Functors on locally finitely presented additive categories

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An additive category with direct limits is said to be locally finitely presented provided that the full subcategory of finitely presented objects is skeletally small and every object is a direct limit of finitely presented objects. Our aim in this paper is to study functors which are defined on locally finitely presented categories. Roughly, most of our results are of the following form. Given a category $\mathcal{C}$ and some property $(*)$, there exists a category $\mathcal{D}$ satisfying $(*)$ and a fully faithful functor $d: \mathcal{C} \to \mathcal{D}$ such that any functor $f: \mathcal{C} \to \mathcal{D}'$ into a category satisfying $(*)$ can be extended uniquely to a functor $f^*: \mathcal{D} \to \mathcal{D}'$ which respects $(*)$ and satisfies $f = f^* \circ d$. The following properties $(*)$ for a category $\mathcal{D}$ and combinations thereof are relevant:

(1) $\mathcal{D}$ has cokernels. (2) $\mathcal{D}$ has kernels. (3) $\mathcal{D}$ is abelian. (4) $\mathcal{D}$ has direct limits.

These "universal property" type results are used to study functors between locally finitely presented categories which have products or which are locally coherent, i.e. the full subcategory of finitely presented objects is abelian. For instance, we show that a functor $f: \mathcal{A} \to \mathrm{Ab}$ from a locally finitely presented category with products into the category of abelian groups commutes with direct limits and products if and only if there exists a presentation $\mathrm{Hom}(Y, \ ) \to \mathrm{Hom}(X, \ ) \to f \to 0$ such that X and Y are finitely presented. Also, we develop various criteria for the existence of left and right adjoints. Of particular interest is the following application. A locally finitely presented category is complete if and only if it is cocomplete. Much of this material can be found in the literature. However, the approach presented here appears to be new and will hopefully serve as a foundation for future work.

1. Preliminaries

We introduce some terminology. Throughout we are working in a fixed *universe* $\mathfrak{U}$ containing an infinite set (e.g. see [6, Numéro 0]). All categories $\mathcal{C}$ are assumed to be $\mathfrak{U}$-*categories* in the sense that for each pair of objects $X, Y \in \mathcal{C}$ the set $\mathrm{Hom}(X, Y)$ is *small*, i.e. bijective with a set in $\mathfrak{U}$. A category $\mathcal{C}$ is *skeletally small* provided that the isomorphism classes of objects in $\mathcal{C}$ form a small set. A non-empty category $\mathcal{I}$ is said to be *filtered* provided that for each pair of objects $\lambda_1, \lambda_2 \in \mathcal{I}$ there are morphisms $\varphi_i: \lambda_i \to \mu$ for some $\mu \in \mathcal{I}$, and for each pair of morphisms $\varphi_1, \varphi_2: \lambda \to \mu$ there is a morphism $\psi: \mu \to \nu$ with $\psi \circ \varphi_1 = \psi \circ \varphi_2$. We call the colimit $\varinjlim_{\lambda \in \mathcal{I}} X_\lambda$ of a functor $X: \mathcal{I} \to \mathcal{A}$, $\lambda \mapsto X_\lambda$ a *direct limit* if $\mathcal{I}$ is a skeletally small filtered category.

Given a category $\mathcal{A}$ with direct limits the finitely presented and finitely generated objects of $\mathcal{A}$ play an important role. Recall that an object $X \in \mathcal{A}$ is *finitely presented* (*finitely generated*) provided that for every direct limit $\varinjlim Y_\lambda$ in $\mathcal{A}$ the natural morphism $\varinjlim \operatorname{Hom}(X, Y_\lambda) \to \operatorname{Hom}(X, \varinjlim Y_\lambda)$ is an isomorphism (a monomorphism). The full subcategory of finitely presented objects of $\mathcal{A}$ is denoted by $\operatorname{fp}(\mathcal{A})$. Following [2], an additive category $\mathcal{A}$ is called *locally finitely presented* if $\operatorname{fp}(\mathcal{A})$ is skeletally small and every object in $\mathcal{A}$ is a direct limit of objects in $\operatorname{fp}(\mathcal{A})$.

Recall that an abelian category $\mathcal{A}$ is a *Grothendieck category* provided that $\mathcal{A}$ has coproducts, exact direct limits and a small generating set of objects. An abelian category $\mathcal{A}$ is locally finitely presented if and only if it is a Grothendieck category with a generating set of finitely presented objects [1, Satz 1.5], [2, 2.4]. Now suppose that $\mathcal{A}$ is locally finitely presented and abelian. An object in $\mathcal{A}$ is finitely generated if and only if it is a quotient of some finitely presented object. The category $\mathcal{A}$ is said to be *locally coherent* provided that finitely generated subobjects of finitely presented objects are finitely presented, equivalently if $\operatorname{fp}(\mathcal{A})$ is abelian [8, Proposition 2.2].

Throughout the paper all functors between pre-additive categories are assumed to be additive. Given two pre-additive categories $\mathcal{C}$ and $\mathcal{D}$, the class of functors $F: \mathcal{C} \to \mathcal{D}$ is denoted by $(\mathcal{C}, \mathcal{D})$ and $\operatorname{Hom}(F, G)$ denotes the class of natural transformations between two functors F and G in $(\mathcal{C}, \mathcal{D})$. If $\mathcal{C}$ is skeletally small, then $(\mathcal{C}, \mathcal{D})$ forms actually a category since the Hom sets are small. The category of abelian groups is denoted by Ab. Given any category $\mathcal{C}$, one defines limits, colimits etc. in $(\mathcal{C}, \mathrm{Ab})$ pointwise and they coincide with the categorical notions if $(\mathcal{C}, \mathrm{Ab})$ is a category.

2. Finitely presented functors and abelian categories

In this section we collect some elementary facts about categories which have cokernels, kernels or which are abelian. Let $\mathcal{C}$ be a pre-additive category. A functor $F: \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is said to be *finitely presented* provided that there exists an exact sequence $\coprod_{i=1}^{n} \operatorname{Hom}(\ ,X_i) \to \coprod_{j=1}^{m} \operatorname{Hom}(\ ,Y_j) \to F \to 0$ of functors in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ such that n and m are finite. Note that the presentation of F could be replaced by $\operatorname{Hom}(\ ,X) \to \operatorname{Hom}(\ ,Y) \to F \to 0$ if $\mathcal{C}$ is additive. Also F is said to be *finitely generated* provided that there exists an exact sequence $\coprod_{i=1}^{n} \operatorname{Hom}(\ ,X_i) \to F \to 0$ in $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ such that n is finite. The finitely presented functors form an additive category with cokernels which we denote by $\operatorname{mod}(\mathcal{C})$. Recall that Yoneda's lemma gives a fully faithful functor $h_{\mathcal{C}}: \mathcal{C} \to \operatorname{mod}(\mathcal{C})$, $X \mapsto \operatorname{Hom}(\ ,X)$. We shall frequently use the category $\operatorname{mod}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ which we denote by $\operatorname{mop}(\mathcal{C})$. In this case Yoneda's lemma gives a fully faithful functor $\mathcal{C} \to \operatorname{mop}(\mathcal{C})$, $X \mapsto \operatorname{Hom}(X,\ )$.

Universal property 2.1 *Let $f: \mathcal{C} \to \mathcal{A}$ be a functor into an additive category with cokernels. Then there exists, up to isomorphism, a unique right exact functor $f^*: \operatorname{mod}(\mathcal{C}) \to \mathcal{A}$ such that $f = f^* \circ h_{\mathcal{C}}$.*

Proof: For any functor $F \in \operatorname{mod}(\mathcal{C})$ choose a presentation

$$\coprod_{i=1}^{n} \operatorname{Hom}(\ ,X_i) \xrightarrow{(\operatorname{Hom}(\ ,\varphi_{ij}))_{i,j}} \coprod_{j=1}^{m} \operatorname{Hom}(\ ,Y_j) \to F \to 0$$

and define $f^*(F)$ to be the cokernel of the morphism

$$\coprod_{i=1}^{n} f(X_i) \xrightarrow{(f(\varphi_{ij}))_{i,j}} \coprod_{j=1}^{m} f(Y_j).$$

Any morphism $\psi: F \to G$ in $\operatorname{mod}(\mathcal{C})$ lifts to a morphism of the presentations, so it induces a unique morphism $f^*(\psi): f^*(F) \to f^*(G)$. It is easily checked that this is well-defined and that $f^*: \operatorname{mod}(\mathcal{C}) \to \mathcal{A}$ is the unique right exact functor extending f.

The preceding result allows to define a *tensor product* $\operatorname{mod}(\mathcal{C}) \times \operatorname{mod}(\mathcal{C}^{\mathrm{op}}) \to \mathrm{Ab}$, $(F,G) \mapsto F \otimes_{\mathcal{C}} G$ which is characterized, up to isomorphism, by the following properties.

(1) There are functorial isomorphism $F \otimes_{\mathcal{C}} \operatorname{Hom}(X,\ ) \cong F(X)$ and $\operatorname{Hom}(\ ,X) \otimes_{\mathcal{C}} G \cong G(X)$ for $X \in \mathcal{C}$.

(2) $F \otimes_{\mathcal{C}} -$ and $- \otimes_{\mathcal{C}} G$ are right exact.

Recall that given a morphism $\psi: Y \to Z$ in $\mathcal{C}$, then a morphism $\varphi: X \to Y$ is a *pseudo-kernel* for ψ provided that the induced sequence of functors $\operatorname{Hom}(\ ,X) \to \operatorname{Hom}(\ ,Y) \to \operatorname{Hom}(\ ,Z)$ is exact. The following result is well-known and easily proved [3, Theorem 1.4].

Lemma 2.2 *The category* $\mathrm{mod}(\mathcal{C})$ *is abelian iff* $\mathcal{C}$ *has pseudo-kernels.*

Let $\mathcal{A}$ be an abelian category. We denote by $\mathrm{proj}(\mathcal{A})$ the full subcategory of projective objects in $\mathcal{A}$ and we say that $\mathcal{A}$ has *sufficently many projectives* provided that for every object $X \in \mathcal{A}$ there exists an epimorphism $Y \to X$ with $Y \in \mathrm{proj}(\mathcal{A})$. Analogous terminology is used for injective objects in $\mathcal{A}$.

Proposition 2.3 *There is, up to equivalence, a bijective correspondence between (skeletally small) additive categories with split idempotents and pseudo-kernels and (skeletally small) abelian categories with sufficently many projectives. The correspondence is given by*

$$\mathcal{C} \mapsto \mathrm{mod}(\mathcal{C}) \qquad \text{and} \qquad \mathcal{A} \mapsto \mathrm{proj}(\mathcal{A}).$$

Proof: If $\mathcal{C}$ is an additive categories with split idempotents and pseudo-kernels, then $\mathrm{mod}(\mathcal{C})$ is abelian and the representable functors $\mathrm{Hom}(\ ,X)$ are precisely the projective objects in $\mathrm{mod}(\mathcal{C})$ by Yoneda's lemma. Conversely, given an abelian category $\mathcal{A}$ with sufficently many projectives, then $\mathrm{proj}(\mathcal{A})$ has pseudo-kernels and the inclusion $\mathrm{proj}(\mathcal{A}) \to \mathcal{A}$ extends to a functor $\mathrm{mod}(\mathrm{proj}(\mathcal{A})) \to \mathcal{A}$ which is an equivalence.

Lemma 2.4 *Let* $\mathcal{A}$ *be an abelian category with sufficently many projectives and suppose that* $f\colon \mathrm{proj}(\mathcal{A}) \to \mathcal{B}$ *is a functor into a category with cokernels. Then there exists, up to isomorphism, a unique right exact functor* $f^*\colon \mathcal{A} \to \mathcal{B}$ *such that* $f^*|_{\mathrm{proj}(\mathcal{A})} = f$.

Proof: Combine Property 2.1 and Proposition 2.3.

Lemma 2.5 *Let* $f\colon \mathcal{A} \to \mathcal{B}$ *be a right exact functor between abelian categories and suppose that* $\mathcal{A}$ *has sufficently many projectives. Then* f *is exact iff* $f(X) \to f(Y) \to f(Z)$ *is exact for each exact sequence* $X \to Y \to Z$ *of projective objects in* $\mathcal{A}$.

Proof: Taking projective presentations of an arbitrary exact sequence in $\mathcal{A}$ and applying f the assertion is an immediate consequence of the snake lemma.

Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between additive categories. We denote by $f^*\colon \mathrm{mod}(\mathcal{C}) \to \mathrm{mod}(\mathcal{D})$ the unique right exact functor which extends f. We list some properties of f^*.

Lemma 2.6 (1) *f is fully faithful iff f^* is fully faithful.*

(2) *Suppose that $\mathcal{C}$ and $\mathcal{D}$ have pseudo-kernels. Then $f: \mathcal{C} \to \mathcal{D}$ preserves pseudo-kernels iff f^* is exact.*

(3) *Suppose that $\mathcal{C}$ has kernels and that $\mathcal{D}$ has pseudo-cokernels. Then f is left exact iff f preserves pseudo-kernels.*

Proof: (1) Straightforward.

(2) Use Lemma 2.5.

(3) If $\mathcal{C}$ has kernels, then a morphism $\varphi: X \to Y$ is a pseudo-kernel for ψ iff φ decomposes as $[\varphi_1\, \varphi_2]: X_1 \coprod X_2 \to Y$ where φ_1 is a kernel and $\psi \circ \varphi_2 = 0$. Thus a left exact functor preserves pseudo-kernels. Conversely, suppose that f preserves pseudo-kernels. It follows from (2) that $f^*: \operatorname{mod}(\mathcal{C}) \to \operatorname{mod}(\mathcal{D})$ is exact. Therefore $f^* \circ h_{\mathcal{C}} = h_{\mathcal{D}} \circ f$ is left exact and we conclude the left exactness of f.

Lemma 2.7 *Let $f: \mathcal{C} \to \mathcal{D}$ be a fully faithful functor between additive categories and suppose that $\mathcal{D}$ has cokernels. Then the following are equivalent.*

(1) *f admits a left adjoint.*

(2) *f admits a right exact functor $g: \mathcal{D} \to \mathcal{C}$ satisfying $g \circ f = \mathrm{id}_{\mathcal{C}}$.*

In this case g is a left adjoint and $\mathcal{C}$ has cokernels.

Proof: Straightforward.

Lemma 2.8 *The Yoneda functor $h_{\mathcal{C}}: \mathcal{C} \to \operatorname{mod}(\mathcal{C})$ has the following properties.*

(1) *If $\mathcal{C}$ has split idempotents and pseudo-kernels, then $h_{\mathcal{C}}$ preserves pseudo-kernels iff $\mathcal{C}$ has kernels.*

(2) *$h_{\mathcal{C}}$ has a left adjoint iff $\mathcal{C}$ has cokernels. A left adjoint is right exact, and it is exact if $\mathcal{C}$ is abelian.*

Proof: Use the above lemmas.

We include the following well-known criterion for a right adjoint functor to be exact.

Lemma 2.9 *Let $f: \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories and let $g: \mathcal{B} \to \mathcal{A}$ be a left adjoint.*

(1) *If f is exact, then g preserves projectives.*

(2) *If $\mathcal{B}$ has sufficently many projectives and g preserves projectives, then f is exact.*

Let $\mathcal{C}$ be a pre-additive category and define $A(\mathcal{C}) = \mathrm{mod}(\mathrm{mop}(\mathcal{C}))$. Further let

$$a_{\mathcal{C}}: \mathcal{C} \to A(\mathcal{C}), \quad X \mapsto \mathrm{Hom}(\ ,\mathrm{Hom}(X,\ ))$$

be the composition of the corresponding Yoneda functors. The category $A(\mathcal{C})$ has the following property.

Universal property 2.10 *The category $A(\mathcal{C})$ is abelian and given any functor $f: \mathcal{C} \to \mathcal{A}$ into an abelian category, there exists, up to isomorphism, a unique exact functor $f^*: A(\mathcal{C}) \to \mathcal{A}$ such that $f = f^* \circ a_{\mathcal{C}}$.*

Proof: The category $A(\mathcal{C})$ is abelian by Lemma 2.2 since $\mathrm{mop}(\mathcal{C})$ has kernels. A functor $f: \mathcal{C} \to \mathcal{A}$ extends uniquely to a left exact functor $f': \mathrm{mop}(\mathcal{C}) \to \mathcal{A}$ and this extends uniquely to a right exact functor $f^*: A(\mathcal{C}) \to \mathcal{A}$. This follows from Property 2.1 and we deduce from Lemma 2.5 and Lemma 2.6 the exactness of f^* since f' is left exact. Any other exact functor $A(\mathcal{C}) \to \mathcal{A}$ extending f needs to be isomorphic to f^* since the restriction to $\mathrm{mop}(\mathcal{C})$ is left exact and therefore isomorphic to f'.

Corollary 2.11 *There exists, up to isomorphism, a unique duality $d: A(\mathcal{C}) \to A(\mathcal{C}^{\mathrm{op}})$ which extends the canonical duality $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$. This duality $F \mapsto d(F)$ is given by*

$$d(F)(X) = \mathrm{Hom}(F, X \otimes_{\mathcal{C}} -).$$

Proof: The functor d is the unique exact functor which makes the following diagram commutative.

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}^{\mathrm{op}} \\ \downarrow{\scriptstyle a_{\mathcal{C}}} & & \downarrow{\scriptstyle a_{\mathcal{C}^{\mathrm{op}}}} \\ A(\mathcal{C}) & \xrightarrow{d} & A(\mathcal{C}^{\mathrm{op}}) \end{array}$$

Corollary 2.12 *There exists, up to isomorphism, a unique equivalence* $\mathrm{mod}(\mathrm{mop}(\mathcal{C})) \to \mathrm{mop}(\mathrm{mod}(\mathcal{C}))$ *which sends* $\mathrm{Hom}(\ ,\mathrm{Hom}(X,\ ))$ *to* $\mathrm{Hom}(\mathrm{Hom}(\ ,X),\ )$ *for every $X \in \mathcal{C}$.*

Corollary 2.13 *The category $A(\mathcal{C})$ has sufficently many projectives which are the functors* $\mathrm{Hom}(\ ,X)$*, $X \in \mathrm{mod}(\mathcal{C}^{\mathrm{op}})$, and sufficently many injectives which are the functors $X \otimes_{\mathcal{C}} -$, $X \in \mathrm{mod}(\mathcal{C})$. Finally, the projective-injective objects are precisely the direct summands of functors* $\coprod_{i=1}^{n} \mathrm{Hom}(\ ,\mathrm{Hom}(X,\ ))$*, $X_i \in \mathcal{C}$.*

Corollary 2.14 *The functor* $d\colon \mathrm{mod}(\mathcal{C}) \to A(\mathcal{C})$, $F \mapsto F \otimes_{\mathcal{C}} -$ *induces an equivalence between* $\mathrm{mod}(\mathcal{C})$ *and* $\mathrm{inj}(A(\mathcal{C}))$. *Moreover, the Yoneda functor* $h\colon \mathcal{C} \to \mathrm{mop}(\mathcal{C})$ *induces a functor* h^* *which is isomorphic to* d.

Remark 2.15 If $\mathcal{C}$ has precisely one object, in other words that it is a ring, then one recovers a case which has been treated in [7] (see also [3]).

3. Contravariantly finite subcategories

Let $\mathcal{A}$ be an additive category. An additive subcategory $\mathcal{C}$ is said to be *contravariantly finite* provided that every object $X \in \mathcal{A}$ has a *right* $\mathcal{C}$*-approximation* i.e. a morphism $Y \to X$ with $Y \in \mathcal{C}$ such that the induced sequence of functors $\mathrm{Hom}(\ ,Y)|_{\mathcal{C}} \to \mathrm{Hom}(\ ,X)|_{\mathcal{C}} \to 0$ from $\mathcal{C}^{\mathrm{op}}$ to Ab is exact. Of course there is the dual notion of covariant finiteness.

Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between additive categories and denote by $f^*\colon \mathrm{mod}(\mathcal{C}) \to \mathrm{mod}(\mathcal{D})$ the unique right exact functor extending f. Also we have for each functor $F\colon \mathcal{D}^{\mathrm{op}} \to$ Ab the restriction $f_*(F) = F \circ f$. We start with some preliminary results.

Lemma 3.1 *The following are equivalent.*

(1) $f_*(F)$ *is finitely presented for all* $F \in \mathrm{mod}(\mathcal{D})$.

(2) f^* *has a right adjoint.*

Proof: (1) $\Rightarrow$ (2) Take f_* as a right adjoint.

(2) $\Rightarrow$ (1) The following isomorphism shows that a right adjoint g takes the same values as f_*.

$$g(F)(X) \cong \mathrm{Hom}(\mathrm{Hom}(\ ,X), g(F)) \cong \mathrm{Hom}(f^*(\mathrm{Hom}(\ ,X)), F) \cong \mathrm{Hom}(\mathrm{Hom}(\ ,f(X)), F) \cong F(f(X)) = f_*(F)(X)$$

Lemma 3.2 *Suppose that* f *is full. Then the following are equivalent.*

(1) *The image* $\mathrm{Im}(f)$ *of* f *is contravariantly finite in* $\mathcal{D}$.

(2) $f_*(\mathrm{Hom}(\ ,X))$ *is finitely generated for all* $X \in \mathcal{D}$.

Moreover, if $\mathcal{D}$ *has pseudo-kernels and* f *is fully faithful, then* (1) – (2) *are equivalent to the following.*

(3) $f_*(F)$ *is finitely presented for all* $F \in \mathrm{mod}(\mathcal{D})$.

Proof: (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3) This is a direct consequence of the definitions involved.

(2) $\Rightarrow$ (3) It suffices to show that $f_*(\mathrm{Hom}(\ ,X))$ is finitely presented for all $X \in \mathcal{D}$ since f_* is exact and $\mathrm{mod}(\mathcal{C})$ is closed under cokernels. To this end let $f(Y) \to X$ be a left $\mathrm{Im}(f)$-approximation of X and let $f(Y') \to X'$ be a left $\mathrm{Im}(f)$-approximation of a pseudo-kernel X' of $f(Y) \to X$. Choose a morphism $\varphi: Y' \to Y$ such that $f(\varphi)$ equals the composition $f(Y') \to X' \to f(Y)$. It is easily seen that φ induces a presentation $\mathrm{Hom}(\ ,Y') \to \mathrm{Hom}(\ ,Y) \to \mathrm{Hom}(\ ,X)\circ f \to 0$ of $f_*(\mathrm{Hom}(\ ,X)) = \mathrm{Hom}(\ ,X)\circ f$. This finishes the proof.

It is well-known that a subcategory $\mathcal{C}$ of an additive category $\mathcal{A}$ is contravariantly finite if the inclusion functor $i: \mathcal{C} \to \mathcal{A}$ has a right adjoint. Consider the following example.

Example 3.3 Let $\mathcal{C}$ be an additive category and $h: \mathcal{C} \to \mathrm{mop}(\mathcal{C})$ be the Yoneda functor. The image of h is contravariantly finite iff $\mathcal{C}$ has pseudo-kernels. The functor h has a right adjoint iff $\mathcal{C}$ has kernels.

Replacing $i: \mathcal{C} \to \mathcal{A}$ by i^* we obtain the following criterion.

Theorem 3.4 *Let $\mathcal{A}$ be an additive category with pseudo-kernels. If $i: \mathcal{C} \to \mathcal{A}$ denotes the inclusion of a full additive subcategory, then the following are equivalent.*

(1) *$\mathcal{C}$ is contravariantly finite.*

(2) *The right exact functor $i^*: \mathrm{mod}(\mathcal{C}) \to \mathrm{mod}(\mathcal{A})$ which extends i has a right adjoint.*

(3) *Restriction induces a functor $i_*: \mathrm{mod}(\mathcal{A}) \to \mathrm{mod}(\mathcal{C})$.*

In this case the category $\mathcal{C}$ has pseudo-kernels and i_* is exact.

Proof: Combine Lemma 3.1 and Lemma 3.2 to prove the equivalence of (1) – (3). To obtain a pseudo-cokernel for a morphism in $\mathcal{C}$ take a right $\mathcal{C}$-approximation of a pseudo-cokernel in $\mathcal{A}$. The exactness of i_* is clear since kernels and cokernels in $\mathrm{mod}(\mathcal{C})$ and $\mathrm{mod}(\mathcal{A})$ are computed pointwise.

4. Module categories and tensor products

Module categories and tensor products are the most important tools for studying locally finitely presented categories and their functors. Let $\mathcal{C}$ be a skeletally small pre-additive category. We denote by $\mathrm{Mod}(\mathcal{C})$ the category of functors from $\mathcal{C}^{\mathrm{op}}$ to the category Ab of abelian groups. The objects in $\mathrm{Mod}(\mathcal{C})$ are called *$\mathcal{C}$-modules*. It is well-known that $\mathrm{Mod}(\mathcal{C})$ is a locally finitely presented abelian category and that $\mathrm{fp}(\mathrm{Mod}(\mathcal{C}))$ and $\mathrm{mod}(\mathcal{C})$ coincide. We denote by $h_{\mathcal{C}}: \mathcal{C} \to \mathrm{Mod}(\mathcal{C})$ the Yoneda functor $X \mapsto \mathrm{Hom}(\ ,X)$. This functor has the following property.

Universal property 4.1 *Let $f: \mathcal{C} \to \mathcal{A}$ be a functor into an additive category with colimits. Then there exists, up to isomorphism, a unique functor $f^*: \mathrm{Mod}(\mathcal{C}) \to \mathcal{A}$ such that $f = f^* \circ h_{\mathcal{C}}$ and f^* commutes with colimits. Moreover, f^* has a right adjoint.*

Proof: The category $\mathrm{Mod}(\mathcal{C})$ has sufficently many projectives and $\mathrm{proj}(\mathrm{Mod}(\mathcal{C}))$ consists precisely of the direct summands of coproducts $\coprod_{i \in I} \mathrm{Hom}(\ ,X_i)$. First we define $f^*(\coprod_i \mathrm{Hom}(\ ,X_i)) = \coprod_i f^*(X_i)$ and for a direct summand M we define $f^*(M)$ to be the cokernel of $f^*(\mathrm{id} - \varepsilon)$ where $\varepsilon \in \mathrm{End}(\coprod_i \mathrm{Hom}(\ ,X_i))$ is the idempotent corresponding to M. Similarly, we define f^* on morphisms between projectives. Thus we obtain a unique functor $\mathrm{proj}(\mathrm{Mod}(\mathcal{C})) \to \mathcal{A}$ extending f and commuting with colimits. Now Property 2.1 allows to extend this functor uniquely to a right exact functor $f^*: \mathrm{Mod}(\mathcal{C}) \to \mathcal{A}$. The right adjoint $f_*: \mathcal{A} \to \mathrm{Mod}(\mathcal{C})$ is obtained by $f_*(M)(X) = \mathrm{Hom}(f(X), M)$ for $M \in \mathcal{A}$. It is routine to check that there is a functorial isomorphism $\mathrm{Hom}(f^*(N), M) \cong \mathrm{Hom}(N, f_*(M))$ for $N \in \mathrm{Mod}(\mathcal{C})$. The uniqueness of f^* follows from the fact that f^* commutes with colimits.

The preceding result allows to define a *tensor product* $\mathrm{Mod}(\mathcal{C}) \times \mathrm{Mod}(\mathcal{C}^{\mathrm{op}}) \to \mathrm{Ab}$, $(M, N) \mapsto M \otimes_{\mathcal{C}} N$ which extends the tensor product defined on $\mathrm{mod}(\mathcal{C}) \times \mathrm{mod}(\mathcal{C}^{\mathrm{op}})$. It is characterized, up to isomorphism, by the following properties.

(1) There are functorial isomorphism $M \otimes_{\mathcal{C}} \mathrm{Hom}(X,\ ) \cong M(X)$ and $\mathrm{Hom}(\ ,X) \otimes_{\mathcal{C}} N \cong N(X)$ for $X \in \mathcal{C}$.

(2) $M \otimes_{\mathcal{C}} -$ and $- \otimes_{\mathcal{C}} N$ have right adjoints.

Let $f: \mathcal{C} \to \mathcal{D}$ be a functor between skeletally small additive categories. The restriction functor $f_*: \mathrm{Mod}(\mathcal{D}) \to \mathrm{Mod}(\mathcal{C})$, $M \mapsto M \circ f$ has a left adjoint

$$f^*: \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{D}),\ M \mapsto N \quad \text{with} \quad N(X) = M \otimes_{\mathcal{C}} (\mathrm{Hom}(X,\ ) \circ f)$$

and a right adjoint

$$^*f: \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{D}),\ M \mapsto N \quad \text{with} \quad N(X) = \mathrm{Hom}(\mathrm{Hom}(\ ,X) \circ f, M).$$

We discuss some properties of these functors.

Lemma 4.2 (1) *$h_{\mathcal{D}} \circ f = f^* \circ h_{\mathcal{C}}$ where $h_{\mathcal{C}}: \mathcal{C} \to \mathrm{Mod}(\mathcal{C})$ and $h_{\mathcal{D}}: \mathcal{D} \to \mathrm{Mod}(\mathcal{D})$ are given by $X \mapsto \mathrm{Hom}(\ ,X)$.*

(2) *f^* is fully faithful iff f is fully faithful.*

Proof: (1) Clear.

(2) It follows from (1) that f is fully faithful if f^* is fully faithful since $h_{\mathcal{C}}$ and $h_{\mathcal{D}}$ are fully faithful. Conversely, suppose that f is fully faithful. The isomorphism

$$\mathrm{Hom}(\textstyle\coprod_i \mathrm{Hom}(\ ,X_i), \coprod_j \mathrm{Hom}(\ ,Y_j)) \cong \prod_i \coprod_j \mathrm{Hom}(X_i, Y_j)$$

shows that f^* induces a fully faithful functor $\mathrm{proj}(\mathrm{Mod}(\mathcal{C})) \to \mathrm{proj}(\mathrm{Mod}(\mathcal{D}))$ since $f^*(\mathrm{Hom}(\ ,X)) = \mathrm{Hom}(\ ,f(X))$ for every $X \in \mathcal{C}$ and f^* commutes with coproducts. It follows from Lemma 2.6 that the induced right exact functor $\mathrm{mod}(\mathrm{proj}(\mathrm{Mod}(\mathcal{C}))) \to \mathrm{mod}(\mathrm{proj}(\mathrm{Mod}(\mathcal{D})))$ is fully faithful. But this functor is isomorphic to f^*.

5. Locally finitely presented categories

Let $\mathcal{C}$ be a skeletally small pre-additive category. A $\mathcal{C}$-module M is said to be *flat* provided that $M \otimes_{\mathcal{C}} -$ is exact. The full subcategory of $\mathrm{Mod}(\mathcal{C})$ formed by the flat modules is denoted by $\mathrm{Flat}(\mathcal{C})$. It is well-known that this is a locally finitely presented category [2, Theorem 1.4]. In fact the category $\mathrm{Flat}(\mathcal{C})$ is closed under direct limits in $\mathrm{Mod}(\mathcal{C})$, so has direct limits itself and a $\mathcal{C}$-module M is flat if and only if it is a direct limit of modules of the form $\mathrm{Hom}(\ ,X)$ with $X \in \mathcal{C}$.

The following description of an arbitrary locally finitely presented category is due to Crawley-Boevey [2, Theorem 1.4] and generalizes results for locally noetherian categories [4, II.4, Théorème 1], locally coherent categories [8, Proposition 2.2], locally finitely presented abelian categories [2, Satz 2.4] and locally finitely presented categories in the sense of Gabriel and Ulmer [5, Korollar 7.9]; see also [6, Numéro 8].

Proposition 5.1 *If $\mathcal{A}$ is a locally finitely presented category, then the functor $\mathcal{A} \to \mathrm{Flat}(\mathrm{fp}(\mathcal{A}))$ given by $X \mapsto \mathrm{Hom}(\ ,X)|_{\mathrm{fp}(\mathcal{A})}$ is an equivalence. There is, up to equivalence, a bijective correspondence between skeletally small additive categories with split idempotents and locally* *finitely presented categories. The correspondence is given by*

$$\mathcal{C} \mapsto \mathrm{Flat}(\mathcal{C}) \qquad \textit{and} \qquad \mathcal{A} \mapsto \mathrm{fp}(\mathcal{A}).$$

This result has some obvious consequences which we now discuss. Let $\mathcal{A}$ be a locally finitely presented category. Denote by $h_{\mathcal{A}}: \mathcal{A} \to \mathrm{Mod}(\mathrm{fp}(\mathcal{A}))$ the fully faithful functor given by $X \mapsto \mathrm{Hom}(\ ,X)|_{\mathrm{fp}(\mathcal{A})}$.

Universal property 5.2 *Let $f: \mathcal{A} \to \mathcal{B}$ be a functor into an additive category with cokernels and direct limits. Suppose also that f commutes with direct limits. Then there exists, up to isomorphism, a unique right exact functor $f^*: \mathrm{Mod}(\mathrm{fp}(\mathcal{A})) \to \mathcal{B}$ such that $f = f^* \circ h_{\mathcal{A}}$.*

Proof: The assumption on $\mathcal{B}$ means that $\mathcal{B}$ has colimits. Thus $f|_{\mathrm{fp}(\mathcal{A})}$ extends uniquely to a functor $f^*: \mathrm{Mod}(\mathrm{fp}(\mathcal{A})) \to \mathcal{B}$ which commutes with colimits by Property 4.1. Using the identification $\mathcal{A} = \mathrm{Flat}(\mathrm{fp}(\mathcal{A}))$ the assertion follows since f commutes with direct limits.

Following Roos we call $\mathcal{A}^{\mathrm{co}} = \mathrm{Flat}(\mathrm{fp}(\mathcal{A})^{\mathrm{op}})$ the *conjugate category* of $\mathcal{A}$. The contravariant functor $\mathrm{fp}(\mathcal{A}) \to \mathcal{A}^{\mathrm{co}}$, $X \mapsto \mathrm{Hom}(X,\ )|_{\mathrm{fp}(\mathcal{A})}$ induces a duality between $\mathrm{fp}(\mathcal{A})$ and $\mathrm{fp}(\mathcal{A}^{\mathrm{co}})$ which we denote by d.

Proposition 5.3 *Let $\mathcal{A}$ be a locally finitely presented category. There exists a tensor product functor $\mathcal{A} \times \mathcal{A}^{\mathrm{co}} \to \mathrm{Ab}$, $(M,N) \mapsto M \otimes_{\mathrm{fp}(\mathcal{A})} N$ which is characterized, up to isomorphism, by the following properties.*

(1) *There are functorial isomorphism $M \otimes_{\mathrm{fp}(\mathcal{A})} Y \cong \mathrm{Hom}(d(Y), M)$ for $Y \in \mathrm{fp}(\mathcal{A}^{\mathrm{co}})$ and $X \otimes_{\mathrm{fp}(\mathcal{A})} N \cong \mathrm{Hom}(d(X), N)$ for $X \in \mathrm{fp}(\mathcal{A})$.*

(2) *$M \otimes_{\mathrm{fp}(\mathcal{A})} -$ and $- \otimes_{\mathrm{fp}(\mathcal{A})} N$ commute with direct limits.*

Proof: Using the identification $\mathcal{A} = \mathrm{Flat}(\mathrm{fp}(\mathcal{A}))$ and $\mathcal{A}^{\mathrm{co}} = \mathrm{Flat}(\mathrm{fp}(\mathcal{A})^{\mathrm{op}})$ the assertion follows if one restricts the tensor product which is defined on $\mathrm{Mod}(\mathrm{fp}(\mathcal{A})) \times \mathrm{Mod}(\mathrm{fp}(\mathcal{A})^{\mathrm{op}})$.

The next few results show how certain properties of a locally finitely presented category $\mathcal{A}$ are reflected by properties of the category $\mathcal{C} = \mathrm{fp}(\mathcal{A})$. We shall need a canonical presentation of any object X and any morphism $\varphi: X \to Y$ in $\mathcal{A}$ as a direct limit of objects and morphisms, respectively, in $\mathcal{C}$. To this end define categories $\mathcal{C}/X$ and $\mathcal{C}/\varphi$ as follows. An object $\lambda \in \mathcal{C}/X$ is given by a morphism $\alpha_\lambda: X_\lambda \to X$ with $X_\lambda \in \mathcal{C}$ and a morphism $\theta: \lambda \to \mu$ is given by a morphism $\alpha_\theta : X_\lambda \to X_\mu$ satisfying $\alpha_\lambda = \alpha_\mu \circ \alpha_\theta$. An object $\lambda \in \mathcal{C}/\varphi$ is given by a commuting diagram of the form

$$\begin{array}{ccc} X_\lambda & \xrightarrow{\varphi_\lambda} & Y_\lambda \\ \Big\downarrow{\scriptstyle \alpha_\lambda} & & \Big\downarrow{\scriptstyle \beta_\lambda} \\ X & \xrightarrow{\varphi} & Y \end{array}$$

with $X_\lambda, Y_\lambda \in \mathcal{C}$ and a morphism $\theta : \lambda \to \mu$ is given by a commuting diagram of the form

$$\begin{array}{ccc} X_\lambda & \xrightarrow{\varphi_\lambda} & Y_\lambda \\ \Big\downarrow{\scriptstyle \alpha_\theta} & & \Big\downarrow{\scriptstyle \beta_\theta} \\ X_\mu & \xrightarrow{\varphi_\mu} & Y_\mu \end{array}$$

with $\alpha_\lambda = \alpha_\mu \circ \alpha_\theta$ and $\beta_\lambda = \beta_\mu \circ \beta_\theta$.

Lemma 5.4 (1) *The category* $\mathcal{C}/X$ *is filtered and* $\varinjlim_{\lambda \in \mathcal{C}/X} X_\lambda = X$.

(2) *The category* $\mathcal{C}/\varphi$ *is filtered and* $\varinjlim_{\lambda \in \mathcal{C}/\varphi} \varphi_\lambda = \varphi$.

(3) φ *induces a natural functor* $\mathcal{C}/X \to \mathcal{C}/Y$.

Proof: Straightforward.

Lemma 5.5 *Let* $F_\lambda : \mathcal{I}_\lambda \to \mathcal{I}$, $\lambda \in \mathcal{J}$, *be a filtered family of functors between filtered categories and suppose that the family is cofinal, i.e.*

(i) *for any* $\nu \in \mathcal{I}$ *there is an object* $\mu \in \mathcal{I}_\lambda$ *for some* $\lambda \in \mathcal{J}$ *and a morphism* $\nu \to F_\lambda(\mu)$*;*

(ii) *for any pair of morphisms* $\varphi_1, \varphi_2 : \nu \to F_\lambda(\mu)$ *there is a morphism* $\psi : \mu \to \chi$ *such that* $F_\lambda(\psi) \circ \varphi_1 = F_\lambda(\psi) \circ \varphi_2$.

Then for any functor X *from* $\mathcal{I}$ *to a category with direct limits, the natural morphism*

$$\varphi : \varinjlim_{\lambda \in \mathcal{J}} (\varinjlim_{\mu \in \mathcal{I}_\lambda} X_{F_\lambda(\mu)}) \longrightarrow \varinjlim_{\nu \in \mathcal{I}} X_\nu$$

is an isomorphism.

Proof: Using (i) – (ii) one constructs for any object Y and any morphism

$$\sigma : \varinjlim_{\lambda \in \mathcal{J}} (\varinjlim_{\mu \in \mathcal{I}_\lambda} X_{F_\lambda(\mu)}) \longrightarrow Y$$

a unique morphism $\tau : \varinjlim_{\nu \in \mathcal{I}} X_\nu \to Y$ satisfying $\sigma = \tau \circ \varphi$. Thus φ induces an isomorphism

$$\mathrm{Hom}(\varinjlim_{\nu \in \mathcal{I}} X_\nu, Y) \longrightarrow \mathrm{Hom}(\varinjlim_{\lambda \in \mathcal{J}} (\varinjlim_{\mu \in \mathcal{I}_\lambda} X_{F_\lambda(\mu)}), Y)$$

which is functorial in Y, and the assertion follows.

We are now in position to prove the following property of a locally finitely presented category.

Universal property 5.6 *Let $f\colon \mathrm{fp}(\mathcal{A}) \to \mathcal{B}$ be a functor into an additive category with direct limits. Then there exists, up to isomorphism, a unique functor $f^*\colon \mathcal{A} \to \mathcal{B}$ such that $f^*|_{\mathrm{fp}(\mathcal{A})} = f$ and f^* commutes with direct limits.*

Proof: Define $f^*(X) = \varinjlim_{\lambda \in \mathcal{C}/X} f(X_\lambda)$ and $f^*(\varphi) = \varinjlim_{\lambda \in \mathcal{C}/\varphi} f(\varphi_\lambda)$ for every object X and every morphism φ in $\mathcal{A}$. We show that f^* commutes with direct limits. To this end let $X = \varinjlim X_\lambda$ be a direct limit in $\mathcal{A}$. The corresponding family of functors $\mathcal{C}/X_\lambda \to \mathcal{C}/X$ satisfies the assumption of Lemma 5.5 and we obtain therefore

$$\varinjlim_\lambda f^*(X_\lambda) = \varinjlim_\lambda (\varinjlim_{\mu \in \mathcal{C}/X_\lambda} f(X_\mu)) \cong \varinjlim_{\nu \in \mathcal{C}/X} f(X_\nu) = f^*(X).$$

The uniqueness follows easily from the requirement that the functor $\mathcal{A} \to \mathcal{B}$ extending f commutes with direct limits.

We proceed with some further properties of $\mathcal{A}$ which are related to properties of $\mathrm{fp}(\mathcal{A})$.

Lemma 5.7 *A locally finitely presented category $\mathcal{A}$ has cokernels iff* $\mathrm{fp}(\mathcal{A})$ *has cokernels. In that case the embedding* $\mathrm{fp}(\mathcal{A}) \to \mathcal{A}$ *is right exact, direct limits in $\mathcal{A}$ are right exact, and any exact sequence $L \to M \to N \to 0$ in $\mathcal{A}$ is a direct limit of exact sequences $L_\lambda \to M_\lambda \to N_\lambda \to 0$ in* $\mathrm{fp}(\mathcal{A})$.

Proof: If $\mathcal{A}$ has cokernels, then $\mathrm{fp}(\mathcal{A})$ is closed under cokernels, so has cokernels itself and the embedding $\mathrm{fp}(\mathcal{A}) \to \mathcal{A}$ is right exact. Conversely, suppose that $\mathrm{fp}(\mathcal{A})$ has cokernels. It is not hard to check that each cokernel in $\mathrm{fp}(\mathcal{A})$ is also a cokernel in $\mathcal{A}$. The fact that direct limits preserve colimits implies that direct limits are right exact. Now let $\varphi\colon L \to M$ be a morphism in $\mathcal{A}$. Write φ as a direct limit of morphisms $\varphi_\lambda\colon L_\lambda \to M_\lambda$ in $\mathrm{fp}(\mathcal{A})$ and consider for each λ a cokernel $M_\lambda \to N_\lambda$ in $\mathrm{fp}(\mathcal{A})$. Again, the direct limit $M \to \varinjlim N_\lambda$ is a cokernel for φ since direct limits preserve colimits. Thus $\mathcal{A}$ has cokernels and the lemma is proven.

Lemma 5.8 *A locally finitely presented category $\mathcal{A}$ has kernels, if* $\mathrm{fp}(\mathcal{A})$ *has kernels. In that case the embedding* $\mathrm{fp}(\mathcal{A}) \to \mathcal{A}$ *is left exact, direct limits in $\mathcal{A}$ are left exact, and any exact sequence $0 \to L \to M \to N$ in $\mathcal{A}$ is a direct limit of exact sequences $0 \to L_\lambda \to M_\lambda \to N_\lambda$ in* $\mathrm{fp}(\mathcal{A})$.

Proof: The fully faithful functor $\mathcal{A} \to \mathrm{Mod}(\mathrm{fp}(\mathcal{A}))$, $X \mapsto \mathrm{Hom}(\ ,X)|_{\mathrm{fp}(\mathcal{A})}$ shows immediately that the embedding $\mathrm{fp}(\mathcal{A}) \to \mathcal{A}$ is left exact and that direct limits in $\mathcal{A}$ are left exact. Now let $\varphi: M \to N$ be a morphism in $\mathcal{A}$. Write φ as a direct limit of morphisms $\varphi_\lambda: M_\lambda \to N_\lambda$ in $\mathrm{fp}(\mathcal{A})$ and consider for each λ a kernel $L_\lambda \to M_\lambda$ in $\mathrm{fp}(\mathcal{A})$. We claim that $\varinjlim L_\lambda \to M$ is a kernel for φ. Using again the functor $\mathcal{A} \to \mathrm{Mod}(\mathrm{fp}(\mathcal{A}))$ this follows since $0 \to \mathrm{Hom}(X, \varinjlim L_\lambda) \to \mathrm{Hom}(X, \varinjlim M_\lambda) \to \mathrm{Hom}(X, \varinjlim N_\lambda)$ is exact for each $X \in \mathrm{fp}(\mathcal{A})$. Thus $\mathcal{A}$ has kernels and the lemma is proven.

Lemma 5.9 *If $\mathcal{A}$ is a locally coherent category, then the embedding* $\mathrm{fp}(\mathcal{A}) \to \mathcal{A}$ *is exact, direct limits in $\mathcal{A}$ are exact, and any exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{A}$ is a direct limit of exact sequences $0 \to L_\lambda \to M_\lambda \to N_\lambda \to 0$ in* $\mathrm{fp}(\mathcal{A})$.

Proof: Combine the previous lemmas.

Proposition 5.10 *Let $\mathcal{A}$ be a locally finitely presented category and let $f: \mathcal{A} \to \mathcal{B}$ be a functor into an additive category which preserves direct limits.*

(1) *If* $\mathrm{fp}(\mathcal{A})$ *and $\mathcal{B}$ have cokernels, then f is right exact if and only if $f|_{\mathrm{fp}(\mathcal{A})}$ is right exact.*

(2) *If* $\mathrm{fp}(\mathcal{A})$ *and $\mathcal{B}$ have kernels and direct limits in $\mathcal{B}$ are left exact, then f is left exact if and only if $f|_{\mathrm{fp}(\mathcal{A})}$ is left exact.*

(3) *If* $\mathrm{fp}(\mathcal{A})$ *and $\mathcal{B}$ are abelian and direct limits in $\mathcal{B}$ are exact, then f is exact if and only if $f|_{\mathrm{fp}(\mathcal{A})}$ is exact.*

Proof: Combine the previous lemmas.

Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of locally finitely presented categories. Any functor $f: \mathrm{fp}(\mathcal{A}) \to \mathrm{fp}(\mathcal{B})$ extends uniquely to a functor $\mathcal{A} \to \mathcal{B}$ which commutes with direct limits by Property 5.6. Alternatively, one could use the induced functor $f^*: \mathrm{Mod}(\mathrm{fp}(\mathcal{A})) \to \mathrm{Mod}(\mathrm{fp}(\mathcal{B}))$

which restricts to a functor $\mathrm{Flat}(\mathrm{fp}(\mathcal{A})) \to \mathrm{Flat}(\mathrm{fp}(\mathcal{B}))$. This observation has some important implication.

Given a full additive subcategory $\mathcal{C}$ of $\mathrm{fp}(\mathcal{A})$ denote by $\vec{\mathcal{C}}$ the full subcategory of $\mathcal{A}$ formed by the objects which are direct limits of objects in $\mathcal{C}$. This subcategory has the following description which is due to Crawley-Boevey [2, Theorem 4.1].

Proposition 5.11 *$\vec{\mathcal{C}}$ is a locally finitely presented category which is closed under direct limits in $\mathcal{A}$, and $\mathrm{fp}(\vec{\mathcal{C}})$ consists of the direct summands of objects in $\mathcal{C}$. An object $X \in \mathcal{A}$ belongs to $\vec{\mathcal{C}}$ if and only if any morphism from an object in $\mathrm{fp}(\mathcal{A})$ to X factors through some object in $\mathcal{C}$.*

Proof: Using the identification $\mathcal{A} = \mathrm{Flat}(\mathrm{fp}(\mathcal{A}))$ the inclusion $\mathcal{C} \to \mathrm{fp}(\mathcal{A})$ induces a fully faithful functor $\mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathrm{fp}(\mathcal{A}))$ by Lemma 4.2, which restricts to an equivalence between $\mathrm{Flat}(\mathcal{C})$ and $\vec{\mathcal{C}}$.

6. Adjoints

We develop various criteria for the existence of left and right adjoints for functors between locally finitely presented categories. This is strongly related to the question whether a functor preserves finitely presented objects. We shall use the following crucial fact.

Lemma 6.1 *Let $f: \mathcal{A} \to \mathcal{B}$ be a functor between categories with direct limits. Suppose there exists a right adjoint which commutes with direct limits. Then $f(X)$ is finitely presented (finitely generated), if X is a finitely presented (finitely generated) object in $\mathcal{A}$.*

Proof: Let $X \in \mathcal{A}$ and $\varinjlim Y_\lambda \in \mathcal{B}$. We have the following sequence of morphisms

$$\varinjlim \mathrm{Hom}(f(X),Y_\lambda) \cong \varinjlim \mathrm{Hom}(X,g(Y_\lambda)) \xrightarrow{\varphi} \mathrm{Hom}(X,\varinjlim g(Y_\lambda)) \cong \mathrm{Hom}(X,g(\varinjlim Y_\lambda)) \cong \mathrm{Hom}(f(X),\varinjlim Y_\lambda)$$

since f and g is a pair of adjoint functors and g commutes with direct limits. Now the assertion follows since φ is an isomorphism (a monomorphism), if X is finitely presented (finitely generated).

Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of locally finitely presented categories. Denote for any functor $f: \mathrm{fp}(\mathcal{A}) \to \mathrm{fp}(\mathcal{B})$ by $f^*: \mathcal{A} \to \mathcal{B}$ the unique functor which extends f and commutes with direct limits.

Theorem 6.2 *For a functor $f\colon \mathrm{fp}(\mathcal{A}) \to \mathrm{fp}(\mathcal{B})$ the following are equivalent.*

(1) *$f^*\colon \mathcal{A} \to \mathcal{B}$ has a left adjoint.*

(2) *$f\colon \mathrm{fp}(\mathcal{A}) \to \mathrm{fp}(\mathcal{B})$ has a left adjoint.*

Moreover, if $g\colon \mathrm{fp}(\mathcal{B}) \to \mathrm{fp}(\mathcal{A})$ is a left adjoint of f, then $g^*\colon \mathcal{B} \to \mathcal{A}$ is a left adjoint of $f^*\colon \mathcal{A} \to \mathcal{B}$.

Proof: (1) $\Rightarrow$ (2) The left adjoint of f^* maps finitely presented to finitely presented objects by Lemma 6.1 and gives therefore a left adjoint of f.

(2) $\Rightarrow$ (1) We shall identify $\mathcal{A} = \mathrm{Flat}(\mathrm{fp}(\mathcal{A}))$ and $\mathcal{B} = \mathrm{Flat}(\mathrm{fp}(\mathcal{B}))$, respectively. Now let $g\colon \mathrm{fp}(\mathcal{B}) \to \mathrm{fp}(\mathcal{A})$ be a left adjoint of f. It is easily verified that $g_*\colon \mathrm{Mod}(\mathrm{fp}(\mathcal{A})) \to \mathrm{Mod}(\mathrm{fp}(\mathcal{B}))$ is a left adjoint of $f_*\colon \mathrm{Mod}(\mathrm{fp}(\mathcal{B})) \to \mathrm{Mod}(\mathrm{fp}(\mathcal{A}))$. This fact implies that $g_*(M) \cong f^*(M)$ is flat for every $M \in \mathcal{A}$. The assertion now follows since $g^*\colon \mathcal{B} \to \mathcal{A}$ is a left adjoint of the functor $\mathcal{A} \to \mathcal{B}$, $M \mapsto g_*(M)$.

We have the following application of which one direction has been proved by Gabriel and Ulmer [5, Korollar 5.8] using some different arguments.

Corollary 6.3 *Let $\mathcal{C}$ be a skeletally small additive category. Then the inclusion* $\mathrm{Flat}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{C})$ *has a left adjoint if and only if $\mathcal{C}$ has cokernels.*

Proof: The inclusion $\mathcal{C} \to \mathrm{mod}(\mathcal{C})$ has a left adjoint iff $\mathcal{C}$ has cokernels by Lemma 2.8. The assertion now follows from the preceding theorem.

Lemma 6.4 (1) *If $\mathcal{C}$ has pseudo-cokernels, then a $\mathcal{C}$-module M is flat iff M maps any sequence $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ in $\mathcal{C}$ where ψ is a pseudo-cokernel for φ, to an exact sequence $M(Z) \to M(Y) \to M(X)$.*

(2) *If $\mathcal{C}$ has cokernels, then a $\mathcal{C}$-module M is flat iff the functor $M\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is leftexact.*

Proof: The category $\mathrm{mod}(\mathcal{C}^{\mathrm{op}})$ is abelian and therefore Proposition 5.10 applies, saying that $M \otimes_{\mathcal{C}} -$ is exact iff the restriction $M \otimes_{\mathcal{C}} -|_{\mathrm{mod}(\mathcal{C}^{\mathrm{op}})}$ is exact. Now the assertions follow from Lemma 2.5 and Lemma 2.6.

Lemma 6.5 *If $\mathcal{C}$ has pseudo-cokernels, then a functor $f\colon \mathcal{C} \to \mathcal{D}$ preserves pseudo-cokernels iff $f_*\colon \mathrm{Mod}(\mathcal{D}) \to \mathrm{Mod}(\mathcal{C})$ preserves flatness.*

Proof: The first implication follows from the characterization of flatness in the previous lemma. Conversely, the exactness of $f_*(\mathrm{Hom}(\ ,Y)) \otimes_{\mathcal{C}} -$ for every $Y \in \mathcal{D}$ implies that f preserves pseudo-cokernels since $f_*(\mathrm{Hom}(\ ,Y)) \otimes_{\mathcal{C}} \mathrm{Hom}(X,\ ) \cong \mathrm{Hom}(f(X),Y)$ for every $X \in \mathcal{C}$.

The characterization of flat modules in the previos lemma allows to deduce the following theorem of Crawley-Boevey [2, Theorem 2.1] which generalizes a well-known result of Chase for modules over a ring.

Proposition 6.6 *For a locally finitely presented category $\mathcal{A}$ the following are equivalent.*

(1) *$\mathcal{A}$ has products.*

(2) $\mathrm{fp}(\mathcal{A})$ *has pseudo-cokernels.*

(3) *A product of flat* $\mathrm{fp}(\mathcal{A})$*-modules is flat.*

The next result gives a criterion for the existence of a right adjoint.

Theorem 6.7 *Let $f: \mathcal{A} \to \mathcal{B}$ be a functor between locally finitely presented categories with products and suppose that f commutes with direct limits. Then the following are equivalent.*

(1) *f has a right adjoint which commutes with direct limits.*

(2) *f restricts to a functor* $\mathrm{fp}(\mathcal{A}) \to \mathrm{fp}(\mathcal{B})$ *which preserves pseudo-cokernels.*

Proof: (1) $\Rightarrow$ (2) Let $g: \mathcal{B} \to \mathcal{A}$ be a right adjoint of f commuting with direct limits. It follows from Lemma 6.1 that f restricts to a functor $\mathrm{fp}(\mathcal{A}) \to \mathrm{fp}(\mathcal{B})$. Let $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ be a sequence in $\mathrm{fp}(\mathcal{A})$ where ψ is a pseudo-cokernel for φ in $\mathrm{fp}(\mathcal{A})$. It is easily checked that ψ is also a pseudo-cokernel in $\mathcal{A}$. It follows that $\mathrm{Hom}(Z, g(M)) \to \mathrm{Hom}(Y, g(M)) \to \mathrm{Hom}(X, g(M))$ is exact for every $M \in \mathcal{B}$ and therefore $\mathrm{Hom}(f(Z), M) \to \mathrm{Hom}(f(Y), M) \to \mathrm{Hom}(f(X), M)$ is exact which shows that $f(\psi)$ is a pseudo-cokernel for $f(\varphi)$.

(2) $\Rightarrow$ (1) Denote by $h: \mathrm{fp}(\mathcal{A}) \to \mathrm{fp}(\mathcal{B})$ the restriction of f. The functor $h_*: \mathrm{Mod}(\mathrm{fp}(\mathcal{B})) \to \mathrm{Mod}(\mathrm{fp}(\mathcal{A}))$ preserves flatness by Lemma 6.5. Thus $\mathrm{Flat}(\mathrm{fp}(\mathcal{B})) \to \mathrm{Flat}(\mathrm{fp}(\mathcal{A}))$, $X \mapsto h_*(X)$ is a right adjoint of f which commutes with direct limits, since f is isomorphic to $\mathrm{Flat}(\mathrm{fp}(\mathcal{A})) \to \mathrm{Flat}(\mathrm{fp}(\mathcal{B}))$, $X \mapsto h^*(X)$.

We finish our discussion of adjoint functors with the following example.

Proposition 6.8 *Let $\mathcal{C}$ and $\mathcal{D}$ be skeletally small additive categories with split idempotents. A functor $f\colon \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{D})$ is isomorphic to g_* for some additive functor $g\colon \mathcal{D} \to \mathcal{C}$ if and only if f commutes with limits and colimits.*

Proof: One direction is clear. So suppose that f commutes with limits and colimits. Thus f has a left adjoint since f commutes with limits and this left adjoint preserves finitely presented objects by Lemma 6.1 and projectives since f is exact. Using the Yoneda embeddings we obtain g by restricting the left adjoint to a functor from $\mathcal{D}$ to $\mathcal{C}$.

7. Products

In this section we discuss the property of certain functors to preserve products. It is well-known that for a ring Λ a right Λ-module M is finitely presented (finitely generated) iff the natural map $M \otimes_\Lambda (\prod_i N_i) \to \prod_i (M \otimes_\Lambda N_i)$ is an isomorphism (epimorphism) for every family $(N_i)_{i\in I}$ of left Λ-modules. This generalizes to arbitrary module categories as follows.

Lemma 7.1 *Let $\mathcal{C}$ be a skeletally small additive category. For $M \in \mathrm{Mod}(\mathcal{C})$ the following are equivalent.*

(1) *M is finitely presented (finitely generated).*

(2) *The natural morphism $M \otimes_\mathcal{C} (\prod_i N_i) \to \prod_i (M \otimes_\mathcal{C} N_i)$ is an isomorphism (epimorphism) for every family $(N_i)_{i\in I}$ in $\mathrm{Mod}(\mathcal{C}^{\mathrm{op}})$.*

(3) *The natural morphism $M \otimes_\mathcal{C} (\prod_i \mathrm{Hom}(X_i, \ )) \to \prod_i (M \otimes_\mathcal{C} \mathrm{Hom}(X_i, \ ))$ is an isomorphism (epimorphism) for every family $(X_i)_{i\in I}$ in $\mathcal{C}$.*

Proof: Adapt the proof for modules over a ring.

Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor between skeletally small additive categories. Recall that f induces the restriction functor $f_*\colon \mathrm{Mod}(\mathcal{D}) \to \mathrm{Mod}(\mathcal{C})$ which has a left adjoint $f^*\colon \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{D})$.

Proposition 7.2 *The following are equivalent.*

(1) *$(f^{\mathrm{op}})_*$ restricts to a functor $\mathrm{mod}(\mathcal{D}^{\mathrm{op}}) \to \mathrm{mod}(\mathcal{C}^{\mathrm{op}})$.*

(2) *The natural morphism $f^*(\prod_i M_i) \to \prod_i f^*(M_i)$ is an isomorphism for all families $(M_i)_{i\in I}$ in $\mathrm{Mod}(\mathcal{C})$.*

(3) *The natural morphism* $f^*(\prod_i \operatorname{Hom}(\ ,X_i)) \to \prod_i f^*(\operatorname{Hom}(\ ,X_i))$ *is an isomorphism for all families* $(X_i)_{i\in I}$ *in* $\mathcal{C}$.

Proof: The assertion follows from the definition of f_* and f^* using the lemma above.

Theorem 7.3 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be locally finitely presented categories with products. Suppose that* $f\colon \operatorname{fp}(\mathcal{A}) \to \operatorname{fp}(\mathcal{B})$ *is a functor and denote by* $f^*\colon \mathcal{A} \to \mathcal{B}$ *the induced functor which commutes with direct limits. Then the following are equivalent.*

(1) $(f^{\mathrm{op}})_*$ *restricts to a functor* $\operatorname{mod}(\operatorname{fp}(\mathcal{B})^{\mathrm{op}}) \to \operatorname{mod}(\operatorname{fp}(\mathcal{A})^{\mathrm{op}})$.

(2) *The natural morphism* $f^*(\prod_i M_i) \to \prod_i f^*(M_i)$ *is an isomorphism for all families* $(M_i)_{i\in I}$ *in* $\mathcal{A}$.

(3) *The natural morphism* $f^*(\prod_i M_i) \to \prod_i f^*(M_i)$ *is an isomorphism for all families* $(M_i)_{i\in I}$ *in* $\operatorname{fp}(\mathcal{A})$.

Proof: The assertion follows immediately from the previous result if one identifies $\mathcal{A}$ and $\mathcal{B}$ with $\operatorname{Flat}(\operatorname{fp}(\mathcal{A}))$ and $\operatorname{Flat}(\operatorname{fp}(\mathcal{B}))$, respectively.

Specializing the preceding theorem to the case that f is an embedding and using Theorem 3.4 we obtain the following result which is due to Crawley-Boevey [2, Theorem 4.2].

Corollary 7.4 *Let* $\mathcal{A}$ *be a locally finitely presented category with products and* $\mathcal{C}$ *a full additive subcategory of* $\operatorname{fp}(\mathcal{A})$. *Then* $\vec{\mathcal{C}}$ *is closed under products in* $\mathcal{A}$ *if and only if* $\mathcal{C}$ *is covariantly finite in* $\operatorname{fp}(\mathcal{A})$.

8. The category $D(\mathcal{A})$

Let $\mathcal{A}$ be a locally finitely presented category. In this section we construct a functor $d_{\mathcal{A}}\colon \mathcal{A} \to D(\mathcal{A})$ into a locally finitely presented category with kernels which is an analogue of the functor $\mathcal{A} \to \operatorname{Mod}(\operatorname{fp}(\mathcal{A}))$, $X \mapsto \operatorname{Hom}(\ ,X)|_{\operatorname{fp}(\mathcal{A})}$. This is based on a construction used by Crawley-Boevey [2].

Let $\mathcal{C} = \operatorname{fp}(\mathcal{A})$ and denote by $h\colon \mathcal{C} \to \operatorname{mop}(\mathcal{C})$ the Yoneda functor. We define $D(\mathcal{A}) = \operatorname{Flat}(\operatorname{mop}(\mathcal{C}))$ and $d_{\mathcal{A}} = h^*$ is the unique functor commuting with direct limits which extends h. Note that $D(\mathcal{A})$ has kernels and left exact direct limits by Lemma 5.8.

Universal property 8.1 *Let $f: \mathcal{A} \to \mathcal{B}$ be a functor into an additive category with kernels and left exact direct limits. Supppose also that f commutes with direct limits. Then there exists, up to isomorphism, a unique left exact functor $f^*: D(\mathcal{A}) \to \mathcal{B}$ such that $f = f^* \circ d_{\mathcal{A}}$.*

Proof: The restriction $f|_{\mathcal{C}}: \mathcal{C} \to \mathcal{B}$ extends uniquely to a left exact functor $f': \mathrm{mop}(\mathcal{C}) \to \mathcal{B}$ by Property 2.1 and this extends uniquely to a functor $f^*: D(\mathcal{A}) \to \mathcal{B}$ commuting with direct limits by Property 5.6. The left exactness of f^* follows from Proposition 5.10.

Corollary 8.2 *The category $\mathcal{A}$ has kernels if and only if $d_{\mathcal{A}}$ admits a right adjoint. A right adjoint commutes with direct limits, and it preserves finitely presented objects if and only if* $\mathrm{fp}(\mathcal{A})$ *has kernels.*

Proof: Combine the above theorem with the dual statements of Lemma 2.7 and Lemma 2.8.

Corollary 8.3 *The category $\mathcal{A}$ has products if and only if $D(\mathcal{A})$ has products. In this case $d_{\mathcal{A}}$ commutes with products.*

Proof: The category $\mathcal{A}$ has products iff $\mathcal{C}$ has pseudo-cokernels by Proposition 6.6. It follows that $\mathcal{A}$ has products iff $D(\mathcal{A})$ has products since $\mathcal{C}$ has pseudo-cokernels iff $\mathrm{mop}(\mathcal{C}) = \mathrm{fp}(D(\mathcal{A}))$ has pseudo-cokernels by Lemma 2.2. It remains to check that the image of $d_{\mathcal{A}}$ is closed under products taken in $D(\mathcal{A})$. This follows with Corollary 7.4 since the image of the Yoneda functor h is covariantly finite in $\mathrm{mop}(\mathcal{C})$.

Corollary 8.4 *Let $\mathcal{A}$ be a locally finitely presented category with products. Then $\mathcal{A}$ has kernels if and only if $\mathcal{A}$ has cokernels.*

Proof: By Theorem 6.7, the functor $d_{\mathcal{A}}: \mathcal{A} \to D(\mathcal{A})$ has a right adjoint commuting with direct limits iff the Yoneda functor $\mathcal{C} \to \mathrm{mop}(\mathcal{C}) = \mathrm{fp}(D(\mathcal{A}))$ preserves pseudo-cokernels. This happens iff $\mathcal{C}$ has cokernels, equivalently if $\mathcal{A}$ has cokernels, by Lemma 2.8 and Lemma 5.7. Now the assertion follows from Corollary 8.2.

Recall that an additive category is *(co)complete* if it has arbitrary (co)limits, equivalently if (co)kernels and (co)products exist.

Corollary 8.5 *A locally finitely presented category is complete if and only if it is cocomplete.*

Our discussion suggests a further construction. We define a functor $c_{\mathcal{A}}\colon \mathcal{A} \to C(\mathcal{A})$ into a locally coherent category as follows. Let $C(\mathcal{A}) = \mathrm{Flat}(A(\mathcal{C}))$ and $c_{\mathcal{A}} = a_{\mathcal{C}}^*$ is the unique functor commuting with direct limits which extends $a_{\mathcal{C}}\colon \mathcal{C} \to A(\mathcal{C})$, $X \mapsto \mathrm{Hom}(\ ,\mathrm{Hom}(X,\ ))$. Note that $C(\mathcal{A}) = D(\mathrm{Mod}(\mathcal{C}))$ and that $c_{\mathcal{A}}$ is isomorphic to the composition of $\mathcal{A} \to \mathrm{Mod}(\mathcal{C})$, $X \mapsto \mathrm{Hom}(\ ,X)|_{\mathcal{C}}$ with $d_{\mathrm{Mod}(\mathcal{C})}$.

Universal property 8.6 *Let $f\colon \mathcal{A} \to \mathcal{B}$ be a functor into an abelian category with exact direct limits. Supppose also that f commutes with direct limts. Then there exists, up to isomorphism, a unique exact functor $f^*\colon C(\mathcal{A}) \to \mathcal{B}$ such that $f = f^* \circ c_{\mathcal{A}}$.*

Proof: Analogous to the proof of Property 8.1.

9. Locally coherent categories

Recall that a locally finitely presented category $\mathcal{A}$ is locally coherent provided that $\mathrm{fp}(\mathcal{A})$ is abelian. Note that in this case the embedding of $\mathrm{fp}(\mathcal{A})$ into $\mathcal{A}$ is exact. In this section we discuss how certain properties of such a category $\mathcal{A}$ are reflected by properties of $\mathrm{fp}(\mathcal{A})$. We need some preparations.

Lemma 9.1 *Let $\mathcal{A}$ be a Grothendieck category. An object $X \in \mathcal{A}$ is finitely presented (finitely generated) if and only if for every direct limit $\varinjlim Y_\lambda$ in $\mathcal{A}$ with $Y_\lambda \in \mathrm{inj}(\mathcal{A})$ for all λ the natural morphism $\varinjlim \mathrm{Hom}(X, Y_\lambda) \to \mathrm{Hom}(X, \varinjlim Y_\lambda)$ is an isomorphism (a monomorphism).*

Proof: See [1, Korollar 1.8].

Recall that an object M of a Grothendieck category $\mathcal{A}$ is *fp-injective* provided that $\mathrm{Ext}^1(Z, M) = 0$ for all $Z \in \mathrm{fp}(\mathcal{A})$. We denote the full subcategory of fp-injective objects in $\mathcal{A}$ by $\mathrm{fpinj}(\mathcal{A})$. Note that $\mathrm{fpinj}(\mathcal{A})$ is automatically closed under products. The following lemma is taken from [2].

Lemma 9.2 *Let $\mathcal{A}$ be a locally finitely presented abelian category. For $Z \in \mathrm{fp}(\mathcal{A})$ and $M \in \mathcal{A}$ the following are equivalent.*

(1) $\mathrm{Ext}^1(Z, M) = 0$.

(2) $0 \to \mathrm{Hom}(Z,M) \to \mathrm{Hom}(Y,M) \to \mathrm{Hom}(X,M) \to 0$ *is exact for all exact sequences* $0 \to X \to Y \to Z \to 0$ *in* $\mathcal{A}$ *with* Y *finitely presented.*

Proof: Straightforward.

Using the above lemmas we obtain the following.

Lemma 9.3 *Let* $\mathcal{A}$ *be a locally finitely presented abelian category.*

(1) *If* $(M_i)_{i\in I}$ *is a filtered family of subobjects* $M_i \in \mathrm{fpinj}(\mathcal{A})$ *of some* $M \in \mathcal{A}$*, then* $\sum_i M_i \in \mathrm{fpinj}(\mathcal{A})$*. In particular,* $\mathrm{fpinj}(\mathcal{A})$ *is closed under coproducts.*

(2) $\mathcal{A}$ *is locally coherent iff* $\mathrm{fpinj}(\mathcal{A})$ *is closed under direct limits.*

Proof: Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathcal{A}$ with $Y \in \mathrm{fp}(\mathcal{A})$. Suppose also that $Z \in \mathrm{fp}(\mathcal{A})$, equivalently that X is finitely generated. Let $\varinjlim M_\lambda$ be a direct limit of fp-injective objects in $\mathcal{A}$. Then we obtain the following commutative diagram

$$\begin{array}{ccccc}
\varinjlim \mathrm{Hom}(Y, M_\lambda) & \longrightarrow & \varinjlim \mathrm{Hom}(X, M_\lambda) & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \varphi} & & \\
\mathrm{Hom}(Y, \varinjlim M_\lambda) & \longrightarrow & \mathrm{Hom}(X, \varinjlim M_\lambda) & \longrightarrow & 0
\end{array}$$

where the upper row is exact and ψ is an isomorphism by our assumptions.

(1) If the M_λ's form a filtered family of subobjects of some $M \in \mathcal{A}$, then φ is an isomorphism since X is finitely generated. Thus the lower row is exact and $\sum M_\lambda = \varinjlim M_\lambda$ is fp-injective.

(2) The map φ is an isomorphism iff the lower row is exact. Thus a finitely generated subobject X of Y is finitely presented iff $\varinjlim M_\lambda$ is fp-injective for any choice of the M_λ's.

Remark 9.4 Let $\mathcal{A}$ be a locally coherent category. The equivalence $\mathcal{A} \to \mathrm{Flat}(\mathrm{fp}(\mathcal{A}))$ given by $X \mapsto \mathrm{Hom}(\ ,X)|_{\mathrm{fp}(\mathcal{A})}$ induces an equivalence between $\mathrm{fpinj}(\mathcal{A})$ and the full subcategory of exact functors from $\mathrm{fp}(\mathcal{A})^{\mathrm{op}}$ to Ab.

The following is another preliminary result.

Lemma 9.5 *Let* $\mathcal{A}$ *be a locally coherent category. Then* $\mathrm{fpinj}(\mathcal{A})$ *has direct limits and* $\mathrm{fp}(\mathrm{fpinj}(\mathcal{A})) = \mathrm{fpinj}(\mathcal{A}) \cap \mathrm{fp}(\mathcal{A}) = \mathrm{inj}(\mathrm{fp}(\mathcal{A}))$.

Proof: The assertion follows from the definitions involved using the above lemmas.

We say that a locally coherent category has *sufficently many fp-injectives* provided that $\mathrm{fpinj}(\mathcal{A})$ is locally finitely presented. The following is a consequence of our discussion of fp-injective objects.

Theorem 9.6 *A locally coherent category has sufficently many fp-injectives if and only if* $\mathrm{fp}(\mathcal{A})$ *has sufficently many injectives. The finitely presented objects in* $\mathrm{fpinj}(\mathcal{A})$ *are precisely the injective objects in* $\mathrm{fp}(\mathcal{A})$.

Proof: We have $\mathcal{C} = \mathrm{inj}(\mathrm{fp}(\mathcal{A})) = \mathrm{fp}(\mathrm{fpinj}(\mathcal{A}))$ by the preceding lemma. Suppose first that $\mathrm{fpinj}(\mathcal{A})$ is locally finitely presented. Thus $\mathrm{fpinj}(\mathcal{A}) = \vec{\mathcal{C}}$. We need to show that for every $X \in \mathrm{fp}(\mathcal{A})$ there exists a monomorphism $X \to Y$ with $Y \in \mathcal{C}$. To this end let $\varphi: X \to \varinjlim Y_\lambda$ be an injective envelope in $\mathcal{A}$ with $Y_\lambda \in \mathcal{C}$ for all λ. The morphism φ factors through some Y_λ since X is finitely presented and the corresponding morphism $X \to Y_\lambda$ is a monomorphism. Now suppose that $\mathrm{fp}(\mathcal{A})$ has sufficently many injectives. We claim that $\vec{\mathcal{C}} = \mathrm{fpinj}(\mathcal{A})$. We have $\vec{\mathcal{C}} \subseteq \mathrm{fpinj}(\mathcal{A})$ since $\mathrm{fpinj}(\mathcal{A})$ is closed under direct limits by Lemma 9.3. Using Proposition 5.11 it remains to show that every morphism $X \to Y$ with $X \in \mathrm{fp}(\mathcal{A})$ and $Y \in \mathrm{fpinj}(\mathcal{A})$ factors through some object $Z \in \mathcal{C}$. But this is clear since we can choose a monomorphism $X \to Z$ and therefore the proof is complete.

Having discussed the case that $\mathrm{fp}(\mathcal{A})$ has sufficently many injectives we turn now to the case that $\mathrm{fp}(\mathcal{A})$ has sufficently many projectives.

Theorem 9.7 *Let* $\mathcal{A}$ *be a locally coherent category. Then* $\mathcal{A}$ *is equivalent to a module category* $\mathrm{Mod}(\mathcal{C})$ *if and only if* $\mathrm{fp}(\mathcal{A})$ *has sufficently many projectives. In this case* $\mathcal{A}$ *and* $\mathrm{Mod}(\mathrm{proj}(\mathrm{fp}(\mathcal{A})))$ *are equivalent.*

Proof: Suppose first that $\mathcal{A} \cong \mathrm{Mod}(\mathcal{C})$. Any finitely presented object in $\mathrm{Mod}(\mathcal{C})$ is a quotient of some representable functor which is projective by Yoneda's lemma. Thus $\mathrm{fp}(\mathcal{A})$ has sufficently many projectives. Conversely, suppose that $\mathrm{fp}(\mathcal{A})$ has sufficently many projectives and let $\mathcal{C} = \mathrm{proj}(\mathrm{fp}(\mathcal{A}))$. Using Proposition 2.3 we deduce that $\mathrm{fp}(\mathcal{A}) \cong \mathrm{mod}(\mathcal{C})$ and this implies $\mathcal{A} \cong \mathrm{Mod}(\mathcal{C})$ by Proposition 5.1. This finishes the proof.

Let $\mathcal{A}$ be a locally finitely presented category with products. It has been shown by Crawley-Boevey that under this assumption the category $D(\mathcal{A})$ is locally coherent and that $d_{\mathcal{A}}: \mathcal{A} \to D(\mathcal{A})$ identifies $\mathcal{A}$ with $\mathrm{fpinj}(D(\mathcal{A}))$ [2, Theorem 3.3]. The next result shows that Crawley-Boevey's construction is essentially unique.

Corollary 9.8 *There is, up to equivalence, a bijective correspondence between locally finitely presented categories with products and locally coherent categories with sufficently many fp-injectives. The correspondence is given by*

$$\mathcal{A} \mapsto D(\mathcal{A}) \qquad \textit{and} \qquad \mathcal{A} \mapsto \mathrm{fpinj}(\mathcal{A}).$$

Proof: Combine the correspondence between locally finitely presented categories with products and skeletally small additive categories with split idempotents and pseudo-cokernels given by $\mathcal{A} \mapsto \mathrm{fp}(\mathcal{A})$, the correspondence between skeletally small additive categories with split idempotents and pseudo-cokernels and skeletally small abelian categories with sufficently many injectives given by $\mathrm{fp}(\mathcal{A}) \mapsto \mathrm{mop}(\mathrm{fp}(\mathcal{A}))$, and finally, the correspondence between skeletally small abelian categories with sufficently many injectives and locally coherent categories with sufficently many fp-injectives given by $\mathrm{mop}(\mathrm{fp}(\mathcal{A})) \mapsto \mathrm{Flat}(\mathrm{mop}(\mathrm{fp}(\mathcal{A})))$.

10. Functors between locally coherent categories

To develop further the theory of locally coherent categories one needs to study their functors. The most natural type of functors, namely those which preserve direct limits, finitely presented objects and exactness, will be the main objective of this section.

Theorem 10.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be locally coherent categories. Suppose that $f: \mathrm{fp}(\mathcal{A}) \to \mathrm{fp}(\mathcal{B})$ is a functor and denote by $f^*: \mathcal{A} \to \mathcal{B}$ the induced functor which commutes with direct limits. Then f^* is exact if and only if f is exact. Moreover, if f is exact, then there exists a right adjoint $f_*: \mathcal{B} \to \mathcal{A}$ which commutes with direct limits and sends fp-injective objects to fp-injective objects.*

Proof: The assertion is a consequence of Proposition 5.10 and Theorem 6.7, except the fact that f_* preserves fp-injectivity. To this end let $M \in \mathrm{fpinj}(\mathcal{B})$ and $0 \to X \to Y \to$

$Z \to 0$ be an exact sequence in $\mathrm{fp}(\mathcal{A})$. It suffices to show that $0 \to \mathrm{Hom}(Z, f_*(M)) \to \mathrm{Hom}(Y, f_*(M)) \to \mathrm{Hom}(X, f_*(M)) \to 0$ is exact by Lemma 9.2. Applying f^* we get an exact sequence $0 \to \mathrm{Hom}(f^*(Z), M) \to \mathrm{Hom}(f^*(Y), M) \to \mathrm{Hom}(f^*(X), M) \to 0$ since f^* is exact and M is fp-injective. But this sequence is isomorphic to $0 \to \mathrm{Hom}(Z, f_*(M)) \to \mathrm{Hom}(Y, f_*(M)) \to \mathrm{Hom}(X, f_*(M)) \to 0$ since f_* is a right adjoint of f^*. Thus $f_*(M)$ is fp-injective.

Remark 10.2 The right adjoint f_* sends injective objects to injective objects since f^* is exact. This follows from Lemma 2.9.

In view of Theorem 10.1 it is natural to ask whether given locally coherent categories $\mathcal{A}$ and $\mathcal{B}$ any functor $\mathrm{fpinj}(\mathcal{B}) \to \mathrm{fpinj}(\mathcal{A})$ commuting with direct limits and products can be extended to a functor $f_*: \mathcal{B} \to \mathcal{A}$ for some exact functor $f: \mathrm{fp}(\mathcal{A}) \to \mathrm{fp}(\mathcal{B})$. We begin our discussion of this question with a preliminary result.

Lemma 10.3 *Let $\mathcal{A}$ be an abelian category with products and sufficently many injectives. For a functor $f: \mathcal{A} \to \mathcal{B}$ into an additive category the following are equivalent.*

(1) *f preserves limits.*

(2) *f is left exact and $f|_{\mathrm{inj}(\mathcal{A})}$ preserves products.*

Proof: (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (1) A limit can be computed as a kernel between a particular map between two products. Thus it suffices to show that f preserves products. To this end let $(X_i)_{i \in I}$ be a family of objects in $\mathcal{A}$ and choose injective copresentations $0 \to X_i \to Y_i \to Z_i$ for each i. We obtain the following exact commutative diagram for every $M \in \mathcal{B}$.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(M, f(\prod_i X_i)) & \longrightarrow & \mathrm{Hom}(M, f(\prod_i Y_i)) & \longrightarrow & \mathrm{Hom}(M, f(\prod_i Z_i)) \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathrm{Hom}(M, \prod_i f(Y_i)) & \longrightarrow & \mathrm{Hom}(M, \prod_i f(Z_i)) \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_i \mathrm{Hom}(M, f(X_i)) & \longrightarrow & \prod_i \mathrm{Hom}(M, f(Y_i)) & \longrightarrow & \prod_i \mathrm{Hom}(M, f(Z_i))
\end{array}
$$

All vertical maps are isomorphisms and therefore the canonical map $\mathrm{Hom}(M, f(\prod_i X_i)) \to \prod_i \mathrm{Hom}(M, f(X_i))$ is an isomorphism. This shows that $f(\prod_i X_i)$ is a product of the $f(X_i)$'s.

Universal property 10.4 *Let $\mathcal{A}$ be a locally finitely presented category with products and let $f: \mathcal{A} \to \mathcal{B}$ be a functor into an additive category with kernels and left exact direct limits. Supppose also that f commutes with direct limits. Then there exists, up to isomorphism, a unique left exact functor $f^*: D(\mathcal{A}) \to \mathcal{B}$ such that $f = f^* \circ d_{\mathcal{A}}$. Moreover, the following are equivalent.*

(1) *f preserves products.*

(2) *f^* preserves products.*

(3) *f^* has a left adjoint.*

Proof: The existence of f^* is Property 8.1. It remais to discuss (1) – (3). If f preserves products, then f^* preserves limits by the lemma above since $d_{\mathcal{A}}$ induces an equivalence between $\mathcal{A}$ and $\mathrm{fpinj}(D(\mathcal{A}))$. The existence of a left adjoint then follows from the adjointness theorem. The other implications are trivial.

We are now in position to answer the above question related to Theorem 10.1.

Corollary 10.5 *Let $\mathcal{A}$ and $\mathcal{B}$ be locally coherent categories and suppose that $\mathcal{A}$ has sufficently many fp-injectives. Let $f: \mathrm{fpinj}(\mathcal{A}) \to \mathrm{fpinj}(\mathcal{B})$ be a functor commuting with direct limits and denote by $f^*: \mathcal{A} \to \mathcal{B}$ the unique left exact functor commuting with direct limits which extends f. Then f^* has a left adjoint if and only if f commutes with products. In that case a left adjoint $\mathcal{B} \to \mathcal{A}$ is exact and preserves finitely presented objects.*

Proof: The first part of the assertion follows from the previous theorem and it remains to verify the properties of a left adjoint. First observe that such a left adjoint restricts to a functor $g: \mathrm{fp}(\mathcal{B}) \to \mathrm{fp}(\mathcal{A})$ since f^* commutes with direct limits. Now it suffices to show that g is exact by Proposition 5.10. Let $0 \to X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \to 0$ be an exact sequence in $\mathrm{fp}(\mathcal{B})$. We need to check that $g(\varphi)$ is a mono since a left adjoint preserves cokernels. Choose a mono $\chi: g(X) \to M$ into some fp-injective object. Using the adjointness, the exactness of $\mathrm{Hom}(Y, f(M)) \to \mathrm{Hom}(X, f(M)) \to 0$ implies that χ factors through $g(\varphi)$. Thus $g(\varphi)$ is a mono and the proof is complete.

11. Functors commuting with direct limits and products

Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of locally finitely presented categories. The functors $f: \mathcal{A} \to \mathcal{B}$ which commute with direct limits form a category which can be identified with $(\mathrm{fp}(\mathcal{A}), \mathcal{B})$ by Property 5.6. If $\mathcal{A}$ and $\mathcal{B}$ have products then we denote by $\mathrm{Fp}(\mathcal{A}, \mathcal{B})$ the category of functors $f: \mathcal{A} \to \mathcal{B}$ which commute with direct limits and products. We shall give a description of this category using results of the previous section. The following well-known fact is needed.

Lemma 11.1 *Let $f_i: \mathcal{A} \to \mathcal{B}$ and $g_i: \mathcal{B} \to \mathcal{A}$, $i = 1, 2$, be pairs of functors such that g_i is a left adjoint for f_i, $i = 1, 2$. If $\varphi: f_1 \to f_2$ is a natural transformation, then there is a unique natural transformation $\psi: g_2 \to g_1$ making the following diagram commutative for all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$.*

$$\begin{array}{ccc} \mathrm{Hom}(g_1(Y), X) & \xrightarrow{\sim} & \mathrm{Hom}(Y, f_1(X)) \\ \Big\downarrow{\scriptstyle \mathrm{Hom}(\psi_Y, X)} & & \Big\downarrow{\scriptstyle \mathrm{Hom}(Y, \varphi_X)} \\ \mathrm{Hom}(g_2(Y), X) & \xrightarrow{\sim} & \mathrm{Hom}(Y, f_2(X)) \end{array}$$

Given a pair of abelian categories $\mathcal{C}$ and $\mathcal{D}$, we denote by $\mathrm{Ext}(\mathcal{C}, \mathcal{D})$ the class of functors $f: \mathcal{C} \to \mathcal{D}$ which are exact, together with its natural transformations. Now fix a pair $\mathcal{A}$ and $\mathcal{B}$ of locally finitely presented categories with products. Any functor $f \in \mathrm{Fp}(\mathcal{A}, \mathcal{B})$ extends by Corollary 10.5 uniquely to a functor $f^*: D(\mathcal{A}) \to D(\mathcal{B})$ which has an exact left adjoint. This functor restricts to an exact functor $g: \mathrm{fp}(D(\mathcal{B})) \to \mathrm{fp}(D(\mathcal{A}))$.

Theorem 11.2 *The assignment $f \mapsto g$ is functorial and induces a duality between* $\mathrm{Fp}(\mathcal{A}, \mathcal{B})$ *and* $\mathrm{Ext}(\mathrm{mop}(\mathrm{fp}(\mathcal{B})), \mathrm{mop}(\mathrm{fp}(\mathcal{A})))$.

Proof: The assignment is functorial and fully faithful by the preceding lemma since the relevant natural transformations are determined by their values on the finitely presented objects. The functor is dense by Theorem 10.1.

The following result is a reformulation of the previous one and makes the description of $\mathrm{Fp}(\mathcal{A}, \mathcal{B})$ more explicit.

Corollary 11.3 *Let $f: \mathcal{A} \to \mathcal{B}$ be a functor which commutes with direct limits and products. Denote for every $X \in \mathrm{fp}(\mathcal{B})$ by F_X the functor $\mathcal{A} \to \mathrm{Ab}$, $M \mapsto \mathrm{Hom}(X, f(M))$. Then the assignment $X \mapsto F_X$ defines a functor $F: \mathrm{fp}(\mathcal{B})^{\mathrm{op}} \to \mathrm{Fp}(\mathcal{A}, \mathrm{Ab})$. The assignment $f \mapsto$*

F is functorial and induces an equivalence between $\mathrm{Fp}(\mathcal{A},\mathcal{B})$ *and the category of functors* $F\colon \mathrm{fp}(\mathcal{B})^{\mathrm{op}} \to \mathrm{Fp}(\mathcal{A},\mathrm{Ab})$ *such that* $F(Z) \stackrel{F(\psi)}{\to} F(Y) \stackrel{F(\varphi)}{\to} F(X)$ *is exact whenever* ψ *is a pseudo-cokernel for* φ.

Proof: An inverse for $f \mapsto F$ is obtained as follows. Given a functor F, one defines a functor $f\colon \mathcal{A} \to \mathcal{B} = \mathrm{Flat}(\mathrm{fp}(\mathcal{B}))$ commuting with direct limits and products by

$$f(M)\colon \mathrm{fp}(\mathcal{B})^{\mathrm{op}} \to \mathrm{Ab}, \quad X \mapsto F(X)(M)$$

for every $M \in \mathcal{A}$.

We give now an interpretation of this result in case that $\mathcal{B} = \mathrm{Ab}$.

Theorem 11.4 *Let $\mathcal{A}$ be a locally finitely presented category with products. A functor* $f\colon \mathcal{A} \to \mathrm{Ab}$ *commutes with direct limits and products if and only if there exists a presentation* $\mathrm{Hom}(Y,\ ) \to \mathrm{Hom}(X,\ ) \to f \to 0$ *with X and Y in* $\mathrm{fp}(\mathcal{A})$. *The category* $\mathrm{Fp}(\mathcal{A},\mathrm{Ab})$ *of these functors is abelian.*

Proof: We use the above notation with $\mathcal{B} = \mathrm{Mod}(\mathbb{Z}) = \mathrm{Ab}$. Moreover, we view $\mathcal{A}$ and $\mathcal{B}$ in the natural way as subcategories of $D(\mathcal{A})$ and $D(\mathcal{B})$, respectively. The adjointness gives an isomorphism $f(M) = \mathrm{Hom}(\mathbb{Z}_{\mathbb{Z}}, f(M)) \cong \mathrm{Hom}(g(\mathbb{Z}_{\mathbb{Z}}), M)$ with $g(\mathbb{Z}_{\mathbb{Z}}) \in \mathrm{fp}(D(\mathcal{A}))$. Thus there is an exact sequence $0 \to g(\mathbb{Z}_{\mathbb{Z}}) \to X \to Y$ in $D(\mathcal{A})$ with $X, Y \in \mathrm{fp}(\mathcal{A})$ and this gives a presentation $\mathrm{Hom}(Y, M) \to \mathrm{Hom}(X, M) \to f(M) \to 0$ since M is fp-injective in $D(\mathcal{A})$. Conversely, any functor having such a presentation commutes with direct limits and products since $\mathrm{Hom}(X,\ )$ has this property for any finitely presented object X and direct limits and products are right exact. Identifying $(\mathrm{fp}(\mathcal{A}), \mathrm{Ab})$ with those functors $f\colon \mathcal{A} \to \mathrm{Ab}$ which commute with direct limits, the above result says that $\mathrm{Fp}(\mathcal{A},\mathrm{Ab}) = \mathrm{fp}(\mathrm{fp}(\mathcal{A}),\mathrm{Ab})$. Thus $\mathrm{Fp}(\mathcal{A},\mathrm{Ab})$ is abelian since $\mathrm{fp}(\mathcal{A})$ has pseudo-cokernels.

Remark 11.5 In case that $\mathcal{A}$ is the category of modules over some ring the characterization of functors commuting with direct limits and product has also been observed by Crawley-Boevey.

12. Functors between module categories

To illustrate the results of the preceding section we give some examples from module theory.

Corollary 12.1 *Let $\mathcal{C}$ and $\mathcal{D}$ be a pair of skeletally small pre-additive categories. Then there is a canonical duality* $\mathrm{Fp}(\mathrm{Mod}(\mathcal{C}),\mathrm{Mod}(\mathcal{D}))\to\mathrm{Fp}(\mathrm{Mod}(\mathcal{C}^{\mathrm{op}}),\mathrm{Mod}(\mathcal{D}^{\mathrm{op}}))$ *between the categories of functors commuting with direct limits and products.*

Proof: Recall from Section 2 that there are canonical dualities $\mathrm{mop}(\mathrm{mod}(\mathcal{C}))\to\mathrm{mop}(\mathrm{mod}(\mathcal{C}^{\mathrm{op}}))$ and $\mathrm{mop}(\mathrm{mod}(\mathcal{D}))\to\mathrm{mop}(\mathrm{mod}(\mathcal{D}^{\mathrm{op}}))$. Using this fact the assertion follows from Theorem 11.2.

Corollary 12.2 *Let Λ be a ring and $\mathcal{A}$ be a locally finitely presented category with products. Suppose that $f\colon\mathcal{A}\to\mathrm{Mod}(\Lambda)$ is a functor which commutes with direct limits and products, and denote by $F\colon\mathcal{A}\to\mathrm{Ab}$ its composition with the forgetful functor. Then there is a presentation $\mathrm{Hom}(Y,\ )\to\mathrm{Hom}(X,\ )\to F\to 0$ with $X,Y\in\mathrm{fp}(\mathcal{A})$ and a ring homomorphism $\varphi\colon\Lambda\to\mathrm{End}(F)^{\mathrm{op}}$ such that $f(M)=\varphi_*(F(M))$ for all $M\in\mathcal{A}$. Conversely, any functor $F\colon\mathcal{A}\to\mathrm{Ab}$ having such a presentation together with a ring homomorphism $\varphi\colon\Lambda\to\mathrm{End}(F)^{\mathrm{op}}$ gives a functor $f\colon\mathcal{A}\to\mathrm{Mod}(\Lambda)$ which commutes with direct limits and products.*

Proof: Adapt the proof of Theorem 11.4 by replacing $\mathbb{Z}$ by Λ. The homomorphism φ is obtained by composing the canonical morphism $\Lambda=\mathrm{End}(\Lambda_\Lambda)\to\mathrm{End}(g(\Lambda_\Lambda))$ with $\mathrm{End}(g(\Lambda_\Lambda))\cong\mathrm{End}(F)^{\mathrm{op}}$.

Given a module M over some ring, we denote by $\ell_{\mathrm{end}}(M)$ its length over $\mathrm{End}(M)^{\mathrm{op}}$.

Corollary 12.3 *Let $f\colon\mathrm{Mod}(\Lambda)\to\mathrm{Mod}(\Gamma)$ be a functor commuting with direct limits and products. Then there exists $c_f\in\mathbb{N}$ such that $\ell_{\mathrm{end}}(f(M))\le c_f\cdot\ell_{\mathrm{end}}(M)$ for all $M\in\mathrm{Mod}(\Lambda)$.*

Proof: There is a presentation $\mathrm{Hom}(Y,\ )\to\mathrm{Hom}(X,\ )\to f\to 0$ of f with $X,Y\in\mathrm{mod}(\Lambda)$. Choosing an epimorphism $\Lambda_\Lambda^n\to X$ we put $c_f=n$.

Let R be a commutative artinian ring and suppose that Λ and Γ are R-algebras. Suppose also that Λ is finitely generated as an R-module.

Corollary 12.4 *Let $f\colon\mathrm{Mod}(\Lambda)\to\mathrm{Mod}(\Gamma)$ be an R-linear functor commuting with direct limits and products. Then $f(M)$ is finitely generated as an R-module for every $M\in\mathrm{mod}(\Lambda)$.*

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