A REALIZATION OF AFFINE LIE ALGEBRAS OF TYPE $\tilde{A}_{n-1}$ VIA THE DERIVED CATEGORIES OF CYCLIC QUIVERS

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Abstract. By using the locally finite-dimensional representations and the derived categories of a linear quiver $\Delta(\infty)$ and a cyclic quiver $\Delta(n)$, we give a realization of infinite rank affine Lie algebra A_∞ and affine Kac-Moody algebras of type $\tilde{A}_{n-1}$. Also their Chevalley bases are well displayed.

1. Introduction

According to the investigations of Ringel ([9],[10] etc.), there are Lie algebra structure arising from the module categories of a quite large class of representation-finite algebras. The most important case is that for a representation-finite hereditary algebra A, he showed that the Lie algebra $K(\mathrm{mod}\ A)$ generated by indecomposable modules is isomorphic to the Chevalley $\mathbf{Z}$-form of $\mathbf{n}_+$, where $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$ is a triangular decomposition of the corresponding complex semisimple Lie algebra $\mathbf{g}$. Now it is sure that Ringel's method is effective not only for representation-finite cases (see [1],[11]). However one should use the composition algebra to replace the Hall algebra in the representation-infinite cases.

Naturally the next question is that whether it is possible to use representations of algebras to recover the whole Lie algebra $\mathbf{g}$ and not only $\mathbf{n}^+$. In [6] we gave a complete realization of complex semisimple Lie algebras in terms of the root categories. Denote by $D^b(B)$ the derived category of a finite dimensional algebra B, this is a triangulated category with the translation functor T. In particular, the category $D^b(B)$ was well described when B is a tilted algebra. Let A be a representation-finite hereditary algebra, Happel[2] pointed out that the isomorphism classes of indecomposable objects in $D^b(A)/T^2$ correspond bijectively to the root system of $\mathbf{g}$ (not only positive roots). For this reason, $\mathcal{R}(A) = D^b(A)/T^2$ is called the root category. Even when A is a representation-infinite hereditary algebra, the root category $\mathcal{R}(A)$ still provide a good framework which is comparable with the root system of the corresponding Kac-Moody algebras. We try to give the realization of affine Kac-Moody algebras by using the root category $\mathcal{R}(k\Delta)$, where Δ is the corresponding Euclidean

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quiver. In this paper, as the first step, we obtain the infinite rank affine Lie algebra A_∞ and affine Kac-Moody algebras of type $\tilde{A}_{n-1}$ from the root categories of locally finite dimensional representations of the linear quiver $\Delta(\infty)$ and the cyclic quiver $\Delta(n)$ respectively. At the same time we set up their Chevalley bases corresponding to the indecomposable objects in the root categories.

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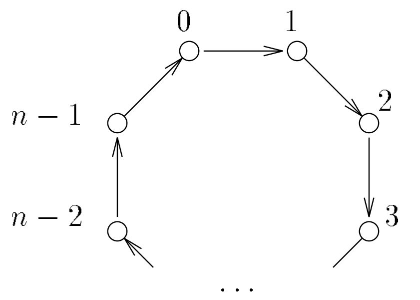
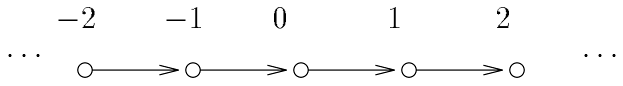
2. REPRESENTATIONS AND DERIVED CATEGORIES OF CYCLIC QUIVERS

A quiver $\Delta = (\Delta_0, \Delta_1) : \Delta_0$ is the set of vertices, Δ_1 the set of arrows. Let k be a field, a k-representation $V = (V_x, V_\alpha)$ of Δ is given by vector spaces V_x for all $x \in \Delta_0$ and linear maps $V_\alpha : V_x \to V_y$ for any arrow $\alpha : x \to y$. If V and V' are two representations of Δ, a map $f = (f_x) : V \to V'$ is given by maps $f_x : V_x \to V'_x$ for $x \in \Delta_0$ such that $f_x V'_\alpha = V_\alpha f_y$ for any arrow $\alpha : x \to y$. In this way we obtain the category of representations of Δ. A representation $V = (V_x, V_\alpha)$ of Δ is finite dimensional if $\sum_{x\in\Delta_0} dim_k V_x < \infty$. Denote by $rep(\Delta, k)$ the category of finite dimensional k-representations of Δ. In the present work we consider the finite dimensional representations of the cyclic quivers and the induced categories. A cyclic quiver $\Delta = \Delta(n)$ is the quiver



with $n \geq 1$; however maybe $n = \infty$, $\Delta(\infty)$ is the quiver



A stable translation quiver $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ is given by a locally finite quiver (Γ_0, Γ_1), where Γ_0 denotes the vertex set and Γ_1 the set of arrows, together with a bijective map $\tau : \Gamma_0 \to \Gamma_0$ such that for any $y, z \in \Gamma_0$ the number of arrows from y to z is equal to the number of arrows from τz to y. In this article the arrows in Γ are called irreducible maps and the map τ is called the AR translation (Auslander-Reiten translation). We consider a special translation quiver in this article, namely a tube $\mathcal{T}$ of rank n which is isomorphic to $\mathbf{Z}A_\infty/\tau^n, n > 0$. The constructure of $\mathcal{T}$ is as follows.

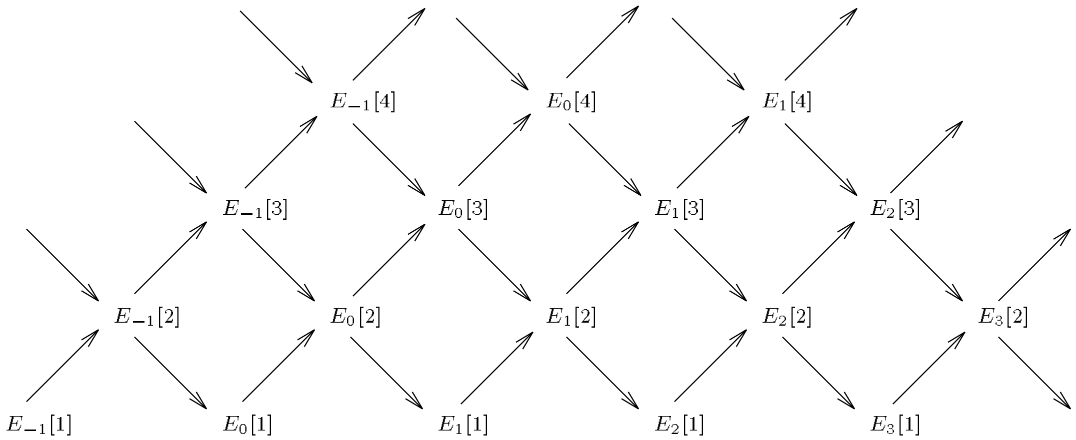


Fig 1

where $E_i[j] = E_l[m]$ if and only if $j = m$ and $i \equiv l(\mathrm{mod}\ \ n)$, $\tau E_i[j] = E_{i-1}[j]$. Particularly we take the translation quiver $\mathbf{Z}A_\infty$ as the tube of rank ∞ for granted, in this situation $E_i[j] = E_l[m]$ if and only if $j = m$ and $i = l$.

The category $rep(\Delta, k)$ when $\Delta = \Delta(n)$ can be completely described.

For $n < \infty$, the irreducible $\Delta(n)$-representations consist of two types. Type 1: for $0 < i < n-1$ $E_i[1] = (V_j, V_\alpha)$, where $V_j = k$ if $j = i$, $V_j = 0$ if $j \neq i$; $V_\alpha = 0$ for any $\alpha \in \Delta_1$. Type 2: for $\lambda \in k^*$, define $E_\lambda[1] = (V_j, V_\alpha)$, where $V_j = k$ for any $j \in \Delta_0$, $V_\alpha = \lambda$ for $\alpha : n-1 \to 0$ and $V_\alpha = 1$ for other α.The full subcategory $\mathcal{T}_0$ of $rep(\Delta(n), k)$ which objects are finite dimensional representations with composition factors belonning to some isomorphism classes in $\{E_i[1] | i = 0, 1, \cdots, n-1\}$ is the categoy of locally nipotent finite dimensional representations of $\Delta(n)$ (see[11]). It is isomorphic to the mesh category of the tube of rank n. The Auslander-Reiten quiver of $\mathcal{T}_0$ is as in Fig 1. Given any $\lambda \in k^*$, the full subcategory $\mathcal{T}_\lambda$ which objects are finite dimensional representations with composition factors isomorphic to $E_\lambda[1]$ is isomorphic to the mesh category of a homogeneous tube (ie, a tube of rank 1). A standard fact in representation theory (for example, see [12]) is that $rep(\Delta(n), k)$ is a standard stable tublar $\mathbb{A}^1(k)-$family of type (n), that is,

$$rep(\Delta(n), k) = k(\mathcal{T}_0) \oplus \bigoplus_{\lambda \in k^*} k(\mathcal{T}_\lambda)$$

where $k(\mathcal{T}_i)$ is the mesh category of $\mathcal{T}_i$, there is no nonzero morphism between different tubes.

The irreducible representations of $\Delta(\infty)$ are, for any $i \in \mathbf{Z}$, $E_i[1] = (V_j, V_\alpha)$ where $V_j = k$ if $j = i$ and $V_j = 0$ if $j \neq i$, $V_\alpha = 0$ for any arrow α. $rep(\Delta(\infty), k)$ is isomorphic to the mesh category of $\mathbf{Z}A_\infty$.

Given an abelian category $\mathcal{A}$, delote by $D^b(\mathcal{A})$ the derived category of bounded complexes over $\mathcal{A}$. The category $D^b(\mathcal{A})$ is a triangulated category with the translation functor T obtained from $K^b(\mathcal{A})$ by localizing with respect to the set of quasi-isomorphisms. For a more detailed account of the derived categories in the representation theory we refer to [2] and [3]. If we take $\mathcal{A} = rep(\Delta(n), k)$, we simply denote their derived categories by $D^b(\Delta(n), k)$. They can be easily described as follows. For $1 \leq n < \infty, D^b(\Delta(n), k)$ consists of the standard components $\{T^i\mathcal{T}_0, T^i\mathcal{T}_\lambda \mid \lambda \in k^*, i \in \mathbf{Z}\}$, the only nonzero morphisms between different components are those from $T^i\mathcal{T}_0$ to $T^{i+1}\mathcal{T}_0$ and from $T^i\mathcal{T}_\lambda$ to $T^{i+1}\mathcal{T}_\lambda$ for $\lambda \in k^*$. Since

$$D^b(\Delta(n), k) \simeq \sum_{i \in \mathbf{Z}} k(T^i\mathcal{T}_0) \oplus \bigoplus_{\lambda \in k^*} \sum_{i \in \mathbf{Z}} k(T^i\mathcal{T}_\lambda),$$

we can consider the full subcategory (also the quotient category) $D_0^b(\Delta(n), k) \simeq \sum_{i \in \mathbf{Z}} k(T^i\mathcal{T}_0)$ of $D^b(\Delta(n), k)$. For $n = \infty$, $D^b(\Delta(\infty), k)$ consists of the standard components $\{T^i\Gamma \mid \Gamma \simeq \mathbf{Z}A_\infty, i \in \mathbf{Z}\}$, the only nonzero morphisms between different components are those from $T^i\Gamma$ to $T^{i+1}\Gamma$. For an uniform terminology we take also $D_0^b(\Delta(\infty), k) = D^b(\Delta(\infty), k)$.

All the facts above can be easily got according to the following two observations.

1. If $n = \infty$ we have the canonical embedding from $rep(A_{2m+1}, k)$ to $rep(\Delta(\infty), k)$, where A_{2m+1} is the quiver

$$\overset{-m}{\circ} \longrightarrow \overset{-(m-1)}{\circ} \cdots \quad \overset{-1}{\circ} \longrightarrow \overset{0}{\circ} \longrightarrow \overset{1}{\circ} \quad \cdots \quad \overset{m-1}{\circ} \longrightarrow \overset{m}{\circ}$$

For any finite dimensional representation V of $\Delta(\infty)$, the support of V is contained in A_{2m+1} for almost all m. Note that the embedding is addmissible with the translation functor T, so we have the canonical embedding from $D^b(k(A_{2m+1}))$ to $D^b(\Delta(\infty), k)$.

2. We can embed $rep(\Delta(n), k)$ into the category of representations of the quiver Δ'

$$\begin{tikzcd}
& 1 \arrow[r] & 2 & \cdots & n-2 \arrow[r] & n-1 \arrow[dr] & \\
0 \arrow[ur] \arrow[rrrrrr, "\gamma"] & & & & & & n
\end{tikzcd}$$

as the full subcategory $\mathcal{U}$ of all representations (V_i, V_α) with V_γ being an identity map. Let $\mathcal{T}(\infty)$ be the component of $rep(\Delta', k)$ containing the quasi-simple module with $V_\gamma = 0$. and let $\mathcal{T}$ be the category of regular representations of Δ'. Then $\mathcal{T} = \mathcal{U} \oplus \mathcal{T}(\infty)$ (see [12], p160).

There exists a canonical Galois covering $\Delta(\infty) \to \Delta(n)$ for any $n \in \mathbf{N}$. Correspondly we have the covering functor $G : rep(\Delta(\infty), k) \to rep(\Delta(n), k)$. It is easy to see that $G(rep(\Delta(\infty), k)) = \mathcal{T}_0$. The Galois group of the covering functor G is the cyclic group generated by τ^n, where τ is the Auslander-Reiten translation in $rep(\Delta(\infty), k)$. Simlarly the covering functor G is admissible with the translation functor T, that is, $GT = TG$. So G induces a covering functor, still denoted by G, $D^b(\Delta(\infty), k) \to D^b(\Delta(n), k)$ and $G(D^b(\Delta(\infty), k)) = D^b_0(\Delta(n), k)$. The Galois group of this covering functor is also the cyclic group generated by τ^n. Therefore $D^b_0(\Delta(n), k)$ is isomorphic to the orbit category $D^b(\Delta(\infty), k)/\tau^n$.

For our porpuse we should consider the orbit categories of $D^b_0(\Delta(n), k)$ under the automorphism T^2, ie, we define $\mathcal{R}(n) = D^b_0(\Delta(n), k)/T^2$ for $n \in \mathbf{N} \cup \{\infty\}$ and $F : D^b_0(\Delta(n), k) \to \mathcal{R}(n)$ is the canonical covering functor. Following Happel[2] in the representation-finite situation, $\mathcal{R}(n)$ is called the root category. The covering functor G induces a covering from $\mathcal{R}(\infty)$ to $\mathcal{R}(n)$, we also denote as $G : \mathcal{R}(\infty) \to \mathcal{R}(n)$.

Let $\mathcal{C}$ be an additive category together with a translation functor T and a set Ω of sextuples in $\mathcal{C}$. We call $(\mathcal{C}, T, \Omega)$ an 'almost'triangulated category and the elements of Ω are then called triangles if the following axioms are satisfied:

(Tr1) Every sextuple isomorphic to a triangle is a triangle. The sextuple $X \xrightarrow{1_X} X \xrightarrow{0} 0 \xrightarrow{0} TX$ is a triangle, where 1_X is the identity morphism.

(Tr2) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is a triangle,then $Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY$ is a triangle.

(Tr3) Given two triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} TX'$ and morphisms $f : X \to X'$, $g : Y \to Y'$ such that $fu' = ug$, then there exists a morphism $h : Z \to Z'$ such that (f, g, h) induces a morphism from the first triangle to the second.

(Tr4) (The octahedral axiom) Suppose given triangles:

$$X \xrightarrow{u} Y \xrightarrow{i} Z' \xrightarrow{i'} TX$$

$$Y \xrightarrow{v} Z \xrightarrow{j} X' \xrightarrow{j'} TY$$

$$X \xrightarrow{uv} Z \xrightarrow{k} Y' \xrightarrow{k'} TX.$$

Then there exist morphisms $f : Z' \to Y', g : Y' \to X'$ such that the following diagram commutes and the third row is a triangle.

$$\begin{array}{ccccccccc}
T^{-1}Y' & \xrightarrow{T^{-1}k'} & X & \xrightarrow{1_X} & X & & & & \\
\downarrow T^{-1}g & & \downarrow u & & \downarrow uv & & & & \\
T^{-1}X' & \xrightarrow{T^{-1}j'} & Y & \xrightarrow{v} & Z & \xrightarrow{j} & X' & \xrightarrow{j'} & TY \\
 & & \downarrow i & & \downarrow k & & \downarrow 1_{X'} & & \downarrow Ti \\
 & & Z' & \xrightarrow{f} & Y' & \xrightarrow{g} & X' & \xrightarrow{j'Ti} & TZ' \\
 & & \downarrow i' & & \downarrow k' & & & & \\
 & & TX & \xrightarrow{1_{TX}} & TX & & & &
\end{array}$$

Now we define a set Ω in $\mathcal{R}(n)$ consisting of all sxtuples isomorphic to

$$FM \xrightarrow{Ff} FN \xrightarrow{Fg} FL \xrightarrow{Fh} TFM$$

with

$$M \xrightarrow{f} N \xrightarrow{g} L \xrightarrow{h} TM$$

being a triangle in $\mathrm{D}_0^b(\Delta(n), k)$. It is not difficult to check that $\mathcal{R}(n)$ together with T and Ω is an 'almost'triangulated category.

3. Triangles and orbit spaces in $\mathcal{R}(n)$

In this section we assume k is a finite field and use $|\ |$ to denote the ordinary cardinality. Given $X, Y, Z \in \mathcal{R}(n)$, put

$$W(X, Y; Z) = \{(f, g, h) \in \mathrm{Hom}_{\mathcal{R}(n)}(X, Z) \times \mathrm{Hom}_{\mathcal{R}(n)}(Z, Y) \times \mathrm{Hom}_{\mathcal{R}(n)}(Y, TX) |$$

$$X \xrightarrow{f} Z \xrightarrow{g} Y \xrightarrow{h} TX \text{ is a triangle}\}.$$

We can introduce the action of $\mathrm{Aut}\, X \times \mathrm{Aut}\, Y$ on $W(X, Y; Z)$ as follows

$$(\alpha, \beta)(f, g, h) = (\alpha f, g\beta, \beta^{-1}h(T\alpha)^{-1})$$

for any $(\alpha, \beta) \in \mathrm{Aut}\, X \times \mathrm{Aut}\, Y$, $(f, g, h) \in W(X, Y; Z)$ and define the orbit space to be

$$V(X, Y; Z) = W(X, Y; Z)/(\mathrm{Aut}\, X \times \mathrm{Aut}\, Y).$$

If X, Y and Z belong to a tube simultaneously, we can take them as representations of the quiver $\Delta(n)$, that is, modules of the hereditary algebra $k\Delta(n)$. Define $V_0(X, Y; Z)$ to be the set of all filtrations $Z \supset U \supset 0$ of Z such that $U \simeq X$ and $Z/U \simeq Y$; $W_0(X, Y; Z)$ to be the set of (f, g) such that the sequence $0 \to X \xrightarrow{f} Z \xrightarrow{g} Y \to 0$ is exact in $\mathrm{mod}\, k\Delta(n)$.

Lemma 3.1. Let X, Y and Z belong to a tube simultaneously. Then the action of $\mathrm{Aut}\, X \times \mathrm{Aut}\, Y$ on $W(X, Y; Z)$ is free.

Proof. Easy. $\square$

Lemma 3.2. Let X, Y and Z belong to a tube simultaneously. Then $|W(X,Y;Z)| = |W_0(X,Y;Z)|$ and $|V(X,Y;Z)| = |V_0(X,Y;Z)|$. In particular, if X, Y and Z are indecomposable, then $|V(X,Y;Z)| = 1$ if and only if there is a triangle of form $X \to Z \to Y \to TX$; otherwise $|V(X,Y;Z)| = 0$.

Proof. The sextuple $X \xrightarrow{f} Z \xrightarrow{g} Y \xrightarrow{h} TX$ is a triangle if and only if $0 \to X \xrightarrow{f} Z \xrightarrow{g} Y \to 0$ is a exact sequence and h is the corresponding element in $\mathrm{Hom}(Y,TX) \simeq \mathrm{Ext}(Y,X)$. So $|W(X,Y;Z)| = |W_0(X,Y;Z)|$. The map $\theta : W_0(X,Y;Z) \to V_0(X,Y;Z)$ defined by $\theta(f,g) = (0 \subset im(f) \subset Z)$ is surjective, and its fibers are all isomorphic to $\mathrm{Aut}\, X \times \mathrm{Aut}\, Y$. Thus $|\mathrm{Aut}\, X|\,|\mathrm{Aut}\, Y|\,|V_0(X,Y;Z)| = |W_0(X,Y;Z)|$. We have $|V(X,Y;Z)| = |V_0(X,Y;Z)|$ in view of Lemma 3.1. If X, Y and Z are indecomposable in a tube, they are uniserial as modules of $k\Delta(n)$. Therefore $|V_0(X,Y;Z)| = 1$ if and only if there is a exact sequence of form $0 \to X \to Z \to Y \to 0$ in $\mathrm{mod}\, k\Delta(n)$. The statement follows. $\square$

Lemma 3.3. Let X, Y, Z be indecomposable and belong to a tube simultaneously. Then

$$|V(Z,TX;Y)| = \frac{|V(X,Y;Z)|\,|\mathrm{Aut}\, Y|\,|\mathrm{Hom}(Z,X)|^2}{|\mathrm{Aut}\, Z|\,|\mathrm{Hom}(Y,X)|}$$

Proof. By Lemma 3.1,

$$|V(X,Y;Z)| = \frac{|W(X,Y;Z)|}{|\mathrm{Aut}\, X|\,|\mathrm{Aut}\, Y|} = \frac{|W(Z,TX;Y)|}{|\mathrm{Aut}\, X|\,|\mathrm{Aut}\, Y|}.$$

Consider the isotropy group of the action of $\mathrm{Aut}\, Z \times \mathrm{Aut}\, TX$ on $W(Z,TX;Y)$. Given any triangle of the form

$$Z \xrightarrow{g} Y \xrightarrow{h} TX \xrightarrow{-Tf} TZ.$$

The corresponding isotropy group is denoted by G_0. Let $\alpha \in \mathrm{Aut}\, Z, \beta \in \mathrm{Aut}\, X$ satisfying that $\alpha g = g, hT\beta = h$ and $(T\beta)^{-1}TfT\alpha = Tf$. Then $(\alpha - 1)g = 0, h(1 - T\beta) = 0$ and $f\alpha = \beta f$. Since

$$X \xrightarrow{f} Z \xrightarrow{g} Y \xrightarrow{h} TX$$

and

$$Y \xrightarrow{h} TX \xrightarrow{-Tf} TZ \xrightarrow{-Tg} TY$$

are triangles, there exist s and t in $\mathrm{Hom}(Z,X)$ such that $\alpha - 1 = sf$ and $1 - T\beta = TfTt$, that is, $\alpha = 1 + sf$ and $\beta = 1 - ft$. Therefore the map

$$\mathrm{Hom}(Z,X) \times \mathrm{Hom}(Z,X) \longrightarrow G_0 \subset \mathrm{Aut}\, Z \times \mathrm{Aut}\, TX$$

given by $(s,t) \mapsto (\alpha, T\beta)$ with $\alpha = 1 + sf$ and $\beta = 1 - ft$ is surjective. If $\alpha = 1 + s_1 f = 1 + s_2 f$, then $(s_1 - s_2)f = 0$. Because we have the exact sequence $0 \to X \xrightarrow{f} Z \xrightarrow{g} Y \to 0$ in $\mathrm{mod}\, k\Delta(n)$, f is injective, then $s_1 = s_2$. If $\beta = 1 - ft_1 = 1 - ft_2$, that is, $f(t_1 - t_2) = 0$. There is $r : Y \to X$ such that $t_1 - t_2 = gr$. Since g is surjective in $\operatorname{mod} k\Delta(n)$, r is uniquely determined. We have proved that $|G_0| = |\operatorname{Hom}(Z,X)|^2/|\operatorname{Hom}(Y,X)|$. So

$$|W(Z,TX;Y)| = \frac{|V(Z,TX;Y)|\,|\operatorname{Aut} Z|\,|\operatorname{Aut} X|\,|\operatorname{Hom}(Y,X)|}{|\operatorname{Hom}(Z,X)|^2}$$

We have

$$|V(Z,TX;Y)| = \frac{|V(X,Y;Z)|\,|\operatorname{Aut} Y|\,|\operatorname{Hom}(Z,X)|^2}{|\operatorname{Aut} Z|\,|\operatorname{Hom}(Y,X)|}$$

□

Simlarly we can prove the following

Lemma 3.4. Let X,Y,Z be indecomposable and belong to a tube simultaneously. Then

$$|V(T^{-1}Y,Z;X)| = \frac{|V(X,Y;Z)|\,|\operatorname{Aut} X|\,|\operatorname{Hom}(Y,Z)|^2}{|\operatorname{Aut} Z|\,|\operatorname{Hom}(Y,X)|}$$

4. Lie algebras arising in $\mathcal{R}(n)$

We consider $D^b_0(\Delta(n),k)$ in case k is a finite field. By our observations (1) and (2) in Section 1, we know that any indecomposable object of $D^b_0(\Delta(n),k)$ is isomorphic to a stalk complex with indecomposable stalk (see [3],5.2). Let E be a finite extension of k, then $X^E = X\otimes_k E$ is well-defined for any indecomposable object X in $D^b_0(\Delta(n),k)$ which is taken as a stalk complex. X^E is still a stalk complex in $D^b_0(\Delta(n),E)$. Let M be any object $D^b_0(\Delta(n),k)$, then $M = X_1\oplus\cdots\oplus X_t$ with X_i indecomposable, we define $M^E = X_1^E\oplus\cdots\oplus X_t^E$. It is easy to see that every object in $D^b_0(\Delta(n),E)$ is isomorphic to some M^E with $M\in D^b_0(\Delta(n),k)$.

If a field E is a finite extension of k, remember from [10] that E is called conserative for $D^b_0(\Delta(n),k)$ provided $(\operatorname{End} X/\operatorname{rad}\operatorname{End} X)^E$ is a field for any indecomposable object $X\in D^b_0(\Delta(n),k)$. Let Ω be the set of isomorphism classes of finite extensions of k which are conserative for $D^b_0(\Delta(n),k)$. Since $\operatorname{End} X/\operatorname{rad}\operatorname{End} X = k$ for any indecomposable object $X\in D^b_0(\Delta(n),k)$, it follows that Ω is an infinite set from [10]. Following [9] we have the definition.

For objects $M,N,L\in D^b_0(\Delta(n),k)$ $(\mathcal{R}(n))$, if there exists a integral polynomial $\varphi^L_{MN}\in\mathbf{Z}[q]$ such that

$$\varphi^L_{MN}(|E|) = |V(M^E,N^E;L^E)|$$

holds for any $E\in\Omega$. Then we say that $D^b_0(\Delta(n),k)$ $(\mathcal{R}(n))$ has the Hall polynomial φ^L_{MN}.

Lemma 4.1. $D^b_0(\Delta(\infty),k)$ $(\mathcal{R}(\infty))$ has the polynomials φ^L_{MN} and φ^L_{NM} for any $M,N,L\in D^b_0(\Delta(\infty),k)$ $(\mathcal{R}(\infty))$ with N indecomposable.

Proof. There is a sufficiently large m such that $M, N, L \in \mathrm{D}^b(kA_{2m+1})$, moreover, the Hall polynomials φ^L_{MN} and φ^L_{NM} for $D^b_0(\Delta(\infty), k)$ are equal to φ^L_{MN} and φ^L_{NM} for $D^b(kA_{2m+1})$ respectively. The existence of the later is known from [9]. $\square$

Lemma 4.2. $D^b_0(\Delta(n), k)$ ($\mathcal{R}(n)$) has the polynomials φ^Z_{XY} for any indecomposable objects $X, Y, Z \in D^b_0(\Delta(n), k)$ ($\mathcal{R}(n)$).

Proof. Let $G : D^b_0(\Delta(\infty), k) \to D^b_0(\Delta(n), k)$ be the Galois covering. Choose an indecomposable object $\tilde{Y} \in D^b_0(\Delta(\infty), k)$ such that $G\tilde{Y} = Y$. If $W(X, Y; Z) \neq \emptyset$, according to the standardity of the component $\mathbf{Z}A_\infty$, there are unique indecomposable objects $\tilde{X}, \tilde{Z} \in D^b_0(\Delta(\infty), k)$ such that G induces a bijection $W(\tilde{X}, \tilde{Y}; \tilde{Z}) \to W(X, Y; Z)$. By Lemma 3.1, 3.3 and 3.4, we have the existence of φ^Z_{XY}. $\square$

Corollary 4.1. Let X, Y and Z be indecomposable in $\mathcal{R}(n)$. If there exists a triangle of form $X \to Z \to Y \to TX$ then $\varphi^Z_{XY}(1) = 1$; otherwise $\varphi^Z_{XY}(1) = 0$.

Proof. Let X, Y and Z belong to a tube simultaneously, we can view them as uniserial modules of $k\Delta(n)$. Obviously $|V(X^E, Y^E; Z^E)| = 1$ if and only if there exists a triangle of form $X \to Z \to Y \to TX$ in $\mathcal{R}(n)$. So $\varphi^Z_{XY} \equiv 1$. The other situations can be reduced to as in Lemma 3.3 and 3.4. For example, we compute $\varphi^Y_{Z\,TX}$. Since

$$|V(Z^E, TX^E; Y^E)| = \frac{|V(X^E, Y^E; Z^E)|\,|\operatorname{Aut} Y^E|\,|\operatorname{Hom}(Z^E, X^E)|^2}{|\operatorname{Aut} Z^E|\,|\operatorname{Hom}(Y^E, X^E)|}$$

by Lemma 3.3 and $\operatorname{Aut} Y^E = E^* \oplus \operatorname{rad}\ \operatorname{End} Y^E$, $\operatorname{Aut} Z^E = E^* \oplus \operatorname{rad}\ \operatorname{End} Z^E$, we have

$$|V(Z^E, TX^E; Y^E)| = \frac{|V(X^E, Y^E; Z^E)|\,|\operatorname{rad}\ \operatorname{End} Y^E|\,|\operatorname{Hom}(Z^E, X^E)|^2}{|\operatorname{rad}\ \operatorname{End} Z^E|\,|\operatorname{Hom}(Y^E, X^E)|}.$$

The polynomials for $\operatorname{rad}\ \operatorname{End} Y^E, \operatorname{rad}\ \operatorname{End} Z^E$ and $\operatorname{Hom}(Z^E, X^E), \operatorname{Hom}(Y^E, X^E)$ are just equal to some powers of q. So $\varphi^Y_{Z\,TX}(1) = \varphi^Z_{XY}(1) = 1$ if and only if there is a triangle of form $Z \to Y \to TX \to TZ$. Otherwise $\varphi^Y_{Z\,TX}(1) = \varphi^Z_{XY}(1) = 0$. $\square$

We consider the Grothendieck group of $\mathcal{R}(n)$ as usual, but denote it by $\mathcal{H}(n)$. It is the quotient of a free abelian group with a basis $\{H_M | M \in \mathcal{R}(n)\}$ modulo the relations generated by $H_X - H_Z + H_Y$ provided there exists a triangle of the form $X \to Z \to Y \to TX$. According to Ringel[12] we define a symmetric bilinear function $< - \mid - >$ on $\mathcal{H}(n) \times \mathcal{H}(n)$ as follows.

$$< H_X \mid H_Y >= dim_k \operatorname{Hom}_{\mathcal{R}(n)}(X, Y) - dim_k \operatorname{Hom}_{\mathcal{R}(n)}(X, TY)$$

$$+ dim_k \operatorname{Hom}_{\mathcal{R}(n)}(Y, X) - dim_k \operatorname{Hom}_{\mathcal{R}(n)}(Y, TX).$$

for $X, Y \in \mathcal{R}(n)$. Then we extend bilinearly this definition onto $\mathcal{H}(n) \times \mathcal{H}(n)$. It is easy to check that this is well-defined.

For $X \in \mathcal{R}(n)$, denoted by $[X]$ the isomorphism class of X. Let $\mathcal{N}(n)$ be the free abelian group with a basis $\{u_{[X]} \mid X \text{is indecomposable in} \mathcal{R}(n)\}$, for a simplicity we write u_X for $u_{[X]}$. We put

$$\mathcal{G}(n) = \mathcal{H}(n) \oplus \mathcal{N}(n)$$

a direct sum as $\mathbf{Z}$-modules. We want to define an operation on $\mathcal{G}(n)$ by using Hall polynomials and the symmetric bilinear function such that $\mathcal{G}(n)$ becomes a Lie algebra. Given indecomposable objects $M, N, L \in \mathcal{R}(n)$, we define $\gamma_{M\,N}^{L} = \varphi_{M\,N}^{L}(1) - \varphi_{N\,M}^{L}(1)$. For any $\mathcal{R}(n)$, $\gamma_{M\,N}^{L}$ is defined according to Lemma 4.2.

Now we define a bilinear operation $[-,-]$ on $\mathcal{G}(n)$ as follows.

(a) For any two indecomposable objects $X, Y \in \mathcal{R}(n)$

$$[u_X, u_Y] = \begin{cases} \sum_{[L], L \in \mathrm{ind}\, \mathcal{R}(n)} \gamma_{X\,Y}^{L} u_L & \text{if } Y \not\simeq TX \\ H_X & \text{if } Y \simeq TX \end{cases}$$

(b) For any objects $X, Y \in \mathcal{R}(n)$ with Y indecomposable

$$[H_X, u_Y] = - < H_X | H_Y > u_Y \text{ and } [u_Y, H_X] = -[H_X, u_Y].$$

(c)

$$[\mathcal{H}(n), \mathcal{H}(n)] = 0.$$

and we extend $[-,-]$ bilinearly onto the whole $\mathcal{G}(n)$. This is a well-defined operation since $< - | - >$ is a bilinear function on $\mathcal{H}(n) \times \mathcal{H}(n)$.

With these notations, we have our first main result.

Theorem 4.1. $\mathcal{G}(n)$ together with the operation $[-,-]$ is a Lie algebra for any $n \geq 1$ and $n = \infty$.

Proof. At first we consider the case $n = \infty$. It is necessary to prove the Jacobi identity

$$[u_X, [u_Y, u_Z]] = [[u_X, u_Y], u_Z] + [u_Y, [u_X, u_Z]] \qquad (1)$$

for any indecomposable objects X, Y and Z in $\mathcal{R}(\infty)$. We can prove this fact by using a similar method as in [6],but the following way is easier. There exists a sufficiently large $m > 0$ such that X, Y and Z belong to $D^b(A_{2m+1})/T^2$, moreover, according to the observation (1) in Section 2, the operation $[-,-]$ for X, Y and Z in $D^b(A_{2m+1})/T^2$ is the same as that for X, Y and Z in $\mathcal{R}(\infty)$. So we have the Jacobi identity (1) by [6]. Completely similarly we have the following

$$[H_X, [u_Y, u_Z]] = [[H_X, u_Y], u_Z] + [u_Y, [H_X, u_Z]] \qquad (2)$$

and

$$[H_X, [H_Y, u_Z]] = [[H_X, H_Y], u_Z] + [H_Y, [H_X, u_Z]] \qquad (3).$$

So the theorem is proved at $n = \infty$. Consider the case $1 \leq n < \infty$. We have the canonical covering functur $G : \mathcal{R}(\infty) \to \mathcal{R}(n)$, the induced morphism $G : \mathcal{G}(\infty) \to \mathcal{G}(n)$ is given by $Gu_X = u_{GX}$ and $GH_X = H_{GX}$. However G may not be a homomorphism about the operation $[-,-]$. Anyway G induces an new operation $[-,-]'$ on $\mathcal{G}(n)$ defined as follows.

$$[u_A, u_B]' = \sum_{Y \in G^{-1}B} G[u_{\tilde{A}}, u_Y]$$

$$[H_A, u_B]' = \sum_{Y \in G^{-1}B} G[H_{\tilde{A}}, u_Y]$$

for any indecomposable objects A, B in $\mathcal{R}(n)$, where $\tilde{A} \in G^{-1}A$. It is easy to check that

$$\sum_{Y \in G^{-1}B} G[u_{\tilde{A}}, u_Y] = \sum_{X \in G^{-1}A} G[u_X, u_{\tilde{B}}]$$

$$\sum_{Y \in G^{-1}B} G[H_{\tilde{A}}, u_Y] = \sum_{X \in G^{-1}A} G[H_X, u_{\tilde{B}}]$$

for arbitrary $\tilde{A} \in G^{-1}A$ and $\tilde{B} \in G^{-1}B$. So the operation $[-,-]'$ is well-defined. We claim that $\mathcal{G}(n)$ together with $[-,-]'$ is a Lie algebra. We have the fact

$$[u_A, [u_B, u_C]']' = \sum_{Z \in G^{-1}C} [u_A, G[u_{\tilde{B}}, u_Z]]' = \sum_{Y \in G^{-1}B, Z \in G^{-1}C} G[u_{\tilde{A}}, [u_Y, u_Z]]$$

and

$$[H_A, [u_B, u_C]']' = \sum_{Y \in G^{-1}B, Z \in G^{-1}C} G[H_{\tilde{A}}, [u_Y, u_Z]]$$

Then the Jacobi identity follows easily. It remains to show that $[-,-]'$ and $[-,-]$ coincide on $\mathcal{G}(n)$. Let A, B be indecomposable objects in $\mathcal{R}(n)$. For $\tilde{A} \in G^{-1}A$, $< H_A | H_B >= \sum_{Y \in G^{-1}B} < H_{\tilde{A}} | H_Y >$. Then

$$[H_X, u_Y]' = [H_X, u_Y].$$

If there is a triangle of the form

$$A \longrightarrow L \longrightarrow B \longrightarrow TA$$

in $\mathcal{R}(n)$ with L indecomposable. Given any $\tilde{A} \in G^{-1}A$, there exists an unique $\tilde{B} \in G^{-1}B$ (up to isomorphism) such that we have a triangle of the form

$$\tilde{A} \longrightarrow \tilde{L} \longrightarrow \tilde{B} \longrightarrow T\tilde{A}$$

in $\mathcal{R}(\infty)$ with $\tilde{L}$ being indecomposable. Of course $\tilde{L} \in G^{-1}L$. So $[u_A, u_B]' = [u_A, u_B]$ follows from Corollary 4.1. Therefore $[-,-]$ is a Lie operation on $\mathcal{G}(n)$. $\square$

5. Chevalley basis of $\mathcal{G}(n)$

In the first part of this section, we only consider the category $\mathcal{R}(\infty)$ and the corresponding Lie algebra $\mathcal{G}(\infty)$.

For any object $M \in \mathcal{R}(\infty)$, $H_{TM} = -H_M$, then $H_M = \sum_{i\in\mathbf{Z}} a_i H_{E_i[1]}$, where $a_i \in \mathbf{Z}$ and only finitely many $a_i \neq 0$. If M is indecomposable, then $H_M = \sum_{i\in\mathbf{Z}} a_i H_{E_i[1]}$ with all $a_i \geq 0$ or all $a_i \leq 0$; in fact, the possible values for a_i are only $0, 1$ or -1. If $M \in rep(\Delta(\infty), k)$, $(a_i)_{i\in\mathbf{Z}}$ is just the dimension vector. It is well-known that the map $[M] \mapsto (a_i)_{i\in\mathbf{Z}}$ where $H_M = \sum_{i\in\mathbf{Z}} a_i H_{E_i[1]}$ induces a bijection between the isomorphism classes of indecomposable objects of $\mathcal{R}(\infty)$ and the root system Φ of type A_∞. Let $\mathcal{G}'(A_\infty)$ be the infinite rank affine Lie algebra of type A_∞ (see [5]). We have the following result.

Theorem 5.1. $\mathcal{G}(\infty) = \mathcal{H}(\infty) \oplus \mathcal{N}(\infty)$ together with $[-,-]$ is the $\mathbf{Z}$-form of $\mathcal{G}'(A_\infty)$. In particular, $\mathcal{G}(\infty) \otimes_{\mathbf{Z}} \mathbf{C}$ is isomorphic to $\mathcal{G}'(A_\infty)$.

Proof. Let

$$\begin{aligned} a_{ij} &= \dim_k \mathrm{Hom}_{\mathcal{R}(\infty)}(E_i[1], E_j[1]) - \dim_k \mathrm{Hom}_{\mathcal{R}(\infty)}(E_i[1], TE_j[1]) \\ &\quad + \dim_k \mathrm{Hom}_{\mathcal{R}(\infty)}(E_j[1], E_i[1]) - \dim_k \mathrm{Hom}_{\mathcal{R}(\infty)}(E_j[1], TE_i[1]) \\ &= \dim_k \mathrm{Hom}_{k\Delta(\infty)}(E_i[1], E_j[1]) - \dim_k \mathrm{Ext}_{k\Delta(\infty)}(E_i[1], E_j[1]) \\ &\quad + \dim_k \mathrm{Hom}_{k\Delta(\infty)}(E_j[1], E_i[1]) - \dim_k \mathrm{Ext}_{k\Delta(\infty)}(E_j[1], E_i[1]). \end{aligned}$$

So we have

$$a_{ij} = \begin{cases} 2 & \text{if } j = i \\ -1 & \text{if } j = i+1 \\ -1 & \text{if } j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

The matrix $(a_{ij})_{i,j\in\mathbf{Z}}$ is just the generalized Cartan matrix of type A_∞. Put $e_i = u_{E_i[1]}, f_i = -u_{TE_i[1]}, h_i = -H_{E_i[1]}$. We check the following relations.

(1) $[h_i, h_j] = 0$ for all $i, j \in \mathbf{Z}$.

(2) $[e_i, f_i] = h_i$ and $[e_i, f_j] = 0$ if $i \neq j$.

Here $[e_i, f_i] = [u_{E_i[1]}, -u_{TE_i[1]}] = -[u_{E_i[1]}, u_{TE_i[1]}] = -H_{E_i[1]} = h_i$. If $i \neq j$, then $\mathrm{Hom}(E_i[1], E_j[1]) = \mathrm{Hom}(E_j[1], E_i[1]) = 0$. Any triangle of the form

$$E_i[1] \xrightarrow{f} M \xrightarrow{g} TE_j[1] \xrightarrow{h} TE_i[1]$$

must be split, i.e $h = 0$ and $M \simeq E_i[1] \oplus TE_j[1]$. Therefore $[u_{E_i[1]}, u_{TE_j[1]}] = 0$, so $[e_i, f_j] = 0$.

(3) $[h_i, e_j] = a_{ij} e_j$ and $[h_i, f_j] = -a_{ij} f_j$.

Obtained from the definition directly.

(4) $(ad\, e_i)^{-a_{ij}+1} e_j = 0$ and $(ad\, f_i)^{-a_{ij}+1} f_j = 0$ for $i \neq j$.

Fix i and j, when m is sufficiently large, the support of $E_i[1]$ and $E_j[1]$ lie in the quiver A_{2m+1}. By the observation (1) in Section 2, we know that the formula (4) is valid if and only if it is valid when $E_i[1]$ and $E_j[1]$ are restricted as modules of A_{2m+1}. Applying to e_i and e_j the Lie operations in this paper and introduced by Ringel are same, so $(ad\, e_i)^{-a_{ij}+1}e_j = 0$ as well as $(ad\, f_i)^{-a_{ij}+1}f_j = 0$ are known from Ringel[8].

By a result of Gabber-Kac ,which is a generalization of the Serre Theorem, the above relations are the define relations of $\mathcal{G}'(A_\infty)$. There is a surjective homomorphism $\mathcal{G}'(A_\infty) \to \mathcal{G}(\infty)$, which maps the root space $\mathcal{G}_\alpha$ of $\mathcal{G}'(A_\infty)$ for any root α of A_∞ to the subspace of $\mathcal{G}(\infty) \otimes_{\mathbf{Z}} \mathcal{C}$ generated by u_X where X is the indecomposable object in $\mathcal{R}(\infty)$ with $\underline{dim}X = \alpha$; and maps h_i to $-H_{E_i[1]}$ for any $i \in \mathbf{Z}$. Obviously this is an isomorphism. $\square$

In the definition of the Lie algebra $\mathcal{G}(\infty)$, the structure constants are always integral. The elements u_M with M indecomposable in $\mathcal{R}(\infty)$ and $H_{E_i[1]}, i \in \mathbf{Z}$ constitute a $\mathbf{Z}$-basis of $\mathcal{G}(\infty)$. Not surprisingly, it relate to the Chevalley basis of $\mathcal{G}'(A_\infty)$ well.

Let $\Phi(\infty)$ be the root system of type A_∞. We define a set $\{x_\alpha | \alpha \in \Phi(\infty)\}$ as follows.

$$x_\alpha = \begin{cases} u_X & \text{if } \alpha \in \Phi^+(\infty), \text{ where } X \text{ is indecomposable and } \underline{dim}X = \alpha \\ -u_{TX} & \text{if } \alpha \in \Phi^-(\infty), \text{ where } X \text{ is indecomposable and } \underline{dim}X = -\alpha \end{cases}$$

Proposition 5.1. The set $\{x_\alpha | \alpha \in \Phi(\infty)\}$ is a Chevalley basis of $\mathcal{G}'(A_\infty)$.

Proof. Let $h_\alpha = [x_\alpha, x_{-\alpha}]$. If $\alpha, \beta, \alpha+\beta \in \Phi(\infty)$, suppose that $[x_\alpha, x_\beta] = c_{\alpha,\beta}x_{\alpha+\beta}$, we need to prove $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$. (1) If $\alpha, \beta, \alpha+\beta \in \Phi^+(\infty)$. Let X, Y, Z be indecomposable in $rep(\Delta(\infty), k)$ such that $\underline{dim}X = \alpha, \underline{dim}Y = \beta, \underline{dim}Z = \alpha+\beta$, then $[u_X, u_Y] = c_{\alpha,\beta}u_Z$ and $[x_{-\alpha}, x_{-\beta}] = [-u_{TX}, -u_{TY}] = [u_{TX}, u_{TY}] = c_{\alpha,\beta}u_{TZ} = -c_{\alpha,\beta}x_{-\alpha,-\beta}$. (2) If $\alpha \in \Phi^+(\infty), \beta \in \Phi^-(\infty)$ but $\alpha+\beta \in \Phi^+(\infty)$. Let X, Y, Z be indecomposable in $rep(\Delta(\infty), k)$ such that $\underline{dim}X = \alpha, \underline{dim}Y = -\beta, \underline{dim}Z = \alpha+\beta$, then $[u_X, -u_{TY}] = [x_\alpha, x_\beta] = c_{\alpha,\beta}x_{\alpha+\beta} = c_{\alpha,\beta}u_Z$ and $[x_{-\alpha}, x_{-\beta}] = [-u_{TX}, u_Y] = -[u_{TX}, u_Y] = c_{\alpha,\beta}u_{TZ} = -c_{\alpha,\beta}x_{-\alpha,-\beta}$. Other cases are similar. So we can claim that $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$ for all $\alpha, \beta, \alpha+\beta \in \Phi(\infty)$. The proof is finished. $\square$

Remark 1. As above we have realized the infinite rank affine Lie algebras of type A_∞. The same argument can be used to give the realizations of the remaining infinite rank affine Lie algebras of type $A_{+\infty}, B_\infty, C_\infty$ and D_∞. For this aim, one should consider the finite dimensional representations and the induced categories of the corresponding quivers or species. Only one thing which should be paid special attentions is that instead of the usual Grothendieck group the weighted Grothendieck group will be used for the cases B_∞ and C_∞ as the Cartan subalgebra (see [6]).

In the rest of this section, we consider the case $2 \le n < \infty$. In fact th Lie algebra $\mathcal{G}(n)$ is not what we want, since the elements $u_{E_i[n]}, i = 0, \cdots, n-1$, can not be generated by the elements $u_{E_i[1]}$ and $u_{TE_i[1]}, i = 0, \cdots, n-1$. Instead of $\mathcal{G}(n)$, we consider the Lie subalgebra of $\mathcal{G}(n)$, denoted by $\mathcal{G}'(n)$, generated by the elements $u_{E_i[1]}$ and $u_{TE_i[1]}, i = 0, \cdots, n-1$. Let

$$\begin{aligned} a_{ij} &= \dim_k \mathrm{Hom}_{\mathcal{R}(n)}(E_i[1], E_j[1]) - \dim_k \mathrm{Hom}_{\mathcal{R}(n)}(E_i[1], TE_j[1]) \\ &+ \dim_k \mathrm{Hom}_{\mathcal{R}(n)}(E_j[1], E_i[1]) - \dim_k \mathrm{Hom}_{\mathcal{R}(n)}(E_j[1], TE_i[1]) \\ &= \dim_k \mathrm{Hom}_{k\Delta(n)}(E_i[1], E_j[1]) - \dim_k \mathrm{Ext}_{k\Delta(n)}(E_i[1], E_j[1]) \\ &+ \dim_k \mathrm{Hom}_{k\Delta(n)}(E_j[1], E_i[1]) - \dim_k \mathrm{Ext}_{k\Delta(n)}(E_j[1], E_i[1]). \end{aligned}$$

An easy calculation yields that

$$a_{ij} = \begin{cases} 2 & \text{if } j = i \\ -1 & \text{if } j = i+1, 0 < j \leq n-1 \\ -1 & \text{if } j = i-1, 0 < i \leq n-1 \\ -1 & \text{if } j = 0, i = n-1 \\ -1 & \text{if } i = 0, j = n-1 \\ 0 & \text{otherwise} \end{cases}$$

So the matrix $(a_{ij})_{n\times n}$ is just the generalized Cartan matrix of type $\tilde{A}_{n-1}$. If we put $e_i = u_{E_i[1]}, f_i = -u_{TE_i[1]}, h_i = -H_{E_i[1]} = -[u_{E_i[1]}, u_{TE_i[1]}]$. As the same as in the proof of Theorem 5.1, we can check the following relations.

(1) $[h_i, h_j] = 0$ for $i, j = 0, \cdots, n-1$.
(2) $[e_i, f_i] = h_i$ and $[e_i, f_j] = 0$ if $i \neq j$.
(3) $[h_i, e_j] = a_{ij} e_j$ and $[h_i, f_j] = -a_{ij} f_j$.
(4) $(ad\, e_i)^{-a_{ij}+1} e_j = 0$ and $(ad\, f_i)^{-a_{ij}+1} f_j = 0$ for $i \neq j$.

Therefore, in view of a theorem of Gabber-Kac (a generalization of the Serre Theorem), there is a surjective homomorphism $\psi : \mathcal{G}'(\tilde{A}_{n-1}) \to \mathcal{G}'(n) \otimes_{\mathbf{Z}} \mathcal{C}$, which maps e_i to e_i, f_i to f_i and h_i to h_i, where $\mathcal{G}'(\tilde{A}_{n-1})$ is the affine Kac-Moody algebra of type $\tilde{A}_{n-1}$ (see [5]). It is easy to see that this is an isomorphism. So we have the following result, if we identify $\mathcal{G}'(\tilde{A}_{n-1})$ to $\mathcal{G}'(n) \otimes_{\mathbf{Z}} \mathcal{C}$ along the natural isomorphism ψ.

Theorem 5.2. For $2 \leq n < \infty$, $\mathcal{G}'(n)$ is a $\mathbf{Z}$-form of the Kac-Moody algebra of type $\tilde{A}_{n-1}$. In particular, $\mathcal{G}'(n) \otimes_{\mathbf{Z}} \mathbf{C}$ is a realization of the Kac-Moody algebra of type $\tilde{A}_{n-1}$. □

By our construction, $\mathcal{G}'(n) = \mathcal{H} \oplus \mathcal{N}^+ \oplus \mathcal{N}^-$, where $\mathcal{H}$ is the Cartan algebra corresponding to the Grothendieck group of $\mathcal{R}(n)$, $\mathcal{N}^+$ is the subalgebra generated by $u_{E_i[1]}, i = 0, \cdots, n-1$, $\mathcal{N}^-$ is the subalgebra generated by $u_{TE_i[1]}, i = 0, \cdots, n-1$. This is a triangular decomposition of $\mathcal{G}'(n)$. We want to point out that the category $\mathcal{R}(n)$ still provide the all informations of $\mathcal{G}'(n) \otimes_{\mathbf{Z}} \mathbf{C}$, for example, its Chevalley basis.

Theorem 5.3. (a) The elements $u_{E_i[j]}$, where $0 \leq i \leq n-1, j \not\equiv 0 (mod\ n)$ in $\mathbf{N}$, and $u_{E_i[kn]} - u_{E_{i+1}[kn]}$, where $0 \leq i \leq n-2, k \geq 1$, form a $\mathbf{C}$-basis of $\mathcal{N}^+ \otimes_{\mathbf{Z}} \mathbf{C}$. (b) The elements $u_{TE_i[j]}$, where $0 \leq i \leq n-1, j \not\equiv 0 (mod\ n)$ in $\mathbf{N}$, and $u_{TE_i[kn]} - u_{TE_{i+1}[kn]}$, where $0 \leq i \leq n-2, k \geq 1$, form a $\mathbf{C}$-basis of $\mathcal{N}^- \otimes_{\mathbf{Z}} \mathbf{C}$.

Proof. The elements $u_{E_i[1]}, 0 \leq i \leq n-1$, are the generators of $\mathcal{N}^+$. For $1 < j < n$ we have the triangle of the form

$$E_i[1] \longrightarrow E_i[j] \longrightarrow E_{i+1}[j-1] \longrightarrow TE_i[1] \tag{5.1}$$

and have no triangle of the form

$$E_{i+1}[j-1] \longrightarrow M \longrightarrow E_i[1] \longrightarrow TE_{i+1}[j-1]$$

with M indecomposable. In view of Corollary 4.1, $[u_{E_i[1]}, u_{E_{i+1}[j-1]}] = u_{E_i[j]}$. It follows that $u_{E_i[j]}$ for all $1 \leq j < n$ belong to $\mathcal{N}^+$ by the induction on j. If $j = n$, we have the triangle of the form (5.1) and also the triangle of the form

$$E_{i+1}[j-1] \longrightarrow E_{i+1}[j] \longrightarrow E_i[1] \longrightarrow TE_{i+1}[j-1] \tag{5.2}$$

since $j = n$ and then $E_{i+j}[1] = E_i[1]$. Therefore $u_{E_i[n]} - u_{E_{i+1}[n]} = [u_{E_i[1]}, u_{E_{i+1}[n-1]}] \in \mathcal{N}^+$. We have the triangle of the form

$$E_i[1] \longrightarrow E_i[kn] \longrightarrow E_{i+1}[kn-1] \longrightarrow TE_i[1] \tag{5.3}$$

and

$$E_{i+1}[kn-1] \longrightarrow E_{i+1}[kn] \longrightarrow E_i[1] \longrightarrow TE_{i+1}[kn-1] \tag{5.4}$$

So $u_{E_i[kn]} - u_{E_{i+1}[kn]} = [u_{E_i[1]}, u_{E_{i+1}[kn-1]}]$. If $j = kn+t, 0 < t < n$, we have the triangles of the form

$$E_i[kn] \longrightarrow E_i[kn+t] \longrightarrow E_i[t] \longrightarrow TE_i[kn] \tag{5.5}$$

$$E_i[t] \longrightarrow E_i[kn+t] \longrightarrow E_{i+t}[kn] \longrightarrow TE_i[t] \tag{5.6}$$

and have no triangle of the form

$$E_i[t] \longrightarrow M \longrightarrow E_i[kn] \longrightarrow TE_i[t]$$

$$E_{i+t}[kn] \longrightarrow N \longrightarrow E_i[t] \longrightarrow TE_{i+t}[kn]$$

with M, N indecomposable. It follows $2u_{E_i[kn+t]} = [u_{E_i[kn]} - u_{E_{i+t}[kn]}, u_{E_i[t]}]$. By the induction on j and k we see that all elements $u_{E_i[j]}$, where $0 \leq i \leq n-1, j \not\equiv 0 (mod\ n)$ in $\mathbf{N}$ and $u_{E_i[kn]} - u_{E_{i+1}[kn]}$ where $0 \leq i \leq n-2, k \geq 1$, belong to $\mathcal{N}^+ \otimes_{\mathbf{Z}} \mathbf{C}$. Obviously those elements are linearly independent over $\mathbf{C}$. Let V be the subspace of $\mathcal{N}^+ \otimes_{\mathbf{Z}} \mathbf{C}$ generated by those elements. V is stable under the actions of $ad\, u_{E_i[1]}$, hence V is a subalgebra of $\mathcal{N}^+ \otimes_{\mathbf{Z}} \mathbf{C}$ which contains all $u_{E_i[1]}$. Then $V = \mathcal{N}^+ \otimes_{\mathbf{Z}} \mathbf{C}$. (a) is proved. The proof of (b) is as the same as that of (a). □

Remark 2. (1) O ne can replace the field $\mathbf{C}$ of complex numbers by the field $\mathbf{Q}$ of rational numbers in Theorem 5.3. (2) If we put the sutable coefficients to the elements $u_{E_i[kn]} - u_{E_{i+1}[kn]}$ and $u_{TE_i[kn]} - u_{TE_{i+1}[kn]}$, the bases in Theorem 5.3 will be the $\mathbf{Z}$-bases of $\mathcal{N}^+ \otimes_{\mathbf{Z}} \mathbf{C}$ and $\mathcal{N}^- \otimes_{\mathbf{Z}} \mathbf{C}$ respectively.

Let $\mathcal{G}'(n) \otimes_{\mathbf{Z}} \mathbf{C} = \mathcal{H} \oplus \sum_{\alpha\in\Phi} \mathcal{G}_\alpha$ be the root space decomposition of $\mathcal{G}'(n) \otimes_{\mathbf{Z}} \mathbf{C}$. Then $u_{E_i[kn]} - u_{E_{i+1}[kn]}$ belong to $\mathcal{G}_{k\alpha_0}$, where α_0 is the minimal positive imaginary root of $\mathcal{G}'(n) \otimes_{\mathbf{Z}} \mathbf{C}$. So we have the following.

Corollary 5.1. The multiplicity of every imaginary root of $\mathcal{G}'(n) \otimes_{\mathbf{Z}} \mathcal{C}$ is $n-1$.

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