$\mathcal{O}$ ALGEBRAS WITH THREE SIMPLE MODULES

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0. Introduction.

In [D], Dyer has introduced the concept of $\mathcal{O}$ algebra, this is a basic quasi-hereditary Koszul algebra with an anti-involution, such that its quadratic dual is also quasi-hereditary. Typical examples of such algebras are ones associated with blocks of category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand [BGG], see also Beilinson-Ginsburg-Soergel [BGS].

The present paper grew out of a desire to understand $\mathcal{O}$ algebras in the framework of the representation theory of finite-dimensional algebras, in particular, inside the one of quasi-hereditary algebras in the sense of Cline-Parshall-Scott [CPS]; see also Dlab-Ringel [DR] and Ringel [R2]. We consider such algebras with three (two) simple modules, this is the simplest and basic case: any $\mathcal{O}$ algebra has a subalgebra of this form.

We recall some definitions and facts in section 1. In section 2 we classify Dyer algebras with three simple modules in terms of quivers and relations (denoted by $\mathcal{O}(l,m,n)$), they are all quadratic. Representation types of $\mathcal{O}(l,m,n)$ are determined in section 3. It is proved in section 4 that algebras $\mathcal{O}(l,m,n)$ are closed under quadratic duals, and that they are all $\mathcal{O}$ algebras; as a corollary the global dimensions are given. In section 5 we give a sufficient condition for the existence of an exact Borel subalgebra (in the sense of König [K]) of a Dyer algebra, and then use this to compute Kazhdan-Lusztig-Vogan polynomials of $\mathcal{O}(l,m,n)$. One of the most important things in representations of a quasi-hereditary algebra is the existence of the characteristic module and the Ringel dual, see [R2]. In section 6 we discuss Ringel duals of $\mathcal{O}(l,m,n,M,B_v)$.

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1. Preliminaries.

1.1. Throughout the paper we consider basic finite-dimensional associative algebras over an algebraically closed field k. By a theorem of Gabriel such an algebra is given by quiver and relations.

Let A-mod denote the category of finite-dimensional left modules over A, and Λ the complete set of orthogonal primitive idempotents of A, which is assumed to be a poset and is called weight poset of A. If $i \in \Lambda$, denote by $L(i)$ the corresponding simple module; by $P(i)$ and $I(i)$ its projective cover and injective hull respectively, then the standard A-module $\Delta(i)$ (the costandard A-modules $\nabla(i)$, resp.) is defined to be the marximal factor module of $P(i)$ with composition factors $L(j)$ for $j \leq i$ (the marximal submodule of $I(i)$ with composition factors $L(j)$ for $j \leq i$, resp.); and A is said to be quasi-hereditary with weight poset Λ provided that first, $L(i)$ occurs with multiplicity one in $\Delta(i)$, and second, all projective A-modules are in $\mathcal{F}(\{\Delta(i), \forall i \in \Lambda\})$, where $\mathcal{F}(\{\Delta(i), \forall i \in \Lambda\})$ denotes the full subcategory of $A-$mod of modules having a filtration with all factors in $\{\Delta(i), \forall i \in \Lambda\}$, see [DR].

Assume that A is a quasi-hereditary algebra with weight poset Λ, then there exists a unique basic (generalized) tilting module T such that $addT = \mathcal{F}(\{\Delta(i), \forall i \in \Lambda\}) \cap \mathcal{F}(\{\nabla(i), \forall i \in \Lambda\})$, where $addT$ is the full subcategory of $A-$mod of direct sums of direct summands of T, and T is then called the characteristic module of (A, Λ); form the endomorphism algebra $B = End_A T$, then B is quasi-hereditary with weight poset Λ^{op}; and B is called the Ringel dual of A; note that the Ringel dual of B is exactly A; see [R2].

1.2. Let $A = k\overrightarrow{\triangle}/I$ with a finite quiver $\overrightarrow{\triangle}$ and an admissible ideal I, we identify the set of vertices of $\overrightarrow{\triangle}$ with the weight poset Λ. Let A_1 be the k-space spanned by arrows of $\overrightarrow{\triangle}$, denote by $A^{\pm}$ the subalgebra of A generated by the union $\Lambda \cup A_1^{\pm}$, where $A_1^+ = \sum\limits_{y>x} yA_1x$, and $A_1^- = \sum\limits_{y<x} yA_1x$. The following is called qh algebra in [D], here we use the name suggested by C.M.Ringel. Note that we don't require that A is a graded algebra.

Definition. Let A be an k-algebra with weight poset Λ, which satisfies the following conditions, then A is called a Dyer algebra with weight poset Λ:

(1) One has $A = A^- A^+$.

(2) One has $dim_k A \geq \sum\limits_{i\in\Lambda} (dim_k \Delta(i))^2$.

(3) There is an involutory algebra anti-automorphism ω of A such that $\omega(i) = i$ for $i \in \Lambda$ and ω sends arrows in $\overrightarrow{\triangle}$ to arrows.

Observe that if A is a Dyer algebra with weight poset Λ, then A is quasi-hereditary with Λ; in this case, the standard module $\Delta(i) = Ai/AA_1^+ i$ for $i \in \Lambda$;

and one has the dimension equality $dim_k A = \sum_{i\in\Lambda}(dim_k\Delta(i))^2$.

1.3. Let $A = k\overrightarrow{\triangle}/I$, if I is generated by relations which are k-linearly combinations of paths of length 2, then A is said to be quadratic. For a quadratic algebra A one can form the quadratic dual $A^!$ of A, see [BGS].

1.4. Let $A = \bigoplus_{n\geq o} A_n$ be a graded algebra with respect to length of paths. A graded A-module M is defined to be a A-module M with a k-space decomposition $M = \bigoplus_{i\in Z} M_i$ such that $A_i M_j \subseteq M_{i+j}$ for all $i \in N, j \in Z$. Let grA denote the the category of graded A-modules, the morphisms $M \longrightarrow N$ in grA are A-morphisms $f : M \longrightarrow N$ with $f(M_i) \subseteq N_i$. For a left graded A-module M, the dual space M^* can be regarded as a right graded A-module with $(M^*)_i = (M_{-i})^*$. If A has an anti-involution ω as graded algebra, then one gets a right graded A-module M^t with operation $ma = \omega(a)m$ for $m \in M, a \in A$; in this case, we obtain a duality δ on grA by $\delta M = (M^t)^*$, which fixing simple modules. For $i \in Z$ let σ^i be the shift functor on grA: $(\sigma^i M)_j = M_{j-i}$.

Definition. Let A be a basic graded k-algebra with weight set Λ, A is called a Koszul algebra provided that $Ext^n_{grA}(L, \sigma^m L) = 0$ unless $n = m$, where $L = \bigoplus_{i\in\Lambda} L(i)$.

Note that a Koszul algebra must be quadratic, see [BGS]. In order to explain equivalent conditions of a Koszul algebra we need the following complex from [D].

Let A be a quadratic graded algebra, N a right graded A-module, M a left graded $A^!$-module, the complex $\mathcal{K}(N, M)$ of graded k-spaces is defined as follows

$$\cdots \longrightarrow \mathcal{K}(N, M)^p \xrightarrow{\partial} \mathcal{K}(N, M)^{p+1} \longrightarrow \cdots$$

where $\mathcal{K}(N, M)^p_q = N_{p+q} \otimes_{A_0} M_p$ for $p, q \in Z$; and $\partial(n \otimes m) = \sum_i n e_i \otimes e_i^* m$, where $\{e_i\}$ is a basis of A_1 and $\{e_i^*\}$ is a dual basis of $A^!_1$. We call $\mathcal{K}(A, (A^!)^*)$ the Koszul complex of A, it is a complex of graded projective left A-modules. Then A is a Koszul algebra if and only if $\mathcal{K}(A, (A^!)^*)$ is a resolution of L (we may identify L with A_0, the k-space spanned by Λ), see [BGS, D].

Defintion 1.5. A Koszul algebra A is called an $\mathcal{O}$ algebra with weight poset Λ if A and $A^!$ are both Dyer algebras with weight poset Λ and Λ^{op} respectively.

Observe that if A is an $\mathcal{O}$ algebra, so is $A^!$. For a quadratic Dyer algebra A, A is an $\mathcal{O}$ algebra if and only if the complex $\mathcal{K}(\Delta(i)^t, \delta\Delta_!(j))$ of k-spaces is acyclic for $i \neq j$ in Λ, where $\Delta_!(j)$ means the standard module of $A^!$, t and δ are in the sense of 1.4, see [D, 2.15].

2. Classification.

2.1. We need a notation of Dlab-Ringel [DR]: let A be a k-algebra, two weight posets Λ and Λ' of A are said to be equivalent provided the sets of standard modules $\{\Delta_\Lambda(\lambda)/\lambda\in\Lambda\}$ and $\{\Delta_{\Lambda'}(\lambda)/\lambda\in\Lambda'\}$ coincide, and sets of costandard modules $\{\nabla_\Lambda(\lambda)/\lambda\in\Lambda\}$ and $\{\nabla_{\Lambda'}(\lambda)/\lambda\in\Lambda'\}$ coincide.

Definition. Let A and A' be two quasi-hereditary algebras with weight posets Λ and Λ' respectively, an algebra isomorphism π from A to A' is called an equivalence provided $\pi(\Lambda)$ and Λ' are equivalent as weight posets of A', where the partial order in $\pi(\Lambda)$ is induced by the one in Λ. In this case, we say (A,Λ) is equivalent to (A',Λ'), or briefly, A is equivalent to A'.

Note that if (A,Λ) is equivalent to (A',Λ'), then Λ and Λ' is not necessarily isomorphic as posets.

Lemma 2.2. Assume that A is a Dyer algebra with weight poset Λ, i and j are not comparable in Λ, then there is no arrow between i and j in the quiver of A. If A is in addtion connected, then the Hasse diagram of Λ is connected.

Proof. Otherwise, let α be an arrow connecting i and j in the quiver of A, then $\alpha\notin A^-A^+=A$, from this assertions follow. $\square$

Proposition 2.3. Let A be a Dyer algebra with weight poset Λ, then Λ can be refined to be a totally ordered set Λ', such that (A,Λ) and (A,Λ') are equivalent.

Proof. Since Λ is finite, Λ can be refined to be a tatolly ordered set Λ'. By Lemma 2.2 we see that for both weight poset Λ and Λ' we obtain the same A_1^+ and A_1^- in 1.2, note that $\Delta(i)=Ai/AA_1^+i$ and $\nabla(i)=D(iA/iA_1^-A)$ for each weight i, where D is the dual $Hom_k(\ ,k)$, so Λ and Λ' are equivalent; and (A,Λ') is also a Dyer algebra. $\square$

Let $\underline{n}$ denote the set $\{1,\cdots n\}$.

Lemma 2.4. Let A be the $k-$algebra given by the following quiver with $l(m^2+n^2)+mn\neq 0$: the set of vertices is $\{x,y,z\}$; there are l arrows from x to z and l arrows from z to x, which are denoted by η_i and ξ_i for $i\in\underline{l}$ respectively; there are m arrows from x to y and m arrows from y to x, which are denoted by α_i and β_i for $i\in\underline{m}$ respectively; there are n arrows from y to z and n arrows from z to y, which are denoted by γ_i and δ_i for $i\in\underline{n}$ respectively. Relations are as follows

$\delta_i\gamma_j$ for $(i,j)\in\underline{n}\times\underline{n}$; $\xi_i\eta_j$ for $(i,j)\in\underline{l}\times\underline{l}$;

$\xi_i\alpha_j$ and $\beta_j\eta_i$ for $(i,j)\in\underline{l}\times\underline{m}$;

$\beta_i\alpha_j-\sum\limits_{s,t}a_{ijst}\gamma_s\delta_t$ for $(i,j)\in\underline{m}\times\underline{m}$, where $a_{ijst}=a_{jits}$ with all a's$\in k$ (if $m=0$, it is understood as there are no such relations; if $n=0$, it reads $\beta_i\alpha_j$);

$2v$ linearly independent $k-$combinations $\sum\limits_{i,j}b_{iju}\alpha_i\gamma_j$, $\sum\limits_{i,j}b_{iju}\delta_j\beta_i$ for $u\in\underline{v}$, where all b's$\in k$ and $0\leqq v\leqq mn$.

Then A is a connected quadratic Dyer algebra with weight poset $\Lambda=\{x<y<z\}$.

Proof. Since $l(m^2+n^2)+mn\neq 0$ and the relations given, A is obviously connected and quadratic. Let ω be the anti-involution of A determined by the followings: $\omega(i) = i$ for $i \in \Lambda$, $\omega(\alpha_i) = \beta_i$ for $i \in \underline{m}$, $\omega(\gamma_i) = \delta_i$ for $i \in \underline{n}$ and $\omega(\eta_i) = \xi_i$ for $i \in \underline{l}$. This is indeed well defined, by the relations given.

Let V be the $k-$space spanned by $\alpha_i\gamma_j$ for $(i,j) \in \underline{m}\times\underline{n}$, and V_1 the $k-$subspace of V spanned by $k-$ combinations $\sum\limits_{i,j} b_{iju}\alpha_i\gamma_j$ for $u \in \underline{v}$. One can choose a subset T of $\underline{m}\times\underline{n}$ with $|T| = mn - v$ such that $\{\alpha_i\gamma_j/(i,j) \in T\}$ is a basis of V/V_1. Then A has a basis consisting of the followings:

$i \in \Lambda$; α_i and β_i for $i \in \underline{m}$; γ_i and δ_i for $i \in \underline{n}$; η_i and ξ_i for $i \in \underline{l}$; $\alpha_i\beta_j$ for $(i,j) \in \underline{m}\times\underline{m}$; $\gamma_i\delta_j$ for $(i,j) \in \underline{n}\times\underline{n}$; $\eta_i\xi_j$ for $(i,j) \in \underline{l}\times\underline{l}$; $\eta_i\delta_j$ and $\gamma_j\xi_i$ for $(i,j) \in \underline{n}\times\underline{l}$; $\alpha_i\gamma_j$ and $\delta_j\beta_i$ for $(i,j) \in T$; $\alpha_i\gamma_j\delta_s$ and $\gamma_s\delta_j\beta_i$ for $(i,j,s) \in T\times\underline{n}$; $\alpha_i\gamma_j\xi_s$ and $\eta_s\delta_j\beta_i$ for $(i,j,s) \in T\times\underline{l}$; $\alpha_i\gamma_j\delta_s\beta_t$ for $(i,j,t,s) \in T\times T$.

So one has $A = A^-A^+$. Consider standard the modules $\Delta(i) = Ai/AA_1^+i$ for $i \in \Lambda$, then $\Delta(x) = L(x)$; $\Delta(y)$ has a basis consisting of y and α_i for $i \in \underline{m}$; and $\Delta(z)$ has a basis consisting of z, η_i for $i \in \underline{l}$, γ_i for $i \in \underline{n}$ and $\alpha_i\gamma_j$ for $(i,j) \in T$. Thus one has the dimensional equality

$$\begin{aligned}
dim_kA &= 3 + 2(l+m+n) + l^2 + m^2 + n^2 + 2ln + 2(mn-v) + 2n(mn-v) \\
&\quad + 2l(mn-v) + (mn-v)^2 \\
&= 1 + (1+m)^2 + (1+l+n+mn-v)^2 \\
&= \sum_{i\in\Lambda}(dim_k\Delta(i))^2
\end{aligned}$$

This completes the proof. $\square$

2.5. For later use we need some notations. For a pair $(i,j) \in \underline{m}\times\underline{m}$, from relations in Lemma 2.4 one forms a $n\times n$ matrix $a_{ij} = (a_{ijst})$, and then gets a $mn \times mn$ symmetric matrix $M = (a_{ij})$. Let B_v denote the $mn\times v$ matrix $(b_1, \cdots, b_v)$, where $b_u = (b_{11u}, \cdots, b_{1nu}, \cdots, b_{iju}, \cdots, b_{mnu})^t$ for $u \in \underline{v}$, t means the transpose and v suggests the rank. We denote the algebra in 2.4 together with the linear order $\{x < y < z\}$ by $\mathcal{O}(l,m,n,M,B_v)$, or briefly, $\mathcal{O}(l,m,n)$, where $l(m^2+n^2) + mn \neq 0$.

By interchanging the positions of a_{ijst} and a_{stij} for $(i,j) \in \underline{m}\times\underline{m}$ and $(s,t) \in \underline{n}\times\underline{n}$ in M, one gets a symmetric matrix, denote by P_M. Since B_v is a $mn\times v$ matrix with rank v, one can find a $(mn-v)\times mn$ matrix $(c_1, \cdots, c_{mn-v})^t$ of rank $mn-v$ such that $c_wb_u = 0$ for $w \in \underline{mn-v}$ and $u \in \underline{v}$. Let $c_w = (c_{11w}, \cdots, c_{n1w}, \cdots, c_{jiw}, \cdots, c_{nmw})$, then

$$\sum_{i,j} c_{jiw}b_{iju} = 0 \quad for \quad u \in \underline{v}, \quad w \in \underline{mn-v}. \tag{1}$$

Denote by L_{B_v} the $mn\times(mn-v)$ matrix $(d_1, \cdots, d_{mn-v})$ with $d_w = (c_{11w}, \cdots, c_{1nw}, \cdots, c_{ijw}, \cdots, c_{mnw})^t$ for $w \in \underline{mn-v}$, where $rank(L_{B_v}) = mn - v$.

Lemma 2.6. Let A be a connected Dyer algebra with weight poset $\Lambda = \{x < y < z\}$. Then there exist deta l, m, n, M and B_v such that $A \simeq \mathcal{O}(l, m, n, M, B_v)$; in particular, A is quadratic.

Proof. Let ω be the anti-involution of A, then ω sends arrows from i to j to arrows from j to i, where $i, j \in \Lambda$. So the number of arrows from i to j is equal to the number of arrows from j to i. Since A is quasi-hereditary, the quiver of A contains no loops, it follows that the quiver of A must be the one in Lemma 2.4, say $\vec{\triangle}$. Let $A = k\vec{\triangle}/I$, where I is an ideal in $k\vec{\triangle}$ generated by some $k-$combinations of paths of length at least two. Since $A = A^-A^+$ and $zA^-A^+z = \langle z\rangle$, $zA^-A^+y = \langle \delta_i$ for $i \in \underline{n}\rangle$, $yA^-A^+z = \langle \gamma_i$ for $i \in \underline{n}\rangle$, $yA^-A^+y = \langle y, \gamma_i\delta_j$ for $(i,j) \in \underline{n}\times\underline{n}\rangle$ (where $\langle\ ,\ \rangle$ denotes the k-space spanned by elements inside), there exist a $mn \times mn$ matrix M and a $mn \times v$ matrix B_v with rank v, where $0 \leqq v \leqq mn$, such that all relations in 2.4 are contained in I. Since $\omega(\beta_i\alpha_j - \sum\limits_{s,t} a_{ijst}\gamma_s\delta_t) = \beta_j\alpha_i - \sum\limits_{s,t} a_{ijst}\gamma_t\delta_s$, one has $a_{ijst} = a_{jits}$, i.e. M is symmetric.

We claim that I is exactly the ideal generated by all relations in 2.4. In fact, note that bases of standard modules $\Delta(i)$ for $i \in \Lambda$ are still as stated in the proof of Lemma 2.4, so, if there are other relations not generated by the ones in Lemma 2.4, then the dimension of A must be less than $\sum\limits_{i\in\Lambda}(dim_k\Delta(i))^2$, this implies that $A \simeq \mathcal{O}(l, m, n, M, B_v)$. $\Box$

Summerizing 2.3, 2.4 and 2.6 we have

Theorem 2.7. Let A be a connected k-algebra with three simple modules. Then A is a Dyer algebra if and only if A is equivalent to some $\mathcal{O}(l, m, n, M, B_v)$ for $l(m^2 + n^2) + mn \neq 0$. In particular, A is quadratic.

Corollary 2.8. (1) Let Λ be the poset $\{x < y, x < z\}$, A a connected $k-$algebra. Then A is a Dyer algebra with weight poset Λ if and only if A is equivalent to some $\mathcal{O}(l, m, 0)$ with $lm \neq 0$.

(2) Let Λ be the poset $\{x < z, y < z\}$, A a connected k-algebra with weight poset Λ if and only if A is equivalent to some $\mathcal{O}(l, 0, n)$ with $ln \neq 0$.

Proof. Let $\Lambda = \{x < y, x < z\}$. By 2.2 and 2.3 we have the necessity of (1); by dropping the order $y < z$ in the weight poset of $\mathcal{O}(l, m, 0)$, one gets also a Dyer algebra with weight poset Λ, which has the same standard modules and costandard modules with $\mathcal{O}(l, m, 0)$, this proves the sufficiency of (1); and (2) is similarly proved. $\Box$

Remark 2.9. A is a basic connected quasi-hereditary algebra with weight poset $\{x < y\}$ if and only if A is given by the quiver which has s arrows from x to y (denoted by α_i for $\alpha \in \underline{s}$) and t arrows from y to x(denoted by β_j for $j \in \underline{t}$), and with relations $\beta_i\alpha_j$ for $(i,j) \in \underline{t} \times \underline{s}$. Such an algebra is a Dyer algebra if and only if $s = t$.

3. Representation types.

With notations in 2.5, we have

Theorem 3.1. Let A be a connected Dyer algebra with three simple modules, then

(i) A is representation-finite if and only if A is isomorphic to $\mathcal{O}(0,1,1,1,0)$ or $\mathcal{O}(0,1,1,\lambda,1)$ for $\lambda \in \{0,1\}$;

(ii) A is of tame representation type if and only if A is isomorphic to one of $\mathcal{O}(1,1,0)$, $\mathcal{O}(1,0,1)$ and $\mathcal{O}(0,1,1,0,0)$;

(iii) A is of wild representation type if and only if A is not the case in **(i)** and **(ii)**.

3.2. In order to prove the Theorem, we need a special case of a result due to C.M.Ringel.

Let Q be the quiver with the set of vertices $\{a, a_1, \cdots, a_{s+t}\}$, and the arrows $\alpha_i : a_i \longrightarrow a$ and $\beta_j : a \longrightarrow a_{s+j}$ for $1 \le i \le s$ and $1 \le j \le t$. Let Γ be the union of the subquiver Q and some subquiver Γ', which is not specfied, and the two subquivers intersect in precisely the set $\{a_1, \cdots, a_{s+t}\}$ of vertices, but in no arrows. We admit that some of a_i coincide. Consider the quiver $\triangle$ obtained as follows: replace the vertex a in Γ by two vertices a' and a'' (note that: $a' \neq a''$); and keep all others fixed (of course, α_i in Γ is replaced by $\alpha_i : a_i \longrightarrow a''$ and β_j is replaced by $\beta_j : a' \longrightarrow a_{s+j}$. Note that $\triangle$ has the same subquiver Γ').

Consider the functor ϕ: $k\triangle - mod \longrightarrow k\Gamma - mod$ which associates to a representation $V = (V_i, \varphi_\alpha)$ of $\triangle$ the representation W with $W|_{\Gamma'} = V|_{\Gamma'}$ and $W_a = V_{a'} \oplus V_{a''}$ and correspondingly forming direct sums of the given linear transformations.

Lemma. The functor ϕ induces a bijection between the indecomposable representations V in $k\triangle - mod$ and in $A - mod$ which are non-zero at at least one of the vertices a_i, where $A = k\Gamma/\langle \alpha_i\beta_j \quad for \quad (i,j) \in \underline{s} \times \underline{t}\rangle$.

Proof. Assume that W is indecomposable in $A - mod$, then $W|_a = X \oplus Y$, where Y is the sum of all images of the maps α_i for $i \in \underline{s}$. Let $V \in k\triangle - mod$: $V|_{\Gamma'} = W|_{\Gamma'}$, $V|_{a'} = X$ and $V|_{a''} = Y$, then $W = \phi(V)$. The rest is easily seen. □

Remark. If A is given by the quiver Γ above and relations $\alpha_i\beta_j$ for $(i,j) \in \underline{s} \times \underline{t}$ and some other relations (not only in Γ', even α_i and β_j are involved), then the corresponding assertion is also true (just reserve the corresponding relations for $\triangle$).

3.3. Proof of Theorem 3.1.

We divide this into several steps

(1) Using Theorem 1 of Reiten-Donkin in [DoRe] one gets (i).

(2) Algebras of the form $\mathcal{O}(l,m,n)$ where there at least one of l,m,n is not less than 2 are of wild representation type, since they have quotient algebras $k(\cdot \rightrightarrows \cdot \leftarrow \cdot)$ or $k(\cdot \leftarrow \cdot \rightrightarrows \cdot)$.

(3) $\mathcal{O}(1,1,0)$ is of tame representation type, since it is a string algebra in the sense of [BR] and not representation-finite.

(4) $\mathcal{O}(1,0,1)$ is of tame representation type. In fact, by using Lemma 2 in [R1,p.153], one sees that $\mathcal{O}(1,0,1)$ has the same representation type with the algebra $k(\cdot \underset{\delta}{\overset{\gamma}{\rightleftarrows}} \overset{\downarrow}{\underset{\downarrow}{\cdot}})/I$, where I is the ideal generated by $\delta\gamma$; repeating this argument, then $\mathcal{O}(1,0,1)$ has the same representation type with the tame quiver $\widetilde{D}_4$.

(5) Consider $A \simeq \mathcal{O}(0,1,1,\lambda,\mu)$ for $\lambda,\mu \in k$. If $\mu \neq 0$, then one can take $\mu = 1$, and normalize the relation such that $\lambda \in \{0,1\}$, this is the case $\mathcal{O}(0,1,1,\lambda,1)$ in (1). If $\mu = 0$ and $\lambda \neq 0$, then again normalizing the relation and reducing this to the case $\mathcal{O}(0,1,1,1,0)$ in (1). Finally, again by using Lemma 2 in [R1, p.153] we know that $\mathcal{O}(0,1,1,0,0)$ has the same representation type with the tame quiver $\widetilde{D}_4$.

(6) It remains to check that algebra A of the type $\mathcal{O}(1,1,1)$ is of wild representation type. Consider the algebra $A' = A/I$, where $A = \mathcal{O}(1,1,1,\lambda,\mu)$ and I is the ideal generated by $\beta\alpha$(with the notation in 2.4). By using twice the Remark in 3.2 one finds that A' has the same representation type with an algebra B, such that B has a quotient algebra which is of wild representation type.

Note that (1)-(6) are all possible cases, the theorem is proved. $\square$

4. $\mathcal{O}$ algebras with three simple modules.

Theorem 4.1. Keep notations in 2.5, then

$$\mathcal{O}(l,m,n,M,B_v)^! = \mathcal{O}(l,n,m,-P_M,L_{B_v})$$

note that the algebra on left is with weight poset $\Lambda = \{x < y < z\}$ and the right one is with weight poset $\Lambda^{op} = \{z < y < x\}$; in particular, $\mathcal{O}(l,m,0)^! = \mathcal{O}(l,0,m)$.

Proof. Write $\mathcal{O}(l,m,n,M,B_v)$ as a quotient of a tensor algebra: let V be the k-space spanned by all arrows of the quiver of $\mathcal{O}(l,m,n,M,B_v)$, A_0 the semisimple subalgebra generated by x,y,z; W the k-space spanned by all relations in Lemma 2.4, then $\mathcal{O}(l,m,n,M,B_v) = T_{A_0}(V)/\langle W\rangle$, where $\langle W\rangle$ denotes the ideal generated by W in $T_{A_0}(V)$, the tensor algebra of V over A_0. Form the dual space V^*, by α^*, denotes the element of the dual basis of V^* corresponding to α. Let U be the k-subspace of $V^* \otimes_{A_0} V^*$ spanned by the following elements:

$\beta_i^* \otimes \alpha_j^*$ for $(i,j) \in \underline{m} \times \underline{m}$; $\xi_i^* \otimes \eta_j^*$ for $(i,j) \in \underline{l} \times \underline{l}$;

$\xi_i^* \otimes \gamma_j^*$ and $\delta_j^* \otimes \eta_i^*$ for $(i,j) \in \underline{l} \times \underline{n}$;

$\delta_i^* \otimes \gamma_j^* + \sum\limits_{s,t} a_{stij}\alpha_s^* \otimes \beta_t^*$ for $(i,j) \in \underline{n} \times \underline{n}$;

$2(mn-v)$ linearly independent k-combinations $\sum\limits_{i,j} c_{ijw}\gamma_i^* \otimes \alpha_j^*$ and $\sum\limits_{i,j} c_{ijw}\beta_j^* \otimes \delta_i^*$ for $(i,j) \in \underline{n} \times \underline{m}$, where $w \in \underline{(mn-v)}$, and c_{ijw} is in the sense of 2.5.

Using the equality (1) in 2.5 and making direct calculations one finds that $f(r) = 0$ for all $f \in U$ and $r \in W$. Since $dim_k W + dim_k U = dim_k(V \otimes_{A_0} V)$, one has $U = W^{\perp}$, where $W^{\perp} := \{f \in V^* \otimes V^*/f(W) = 0\}$ (here we identify

$V^* \otimes V^*$ with $(V \otimes V)^*$), this proves that $\mathcal{O}(l,m,n,M,B_v)^! := T_{A_0}(V^*)/\langle W^\perp\rangle \simeq \mathcal{O}(l,n,m,-P_M,L_{B_v})$. $\square$

Now we can give a classification of $\mathcal{O}$ algebras with three simple modules

Theorem 4.2. Let A be a connected k-algebra with three simple modules. Then A is $\mathcal{O}$ algebra if and only if A isomorphic to some $\mathcal{O}(l,m,n,M,L_{B_v})$ for $l(m^2+n^2)+mn \neq 0$.

Proof. By Theorem 2.7, 4.1 and the definition of $\mathcal{O}$ algebra, it suffices to prove that $\mathcal{O}(l,m,n,M,B_v)$ is a Koszul algebra. Let $A = \mathcal{O}(l,m,n,M,B_v)$ with weight poset $\Lambda = \{x < y < z\}$, using Proposition 2.15 in [D], we need to verify that the complex $\mathcal{K}(\Delta(i)^t, \delta\Delta_!(j))$ of graded vector spaces is acyclic for $i \neq j$ in Λ.

Keep notations in Lemma 2.4, then grounded k-spaces of $\Delta(i)$ and $\Delta_!(j)$ are as follows

$\Delta(x) = \langle x\rangle$; $\Delta(y) = \langle y,\ \ \alpha_i\ \ for\ \ i \in \underline{m}\rangle$; $\Delta(z) = \langle z,\ \ \eta_i\ \ for\ \ i \in \underline{l},\ \ \gamma_i\ \ for\ \ i \in \underline{n},\ \ \alpha_i\gamma_j\ \ for\ \ (i,j) \in \underline{m}\times\underline{n}\rangle$; $\Delta_!(x) = \langle x,\ \ \eta_i^*\ \ for\ \ i \in \underline{l},\ \ \alpha_i^*\ \ for\ \ i \in \underline{m},\ \ \gamma_i^*\alpha_j^*\ \ for\ \ (i,j) \in \underline{n}\times\underline{m}\rangle$; $\Delta_!(y) = \langle y,\ \ \gamma_i^*\ \ for\ \ i \in \underline{n}\rangle$; $\Delta_!(z) = \langle z\rangle$.

Note that A and $A_!$ are graded algebra respect to lengths of paths; and standard modules are graded modules in the natural way. Since $\Delta_!(i)_p = 0$ for $p > 2, i \in \Lambda$, where $\Delta_!(i)_p$ denotes the $p-$th homogenous component of $\Delta_!(i)_p$ as a graded $A^!$-module, it suffices to prove that the following complex is acyclic for $i \neq j$ in Λ

$$0 \longrightarrow \mathcal{K}(\Delta(i)^t, \delta\Delta_!(j))^{-2} \xrightarrow{\partial} \mathcal{K}(\Delta(i)^t, \delta\Delta_!(j))^{-1} \xrightarrow{\partial} \mathcal{K}(\Delta(i)^t, \delta\Delta_!(j))^{0} \longrightarrow 0$$

i.e.

$$0 \longrightarrow \Delta(i)^t \otimes (\Delta_!(j)_2^t)^* \xrightarrow{\partial} \Delta(i)^t \otimes (\Delta_!(j)_1^t)^* \xrightarrow{\partial} \Delta(i)^t \otimes j \longrightarrow 0$$

where the tensor product is over A_0, the semisimple subalgebra of A generated by x, y, z. Writing out this graded complex in terms of the homogenous components and noting that $i \otimes j = 0 = \Delta(i)_2^t \otimes (\Delta_!(j)_2^t)^*$ for $i \neq j$, it reduces to check the following isomorphisms under ∂ for $i \neq j$

$$\Delta(i)_1^t \otimes (\Delta_!(j)_2^t)^* \simeq \Delta(i)_2^t \otimes (\Delta_!(j)_1^t)^*;\ \ i \otimes (\Delta_!(j)_1^t)^* \simeq \Delta(i)_1^t \otimes j.$$

and the exactness of the following complex for $i \neq j$

$$0 \longrightarrow i \otimes (\Delta_!(j)_2^t)^* \xrightarrow{\partial} \Delta(i)_1^t \otimes (\Delta_!(j)_1^t)^* \xrightarrow{\partial} \Delta(i)_2^t \otimes j \longrightarrow 0$$

We make the verification for $i = z, j = x$ in the above complex, the rest can be done by direct calculations. Let $(\langle\gamma_i^*\alpha_j^*\rangle^t)^*$ denote the k-space $(\Delta_!(x)_2^t)^*$, and by $((\gamma_i^*\alpha_j^*)^t)^*$ denote the corresponding element in it, using grounded spaces of $\Delta(i)$ and $\Delta_!(j)$ listed above and the operation of t and $*$, then the complex above reads

$$0 \longrightarrow z \otimes (\langle\gamma_i^*\alpha_j^*\rangle^t)^* \xrightarrow{\partial} \langle\eta_i, \gamma_j\rangle^t \otimes (\langle\eta_i^*, \alpha_j^*\rangle^t)^* \xrightarrow{\partial} \langle\alpha_i\gamma_j\rangle^t \otimes x \longrightarrow 0$$

or

$$0 \longrightarrow ((\langle \gamma_i^* \alpha_j^* \rangle^t)^* \xrightarrow{\partial} \langle \gamma_i \rangle^t \otimes ((\langle \alpha_j^* \rangle^t)^* \xrightarrow{\partial} \langle \alpha_i \gamma_j \rangle^t \longrightarrow 0 \qquad (2)$$

where $\partial((\gamma_i^* \alpha_j^*)^t)^* = \gamma_i^t \otimes ((\alpha_j^*)^t)^*$; $\partial(\gamma_i^t \otimes ((\alpha_j^*)^t)^*) = (\alpha_j \gamma_i)^t$ for $(i,j) \in \underline{n} \times \underline{m}$.

Note that the first ∂ is injective and the second one is surjective; and the dimension equality

$$dim_k((\langle \gamma_i^* \alpha_j^* \rangle^t)^* + dim_k \langle \alpha_i \gamma_j \rangle^t = v + (mn - v) = dim_k \langle \gamma_i \rangle^t \otimes ((\langle \alpha_j^* \rangle^t)^*.$$

From this it follows that (2) must be exact. This completes the proof. $\square$

Remark 4.3. (1) Let $\Lambda = \{x < y\}$, then A is $\mathcal{O}$ algebra with weight poset Λ if and only if A is Dyer algebra with weight poset Λ.

(2) Let A be an $\mathcal{O}$ algebra with weight poset Λ, where $|\Lambda| \geq 3$. Assume that $\Lambda' = \{x_i / i = 1, 2, 3\}$ is a coideal of Λ, i.e. if $y \in \Lambda$, $y \geq$ some x_i, then $y \in \Lambda'$. Then by [D,Proposition 2.13] and 4.2, eAe is isomorphic to some $\mathcal{O}(l, m, n, M, B_v)$, where $e = x_1 + x_2 + x_3$.

Proposition 4.4. Assume that A is an arbitrary Koszul algebra and $A^!$ its quadratic dual. Then the global dimension of A is equal to the Loewy length of $A^!$.

Proof. Since A is Koszul, there is an isomorphism $(A^!)^{op} \simeq Ext_A^\bullet(L, L)$ of graded k-algebra, where $Ext_A^\bullet$ denotes the Yoneda Ext-algebra in ungraded A-modules, L is the direct sum of all non-isomorphic simple A-modules, see [BGS]. Let $A^! = \bigoplus_{i \geq 0} A_i^!$ as graded algebra, then $rad^m A^! = \bigoplus_{i \geq m} A_i^!$ for $m \geq 1$, and $A_i^! \simeq Ext_A^i(L, L)$ as k-space for $i \geq 0$, from this the assertion follows. $\square$

From Theorem 4.1 one can see the Loewy length of $\mathcal{O}(l, m, n, M, B_v)^!$, so we have

Corollary 4.5. Let l, m, n be non-negative integers such that $l(m^2+n^2)+mn \neq 0$, then

$$gl.dim.\mathcal{O}(l, m, n, M, B_v) = \begin{cases} 4, & mn \neq 0 \\ 2, & mn = 0 \end{cases}$$

5. Kazhdan-Lusztig-Vogan polynomials.

5.1. For definitions and criterions for an exact Borel subalgebra and a Δ-subalgebra of a quasi-hereditary algebra we refer to [K]. Since we consider basic algebra, both notations of exact Borel($\Delta-$, resp.) and strong exact Borel(strong $\Delta-$ resp.) subalgebras coincide. Let $A = k\overrightarrow{\Delta}/I$ be a Dyer algebra with weight poset Λ, we keep notions in 1.2, i.e. let A_1 be the k-space of A spanned by arrows of $\overrightarrow{\Delta}$, $A^{\pm}$ the subalgebra of A generated by the union $\Lambda \cup A_1^{\pm}$, where $A_1^+ = \sum_{y>x} y A_1 x$ and $A_1^- = \sum_{y<x} y A_1 x$. In this case, if B is an exact Borel subalgebra of A, then $\omega(B)$ is a Δ-subalgebra of A, where ω is the anti-involution of A. The

following proposition give a sufficient condition for the existence for an exact Borel subalgebra of A in terms of usual quiver and relations.

Proposition. Keeping notions above, then A^+ is an exact Borel subalgebra of A if and only if $iA^+ \cap iA_1^- A = 0$ for every weight $i \in \Lambda$.

Proof. It is clear that A^+ is directed, i.e. the usual quiver of $B := A^+$ has no oriented cycles. By Theorem A in [K] it suffices to verify that the given condition is equivalent to the following: for each weight $i \in \Lambda$, the restriction from $A-$modules to $B-$modules gives an isomorphism of $B-$modules: $\nabla_A(i) \simeq \nabla_B(i)$. Since A^+ is directed, the costandard module $\nabla_B(i)$ is just the indecomposable injective $B-$module $D(iB) = D(iA^+)$, where D is the dual functor $Hom_k(\ ,k)$. By definition one has $\nabla_A(i) = D(iA/iA_1^- A)$. Note that $iA = iA^- A^+ = iA^+ + iA_1^- A^- A^+ = iA^+ + iA_1^- A$, and $\nabla_A(i) \simeq D(iA^+/iA^+ \cap iA_1^- A)$, from this one sees the assertion. $\square$

Corollary 5.2. Let $A = \mathcal{O}(l,m,n,M,B_v)$, then A^+ and A^- are respectively an exact Borel subalgebra and Δ-subalgebra of A.

5.3. Let A be a quasi-hereditary algebra with weight poset Λ. Given $y \leq w$ in Λ, the Kazhdan-Lusztig-Vogan polynomial associated to y and w (see [I]) is

$$P_{y,w}(q) = \sum_i (-1)^{d(y,w)-i} dim_k Ext_A^i(\Delta(y), L(w)) q^{(d(y,w)-i)/2}$$

where $d(y,w)$ is the maximum of the non-negative integer n such that there is a sequence $y = y_0 < y_1 < \cdots < y_n = w$ in Λ. The following proposition was given in [I] in case A is a BGG algebra, since BGG-reciprocity holds for any quasi-hereditary algebra([DR, Lemma 2.5]), one has also the following

Proposition. Let A be a quasi-hereditary algebra with weight poset Λ, y and w be elements of Λ, then

(i) $P_{w,w}(q) = 1$;

(ii) If $y < w$, $P_{y,w}(1) = (-1)^{d(y,w)}[L(w) : \Delta(y)]$, where $[L(w) : \Delta(y)]$ denotes the $\Delta(y)-$coefficient of $L(w)$ in $Gr(A)$.

Proposition 5.4. Let $A = \mathcal{O}(l,m,n,M,B_v)$ with weight poset $\Lambda = \{x < y < z\}$, then $P_{x,y}(q) = m$; $P_{y,z}(q) = n$; $P_{x,z}(q) = -lq^{1/2} + v$.

Proof. We only compute $P_{x,z}(q)$, the rest is easy. By 5.2(ii) $B := A^+$ is an exact Borel subalgebra of A, so one has $Ext_A^i(\Delta(x), L(z)) = Ext_A^i(A \otimes_B L(x), L(z)) \simeq Ext_B^i((L(x), L(z))$, for the last isomorphism we refer to [DX, 3.5]. It is clear that $radP_B(x) = P(z)^l \oplus N$, where N is spanned by β_i for $i \in \underline{m}$ and $\delta_j\beta_i$ for $(i,j) \in T$(we keep notations in Lemma 2.4). Let $\Theta = \{i \in \underline{m}/\delta_j\beta_i = 0\ \ in\ \ A,\ \ \forall j \in \underline{n}\}$, and $|\Theta| = t$, then $N = L(y)^t \oplus (\bigoplus_{i \in \underline{m}-\Theta} N_i)$, where N_i is the $B-$module $B\beta_i$. Denote by n_i the dimension of the $k-$space $\langle \delta_j\beta_i, \forall j \in \underline{n}\rangle$, then we have the exact sequence for each $i \in \underline{m} - \Theta$

$$0 \longrightarrow L(z)^{n_i} \longrightarrow N_i \longrightarrow L(y) \longrightarrow 0 \tag{3}$$

Since $L(z) = P_B(z)$ and $p.d._B L(y) = 1$, one has $p.d._B L(x) = 2$. Applying $Hom_B(\ \ , L(z))$ to the exact sequence

$$0 \longrightarrow P_B(z)^l \oplus N \longrightarrow P(x) \longrightarrow L(x) \longrightarrow 0$$

one gets

$$Hom_B(P_B(z)^l \oplus N, L(z)) = Ext^1_B(L(x), L(z)) \simeq k^l;$$
$$Ext^1_B(N, L(z)) = Ext^2_B(L(x), L(z)).$$

From the first equality above one sees $Hom_B(N, L(z)) = 0$, using this and applying $Hom_B(\ \ , L(z))$ to (2) one finds $Ext^1_B(N_i, L(z)) \simeq k^{n-n_i}$. Thus

$$Ext^2_B(L(x), L(z)) \simeq Ext^1_B(L(y)^t, L(z)) \oplus (\bigoplus_{i \in \underline{m}-\Theta} Ext^1_B(N_i, L(z))$$

so

$$dim Ext^2_B(L(x), L(z)) = nt + n(m-t) - \sum_{i \in \underline{m}-\Theta} n_i = nm - (mn - v) = v$$

it follows that

$$\begin{aligned} P_{x,z}(q) &= -dim_k Ext^1_A(\Delta(x), L(z))q^{1/2} + dim_k Ext^2_A(\Delta(x), L(z)) \\ &= -dim_k Ext^1_B(L(x), L(z))q^{1/2} + dim_k Ext^2_B(L(x), L(z)) \\ &= -lq^{1/2} + v.\square \end{aligned}$$

6. Ringel duals.

6.1. We have the following simple observation from the definition of the characteristic module.

Lemma. Let (A, Λ) and (A, Λ') be equivalent as quasi-hereditary algebras, then they have the same characteristic module.

6.2. Let A be an algebra of the form $\mathcal{O}(m, 0, n)$ for $mn \neq 0$, we want determine the characteristic A-module. By Lemma 6.1 we may assume that the weight poset $\Lambda = \{x < z, y < z\}$ (keep notations in 2.4). Then

$$radP(x) = P(z)^m; radP(y) = P(z)^n; radP(z) = L(x)^m \oplus L(y)^n$$

let $T = T(x) \oplus T(y) \oplus T(z)$ be the characteristic module of A, since both x, y are minimal in Λ, one has $T(x) = \Delta(x) = L(x)$, $T(y) = \Delta(y) = L(y)$, we need to determine $T(z)$. By applying $Hom_A(\ \ ,P(z))$ to the exact sequence $0 \longrightarrow P(z)^m \longrightarrow P(x) \longrightarrow L(x) \longrightarrow 0$ one gets $dim_k Ext^1_A(L(x),P(z)) = m$; similarly, $dim_k Ext^1_A(L(y),P(z)) = n$. Let e_i for $i \in \underline{m}$ and ϵ_j for $j \in \underline{n}$ be bases of $Ext^1_A(L(x),P(z))$ and $Ext^1_A(L(y),P(z))$ respectively, by making the push-out of the exact sequence $(\bigoplus_{i=1}^{m} e_i) \oplus (\bigoplus_{j=1}^{n} \epsilon_j)$ along the diagonal map $P(z)^{m+n} \longrightarrow P(z)$ which sends $(x_1,\cdots,x_{m+n})$ to $x_1+\cdots+x_{m+n}$ one obtains the exact sequence

$$0 \longrightarrow P(z) \longrightarrow C_z \longrightarrow L(x)^m \oplus L(y)^n \longrightarrow 0$$

We call this push-out the universal sequence. Applying $Hom_A(L(x)\oplus L(y),\ \ )$ to the universal sequence one has

$$\begin{aligned} Hom_A(L(x)\oplus L(y), L(x)^m \oplus L(y)^n) &\longrightarrow Ext^1_A(L(x)\oplus L(y), P(z)) \\ &\longrightarrow Ext^1_A(L(x)\oplus L(y), C_z) \longrightarrow 0 \end{aligned}$$

the first map is surjective by the construction of the universal sequence, so $Ext^1_A(L(x)\oplus L(y), C_z) = 0$. This means that $C_z \in \mathcal{F}(\Delta)\cap\mathcal{Y}(\Delta)) = \mathcal{F}(\Delta)\cap\mathcal{F}(\nabla))$. We claim that $T(z) = C_z$. In fact, $C_z \in addT$, C_z is of the form $L(x)^a \oplus L(y)^b \oplus T(z)^c$ with $a,b \geq 0$, $c \geq 1$. Since $Hom_A(\Delta(z), C_z) = Hom_A(\Delta(z), T(z)^c) \simeq End\Delta(z) = 1$, one has $c = 1$. From the universal sequence one gets the exact sequence $0 \longrightarrow P(z) \longrightarrow T(z) \longrightarrow L(x)^{m-a} \oplus L(y)^{n-b} \longrightarrow 0$, then one has a sujection $Hom_A(L(x), L(x)^{m-a} \oplus L(y)^{n-b}) \longrightarrow Ext^1_A(L(x),P(z)) \longrightarrow 0$, comparing dimensions one gets that $a = 0 = b$. This proves $T(z) = C_z$.

The same arguments can be applied to $\mathcal{O}(m,n,0)$, so we obtain

Proposition. (1) Let A be an algebra of the form $\mathcal{O}(m,n,0)$, then the characteristic A-module $T = L(x) \oplus T(y) \oplus T(z)$, where $T(y)$ and $T(z)$ are given by the following universal sequences

$$0 \longrightarrow P(y) \longrightarrow T(y) \longrightarrow L(x)^n \longrightarrow 0;$$

$$0 \longrightarrow P(z) \longrightarrow T(z) \longrightarrow L(x)^m \longrightarrow 0.$$

(2) Let A be an algebra of the form $\mathcal{O}(m,0,n)$, then the characteristic A-module $T = L(x) \oplus L(y) \oplus T(z)$, where $T(z)$ is given by the universal sequence

$$0 \longrightarrow P(z) \xrightarrow{\sigma} T(z) \xrightarrow{(\rho_1,\rho_2)} L(x)^m \oplus L(y)^n \longrightarrow 0 \qquad (4)$$

Theorem 6.3. $\mathcal{RO}(m,n,0) = \mathcal{O}(m,0,n)$; $\mathcal{RO}(m,0,n) = \mathcal{O}(m,n,0)$, where $\mathcal{R}$ denotes the Ringel dual.

Proof. It suffices to prove the second equality above. Let $A = \mathcal{O}(m,0,n)$ with weight poset $\Lambda = \{x < z, y < z\}$, and $B := \mathcal{RO}(m,0,n) = End_A T$, where

$T = L(x) \oplus L(y) \oplus T(z)$ is the characteristic A-module. Denote by $P_B(i)$ the indecomposable projective B-module corresponding to $T(i)$ for $i \in \Lambda$. Then by using the tilting equivalence one sees

$$Hom_B(P_B(x), P_B(y)) = 0 = Hom_B(P_B(y), P_B(x)).$$

Applying some suitable functors to (4) one finds

$$dim_k Hom_B(P_B(x), P_B(z)) = m = dim_k Hom_B(P_B(z), P_B(x));$$

$$dim_k Hom_B(P_B(y), P_B(z)) = n = dim_k Hom_B(P_B(z), P_B(y)).$$

Since $radP_A(z) = L(x)^m \oplus L(y)^n$, one gets m canonical injections λ_i: $L(x) \hookrightarrow radP_A(z) \hookrightarrow P_A(z) \xrightarrow{\sigma} T(z)$ and n canonical injections λ'_i from $L(y)$ to $T(z)$, where $i \in \underline{n}$. Then the corresponding maps in $Hom_B(P_B(x), P_B(z))$ induced by λ_i for $i \in \underline{n}$ form a base of $Hom_B(P_B(x), P_B(z))$, by construction it is clearly a base of $Irr(P_B(x), P_B(z)) = radHom_B(P_B(x), P_B(z))/rad^2 Hom_B(P_B(x), P_B(z))$. Similarly, one gets a basis of $Irr(P_B(y), P_B(z))$. Dually, by using canonical projections $\mu_i : T(z) \xrightarrow{\rho_1} L(x)^m \xrightarrow{p_i} L(x)$ and $\mu'_j : T(z) \xrightarrow{\rho_2} L(y)^n \xrightarrow{p_j} L(y)$ for $i \in \underline{m}$ and $j \in \underline{n}$, one gets the corresponding bases of $Irr(P_B(z), P_B(x))$ and $Irr(P_B(z), P_B(y))$. It follows that the quiver of B must be the one of $\mathcal{O}(m, n, 0)$ (note that B has no loop since B is quasi-hereditary). Since $\lambda_i \mu_j = 0$ for $(i, j) \in \underline{m} \times \underline{m}$; $\lambda'_i \mu'_j = 0$ for $(i; j) \in \underline{n} \times \underline{n}$; $\lambda_i \mu'_j = 0 = \lambda'_j \mu_i$ for $(i, j) \in \underline{m} \times \underline{n}$; and note that there is no other relations, B is of the form $\mathcal{O}(m, n, 0)$ with weight poset $\Lambda^{op} = \{z < x, z < y\}$. This completes the proof. □

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