QUASI-HEREDITARY ALGEBRAS AND TILTING MODULES

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Quasi-hereditary algebras, which were introduced by L.L.Scott [S], provide a connection between the representations in the sense of Lie theory and that in the sense of finite dimensional algebras, see [CPS]. V. Dlab and C. M. Ringel have made extensive studies to the ring and module theoretical approachs to this class of algebras. One of the most important results is the existence of the characteristic module, which was given by Ringel [R1] via Auslander-Reiten [AR]. This is both a tilting and cotilting module, and hence it gives a connection with tilting theory which has enjoyed a vigorous progress in the last decade.

This paper is organised as follows : we first fix some conventions and recall some preliminaries. By a result of Ringel, an algebra is quasi-hererditary if and only if there exists a tilting (or, cotilting) module having certain special properties, see [R1]; in the second section, we prove that a tilting module satisfying Ringel's original conditions must be the characteristic module, and hence it is unique; here we have used results of Coelho-Happel-Unger [CHU] on finding complements for partial tilting modules. Given a classical tilting triple (A, T, B), then B need not be quasi-hereditary, even if A is so (see [DR1]); in the third section we give a sufficient condition for B being quasi-hereditary. Finally, the so-called qht-algebras (one class of algebras which can be obtained from quasi-hereditary algebras after classical tilting functors, see section 4 below) are studied, we will see that this class of algebras has some similar properties to the class of quasi-hereditary algebras.

1. Preliminaries

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We assume that A is an artin algebra, $A-mod$ is the category of finitely generated left $A-$modules; $E(1),\cdots,E(n)$ is an ordering of simple A-modules, this amounts to an ordering of a complete set of orthogonal primitive idempotents $e_1,\cdots,e_n$ of A, which we simply denote by $\underline{\mathbf{e}}=(e_1,\cdots,e_n)$. Let $P(i)$ and $I(i)$ be the indecomposable projective and injective modules respectively with $topP(i)\cong E(i)\cong socI(i), 1\leq i\leq n; \triangle=\{\triangle(1),\cdots\triangle(n)\}(resp.\bigtriangledown=\{\bigtriangledown(1),\cdots,\bigtriangledown(n)\})$ be the set of the standard (resp. costandard) modules, where $\triangle(i)$ is the maximal factor module of $P(i)$ with composition factors of the form $E(j)$ for $j\leq i$; and $\bigtriangledown(i)$ is the maximal submodule of $I(i)$ with composition factors of the form $E(j)$ for $j\leq i$. Denote by J_i the ideal $A(e_i+\cdots+e_n)A$ for $1\leq i\leq n$. A is said to be quasi-hereditary with respect to the ordering $\underline{\mathbf{e}}$ provided first, ${}_AA\in\mathcal{F}(\triangle)$, and second, $E(i)$ occurs with multiplicity one in $\triangle(i)$ (or, equivalently, $End_A(\triangle(i))$ is a division ring) for $1\leq i\leq n$. In the following we denote a quasi-hereditary algebra by a pair $(A,\underline{\mathbf{e}})$. Assume that $(A,\underline{\mathbf{e}})$ is quasi-hereditary, then there exists a uniquely defined basic tilting and cotilting module T such that $addT=\mathcal{F}(\triangle)\cap\mathcal{F}(\bigtriangledown)$, where $\mathcal{F}(\triangle)$ (resp. $\mathcal{F}(\bigtriangledown)$) is the full subcategory of $A-mod$ consisting of the modules having a filtration with the factors in $\triangle(resp.\bigtriangledown)$; and $addT$ is the additive category of direct sums of direct summands of T. Following [R1], T is then called the characteristic module of the quasi-hereditary algebra $(A,\underline{\mathbf{e}})$.

A module is said to be basic provided it has no direct summand of the form $N\oplus N$, with N non-zero. By definition an $A-$module T is a partial tilting module if T is of finite projective dimension and $Ext^i_A(T,T)=0$ for $i\geq 1$; a partial tilting module T is said to be an almost complete tilting module provided T has $n-1$ indecomposable direct summands up to isomorphism, where n is the number of isomorphism classes of simple A-modules. A partial tilting module T is called a tilting module provided there exists an exact sequence $0\longrightarrow_A A\longrightarrow T_0\longrightarrow\cdots\longrightarrow T_m\longrightarrow 0$ with all $T_i\in addT$. Dually, we have the notation of cotilting module. Let M be a partial tilting module, if N is an $A-$module such that $M\oplus N$ is a tilting module and $addM\cap addN=0$, then N will be called a complement to M. A tilting (or, partial tilting) module with the projective dimension less or equal to 1 is called a classical tilting (resp. partial tilting) module. It is well known that a classical partial tilting module always admits a complement [B]. However, this is not true for partial tilting module in general, as pointed out in [RS]. Further, conditions for the existence and uniqueness of a complement to a partial tilting module were studied in [RS], [HU] and [CHU]. For a classical tilting triple (A,T,B)(i.e. T is a classical tilting A-module, $B=End_AT$), we denote by $(\mathcal{G}({}_AT),\mathcal{F}({}_AT))$ the torsion pair in $A-mod$ and by $(\mathcal{X}({}_AT),\mathcal{Y}({}_AT))$ the torsion pair in $B-mod$, both induced by T, see [HR].

2. The characteristic module

Given an A-module M, we denote by $\underline{dim}M \in Z^n$ its dimension vector, i.e. $(\underline{dim}M)_i$ is the Jordan-Hölder multiplicity of $E(i)$ in M; assume that J is an ideal of A, in the following, when we say that M is an $A/J-$module, we always mean that $JM = 0$ and the action is via the quotient mapping $A \to A/J$.

Theorem 1 Let A be an artin algebra with an ordering $\underline{\mathbf{e}}$. If there exists an A-module $T = \oplus_{i=1}^n T(i)$ with $T(i)$ indecomposable for $1 \le i \le n$ such that

(i) $(\underline{dim}T(i))_i = 1$ for $1 \le i \le n$;

(ii) $\oplus_{j=1}^i T(j)$ is a tilting A/J_{i+1} module for $1 \le i \le n$.

Then T is the characteristic module of the quasi-hereditary algebra $(A, \underline{\mathbf{e}})$.

In order to prove the theorem, we first establish several lemmas.

Lemma 1 Let A be an artin algebra, ${}_AM$ be a partial tilting module and Y a complement to M. If Y is cogenerated by M, then M is faithful.

Proof Let $f :_A A \to M_0 \oplus Y_0$ be a monomorphism with $M_0 \in addM$ and $Y_0 \in add\ Y$. Since Y is cogenerated by M, Y_0 is cogenerated by M. It follows that ${}_AA$ is cogenerated by M, M is faithful. □

Lemma 2 Let A be an artin algebra. Assume that ${}_AM$ is an almost complete tilting module, C an indecomposable complement to ${}_AM$ such that C is ${}^{\perp}M$-injective, i.e., $Ext_A^j(X, C) = 0$ for $X \in^{\perp} M, j > 0$; where ${}^{\perp}M = \{X \in A-mod/p.d._AX < \infty; Ext_A^i(X, M) = 0, \forall i > 0\}$. If Y is an indecomposable complement to M not isomorphic to C, then Y is cogenerated by M.

Proof Using the similar method to 3.2 in [CHU]. Note that the proof there just used the fact that X_r is indecomposable complement to M and X_r is ${}^{\perp}M$-injective, the assumption on the base field was not used. □

The following Lemma gives a sufficient condition for the uniqueness of the complement to a partial tilting module.

Lemma 3 Let A be an artin algebra with $gl.d.A < \infty$. Let ${}_AM$ be an almost complete tilting module which is not faithful, C an indecomposable complement to M such that C is ${}^{\perp}M$-injective, then C is the unique indecomposable complement to ${}_AM$.

Proof Otherwise, let Y be an indecomposable complement not isomorphic to C, then Y is cogenerated by M by Lemma 2, thus M is faithful by Lemma 1, which contradicts the assumption. □

Lemma 4 Let $(B, \underline{\mathbf{e}})$ be a quasi-hereditary algebra with $\underline{\mathbf{e}} = (e_1, \cdots, e_n, e_{n+1})$, $A = B/J_{n+1}$ with $J_{n+1} = Be_{n+1}B$. Let ${}_AT$ be the characteristic module of $(A, \underline{\mathrm{e}}')$ with $\underline{\mathrm{e}}' = (e_1, \cdots, e_n)$. Then ${}_BT$ is an almost complete tilting module with a unique indecomposable complement.

Proof Frist, we see that $(A, \underline{\mathrm{e}}')$ is quasi-hereditary; and the characteristic module of $(B, \underline{\mathrm{e}})$ is of the form $T \oplus T(n+1)$, where $T(n+1)$ is indecomposable. By Lemma 3 we only need to prove that $T(n+1)$ is ${}^{\perp}T$-injective. By Proposition 2 in [R1] we have the exact sequence

$$0 \to X(n+1) \to T(n+1) \to \bigtriangledown(n+1) = I(n+1) \to 0 \qquad (*)$$

with $X(n+1) \in \mathcal{F}(\bigtriangledown(1), \cdots, \bigtriangledown(n)) \subseteq A$-mod. By the dual of the Lemma in [R1] Part II, we have the exact sequence

$$0 \to T_d \to T_{d-1} \to \cdots \to T_0 \to X(n+1) \to 0$$

with $T_i \in addT$ for $0 \le i \le d$. Now for any $X \in^{\perp} T$, we have

$$Ext_B^j(X, X(n+1)) \cong Ext_B^{j+d}(X, T_d) = 0, \quad \forall j > 0$$

Applying $\mathrm{Hom}_B(X, -)$ to $(*)$, then

$$Ext_B^j(X, T(n+1)) \cong Ext_B^j(X, X(n+1)) = 0, \quad \forall j > 0$$

this implies that $T(n+1)$ is ${}^{\perp}T$-injective. $\square$

Proof of Theorem 1

By the Appendix of [R1], $(A, \underline{\mathrm{e}})$ is quasi-hereditary. In particular, $gl.d.A < \infty$. Using induction on the number of isomorphism class of simple A-modules. By the inductive hypothesis A/J_n is quasi-hereditary with the characteristic module $\oplus_{j=1}^{n-1} T(j)$. By Lemma 4, $T(n)$ is the unique indecomposable complement of $\oplus_{j=1}^{n-1} T(j)$. On the other hand, the characteristic A-module has $\oplus_{j=1}^{n-1} T(j)$ as a direct summand, it follows that the tilting module ${}_AT$ must be the characteristic A-module. $\square$

By the duality, we have

Theorem 1' Let A and $T = \oplus_{i=1}^{n} T(i)$ be same as in Theorem 1. If the following conditions are satisfied

(i) $(\underline{dim} T(i))_i = 1$ for $1 \le i \le n$;

(ii) $\oplus_{j=1}^{i} T(j)$ is a cotilting $A/J_{i+1}-$module for $1 \le i \le n$.

Then T is the characterististic module of the quasi-hereditary algebra $(A, \underline{\mathrm{e}})$.

Remark If we assume in (ii) of Theorem 1 that $\oplus_{j=1}^{i} T(j)$ is both tilting and cotilting as A/J_{i+1}-module for $1 \le i \le n$, then the proof can be simplified in the sense that Lemma 4 need not be used.

As shown in [H], any tilting A-module must have n indecomposable direct summands (up to isomorphism), where n is the number of isomorphism classes of simple A-modules; however, it is still open that whether a partial tilting A-module M with exactly n indecomposable direct summands (up to isomorphism) is a tilting module, the answer is positive if M is a classical partial tilting module [B], or, if A is representation-finite [RS], i.e. A admits only finitely many isomorphism class of indecomposable modules. From these facts we obtain the following direct consequences of Theorem 1, for the proofs of these two corollaries, we only need to note that in both cases, $\oplus_{j=1}^{i} T(j)$ is also a tilting A/J_{i+1}-module for $1 \leq i \leq n$ by the above comments, then Theorem 1 is applied.

Corollary 1 Let A be an artin algebra and $T = \oplus_{i=1}^{n} T(i)$ be a classical tilting A-module, with $T(i)$ indecomposable for $1 \leq i \leq n$. If $(\underline{dim} T(i))_i = 1; (\underline{dim} T(i))_j = 0, j > i, 1 \leq i \leq n$, then T is the characteristic module of the quasi-hereditary algebra $(A, \underline{\mathbf{e}})$.

Corollary 2 Let A is a representation-finite algebra and $T = \oplus_{i=1}^{n} T(i)$ a tilting A-module, with $T(i)$ indecomposable for $1 \leq i \leq n$. If $(\underline{dim} T(i))_i = 1; (\underline{dim} T(i))_j = 0, j > i, 1 \leq i \leq n$, then T is the characteristic module of the quasi-hereditary algebra $(A, \underline{\mathbf{e}})$.

3. Tilting triple

For simplicity, in this section we assume that A is a finite dimensional $k-$algebra with k an algebraically closed field. Let (A, T, B) be a classical tilting triple with T basic, F be the tilting functor $Hom_A(T, -)$. We fix an ordering of the indecomposable direct summands of T, say, $T = T(1) \oplus \cdots \oplus T(n)$, then we get the induced ordering of the simple $B-$modules; and for this ordering we define the modules $U(i), V(i), K(i)$, and $U'(i)$ as follows: let $U(i)$ be the sum of all images of $A-$mappings $T(j) \longrightarrow T(i)$ with $j > i$; by $V(i)$ we denote the quotient $T(i)/U(i)$; thus $U(i)$ and $V(i)$ all lie in $\mathcal{G}({}_AT)$, the torsion class induced by T. And then define $K(i)$ to be the kernel of the universal mapping f_i, hence there is the exact sequence

$$0 \longrightarrow K(i) \longrightarrow \oplus_{j>i} T(j)^{r_{ji}} \xrightarrow{f_i} U(i) \longrightarrow 0 \tag{1}$$

In general, $K(i) \notin \mathcal{G}({}_AT)$. Since $(\mathcal{G}({}_AT), \mathcal{F}({}_AT))$ is the torsion pair in $A-$mod, we get the torsion decomposition

$$0 \longrightarrow Z(i) \longrightarrow K(i) \longrightarrow W(i) \longrightarrow 0 \tag{2}$$

with $Z(i) \in \mathcal{G}({}_AT), W(i) \in \mathcal{F}({}_AT)$. Let $U'(i)$ be the module determined by the following exact sequence in $\mathcal{G}({}_AT)$ induced by (1) and (2)

$$0 \longrightarrow Z(i) \longrightarrow \oplus_{j>i} T(j)^{r_{ji}} \xrightarrow{g_i} U'(i) \longrightarrow 0 \tag{3}$$

thus we have the exact sequence

$$0 \longrightarrow W(i) \longrightarrow U'(i) \xrightarrow{\pi_i} U(i) \longrightarrow 0 \tag{4}$$

Using the above constructons we have

Lemma 5 Let (A, T, B) be a classical tilting triple, for the ordering of B induced by an ordering of indecomposable direct summands of T, we have the standard $B-$module $\triangle_B(i) = Hom_A(T, T(i))/Hom_A(T, U'(i)), 1 \leq i \leq n$.

Proof Applying the tilting functor F to the following commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & Z(i) & \longrightarrow & \oplus_{j>i}T(j)^{r_{ji}} & \xrightarrow{g_i} & U'(i) & \longrightarrow 0 \\ & \downarrow & & \| & & \downarrow \pi_i & \\ 0 \longrightarrow & K(i) & \longrightarrow & \oplus_{j>i}T(i)^{r_{ji}} & \xrightarrow{f_i} & U(i) & \longrightarrow 0 \end{array}$$

we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & FZ(i) & \longrightarrow & \oplus_{j>i}P(j)^{r_{ji}} & \xrightarrow{Fg_i} & FU'(i) & \longrightarrow 0 = & Ext^1_A(T, Z(i)) & \longrightarrow 0 \\ & \downarrow & & \| & & \downarrow F\pi_i & & \downarrow & \\ 0 \longrightarrow & FK(i) & \longrightarrow & \oplus_{j>i}P(j)^{r_{ji}} & \xrightarrow{Ff_i} & FU(i) & \longrightarrow & Ext^1_A(T, K(i)) & \longrightarrow 0 \end{array}$$

since $W(i) \in \mathcal{F}(_AT)$, we see from (4) that $F\pi_i$ is mono, thus $ImFf_i = ImF\pi_i \cong FU'(i)$. On the other hand, since the functor F preserves direct sums and $Hom_B(P(j), P(i)) \cong Hom_A(T(j), T(i))$, by the equivalence of the categories $\mathcal{G}(_AT)$ and $\mathcal{Y}(_AT)$ we can see that $ImFf_i$ is just the sum of all images of mappings $P_B(j) \rightarrow P_B(i)$ with $j > i$, consequently, $\triangle_B(i) = P_B(i)/ImFf_i \cong Hom_A(T, T(i))/Hom_A(T, U'(i))$. $\square$

We denote by $\mathcal{F}(\mathcal{V})$ the full subcatgory of $A-mod$ with objects having a filtration with factors in $\mathcal{V} = \{V(1), \cdots, V(n)\}$. As a consequence of Lemma 5 we have

Collorary 3 Let (A, T, B) be a classical tilting triple with $K(i) \in \mathcal{G}(_AT)$ for $1 \leq i \leq n$, then $\mathcal{F}(\mathcal{V})$ is functorially finite in $A - mod$ and has almost split sequences.

Proof Since $K(i) \in \mathcal{G}(_AT)$, by Lemma 5 we have $Ext^1_A(V(j), V(i)) \cong Ext^1_A(FV(j), FV(i)) = Ext^1_A(\triangle_B(j), \triangle_B(i)) = 0$ for $j \geq i$, it follows from [R1] that $\mathcal{F}(\mathcal{V})$ is functorially finite in $A - mod$. Since $\mathcal{F}(\mathcal{V}) \subseteq \mathcal{G}(_AT)$ and $\mathcal{F}(\triangle)$ is closed under direct summands, by the tilting functor F we know that $\mathcal{F}(\mathcal{V})$ is also closed under direct summands. In [AS] (Thm.2.4) it has been shown that a functorially finite subcategory which is closed under extensions and direct direct summands has relative almost split sequences, thus the last assertion follows. $\square$

With these notions we have

Theorem 2 Let (A, T, B) be a classical tilting triple and $\underline{\mathbf{e}}$ the ordering of B induced by an ordering of indecomposable direct summands of T. If $K(i) \in \mathcal{G}(_AT)$ for $1 \leq i \leq n$, then $(B, \underline{\mathbf{e}})$ is quasi-hereditary if and only if the following conditions are satisfied

(1) $_AT \in \mathcal{F}(\mathcal{V})$;

(2) $Hom_A(T(i), V(i))$ is one dimensional as $k-$space, $1 \leq i \leq n$.

Proof For the ordering $P_B(i) = FT(i), i = 1, \cdots, n$, by Lemma 5 we have $\triangle_B(i) = Hom_A(T, T(i))/Hom_A(T, U'(i))$. Since $K(i) \in \mathcal{G}(_AT)$, $U'(i) = U(i)$, it follows that $\triangle_B(i) = Hom_A(T, T(i))/Hom_A(T, U(i)) \cong FV(i) \in \mathcal{Y}(_AT)$. Since both $\mathcal{G}(_AT)$ and $\mathcal{Y}(_AT)$ are closed under extensions and they are equivalent as categories by the tilting functor F, we see that $_BB \in \mathcal{F}(\triangle_B)$ if and only if $_AT \in \mathcal{F}(\mathcal{V})$; and also it is well known that $End\triangle_B(i) \cong k$ if and only if $Hom_B(P(i), \triangle_B(i)) \cong Hom_A(T(i), V(i))$ is an one-dimensional k-space; this completes the proof. $\square$

We further investigate the conditions in Theorem 2.

Proposition 1 Keep the notations as in Theorem 2, then the following conditions are equivalent for each i, $1 \leq i \leq n$

(1) $K(i) \in \mathcal{G}(_AT)$;

(2) $\triangle_B(i) = FV(i)$;

(3) $\triangle_B(i) \in \mathcal{Y}(_AT)$.

Proof The implication (1) $\Longrightarrow$ (2) is a direct consequence of Lemma 5 and (2) $\Longrightarrow$ (3) is trivial. For (3) $\Longrightarrow$ (1), let $\triangle_B(i) = FX(i)$ with $X(i) \in \mathcal{G}(_AT)$, then by the Lemma 5 we have the exact sequence $0 \longrightarrow FU'(i) \longrightarrow FT(i) \longrightarrow FX(i) \longrightarrow 0$, where the injective map coincides with the F-image of the product of $\pi_i : U'(i) \longrightarrow U(i)$ and $\sigma_i : U(i) \longrightarrow T(i)$, applying the functor $T_B \otimes -$ to the above exact sequence, we then get the exact sequence

$$0 = Tor_1^B(T, FX(i)) \longrightarrow U'(i) \xrightarrow{\pi_i\sigma_i} T(i) \longrightarrow X(i) \longrightarrow 0$$

this implies that π_i is injective, thus $K(i) = Z(i) \in \mathcal{G}(_AT)$. $\square$

Proposion 2 Let (A, T, B) be a classical tilting triple and $K(i) \in \mathcal{G}(_AT)$ for some $1 \leq i \leq n$. Then the following are equivalent

(1) $Hom_A(T(i), V(i))$ is an one dimensional k-space;

(2) For any $f \in EndT(i)$, then either f is an isomorphism, or $Imf \subseteq U(i)$.

Proof (1) $\Longrightarrow$ (2): let $f \in EndT(i)$ and $Imf \not\subseteq U(i)$, denote by p_i the canonical projection $T(i) \to V(i)$, then $fp_i = ap_i$ for some $a \in k - \{0\}$, thus $(Ff)(Fp_i) = aFp_i$ is surjective, so $ImFf + KerFp_i = ImFf + FU(i) = P(i)$. Since $FU(i) \subseteq radP(i)$, Ff is surjective, so is f, thus f is an isomorphism.

(2) $\Longrightarrow$ (1): For any $g \in Hom_B(P(i), \triangle(i))$, then $g = hg_i$ for some $h \in EndP(i)$, where g_i is the cononical projection. Let $h = Ff$ for $f \in EndT(i)$, if f is not isomorphic, then $Imf \subseteq U(i)$, $ImFf \subseteq FU(i)$, hence $g = 0$. This implies that $Hom_A(T(i), V(i)) \cong Hom_B(P(i), \triangle(i))$ is an one dimensional k-space. $\square$

Inspired by the proof of Theorem 6 in [R1] we have the following proposition, for completeness we include the proof.

Proposition 3 Let $M = \oplus_{i=1}^{n} M(i)$ be a basic tilting $A-$module with $M(i)$ indecomposable and $B = End_A M$. If there exists a set of A-modules $\theta = \{\theta(1), \cdots, \theta(n)\}$ with properties

(1) $Hom_A(M(i), \theta(i)) = k; Hom_A(M(j), \theta(i)) = 0; Ext_A^d(M, \theta(i)) = 0$, for $1 \le i \le n, j > i, d \ge 1$.

(2) There is an exact sequence $0 \longrightarrow X(i) \longrightarrow M(i) \longrightarrow \theta(i) \longrightarrow 0$ with $X(i) \in \mathcal{F}(\{\theta(j)/j > i\})$ for $1 \le i \le n$; or with $X(i) \in \mathcal{F}(\{\theta(j)/j < i\})$ for $1 \le i \le n$.

Then B is quasi-hereditary with the ordering induced by $M(i), 1 \le i \le n$.

Proof We first assume that $X(i) \in \mathcal{F}(\{\theta(j)/j > i\})$ for $1 \le i \le n$. By the assumption we have the exact sequence $0 \longrightarrow FX(i) \longrightarrow P_B(i) \longrightarrow F\theta(i) \longrightarrow 0$, where $F = Hom_A(M, -)$. We claim that $\triangle_B(i) = F\theta(i)$ for $1 \le i \le n$. Let $Z(i)$ be the sum of all images of maps $P_B(j) \to P_B(i)$ with $j > i$, then $\triangle_B(i) = P_B(i)/Z(i)$. Since $Hom_B(P_B(j), F\theta(i)) = Hom_A(M(j), \theta(i)) = 0$ for $j > i$, there is an injective $A-$mapping $Z(i) \to FX(i)$. On the other hand, from the surjective mapping $P_B(j) \to F\theta(j)$ and $FX(i) \in \mathcal{F}(\{F\theta(j)/j > i\})$, we may have a surjective mapping $\oplus P_B(j)^{r_{ji}} \to FX(i)$, hence we get an injective mapping $FX(i) \to Z(i)$, so, $\triangle_B(i) \cong F\theta(i), 1 \le i \le n$, from this and the assumptions it follows that B is quasi-hereditary.

For $X(i) \in \mathcal{F}(\{\theta(j)/j < i\})$, $1 \le i \le n$, using the same arguments we can prove that B is quasi-hereditary with $\triangle(i) = F\theta(n - i + 1)$ for $1 \le i \le n$. $\square$

4. The qht-algebras

We would like to study the properties of the class of algebras which can be obtained from quasi-hereditary algebras after classical tilting functors, using tilting theory, this amounts to the study of the algebra A over which there exists a classical tilting module T such that $EndT$ is quasi-hereditary. We only do this for so-called qht-algebras. Keeping the notations of section 3, let (A, T, B) be a classical tilting triple with a fixed ordering of indecomposable direct summands of T, such that the corresponding $K(i) \in \mathcal{G}(_A T)$ for $1 \le i \le n$ and B is quasi-hereditary with the induced ordering, for convenience, we call A a qht-algebra.

Denote by $\mathcal{Y}(\mathcal{V})$ the full subcategory of $A-mod$ with objects Y satisfying $Ext^1_A(X,Y)=0\ \forall X\in\mathcal{F}(\mathcal{V})$, i.e. $\mathcal{Y}(\mathcal{V})=\{Y\in A-mod/Ext^1_A(V(i),Y)=0, 1\leq i\leq n\}$; let $\Theta=\mathcal{F}(\mathcal{V})\cap\mathcal{Y}(\mathcal{V})$.

Theorem 3 Let (A,T,B) be a classical tilting triple and A a $qht-$algebra in the above sense. Then there exists an uniquely determined basic $A-$module M such that $\Theta=addM$ with the following properties

(1) $Ext^i_A(X,Y)=0$ for $X\in\mathcal{F}(\mathcal{V}), Y\in\mathcal{Y}(\mathcal{V})$, $i\geq 1$. In particular, $Ext^i_A(M,M)=0$ for $i\geq 1$;

(2) M has exactly n indecomposable direct summands (up to isomorphism), where n is the number of isomorphism class of simple A-modules; $\forall X\in\mathcal{F}(\mathcal{V}), M-codim({}_AX)<\infty$, i.e. there exists an exact sequence $0\longrightarrow X\longrightarrow M_0\longrightarrow\cdots\longrightarrow M_r\longrightarrow 0$ with $M_i\in addM$ for $0\leq i\leq r$. In particular, $M-codim({}_AT)<\infty$.

(3) $\mathcal{F}(\mathcal{V})=\{X\in A-mod/Ext^1_A(T,X)=0; Ext^i_A(X,M)=0,\forall i\geq 1\}$; and $\mathcal{F}(\mathcal{V})$ is closed under kernels of surjective mappings;

(4) $A_1=End_AM$ is quasi-hereditary; let ${}_{A_1}H$ be the characteristic module of A_1, then $End_{A_1}H\cong B$;

(5) M can be decomposed as $M=\oplus_1^nM(i)$ with $M(i)$ indecomposable, such that there exists an exact sequence $0\longrightarrow V(i)\longrightarrow M(i)\longrightarrow X(i)\longrightarrow 0$ with $X(i)\in\mathcal{F}(\{V(j)/j\geq i\})$ for $1\leq i\leq n$.

Proof By Proposition 1 we see that $\mathcal{F}(\mathcal{V})\subseteq\mathcal{G}({}_AT)$ and $\mathcal{F}(\triangle_B)\subseteq\mathcal{Y}({}_AT)$, so, $\mathcal{F}(\mathcal{V})$ and $\mathcal{F}(\triangle_B)$ are equivalent subcategories under the restrictions of the functors $F=Hom_A(T,-)$ and $T_B\otimes -$. Let $X\in\Theta=\mathcal{F}(\mathcal{V})\cap\mathcal{Y}(\mathcal{V})$, then $Ext^1_B(\triangle_B(i),FX)=Ext^1_A(V(i),X)=0$, thus $FX\in\mathcal{F}(\triangle_B)\cap\mathcal{Y}(\triangle_B)=\mathcal{F}(\triangle_B)\cap\mathcal{F}(\bigtriangledown_B)=\omega=addN$, where N is the characteristic B-module. Let $N=FM$, then $M\in\Theta$, and $X\in addM$, this implies that $\Theta=addM$.

(1) Since A is a $qht-$algebra, $T\in\mathcal{F}(\mathcal{V})$. So, $\forall Y\in\mathcal{Y}(\mathcal{V})=\{Y\in A-mod/Ext^1_A(V(i),Y)=0,1\leq i\leq n\}$, $Ext^1_A(T,Y)=0$, i.e. $Y\in\mathcal{G}({}_AT)$. Thus, we by applying the tilting functor F we have: for $X\in\mathcal{F}(\mathcal{V}),Y\in\mathcal{Y}(\mathcal{V})$, $Ext^i_A(X,Y)\cong Ext^i_A(FX,FY)=0$ by Corollary 3 in [R1].

(2) Since the characteristic module N has n indecomposable direct summands, so does M. For $X\in\mathcal{F}(\mathcal{V})$, then by the Lemma in [R1] we have the exact sequence

$$0\longrightarrow FX\xrightarrow{g_{-1}}N_0\xrightarrow{g_0}\cdots\xrightarrow{g_{r-1}}N_r\longrightarrow 0$$

with $N_i\in addN$ for $0\leq i\leq n$. Let $L_i=kerg_i$, then we get the exact sequences in $\mathcal{Y}({}_AT)$: $0\longrightarrow L_i\longrightarrow N_i\longrightarrow L_{i+1}\longrightarrow 0$, $0\leq i\leq r$, applying $T_B\otimes -$ to these exact sequences and then putting together these sequences, we have the exact sequence

$$0 \longrightarrow X \longrightarrow M_0 \longrightarrow \cdots \longrightarrow M_1 \longrightarrow 0$$

where $M_i = T_B \otimes N_i \in addM, 0 \leq i \leq r$.

(3) Using $\mathcal{F}(\triangle_B) = \{Y/Ext^i_B(Y,N) = 0, \forall i \geq 1\}$ and the tilting functor.

(4) By Theorem 6 and 7 in [R1], $End_A M \cong End_B N$ is quasi-hereditary and $End_{A_1} H \cong B$, where H is the characteristic module of $A_1 = End_B N$.

(5) This follows from [R1, Proposition 2] by applying the tilting functor. $\square$

If A is a finite dimensional k-algebra with k an algebraically closed field and A is a qht-algebra, let α_i be the vector in Z^n whose j-th component is $dim_k(T(j), V(i))$ for $1 \leq i, j \leq n$, hence α_i is just the dimension vector of $\triangle_B(i)$; denote by $C(\mathcal{V})$ the $n \times n$ matrix whose i-th row is α_i, then $C(\mathcal{V})$ is unipotent lower trianglar matrix, so, $\alpha_i, 1 \leq i \leq n$, form a $Z-$basis of the Grothendieck group $K_0(A) = Z^n$. In particular, let $\underline{dim}T(j) = \Sigma_{i=1}^n d_{ij}\alpha_i$, then d_{ij} is just the number of copies of $V(i)$ in any $\mathcal{V}$-filtration of $T(j)$. We denote by d_i the integral vector $(d_{i1}, \cdots, d_{in})$, then d_i is just the dimension vector of $\bigtriangledown_B(i)$; Let D be the matrix whose i-th row is d_i, so D is also unipotent lower triangular, moreover, by using Brauer-Humphreys (or, Bernstein-Gelfand-Gelfand) reciprocity (see [CPS, 3.11], or [DR2, 2.5]) and the relation between Cartan matrices C_A and C_B (see [R2],p.174), we can write out C_A in terms of $T(i)$ and $V(i)$ (recall that C_A is, by definition, the matrix whose i-th column is the transpose of $\underline{dim}P_A(i)$). We formulate these as follows

Corollary 4 In the above setting, we have

(1) $C(\mathcal{V})$ and D are both unipotent lower triangular matrices; in particular, $\alpha_i, 1 \leq i \leq n$; and $d_i, 1 \leq i \leq n$, form $Z-$bases of $K_0(A)$.

(2) Let $\mathbf{dim}\mathcal{V}$ be the matrix whose $i-$th row is $\underline{dim}V(i)$, then the Cartan matrix $C_A = (\mathbf{dim}\mathcal{V})^t \cdot D \cdot C(\mathcal{V})^{-1} \cdot (\mathbf{dim}\mathcal{V})$, where t is the transpose;

(3) The determinant of $\mathbf{dim}\mathcal{V}$ is 1, or -1.

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