Z–matrices and inverse Z–matrices

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Abstract

We consider Z–matrices and inverse Z–matrices, i.e. those nonsingular matrices, whose inverse is a Z–matrix. Recently Fiedler and Markham introduced a classification of Z–matrices. This classification directly leads to a classification of inverse Z–matrices. Among all classes of Z–matrices and inverse Z–matrices the classes of M–matrices, N_0–matrices, F_0–matrices and inverse M–matrices, inverse N_0–matrices and inverse F_0–matrices respectively, had been studied in detail. Here we discuss each single class of Z–matrices and inverse Z–matrices as well as we consider the whole classes of Z–matrices and inverse Z–matrices. We establish some common properties of the classes like eigenvalue bounds and determinant inequalities and we give a new characterization of all classes of Z–matrices and inverse Z–matrices. Some of the results generalize known results for M–matrices, N_0–matrices, and F_0–matrices, and inverse M–matrices, inverse N_0–matrices, and inverse F_0–matrices. However, we will also show that some properties of the specific classes mentioned above do not hold for all classes of Z–matrices and inverse Z–matrices.

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1 Introduction and Notation

Throughout we deal with $n \times n$ Z–matrices, i.e. real matrices whose off-diagonal entries are nonpositive. These matrices arise in many problems in the mathematical and physical sciences. Some of the best-known subclasses of Z–matrices are the class of M–matrices (introduced by Ostrowski), the class of N_0–matrices (introduced by Ky Fan and G.A. Johnson) and the class of F_0–matrices (introduced by G.A. Johnson). Especially for M–matrices a large amount of properties and characterizations exists. However, the other classes of Z–matrices are also of great interest. The first systematical effort in considering the whole class of Z–matrices was made by Fiedler and Markham in [FM]. They introduced a classification of Z–matrices by generalizing the ideas of the definitions of M–matrices, N_0–matrices and F_0–matrices and gave some common properties of all classes. Here we continue discussing the whole class of Z–matrices as well as considering all subclasses of Z–matrices. We summarize some known results and establish new results like a new characterization of all classes of Z–matrices, eigenvalue bounds and determinant inequalities.

On the other hand, there has been interest in inverse Z–matrices, i.e. any nonsingular matrices whose inverse is a Z–matrix. The classification of Z–matrices directly leads to a classification of inverse Z–matrices. All matrices, whose inverse belongs to the same class of Z–matrices, build a subclass of the class of all inverse Z–matrices. The problem to determine whether or whether not a matrix is an inverse Z–matrix or more preciously, is an inverse Z–matrix of a specific class, is called the inverse Z–matrix problem. This problem contains the inverse M–matrix problem, i.e. characterizing all (nonnegative) matrices whose inverse is an M–matrix. Similar to the M–matrix case, there exists an amount of papers considering inverse M–matrices. A survey of this topic is given by C.R. Johnson [J1]. However a complete solution of this problem is still missing. Also inverse N_0–matrices and inverse F_0–matrices has been considered, e.g. G.A. Johnson [J3], Smith [S2], and Ying Chen [C]. The first systematically approach in considering all classes of inverse Z–matrices was made by Nabben and Varga [NV3]. There the authors gave sufficient conditions for matrices to be an inverse Z–matrix for each specific class. Here we give new results for the whole class of inverse Z–matrices as well as for each single subclass. We establish a new characterization and give some new results of the structure of an inverse Z–matrix. Moreover, we give a new set of sufficient conditions of all classes of inverse Z–matrices.

The paper is organized as follows. The next section deals with Z–matrices. This section includes the classification of Z–matrices, which we also need in Section 3. In Section 3 we consider inverse Z–matrices. At the end of the section we give examples of inverse Z–matrices, the so-called type D matrices, which will be studied in detail.

We will use the following notation:
By I_r we denote the $r \times r$ identity matrix. The index r is omitted if $r = n$. If $A = [a_{ij}] \in \mathbb{R}^{n,n}$ is nonnegative, i.e. $a_{ij} \geq 0$, we write $A \geq 0$. Similarly, we write $A \geq B$, for $A, B \in \mathbb{R}^{n,n}$, if and only if $A - B \geq 0$. The spectral radius of $A \in \mathbb{R}^{n,n}$ is denoted by $\rho(A)$. The vector

$e_i \in \mathbb{R}^n$ denotes the vector whose entries are zero except the i-th entry which is 1. The vector of all ones with n entries is denoted by ξ_n.

2 Z–matrices

In 1992 Fiedler and Markham [FM] introduced the following classification of Z–matrices

Definition 2.1 *Let L_s (for $s = 0, \ldots, n$) denotes the class of matrices consisting of real $n \times n$ matrices which have the form*

$$A = tI - B \quad \textit{where} \qquad (2.1)$$
$$B \geq 0 \quad \textit{and} \quad \rho_s(B) \leq t < \rho_{s+1}(B).$$

Here,

$$\rho_s(B) := max\{\rho(\tilde{B}) : \tilde{B} \;\; \textit{is a} \;\; s \times s \;\textit{principal submatrices of}\; B\},$$

and we set $\rho_0(B) := -\infty$ and $\rho_{n+1}(B) := \infty$.

The scalar t and the matrix B in (2.1) are not unique, but every Z–matrix belongs to exactly one set L_s. Moreover none of the classes L_s is void.

Example 2.2 [FM] *Let J denotes the $n \times n$ matrix of all ones. Then, $\rho_s(J) = s$ for all $s = 1, 2, \ldots, n$. If t satisfies $s \leq t < s + 1$, then $tI - J \in L_s$ for $s = 1, \ldots, n$. If $t < 1$, then $tI - J \in L_0$.*

The class L_n is just the class of $n \times n$ (singular and nonsingular) M–matrices. The class L_{n-1} is the class of $n \times n$ N_0–matrices introduced by G. Johnson [J2], and this class contains the N–matrices defined by K. Fan [F]. Moreover, the class of $n \times n$ F_0–matrices introduced by Johnson [J2] is just L_{n-2}. Here we should mention that the classification of Z–matrices given above inherits the dimension of the matrices one consider. If we deal with $n \times n$ matrices we have $n + 1$ classes of Z–matrices each consisting of matrices of the same dimension. In contrast the expression *class of M–matrices* includes always matrices of different dimensions. Hence, we summarize

Remark 2.3 If one consider matrices of different dimensions, one should introduce another index which gives the dimension of the matrices. Hence one should use $L_{s,n}$ in Definition 2.1. Then two classes $L_{s,n}$ and $L_{t,m}$ consist of matrices of the same type, e.g. M–matrices, N_0–matrices, if and only if $n - s = m - t$. However, since we deal in general with $n \times n$ matrices we just use one index.

Fiedler and Markham also give an equivalent characterization of the classes L_s [FM]:

Theorem 2.4 *Let $A \in \mathbb{R}^{n,n}$ be a Z–matrix. Then, $A \in L_s$ for one s, $s = 0, \ldots, n-1$, if and only if all principal submatrices of order s are M–matrices, but there exists a principal submatrix of order $s+1$ which is not an M–matrix.*

For N_0–matrices, Theorem 2.4 was proved by G.A. Johnson [J2] and Smith established this result for F_0–matrices [S3]. For the proof of Theorem 2.4 we refer to [FM].

It is well-know that DA and AD are again M–matrices, if A is an M–matrix and if D is a positive diagonal matrix. Hence, we obtain

Corollary 2.5 [FM] *If $A \in L_s$ and D is an $n \times n$ positive diagonal matrix, then DA and AD belong to L_s for $s = 0, \ldots, n$.*

Moreover, the class L_s is invariant under permutational similarity [FM]. The next Theorem, which was proved in [FM] also, shows the interplay of monotonicity with the class L_s.

Theorem 2.6 *Let $A \in L_s$, $B \in L_t$ with $A \leq B$. Then,*
i) $s \leq t$, and
ii) whenever a matrix C satisfied $A \leq C \leq B$, then $C \in L_q$ where $s \leq q \leq t$.

In the following we consider the sign of the determinant of $A \in L_s$. Again, it is well-known, that $det A \geq 0$, if $A \in L_n$, i.e. A is an M–matrix. G.A. Johnson [J2] proved that $det A \leq 0$ if $A \in N_0$. Moreover, the determinant of a F_0–matrix is nonpositive also [J2]. However, G.A. Johnson gave an example that the sign of the determinant of matrices in L_{n-3} need not to be the same [J2]. Here we give a more detailed analysis of this question. Therefore we need the following Theorem, which was proved by Friedland and Nabben in [FN].

Theorem 2.7 *Let $A \in \mathbb{R}^{n,n}$ be a nonnegative matrix and let λ be a real eigenvalue of A with $\lambda \neq \rho(A)$. Then*

$$\lambda \leq \rho_{\lfloor \frac{n}{2} \rfloor}(A). \tag{2.2}$$

Since Z–matrices are closely related with nonnegative matrices we now obtain

Theorem 2.8 *Let $A \in L_s$. Then*
i) $det A \geq 0$, if $s = n$,
ii) $det A \leq 0$, if $\lfloor \frac{n}{2} \rfloor \leq s < n$.

Proof. It is well-known that i) holds, [BP]. Let $A \in L_s$ with $A = tI - B$, $B \geq 0$ and a fixed $t \in \mathbb{R}$. Now consider the characteristic polynomial of B

$$\chi(z) = det(zI - B).$$

The zeros of $\chi(z)$ indicate the changes of the sign of the determinant of Z–matrices corresponding to the matrix B. But the zeros of $\chi(z)$ are the eigenvalues of B. Hence, with Theorem 2.7, ii) follows immediately. □

The counterexamples G.A. Johnson gave in [J2] were 4×4 matrices in L_1. So these counterexamples do not disprove Theorem 2.8.

As mentioned in [FN] one can not improve (2.2). Thus, matrices in L_s with $s < \lfloor \frac{n}{2} \rfloor$ need not have the same sign of their determinant. A set of Z–matrices of each class L_s with negative determinant is given in Section 4. The following example shows how one can construct Z–matrices with positive determinant.

Example 2.9 *Let J_k denotes the $k \times k$ matrix of all ones. First assume that n is even. Let $r := \frac{n}{2}$. Then, let $A_{11} := \alpha I_r - J_r$ and $A_{22} := \beta I_{n-r} - J_{n-r}$. If $s \leq \alpha, \beta < s+1$, then $\alpha I_r - J_r$ and $\beta I_{n-r} - J_{n-r}$ are in L_s. Now, if $s < \frac{n}{2}$, then $det A_{11} < 0$ and $det A_{22} < 0$. However, $A := A_{11} \oplus A_{22} \in L_s$ and $det A > 0$. Similar one can construct examples for an odd dimension n.*

Thus all classes of Z–matrices consisting of matrices of different(!) dimensions, except the classes of M–matrices N_0–matrices, and F_0–matrices include matrices with different signs of their determinants. However, if we consider the classes L_s which consist by definition of matrices of the same dimension n, then $det A \leq 0$, if $A \in L_s$ and $\lfloor \frac{n}{2} \rfloor \leq s < n$.

Obviously, every Z–matrix A, except an M–matrix, has at least one negative eigenvalue. However, if $A \in L_s$ with $\lfloor \frac{n}{2} \rfloor \leq s < n$, then the following theorem states that A has exactly one negative eigenvalue.

Theorem 2.10 *Let $A \in L_s$ with $\lfloor \frac{n}{2} \rfloor \leq s < n$. Then A has exactly one negative eigenvalue.*

Proof. Suppose

$$A = tI - B,$$

with $\rho_s(B) \leq t < \rho_{s+1}(B)$. If $\mu \in \sigma(B)$, where $\sigma(B)$ denotes the spectrum of B, then $t - \mu \in \sigma(A)$. But with Theorem 2.7 any real eigenvalue μ of B with $\mu \neq \rho(B)$ satisfy

$$\mu \leq \rho_{\lfloor \frac{n}{2} \rfloor}(B).$$

Hence, $(t - \rho(B))$ is the only negative eigenvalue of A. $\square$

The statement of Theorem 2.10 was mentioned for N_0–matrices by G.A. Johnson [J2]. Smith established the same result for F_0–matrices [S2]. Since the determinant of A is the product of the eigenvalues of A, the Example 2.9 shows that one can not extend Theorem 2.10.

In the following we will denote the smallest real eigenvalue by $n(A)$. As mention above, $n(A) < 0$, if $A \in L_s$ for all $s = 0, \dots, n-1$. The smallest real eigenvalue $n(A)$ can be used to determine a circle which contains the eigenvalues of A.

Theorem 2.11 *Let $A = [a_{ij}] \in L_s$, $s = 0, \dots, n-1$ be irreducible. Let $n(A)$ denote the smallest real eigenvalue of A, let $a = \max\{a_{ii}\}$. Then, the circle $c(A) := \{z \in \mathbb{C}^n : |z - a| \leq a - n(A)\}$ contains all eigenvalues of A.*

Proof. Since $a = \max\{a_{ii}\}$, the matrix $aI - A$ is nonnegative. Hence, the modulus of each eigenvalues λ of $aI - A$ is less than $\rho(aI - A)$. Thus, $|a - \lambda| \leq \rho(aI - A) = a - n(A)$. $\square$

For M–matrices and for N_0–matrices, $n(A)$ is also the eigenvalue of minimum modulus, see [FP], [S2]. However, this is not the case for all Z–matrices.

Example 2.12

$$A = \begin{bmatrix} 3 & -2 & -2 \\ -2 & -1 & -1 \\ -2 & -1 & 0 \end{bmatrix}.$$

We have $A \in L_0$ and $\sigma(A) = \{-0.3420, 0.2285, 1.8278\}$.

Fiedler and Pták [FP] and Smith [S2] established comparison results for the eigenvalues of minimum modulus of different M–matrices and N_0–matrices. Here we parallel the results for the smallest real eigenvalues.

Theorem 2.13 *Let $A, B \in L_s$ be irreducible with $A \leq B$. Then, $n(A) \leq n(B)$.*

Proof. Suppose $A = tI - B$ with $B \geq 0$. Then, $n(A) = t - \rho(B)$. Hence, we have $Re(\lambda) \geq n(A)$ for all eigenvalues λ of A. Now let $\alpha < n(A) < 0$. Then obviously $A - \alpha I$ is a Z–matrix and $Re(\lambda - \alpha) \geq n(A) - \alpha > 0$. Thus, $A - \alpha I \in L_n$, see [BP]. Fiedler and Pták proved that for $M \in L_n$, $N \in Z$ with $M \leq N$, N is an M–matrix and $q(N) \geq q(M)$, where $q(M)$ denotes the eigenvalue of M with minimum modulus. Therefore, since $B - \alpha I \geq A - \alpha I$

and $B-\alpha I \in Z$, we obtain $B-\alpha I \in L_n$ and $n(B)-\alpha = q(B-\alpha I) \geq q(A-\alpha I) = n(A)-\alpha$. Thus, $n(B) \geq n(A)$. $\square$

It follows by the definition of N_0–matrices that they are nonsingular. However, matrices of the other classes could be of course singular. If we consider the representation (2.1) the next theorem states that $t = \rho_s(B)$ is a necessary condition for singularity. However, it is not sufficient.

Theorem 2.14 *Let $A \in L_s$ with $\lfloor \frac{n}{2} \rfloor \leq s < n$ be of form* (2.1)*, i.e. $A = tI - B$ with $B \geq 0$ and $\rho_s(B) \leq t < \rho_{s+1}(B)$. If A is singular then $t = \rho_s(B)$. Conversely, let $t = \rho_s(B)$, then A is singular if and only if $\rho_s(B)$ is an eigenvalue of A.*

Proof. Assume that A is singular, then $t \in \sigma(B)$. If t satisfies $t > \rho_s(B)$, then $\rho_s(B)I - B$ has two negative eigenvalues, $\rho_s(B) - \rho(B)$ and $\rho_s(B) - t$. But $\rho_s(B)I - B \in L_s$ and Theorem 2.10 yields to a contradiction. The second part of Theorem 2.14 is obviously. $\square$

In the following we consider Schur complements of Z–matrices. If A is partioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{2.3}$$

where A_{11} and A_{22} are square matrices, and if A_{11} is nonsingular, then the Schur complement of A_{11} in A is

$$A/A_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}. \tag{2.4}$$

Fiedler and Markham called in [FM] a matrix A in L_s *pure* if every principal submatrix of order $s+1$ is negative. Additionally we call a matrix in L_s *strictly pure* if every principal submatrix of order greater or equal $s+1$ is negative. In general Schur complements of Z–matrices need not to be Z–matrices. However one have

Theorem 2.15 *Let $A \in L_s$ be nonsingular and let A_{11} be a $k \times k$ nonsingular principal submatrix of A as in* (2.3)*.*
If $k \leq s$, then
i) $A/A_{11} \in L_j$ for some j with $s-k < j \leq n-k$;
ii) if A is pure, then $A/A_{11} \in L_{s-k}$.
If $k > s$, A is strictly pure, and A/A_{11} is a Z–matrix, then A/A_{11} is in L_{n-k}, i.e. A/A_{11} is an M–matrix.

Proof. If i) or ii) holds, we obviously have that $A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is a Z–matrix, since $A_{11}^{-1} \geq 0$.

Now, let us index the rows and columns of A/A_{11} by $k+1, \ldots, n$. Moreover, with $(A/A_{11})(k+1, \ldots, t)$, $k+1 \leq t \leq n$, we denote the principal submatrix of A/A_{11} consisting of the rows and columns $k+1, \ldots, t$. As well–known we have

$$det(A/A_{11})(k+1, \ldots, t) = \frac{det A(1, \ldots, k, k+1, \ldots, t)}{det A_{11}}. \tag{2.5}$$

Thus, if i) or ii) holds all principal minors of A/A_{11} of order less or equal $s-k$ are nonnegative since both nominator and dominator of the right hand side of (2.5) are nonnegative. Moreover, if ii) holds then there exists a principal minor of A/A_{11} of order $s-k+1$ which is negative. Hence, $A/A_{11} \in L_{s-k}$. However, if $k > s$ holds, the nominator and dominator of the right hand side of (2.5) are negative. Thus, every principal minor of A/A_{11} is positive. Hence, $A/A_{11} \in L_{n-k}$. $\square$

Statement i) and ii) of Theorem 2.15 are due to Fiedler and Markham [FM]. The last statement gives some information about the structure of inverse Z–matrices, which we describe in the next section.

The next Theorem gives a characterization of matrices in L_s using their inverses.

Theorem 2.16 *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular Z–matrix. Then $A \in L_s$ if and only if one of the following alternative cases a) or b) holds*

a) *i) $det A < 0$,*
ii) all principal minors of A^{-1} of order greater or equal $n-s$ are nonpositive,
iii) there exists a principal minor of A^{-1} of order $n-s-1$, which is positive;

b) *i) $det A > 0$,*
ii) all principal minors of A^{-1} of order greater or equal $n-s$ are nonnegative,
iii) there exists a principal minor of A^{-1} of order $n-s-1$, which is negative.

Proof. If A is partitioned as in (2.3), and if $C := A^{-1}$ is conformally partitioned

$$A^{-1} = C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

we have

$$det(A_{22}) = det(C_{11})(det\ A^{-1})^{-1}. \tag{2.6}$$

Equation (2.6) gives a coupling of a principal minor of A with its associated complementary minor of A^{-1}. Now, let $detA < 0$, it follows that each principal minor of A, of order j with $1 \leq j \leq s$, is nonnegative and there exists a principal minor of A, of order $s+1$, which is negative, if and only if each principal minor of A^{-1}, of order k with $n-s \leq k \leq n$, is nonpositive and there exists a principal minor of A^{-1}, of order $n-s-1$, which is positive. Then, Theorem 2.4 completes the proof for a). Similarly, one obtain b). □

Here we should mention again, that a $n \times n$ Z–matrix with positive determinant is a Z–matrix of one of the classes L_s with $0 \leq s < \lfloor \frac{n}{2} \rfloor$, since the matrices of the other classes L_s have nonpositive determinants.

Theorem 2.16 gives some information about the structure of the inverse of a nonsingular matrix in L_s. However, for special classes L_s one have further results. It is well–know that the inverse of a nonsingular M–matrix is a nonnegative matrix, while the inverse of a N_0–matrix is a nonpositive matrix [J2]. Even more, Johnson [J2] proved that the inverse of a nonsingular F_0–matrix is a Z–matrix and Theorem 2.16 says that at least one diagonal entry of the inverse is positive, (see also [J2]). However, we will see in the next section that one can not obtain such results for the other classes of Z–matrices.

3 Inverse Z–matrices

In this section we consider inverse Z–matrices. A nonsingular matrix C is called *inverse Z–matrix* if C^{-1} is a Z–matrix. More preciously, a nonsingular matrix $C \in \mathbb{R}^{n,n}$ is called *inverse L_s–matrix* if $C^{-1} \in L_s$ for one s with $s \in \{0, \ldots, n\}$. Thus, inverse M–matrices are inverse L_n–matrices, inverse N_0–matrices are inverse L_{n-1}–matrices, and inverse F_0–matrices are inverse L_{n-2}–matrices.

Moreover, we denote by $\mathcal{L}_s$ the class of inverse L_s–matrices for $s = 0, \ldots, n.$, while $\mathcal{Z}$ denotes the whole class of inverse Z–matrices. Here we state properties of inverse Z–matrices in general, i.e. for the class $\mathcal{Z}$ as well as for all classes $\mathcal{L}_s$.

Obviously, one can deduce a lot of properties of inverse L_s–matrices from properties of Z–matrices. Here we just mention those which we need in the sequel.

Theorem 3.1 *Let $C \in \mathcal{L}_s$. Then*
i) $detC \geq 0$, if $s = n$,
ii) $detC \leq 0$, if $\lfloor \frac{n}{2} \rfloor \leq s < n$.

Proof. The proof follows immediately form Theorem 2.8. □

Theorem 3.2 *Let $C \in \mathcal{L}_s$ with $\lfloor \frac{n}{2} \rfloor \leq s < n$. Then C has exactly one negative eigenvalue.*

Proof. The proof follows immediately form Theorem 2.10. □

Furthermore, Theorem 2.16 can be formulated for inverse Z–matrices:

Theorem 3.3 *Let $C \in \mathrm{I\!R}^{n,n}$ be a nonsingular inverse Z–matrix. Then $C \in \mathcal{L}_s$ if and only if one of the following alternative cases a) or b) holds*

a) *i) $detC < 0$,*

ii) all principal minors of C of order greater or equal $n - s$ are nonpositive,

iii) there exists a principal minor of C of order $n - s - 1$, which is positive;

b) *i) $detC > 0$,*

ii) all principal minors of C of order greater or equal $n - s$ are nonnegative,

iii) there exists a principal minor of C of order $n - s - 1$, which is negative.

Since the sign of a determinant of a matrix is invariant under multiplication with a positive diagonal matrix we obtain the invariance of the classes $\mathcal{L}_s$ under this operation.

Corollary 3.4 *If $C \in \mathcal{L}_s$, $s \in \{0, \ldots, n\}$, and if D is an $n \times n$ positive diagonal matrix, then DA and AD belong to $\mathcal{L}_s$.*

The class of inverse M–matrices is invariant under the addition of a positive diagonal matrix [J1]. However, this is not true for $\mathcal{Z}$. Consider for example inverse N_0–matrices. Inverse N_0–matrices are nonpositive matrices, hence Z–matrices. If we add sI, $s \in \mathrm{I\!R}$ sufficiently large, the inverse N_0–matrix becomes an M–matrix, which is not an inverse Z–matrix.

Obviously, every principal submatrix of a Z–matrix is again a Z–matrix. But if A is in $\mathcal{L}_s$ and $s \geq \lfloor \frac{n}{2} \rfloor$, then there exist principal submatrices which are again inverse Z–matrices. Even more, there exists at least one principal submatrix of an inverse L_s–matrix which is of the same type. Here we should use the notation introduced in Remark 2.3, i.e. we will use another index to mark the dimension of the matrices we consider.

Theorem 3.5 *Let the $n \times n$ matrix C be in $\mathcal{L}_s$ with $s \geq \lfloor \frac{n}{2} \rfloor$, i.e. $C^{-1} \in L_{s,n}$. Let $\tilde{C}$ be a nonsingular principal submatrix of C of order k, with $k \geq n - s$, then $\tilde{C}$ is an inverse Z–matrix in $\mathcal{L}_{j-n+k,k}$ with $j \geq k$. If $\tilde{C}$ contains at least one positive principal minor of order $n - s - 1$, then the $k \times k$ matrix $\tilde{C}$ satisfies $\tilde{C}^{-1} \in L_{s-n+k,k}$*

Proof. W.l.o.g. we may assume that $\tilde{C} \in \mathbb{R}^{k,k}$ is a leading principal submatrix of C which we denote in following by C_{11}. Then we partition C and C^{-1} as

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \qquad C^{-1} := A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{3.1}$$

Since $C^{-1} \in L_{s,n}$ and the order of A_{22} is less or equal s, we have $det A_{22} \geq 0$. Moreover, we have $det C < 0$ since $s \geq \lfloor \frac{n}{2} \rfloor$. However, since $det(C_{11}) = det(A_{22})(det\ A^{-1})$ we get $det A_{22} > 0$ and $det C_{11} < 0$. The explicit form of A^{-1} is given by

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}(A/A_{11})^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}(A/A_{11})^{-1} \\ -(A/A_{11})^{-1}A_{21}A_{11}^{-1} & (A/A_{11})^{-1} \end{bmatrix}. \tag{3.2}$$

Hence, $C_{11} = (A/A_{22})^{-1}$. But (A/A_{22}) is a Z–matrix. Therefore C_{11} is an inverse Z–matrix. Moreover, since C_{11} is a principal submatrix of C, all principal minors of C_{11} of order greater or equal to $n-s$ are nonpositive. Theorem 3.3 guarantees the existence of a principal minor of order $n-s-1$ of C. If C_{11} contains this principal submatrix, we apply Theorem 3.3 and replace there n by k. Thus the $k \times k$ matrix $\tilde{C}^{-1}$ satisfies $\tilde{C} \in L_{t,m}$ with $t = s-n+k$ and $m = k$. If C_{11} does not contain such a minor, we have $C_{11} \in \mathcal{L}_{j-n+k,k}$ for $j > s$.

□

As special cases Theorem 3.5 says that inverse N_0–matrices and inverse F_0–matrices have at least one principal submatrix which is again an inverse N_0–matrix or an inverse F_0–matrix respectively. This was first proved by Ying Chen [C].

Theorem 3.6 *Let C be an inverse Z–matrix partitioned as*

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

where C_{11} is a nonsingular principal submatrix of C of arbitrary order. Then, C/C_{11} is also an inverse Z–matrix.

Proof. Let $A := C^{-1}$ be partitioned conformally with C such that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Using the explicit form of C^{-1} as in (3.2) we have $A_{22} = (C/C_{11})^{-1}$. Hence, A_{22} is nonsingular. But A_{22} is a principal submatrix of a Z–matrix, therefore A_{22} itself is a Z–matrix. Thus, $(C/C_{11}) = A_{22}^{-1}$ is an inverse Z–matrix. □

In the following we give some examples for matrices in $\mathcal{L}_s$. The first examples of matrices in $\mathcal{L}_s$ for all s were given in [NV3]. There the authors defined so called shifted generalized ultrametric matrices. These matrices contains special classes of inverse M–matrices, so called ultrametric matrices and generalized ultrametric matrices, see [NV1], [NV2]. Shifted generalized ultrametric matrices of a specific type are Z–matrices and have inverses which are again Z–matrices. Our examples for matrices in $\mathcal{L}_s$ will not be Z–matrices in general.

In 1972 Markham [Mar] defined the class of type D matrices

Definition 3.7 *$A = [a_{i,j}] \in \mathbb{R}^{n,n}$ is of* type D *if*

$$a_{ij} = \begin{cases} a_i, & i \leq j, \\ a_j, & i > j, \end{cases} \qquad \textit{where } a_n > a_{n-1} > \ldots > a_1.$$

Markham proved that the inverse of a type D matrix A, satisfying $a_1 > 0$, is a tridiagonal M–matrix. Later Johnson [J3] established that for $a_n < 0$ the inverse of a type D matrix is a N_0–matrix. Here we generalize these results. We will show that different type D matrices can lead to examples for matrices of all classes of inverse Z–matrices.

First we need the following observation and propositions.

Observation 3.8 *Let A be a nonsingular matrix of type D, then*

$$\xi_n^T A^{-1} = [a_1^{-1}, 0, \ldots, 0] \in \mathbb{R}^n. \tag{3.3}$$

Proof. The proof follows immediately from the special structure of A and the fact that $AA^{-1} = I$. □

Proposition 3.9 *Let $A \in \mathbb{R}^{n,n}$ be of type D partitioned as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{r,r}$ and $A_{22} \in \mathbb{R}^{n-r,n-r}$ for $1 \leq r < n$. If A_{11} is nonsingular the Schur complement A/A_{11} is again a matrix of type D and

$$A/A_{11} = A_{22} - a_r \xi_{n-r} \xi_{n-r}^T.$$

Proof. Obviously, A_{11} is also of type D. Now consider $A_{11}^{-1} A_{12}$. The columns of A_{12} are all the same and equal to the last row and column of A_{11}. Since $A_{11}^{-1} A_{11} = I$, we obtain

$$A_{11}^{-1} A_{12} = e_r \xi_{n-r}^T. \tag{3.4}$$

Therefore

$$A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12} = A_{22} - a_r\xi_{n-r}\xi_{n-r}^T,$$

which is of type D. □

Observation 3.10 *Let $A \in \mathbb{R}^{n,n}$ be of type D. Then*

$$det A = a_1 \prod_{i=2}^{n}(a_i - a_{i-1}). \tag{3.5}$$

Proof. If $a_1 = 0$, then $det A = 0$; thus (3.5) holds. Otherwise we have

$$det A = a_1 det(A/a_1).$$

But with Proposition 3.9 the Schur complement A/a_1 is again of type D matrix and its first diagonal entry is $a_2 - a_1$. Hence, applying an inductive argument, we obtain (3.5). □

Corollary 3.11 *Let $A \in \mathbb{R}^{n,n}$ of type D. Then $det A$ has the same sign as a_1.*

In the following theorem we show that the inverse of a nonsingular type D matrix is a Z–matrix for all sequences $a_n > \ldots > a_1$ in (3.3). Moreover, the number of nonpositive parameters a_i subtracted by one gives the class L_s to which the inverse belongs.

Theorem 3.12 *Suppose $A = [a_{ij}] \in \mathbb{R}^{n,n}$ is a matrix of type D with $a_1 \neq 0$. Let s denote the number of nonpositive parameters in the sequence $a_n > \ldots > a_1$. Then A^{-1} is a tridiagonal Z–matrix and*

$$A^{-1} \in L_{s-1},$$

where $L_{-1} := L_n$.

Proof. Since $a_1 \neq 0$, A is nonsingular. In the following we prove by induction on the dimension n that the inverse of A is a tridiagonal Z–matrix. For $n = 1$ and $n = 2$ this is obvious. Now we partition A as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

such that $A_{11} \in \mathbb{R}^{r,r}$ and $A_{22} \in \mathbb{R}^{n-r,n-r}$, $1 \leq r < n$, are nonsingular. The inverse of A is given by

$$A^{-1} = \begin{bmatrix} (A/A_{22})^{-1} & -A_{11}^{-1}A_{12}(A/A_{11})^{-1} \\ -(A/A_{11})^{-1}A_{21}A_{11}^{-1} & (A/A_{11})^{-1} \end{bmatrix}.$$

Proposition 3.9 and the inductive hypothesis yield that $(A/A_{11})^{-1}$ and $(A/A_{22})^{-1}$ are tridiagonal Z–matrices. As seen in (3.4) we have $A_{11}^{-1}A_{12} = e_r\xi_{n-r}^T$. Moreover, since the first diagonal entry of A/A_{11} is $a_{r+1} - a_r > 0$ we obtain with (3.3) that only the entry at position $(r, 1)$ of $-A_{11}^{-1}A_{12}(A/A_{11})^{-1}$ is different from zero and that this entry is negative. Then, with the symmetry of A, we have that A^{-1} is a tridiagonal Z–matrix.

If $s = 0$, A is nonnegative. Thus A^{-1} is an M–matrix. If $s \geq 1$, if follows from (3.5) that $detA < 0$. Moreover all principal minors of order greater than $n - s$ are nonpositive. However, the determinant of the principal submatrix consisting of the rows and columns $s + 1, \ldots, n$ is positive. But then Theorem 3.3 gives that $A^{-1} \in L_{s-1}$. □

Theorem 3.12 generalizes the results by Markham and Johnson mentioned above. If $a_1 > 0$ we have $s = 0$ and Theorem 3.12 says that the inverse is an M–matrix. If $a_n < 0$ we have $s = n$ and Theorem 3.12 gives that the inverse is in L_{n-1}, i.e. the class of N_0–matrices. Moreover, if $s = n-1$, the type D matrix is a Z–matrix with the positive diagonal entry a_n. This meets the structure of inverse F_0–matrices Johnson [J2] proved in general.

However, one can see from our examples that inverse Z–matrices are not in general Z–matrices. If $s = n - t$, we obtain a type D matrix which is an inverse Z–matrix with a $t \times t$ positive principal submatrix. On the other hand the examples for inverse L_s matrices given in [NV3] were Z–matrices.

Hence, one can not say something about the sign of the off–diagonal entries of an inverse Z–matrix in general, except for the classes of inverse M–matrices, inverse N_0–matrices, and inverse F_0–matrices. But we have

Theorem 3.13 *Let $A \in L_s$ be nonsingular and assume that there exists a nonsingular principal submatrix $A_{11} \in \mathbb{R}^{s+1,s+1}$ of A such that (A/A_{11}) is a Z–matrix. Then, $(A/A_{11})^{-1}$ is nonnegative, i.e there exists a nonsingular nonnegative principal submatrix of order $n-s-1$ of A^{-1}, if and only if A is strictly pure.*

Proof. First let us assume w.l.o.g. that the principal submatrix A_{11} is a leading principal submatrix. If we then partition $C := A^{-1}$ and A as in (3.1) with $C_{22} \in \mathbb{R}^{n-s-1,n-s-1}$, we have $C_{22} = (A/A_{11})^{-1}$. Since A/A_{11} is a Z–matrix and since $(A/A_{11})^{-1}$ is nonnegative, A/A_{11} is an M–matrix. With (2.5) we obtain that all principal minors of A of order greater or equal $s + 1$ are negative. Hence A is strictly pure. Now assume that A is strictly pure. Let $A_{11} \in \mathbb{R}^{s+1,s+1}$ be a leading principal submatrix of A. Then $(A/A_{11})^{-1} \in \mathbb{R}^{n-s-1,n-s-1}$ is a leading principal submatrix of C. But with Theorem 2.15 A/A_{11} is an M–matrix. Hence $(A/A_{11})^{-1}$ is nonnegative. □

Because of the symmetry and the special structure of type D matrices one can extend the statement of Theorem 3.2 for type D matrices.

Theorem 3.14 *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$ be a matrix of type D with $a_1 < 0$. Then A has exactly one negative eigenvalue.*

Proof. If a_1 is the only nonpositive diagonal entry of A, then A is an inverse N_0–matrix and the theorem is proved using Theorem 3.2. If there are more then one nonpositive diagonal entry one can find a leading principal submatrix A_{11} of A which is an inverse N_0–matrix. Let $\tilde{A}_{11}$ be the leading principal submatrix which order is one greater than the order of A_{11}.

Now, let λ_i and μ_j denote the eigenvalues of A_{11} and $\tilde{A}_{11}$ respectively, arranged in increasing order. We have $\lambda_1 < 0$ and $\lambda_2 \geq 0$. Cauchy's Interlacing Theorem, [HJ], gives

$$\mu_i \leq \lambda_i \leq \mu_{i+1}, \quad \text{for all } i.$$

Hence, $\tilde{A}_{11}$ does not have three or more negative eigenvalues. Moreover, since $a_1 < 0$ each leading principal minor of A is negative, thus $\tilde{A}_{11}$ does not have an even number of negative eigenvalues. Hence, $\tilde{A}_{11}$ has exactly one negative eigenvalue. Using an inductive argument, we conclude the proof. □

For completeness we should mention again that a type D matrix with $a_1 > 0$ has only positive eigenvalues. If $a_1 = 0$, the matrix is singular and has only nonnegative eigenvalues.

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