the module P and R^- the radical of the indecomposable projective $(\tilde{\Gamma}^{n+1}_{m,A})^{op}$-module induced by the vertex $s = b_n$. Note that R^+ is a $\tilde{\Gamma}^s$-module and R^- is a $\tilde{\Gamma}^{op}_s$-module. We will be interested in the vector space categories $(\tilde{\Gamma}^s\text{-mod}, \mathrm{Hom}_{\tilde{\Gamma}^s}(R^+, -))$ and $(\tilde{\Gamma}_s\text{-mod}, R^- \otimes_{\tilde{\Gamma}_s} -)$ respectively their faithful versions.

It is shown in [Dr3, 3.3] that ind K_{red} is the disjoint union of $\{P\}$, ind K^+_{red} and ind K^-_{red} where $(K^+_{red}, \mathrm{Hom}(P, -)) \cong (\tilde{\Gamma}^s\text{-mod}_{red}, \mathrm{Hom}_{\tilde{\Gamma}^s}(R^+, -))$ and $(K^-_{red}, \mathrm{Hom}(P, -)) \cong (\tilde{\Gamma}_s\text{-mod}_{red}, R^- \otimes_{\tilde{\Gamma}_s} -)$. Moreover there do not exist any non-zero maps from K^-_{red} to K^+_{red}.

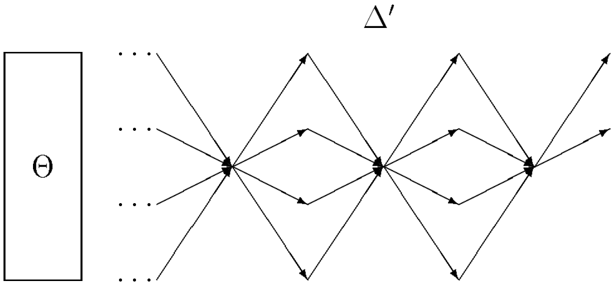
We point out that the preceding isomorphisms of vector space categories will also enable us to evaluate the funktion $\dim_k \mathrm{Hom}(P, -)$ on the indecomposable objects of K_{red}.

4.4. Since P is a maximal object in K_{red} for proving the tamenss of $(K_{red}, \mathrm{Hom}(P, -))$ we only need to prove the tameness of $(K'_{red}, \mathrm{Hom}(P, -))$ where $K' := K^+ \vee K^-$. Let us first calculate the quiver of K'_{red} which has to be the union of two convex subquivers which are just the quivers of $\tilde{\Gamma}^s$-mod$_{red}$ and $\tilde{\Gamma}_s$-mod$_{red}$. This means that we have to calculate the full subquivers of the Auslander-Reiten quivers of the algebras $\tilde{\Gamma}^s$ resp. $\tilde{\Gamma}_s$ supported by the indecomposable modules on which the functors $\mathrm{Hom}_{\tilde{\Gamma}^s}(R^+, -)$ resp. and $R^- \otimes_{\tilde{\Gamma}_s} -$ do not vanish. In principle we might have to cancel some arrows as well but using [Dr3, 4.5] in our situation this turns out not to be necessary. In fact for the representation finite algebra $\tilde{\Gamma}_s$ the resulting quiver Δ'' easily turns out to be:

Δ''

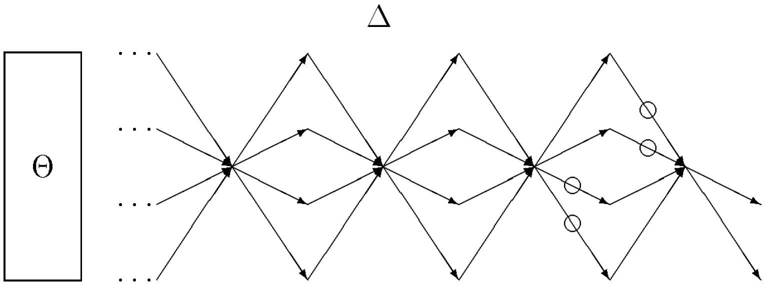


The Auslander-Reiten quiver of the algebra $\tilde{\Gamma}^s$ has a preinjective component which gives rise to a component Δ' of the quiver of $\tilde{\Gamma}^s$-mod$_{red}$ which altogether has the shape



where Θ comprises all components different from Δ'.

[Dr3, 4.4] gives the rule how we have to draw arrows beteen the two quivers constructed above to obtain the quiver of K'_{red}. it turns out that we only have to insert arrows from Δ' to Δ'' yielding a new component Δ whereas the components in Θ remain unchanged. Altogether we arrive at the following picture where we distinguish the inserted arrows by circles.



4.5. We encounter the quiver Δ considered in section 3. Moreover, since Δ' comes from a preinjective component, the relations in R_Δ are obviously satisfied inside this part of Δ. To check that these relations are also satisfied at the right end, we have to calculate the modules there explicitely which is an easy exercise. As mentioned above, we can calculate the value of $\dim_k \mathrm{Hom}(P,-)$ on the vertices of Δ and obtain the value 2 on all nodes and 1 else.

The algebra $\tilde{\Gamma}^n_{m,A}$ is the 1-point extension of the algebra $\tilde{\Gamma}^s$ by the module R^+. This implies that the vector space category $(K^+_{red}, \mathrm{Hom}(P,-)) \cong (\tilde{\Gamma}^s\text{-mod}_{red}, \mathrm{Hom}_{\tilde{\Gamma}^s}(R^+,-))$ has to be tame as well. We will see that this shows the tameness of $(K'_{red}, \mathrm{Hom}(P,-))$. In fact, using lemma 3.4 we will prove that any vector space category $(W, \mathrm{Hom}(P,-))$ where W is a subaggregate of K'_{red} with finite spectroid is isomorphic to some $(W', \mathrm{Hom}(P,-))$ where W' happens to be a subaggregate of K^+_{red}.

More precisely, we intersect W with the subaggregate D of K'_{red} supported by the vertices of Δ to obtain the category N appearing in lemma 3.4 and put L as the complement of N in W. Observe that any display functor of D is full since by [Dr3, 4.3] the category K'_{red} has almost split maps and the quiver Δ is interval finite. Hence, by lemma 3.3 we know D precisely. The category M will be the image of N under the canonical shift functor of D mentioned at the beginning of 3.4 and τ will be the inverse of this shift. In order to apply lemma 3.4 we possibly have to increase N by finitely many indecompsables from D in order to dispose of the needed objects $x_1, \ldots, x_n$.

Finally we wish to prove the validity of the required property concerning the dimensions of the indecomposables in K_{red}. But the tame algebra $\tilde{\Gamma}^n_{m,A}$ is a 1-point extension and hence also a generalized 1-point extension with respect to the vertex b_n. Thus we can use the converse implication of our tameness criterion in [Dr1, 3.3] to establish that the corresponding K^+_{red} has only finitely many indecomposable objects whose dimensions are bounded by a given integer n_0. On the other hand by [Dr3, 2.4] the categories K^+ do not differ if considered with respect to $\tilde{\Gamma}^n_{m,A}$ or with respect to $\tilde{\Gamma}^{n+1}_{m,A}$. Since K_{red} differs from K^+_{red} only by finitely many indecomposable objects, this finishes the proof.

4.6. **Remark.** The factor algebra $\tilde{\Gamma}_{m,A}/\langle \sigma_i\sigma_{i+1}, \sigma_i\nu_{i+1}, \rho_i\gamma_{i+1} : i \in \mathbb{Z}\rangle$ is special biserial. Hence the infinite word $\ldots\rho_i\nu_i^{-1}\sigma_i\gamma_i^{-1}\rho_i\rho_{i+1}\nu_{i+1}^{-1}\sigma_{i+1}\ldots$ gives rise to indecomposable $\tilde{\Gamma}_{m,A}$-modules with arbitrarily large support. Thus $\tilde{\Gamma}_{m,A}$ is not locally support finite and we cannot invoke the classical results to push tameness down to $\Gamma_{m,A}$ (see e.g. [Pe]) but have to invoke more general results announced in [Dd].

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