PROBABILITY MEASURES ON THE SPACE OF TOPOLOGICAL MEASURABLE FUNCTIONS

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spac . The teopolo$\mathcal{M}$n $_1(S)$ appears a e th ral tnen since it is the
coar estitopos ly for w chathmap $Q, \to Q($ i $QB)$), is$\mathcal{M}$tinuous for all
of V $(\nu)^{\chi}_{i}$; $\mathcal{M}$ $_{\mathfrak{q}}$) $\to S$ $\mathcal{M}$ with $,\nu := ()\nu$ $, \nu \ldots \quad {}^{n}_{n} \to \nu \otimes \nu$ under
Borel sets $\mathcal{B}$n $\mathcal{S}$, hand h ore does not no e no topology S
T see LDBs have $\mathcal{S}$, hand he exited and shdw n the sequence $\{X_i, i \in$
obeyin gsonteorm infin mcgndition) (see . [ ]
T eantt dois pape ist isdoss the large viated cipherfor $\{L_n \times L \quad , n \in$
^{2} ^{1} ^{n}
Inthe[2]ae .detainin pe inite (LDB) lo e,t ewspecorresponding to
$\mathcal{M}(q, \mathcal{S})$ muscovered wavebreak topologs $\mathcal{S}$ is S dwith the $_2\tau$-topology.
$\mathbb{N}\}$,the*pru tdn emptrie casum*, when $\mathcal{M}$d initr the mweak topology on
$\{f(\cdot, g(\cdot); f, \in C\}(\mathcal{S}$)onvergeme def lds $h \nu$ ell$\mathcal{M}_1(\quad)$, where $\otimes$
$\mathcal{M}_1(S$) ellc en the c i to y rod $\nu \mapsto o \nu \in^n$ rves ceMations mapping,
holdsof $L \in \mathbb{N}$ $\mathcal{S}$ Spli thesp ca lebind the collection
denoseupe ofoas ne sa Sure. Bsp rs p
thecland ipso ven. Bui tagpdoxr s meas ca eiscondhen us operation
$\mathbb{N}\}$ i $_2$reads zion ofaak Mo ain (see Bol]) sen a of aostationary process

this oserva^{b}t in Tp at bjsimethbd, it w. be ins y teanbeck the special
m mentcd nition (M fohethaw of $L_n d \times d$, r in etas diagonal terms
$\delta_{i,}\otimes$. M eover wa es l sh a LDP fou e h re crodopts. Therefore we
give sn e fr enmentsh f exponential ap xinp onnepti in LD Theory.
Tis oncept as boena b edh by Baxter 2 Jaid e]imProve a result of
Schmok [] 3which bn y aof interest ini th wn Actualy we prove our
resutsnotnl of for$\times L$ $_n$-type vartahlesi a slightly more general

A apaurtslar ap$_{\mathcal{T}}$ o l cme provea LDa r P fictsbnd for von, Mises-
$R(n)$.T hus ewill prD eoP-sLfor $\{R$, ne , $\in \mathbb{N}_1(S\mathcal{M}$ in e dowed with
^{2} ^{ij} ^{1} ^{i} ^{j}
^{2} ^{i} =
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a sIgh ty B reaEnzeall δ no Uf-stastatdcs i . u e
T s i g a p d a i r d i r i d t f v e s d c t i o n s i n S e n w e p r e s e n t t h e m a i n
results In e t h S i o n 3 w c e l a s o m e n t e s u l t s e a r g t i o n a l t h e o r y w h i c h g e n -
eralize result sIn [8] an [8 a] J e o v e r t w e s t o f f t h a e n g d f f e c o m p o s i t i o n . W e r
applicatiomof s our l r e s t s a o b t a i n L D P ' s t f -s d t i c a . i

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continuous tf tionsB in (eq Bppd with sup enum.m $\mathcal{M}$ (S) denotes
$_A$
ole $\mathcal{M}$ (S)p er Dlnitdtm be esbar(S $\mathcal{S}$ e $\mathcal{S}$ e Es $\mathcal{M}$ (S) wel $\mathcal{S}$ e
T ewealopology o $\mathcal{M}$ S) is d fnedr be go d bym φ (·), $\varphi \in C^{\infty}(S)$.
applyt heses uds b i to Dle LDaTheort ine s 4.iorDection 5 contains
fa ilin $\{\rho(\cdot, A) \in \mathcal{S}\}$ eso ines called the ind yebraS The r-topology
$\rho_\varphi(\mu = S \varphi d\mu$. Bet i ($\mathcal{SM}$ bd ehe o- bra g ($\mathcal{SM}$ enerated by the
B^w ( ) dnd the Bore e elga gdmentech e w polog. Since r makes
$\mathcal{M}$ (S)intoP olisp ea isP olish, $B(\mathcal{M}$ (μ)| φ S (see Demma 2.1,[5]).

$\in \nu \{ \mathcal{M}) : \int \varphi (\mu)^{<\tau}$

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page 5 8 lo d er t d s ea ines tr,ranotμf oun nitdeln is required.

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and th dAm etric. eVa $\|\mu\|$ toobe tr take t nia $\|\mu\|_{\text{var}}$, the usual
variatoin-nor for $\mathcal{M}$ S . We are now in heis f ondition ( ) in [9],

L t $\mathcal{P}$ e $\mathcal{M}$ ($\mathcal{M}$) From now ow ll e w dyscaswhich topology
on $\mathcal{M}$ (S) e prectivM( )Sis given= Fo r(ν r) $\in \mathcal{M}$ (S) let
La b e B S ish epad a $\mathcal{S}$ the Boreal σ eb-al write. BW S) for the set

h

be the *associated product measure* of $M_n(\nu)$ $\frac{1}{n}\sum$ = Moreover let

$$U(\nu := \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} \frac{\nu\nu \quad \nu_j{}^{i} + {}_{i} \otimes \nu^{j}}{2}$$

be t associated *symmetrized product* of e indts smeDenote by $\sim \in$

$\mathcal{M}(\mathcal{M}_1(S)$e spedtiv$_n$ ey $\mathcal{M}_1(\mathcal{M}_1(S^2)$l spec$_n$ $\in$ the $\mathcal{M}_1(\mathcal{M}_1(S^2))$ the $R(n)$ no $\mathcal{M}$ $(_1)S$. If $\mathcal{M}$ (isendowed wip het gy ovν) respectively distib uion $M_{(s)}$ rtt pe ely U (·yrespe: ely i derthe gi en measure U ( ) is $\mathcal{B}(\nu \mathcal{M}(S)) \mathcal{M}\mathcal{B}(\quad)$ measurable e, s n))$\mathcal{B}\mathcal{B}(S^2)$ and the

measurabilif tollows e mh ν) is clearl $(\mathcal{M}\ \mathcal{B}\mathcal{B}(\mathcal{M}_1(S^2))$ measurable and $V_n(\nu) = M_n(\nu) \otimes M_n(\nu)$ spac s aren eodowed wh hi wea tspol3g eey (orem[,3.2]). Thus the k

$$\frac{1}{2} \quad {}_{i_1}$$

$$(V \quad \frac{2}{n^2} 2U_n(\nu) + \otimes \qquad \nu\nu.$$

Let $X_n n \{ \in$ Nea n ue of rand dm var esi b apiity a space $(\Omega, \mathcal{A}, P)$

$A \in \mathcal{B}(X$:

be th eassac ted *pro al ptirical measu* m ly odefine $U(\delta\ )$ and

$L_n := M_n(\delta) = \frac{1}{n}\sum^{n}$ $\delta L_n \times L_n$) a $\delta(\mathcal{M}\ \mathcal{B}(S^2)$-measurable. where int Ae notes t hrior of A, with o pecte to the ology t on X and cl A the cl roe u By d ren ar tr functi r s to micsemicontinuous and has spac $(X, \mathcal{B}(\ ))$ satisfL e PD on X ewith o fmt : X i $[0, \infty)$ if for all recompact re se ts $\{\pm\ )(\ j$ for all $\} \in [$ e. 0∞. x i

T e left-o hs inequi ty alled the large iatie r bounds and the right-

m st sical the uppu und. oThe *weak* h d r e to responds to the

upper b oun d lding for yn t-compact $\mathcal{B}(X)$ An $\in$ $(h \{ \text{Var}x \le L\}$ required well know nc ta t $\mathcal{M}$ is continuous, if both to be ndy co-sed (wh hi qus valent to I ng oe wet-key micontinuous).

$$- \inf_{x \in \text{int}\ A} (\ x) \le \liminf \log P_n(A_n) \le \limsup_{n \to} A) P \le (- \inf_{x \in \text{cl}\ A} I \quad),$$

with stat a pe S. oDe tn $\delta = (\delta_{X}, \ldots, \delta_{X})$. L e

$$\times \mathbb{L}_n := V\ (\delta\ ) = \otimes \qquad \delta \qquad \delta_{X_i}$$

X n

N wwea ostnate oi m rsault:

Theorem 2.1 *A sequence $\{\tilde{R}_n, n \in \mathbb{N}\}$ satisfies a weak form of the LDP with rate function $I : \mathcal{M}_1(S) \to [0, \infty]$ if for all $\mathcal{B}(\mathcal{M}_1(S))$-measurable functions $V : S \to [0, \infty)$*

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E}\left[\exp\left(n \int_S V \, d\nu\right) \tilde{R}_n(d\nu)\right] \leq \sup_{\nu \in \mathcal{M}_1(S)} \left(\int_S V \, d\nu - I(\nu)\right) \quad \text{(M)}$$

for all $\mu \in \mathcal{M}_1(S)$. Here $\{\tilde{R}_n, n \in \mathbb{N}\}$ is called exponentially tight if $\{\nu \in \mathcal{M}_1(S) : \ldots\}$ satisfies a

with $\nu \in \mathcal{M}_1(S)$. Every σ-compact set $\ldots$ $\nu_i \in \mathcal{M}_1(S)$ $\ldots$

Remark 2.2 The easy to recognize that the statement $\ldots$ with $m \in \mathbb{N}$ we use the following $\ldots$

Let us now consider the problem of finding a good upper bound for the sequence $\{\tilde{R}_n, n \in \mathbb{N}\}$ of distributions R_n of L_n under Q^n, where Q^n denotes the n-fold product measure $Q \times \cdots \times Q$.

We recall how the definition of the relative entropy $H(\nu|\mu)$ of ν with respect to μ is related to Kullback-Leibler information. Let $\mathcal{M}_1(S)$ be equipped with the weak topology and $\mathcal{B}(\mathcal{M}_1(S))$ the corresponding Borel σ-field. Then $H(\nu|\mu) = \int \log \frac{d\nu}{d\mu} \, d\nu$ if $\nu \ll \mu$ and $H(\nu|\mu) = \infty$ otherwise. Furthermore, remember

$$H(\nu|\mu) = \sup_{\varphi \in C_b(S)} \left(\int \varphi \, d\nu - \log \int \exp \varphi \, d\mu \right) \quad (2.3)$$

is fulfilled, if the sequence $\{\ldots, n \in \mathbb{N}\}$ in $\mathcal{M}_1(\ldots)$ is $\ldots$ sequence. Therefore

Q

wi h $Q \in \mathcal{M}(\mathcal{M}_1(S))$ d n fi ee the r Legene Fe ne ns for m of Λ_Q by

$$\nu\tilde{I}_Q(=s\quad)\,p\,u\left\{\int_S \varphi(d\nu) - \in\Lambda C\quad ;\overset{Q}{\varphi}\ \varphi)\right\}_b(S \tag{2.4}$$

fo u $\nu\in\mathcal{M}($ $S.)$ Reme b m $(\mathcal{M})_* = wC_b(S)$, e re he s n e d eal topological

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Q

I f $\nu \in \mathcal{M}S$ (), we fi d h e

Q I $(u$:= $\{\inf_Q(\tilde{\nu}=:\nu\tilde{\nu}\otimes\in\tilde{\mathcal{M}}\quad,\ \tilde{\ }\quad S)\}$?(

The em. 2 5 $\{Q\in,\mathbb{N}$ *r espectively.* $\mathbb{N}Q$ $n \in$ *isfi s a* $\}$ *LDP with rate fu c in n I i t* $\mathcal{M}_1(S^2$ *n s o ded u ith th p l gs ro. Lo y is* τ*-lower semi-*

Co l ary. o 2 6 *Le t* Ω *be Equi pe wi u e hrt -ad d a* $\mathcal{A}$ *and let* X_n*;*

c n t n u s n i a the set $\nu\{\quad(S\ ):\Pi_Q(\nu)\ \overline{n}\ \}$ *is t i p d o e strong topology*

n 1 $\mathbb{N}$ 1

fo e a h c $L\ .\geq 0$

$\mathbb{N}$

n -1 1 X -1

so $\{\ n\ \mathbb{N}X\ \}$ *is a s e c n o f* $\mathcal{A}$*-S h m a s u r n , i n l n t. n d p s with common*

z a n i l D k i h e -t o g h s t i n g u n t f l a o p q

n $\in$ *N; e a p n r j e t i o s: F i e i n r e :=* μ*,* $\mu \in \mathcal{M}\ (S)$,

Remak r 2.7 . A we hm ea v n t o n e d i n t h e s i od n t i o n m t h e t weak-topology

setn g th i t s e a m e n t o o Th e m e 2.5 (a n d th l y u C h) i k a n o e a s y conclusion

of t e co n a t r i c o n p r i e i p. If l $\mathcal{M}$ $\in \mathcal{M}_1$ e (S) tes n t d d i s t r i b u t i o n of

H is τ*-lo w s e e m i-i c o n u n t u s a n d th e s l e v e t* $\mathcal{M}$ $\{s\ t(S^2) : H\ \nu) \leq L\}$ *are*

I . Th e o r e m 3 1.6 a n r C d l l a o y 3.1.7 a i n [9 p n] b c e d, p l e c a u s e the weak-

to p l o g y s t e n t g i g u a r a s n h a t $\mathcal{M}_1(\mathcal{M}$) a n d (S) th i f y h y p o t h e s i s (C) in

weak-co n t i n o u s u n o n $\mathcal{M}$ $(S_1$ e t) LDP of $\{Q$ o $\in_n \mathbb{N}n$ s h o w c o n t r a c t i o n, s e e h

e s p e i l d y i n c i e s th e P L D h e w e a k y to p o w . B g i l l e t r a n s f o r m the weak

n n

$\tilde{Q}$

n $\hat{\ }$

$M(\nu)$ u n d Q e $\tilde{\ }$, $\{Q$ $\mathbb{N}$ $\in$ r e s $\}$ t i s fi s o a w e f a r m k f D P e d i t h r a t e f u n c t i o n $Q\tilde{\ }$

fo e m a l p l e r e m e n a 4 2 9 , . M p r e c o v e r, n o t e a i t e th e s t e a h of the T e o r e m

[9]. W i h m e a k a f t e m L e a 3 2.7, e [9], w l t e a g P. S D d t e $\nu \mapsto \nu \otimes \nu$ is

d i s t r i b u t i o n. T h e n $\{L\quad P \times (L\quad)\quad, n\ \in t \mathbb{N}\quad e s p\}$ *r* $\{P e b y U i(\delta\quad)\quad, n\ \in \mathbb{N}\}$

n

$(\ H: \nu)n\quad = H\{\ \nu\ |\mu) =: \tilde{\nu}\nu \in \nu\mathcal{M}\quad \nu,\tilde{\ }\tilde{\ }\ (S)\ \}_1$ $(\quad\tilde{\ }$

result o th l e r s n g l t o i p o l c g e t t i n g.

First . Wasqe uences o ib odun y sures on spac . Two qe uences o ib odun y sures on are cabl ed p- omentia y l ivadentifi thea st pxity bihabes $(\Omega, \mathcal{B}, P)$ ion ians $\{P_n \in \mathbb{N}\}$ a vabinale $\mathbb{N}$. in and tw ued u es on a vabinale vari ab $\{\mathbb{E}[\cdot] \text{ resp eti ely with } \{Z\text{l s} \text{ and } \mathbb{E}_n, n \in \mathbb{N}\}$ wlar

Lemm a3: l2 et $\{(Z)\}$] 3 hes proof ind o $\{(d\text{-})\}$ eor Do e μn, $\text{and}\mathbb{N}(\mathbb{E}\text{-}d)$ be nmetric $\text{as}\mathbb{E}\text{m}$. ea $\text{fw}\mathbb{N}$ el > 0 and $j \in J$. If an $\mathbb{D}$ P i hw trate Is ol org $\{\mu \text{ mt} \in \mathbb{N}\}$ h r io tend e $\{\mu_n, n \in \mathbb{N}\}$ is

$$\{\tilde{\mu}_{\text{, }n}{}^{j}{}_{j}{}^{\dagger}\mathbb{N}\} \quad . \qquad \in \qquad \delta$$

$Y \setminus \text{cl} \quad \text{Ah} \quad \text{eexis} \in j_{y}\text{a} \quad J \; \text{dr} \quad > \text{lOtsuch} \qquad \text{t a}$

$$\text{such that fea ch } 0 > e \quad \text{t elis}\{(Z\Gamma, (\tilde{Z}_n) \in \quad \text{in.B and}$$

$$\text{T éollo lwg lemma} \quad \text{m .l uis} \rightarrow \text{Tl}(\text{oe})P_n(\infty \quad \text{VVe .at'aksequence holds an}$$

$$\text{LD} \quad \text{.wher.t s eqt to } d\text{n wh it} \text{ olds .an LD)} \quad \text{wher} \sigma(d=) \; ;(x, y), \; d \; ;) \; \text{gx } \delta \; (\text{t d} \; \text{x}) \; Y \quad \text{e}\{\text{s} \quad \cdot \quad 8'\}^{d}$$

Lem a 3. 1 *e d' b de ee dm metri on L*

$$d(x_{,}{}_{yc})_{\text{l}} \; \text{fo l rl } \alpha, \; \text{y} \in \quad \text{e}\mathbb{L}\{(Z \quad \tilde{,}Z_n))_{/} \quad (\text{d}\Gamma \quad \text{-mdsbrable for each}$$

$$\rightarrow \infty \qquad n \rightarrow \infty \qquad \qquad \qquad

The ε-rate f ction in ded fas $I(x) := \sup\{\varepsilon : x \notin A^{\varepsilon}\}$ and let $a^{\varepsilon} := \inf\{I(x) : x \in A^{\varepsilon}\}$. Assume that $a^{\varepsilon} > 0$ for some $\varepsilon > 0$. Let $\{\delta_y\}_{y \in Y}$ be a collection of positive numbers such that $\delta_y < \varepsilon$ and $I(y) - a^{\varepsilon} < \delta_y$ for all $y \in Y$. Then for each $y \in Y$ there exists $x \in A^{\varepsilon}$ such that $I(x) \le I(y) + \delta_y$. Since A^{ε} is open, there exists $\delta > 0$ such that $U(x, 2\delta) \subset A^{\varepsilon}$. Note that $d(y, x) > \varepsilon$ and thus $U(y, \delta) \cap U(x, \delta) = \emptyset$. Hence for each $y \in Y$ there exist $x \in A^{\varepsilon}$ and $\delta > 0$ such that $I(x) \le I(y) + \delta_y$ and $U(y, \delta) \cap U(x, \delta) = \emptyset$. Since Y is compact, there exists a finite subset E of Y such that $Y \subset \bigcup_{y \in E} U(y, \delta)$. Let $\tilde{A} := \bigcap_{y \in E} A^{\varepsilon}(y, \delta)$. Then $\tilde{A}$ is open and $\tilde{A} \subset A$. Moreover, $\inf_{z \in \tilde{A}} I(z) \le \min_{y \in E} I(y) + \min_{y \in E} \delta_y$. Since $\inf_{z \in \tilde{A}} I(z) \ge a^{\varepsilon}$ and $\min_{y \in E} \delta_y < \varepsilon - a^{\varepsilon}$, we have $\min_{y \in E} I(y) > a^{\varepsilon} - \min_{y \in E} \delta_y$. Therefore, $\min_{y \in E} I(y) > a^{\varepsilon} - (\varepsilon - a^{\varepsilon}) = 2a^{\varepsilon} - \varepsilon$. Hence $\inf_{z \in \tilde{A}} I(z) > a^{\varepsilon} - \varepsilon$. Since $\tilde{A}$ is open, $\inf_{z \in \tilde{A}} I(z) \le \liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(\tilde{A})$. Thus $\liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(\tilde{A}) > a^{\varepsilon} - \varepsilon$. Since $\tilde{A} \subset A$, $\liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(A) > a^{\varepsilon} - \varepsilon$. Since ε is arbitrary, $\liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(A) \ge a^{\varepsilon}$. Hence $\liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(A) \ge \sup_{\varepsilon > 0} a^{\varepsilon}$. Since $\sup_{\varepsilon > 0} a^{\varepsilon} = \inf_{x \in A} I(x)$, we have $\liminf_{n \to \infty} -\frac{1}{n} \log \mu_n(A) \ge \inf_{x \in A} I(x)$.

Not ht t e a

$$\{\tilde{Z}_n \in C A\}_n \left\{ \in Y \setminus \delta U(y \quad y, j, 2 \quad \in Y \setminus U(\cup \bigcup_{y \in E} \quad Z, \{ \quad (d_j \Gamma)) \right.$$

$$\left. \frac{Z}{n \to \infty} \quad n \{ \quad n \quad i \quad n^{c} \quad \delta \, m \quad y \quad n \right\}^{t} \quad y$$

$$\overline{\lim}_{n \to \infty} \frac{1}{n} \log \tilde{\mu}_n(A) \le \left\{ \lim_{\delta \to 0} \frac{1}{n} \log \mu_n(\quad ; \, m \quad \frac{1}{n} \log P \quad (\Gamma_\delta \quad (d)) \right\}. \quad i$$

$$\left. \bigcup \right\}^{t}$$

Because $\{\tilde{\mu}_n, n \in \mathbb{N}\}$ $\frac{1}{n}$ d δ y μ_n, $n \in \mathbb{N}\}$, $\delta_y > 0$ and j d su ht th y $U, 2\delta$ (j, y) $A \subset$ in h Not. ate t and th u th ppernbois du proven e (see pa , $\{$ y E lint A there exist $\{Z_n \in U \quad (y \quad \in \quad) \quad Z_n \in U(y, \quad Z \quad , \quad j \quad n) \in A(\delta \quad (d_{j_y})\}$,

thus w ge t $(U(y, \delta) \quad j_y \quad \tilde{\mu}_n(A) + P$ a (d $(d$ I) ence h $\{\tilde{\mu}_n, n \in \mathbb{N}\}$ and d $\{\mu_n, n \in \mathbb{N}\}$ e d-expo m ly ke quivalent n w tae o i

on $\mathcal{M}_1(S)$ i h w respect to d $_n$ hee ak ont to th ro n s g pge transformed to e ery d_v-ex p ntially qu ent sequen e aly ob th p asumi s on $\mathcal{M}_1(S)$.

$$\geq x \to \inf\{I y \quad x \in \quad , j_y, \delta\} \quad ($$

Thu pply m e a 3h T strong topolog t y i by e de basis elements Since T (d $\subset$ T (d $\forall$, e wet

Corl ary. 3 *Le Y $_1$ $(S, \mathcal{M}_S$ Polish, d a dri ind duced by the total variato in nor $\|\mu\|_{\mathcal{M}_1 S}$. Then a LDP n a se of p b ability measures*

$$(\geq -I \ y)$$

Proof. $\mathcal{M}_1(\ S)$ ndowe w th e wea y topol is g ïsh Pro e Lévy metric.

$_1$

$_y$ y t

$\sim$ $\sim$

$_y$

$\{\Psi \mathcal{M} \quad (\quad) : d_{\mathcal{P}}(\quad \mathcal{K}, \mu \quad \delta \}$

$\mu \in \mathcal{M}_1(S, \delta)$ > 0 a n umb P thro ough par nite ions it of a S of the form $\mathcal{P} = \{B_1, \ldots, B_m\} \in B_S$ $\mathcal{B}$; j (= 1n . . . , (m) d an d max = μ $|\nu(B_j) -$ $\mu(B_j)|$ Sin d c P (ν, μ) $\|\nu - \mu\|_\nu$ apply Lem Sm 2. a 3 of pr [13] for empirical mea uresep ces:)

Lem a3: {5 R_n, N d } $d(R_n, n \in \mathbb{N}$ p n *dv eetialequivalent, where y*

den ob tmsetnic t diu bdo the itatolnca; ion r d) $\|\cdot\|_{\mathrm{var}}$ $\mathcal{M}_1(S)$ ·

Proof. I suc ds to pr ta ot for n every $\varepsilon >$ $\lim_{n\to\infty} \frac{1}{n}$ $\log$ $\sum_{i=1}^{n}$ $\frac{1}{2}$ i

$$\frac{1}{n} \| U \quad (\nu \quad \frac{\binom{n}{n^2}}{n^2} \quad (\nu) \| \otimes \quad \sum \frac{1}{n} \nu \cdot = \nu_j \quad \frac{1}{} \tag{3.6}$$

n $\frac{1}{}$ $\frac{2}{}$

and th eefo r r

Thu of er se $\varepsilon > 0$ $R_n(R$)(($\|(i) \to V_n(\nu)\|$ f r) $(= \in \mathbb{N}$ large enough and

under R ($n \in \mathcal{M}$ ($\mathcal{M}$)) Sp Tn here

m n $\frac{1}{}$ $\frac{1}{}$ i

m $\frac{1}{\binom{n}{m}}$ $\sum^{p}$ i i 1

Not hat e

1 $1m$ 1 $\frac{\binom{n}{}}{(}$ $\sum$ i_i

ident iacl. $\limsup_{n} R(h) \|V (\nu) \sum \in (\nu)\|_n - = {}^{1}$ ∞

and p den o t the sum in m over all m! pe s at u . . . , (i i_m) of $(1, \ldots m)$.

$$f \quad \nu, \quad (, \nu_n) = i \otimes \quad 1 \quad \nu \quad , \quad \nu \cdot \cdot \cdot \otimes$$

m $m!$ m m

$$_n = (\nu) \quad m! U_{nm}(\nu) + \quad V \quad \nu \quad \otimes \nu \cdots \otimes \tag{3.8}$$

m n

We use t f hollowinul rg which is an iom fve p o The 6 rem 3.2.21 in [9].

wh e the sm u $\sum_r$ n ico sts ad realizatione ths aw two s to fl i , . . . , i_m being

the prof is v en. □

*Remakr 3.7*t . I t asy e o en gize 3 Le mm a m s to t ariat k case, that is, let

$\{R_n^m, n \in \mathbb{N}$ } esp ctiv R ye , $n\{ \in \mathbb{N}\}$, m t ∈ be ue s ed n of distributions

of V (ν) (s e R m 2 spe ctively of

Let b e P s Σ o h an d μ $\mathcal{M}_1(\Sigma)$ be fixed.

Theorem 3.9 *Let $\{P_n, n \in \mathbb{N}\}$ be a sequence of probability measures on $(\mathcal{M}(\Sigma), \mathcal{B}(\mathcal{M}(\Sigma)))$ which satisfies the following conditions:*

(i) *$\{P_n, n \in \mathbb{N}\}$ is exponentially tight and satisfies a weak form of the LDP with respect to the weak topology on $\mathcal{M}(\Sigma)$.*

(ii) *For all $\beta \in M_+(\Sigma)$ and all $\mathcal{B}(\Sigma)$ measurable, bounded functions $V : \Sigma \to \mathbb{R}$ the following limit exists:*

$$\lim_{n\to\infty} \frac{1}{n} \log \int_{\mathcal{M}(\Sigma)} \exp\left[n \int_\Sigma V \, d\nu \right] P_n(d\nu) = \Lambda(V) \quad (3.10)$$

and $\Lambda(V) \leq \sup_{\nu \in \mathcal{M}(\Sigma)} \left(\int_\Sigma V \, d\nu - I(\nu) \right)$.

Then $\{P_n, n \in \mathbb{N}\}$ satisfies the LDP with rate function I in the strong topology on $\mathcal{M}(\Sigma)$.

Proof. This is a restatement of Corollary 3.2.5 and Lemma 3.2.6 in [9]. These are valid under the stronger condition

the proof of the theorem follows the lines of the proof of Theorem 3.2.21 in [9]. Notice that in the proof of Theorem 3.2.21, [9], although the τ-topology ($\mathcal{M}(\Sigma)$ is a Polish space) is sometimes used, since $\mathcal{M}(\Sigma)$ is actually not necessarily separable (and $\mathcal{M}(\Sigma)$ is separable if Σ is countable) it is sometimes necessary to use the strong topology.

Remark 3.11 If $\{P_n, n \in \mathbb{N}\}$ satisfies (3.10) in the strong topology, then it does so in the weak topology. Check step by step that all arguments hold true under the weaker condition (3.10). This follows immediately since, thus, $\{P_n, n \in \mathbb{N}\}$ satisfies a LDP with respect to the weak topology on $\mathcal{M}(\Sigma)$. On the other hand if $\{P_n, n \in \mathbb{N}\}$ is exponentially tight, then the sequence is exponentially tight in the weak topology on $\mathcal{M}(\Sigma)$ which is a Polish space. That is true in the weak topology on $\mathcal{M}(\Sigma)$ is a local compact topological space. That

In our notation, let $X_n \in \mathbb{N}$ be an S valued exchangeable sequence with common law μ^m, $S \subset \mathbb{R}^m$ be measurable, $\mathbb{N}$ be measurable, $\mathbb{R}^m$ a complete separable metric vector space with norm $\|\cdot\|$, endowed with $\mathcal{B}$ the Borel σ-algebra. Let h be symmetric in its arguments. We need the following lemma, due to

$$\sup_{n} \int \exp\left[n \int V \, d\nu \right] P_n(d\mathcal{M}) \leq \exp\left[\beta V \right] d\mu.$$

Lem a 3. 4 *et* L

$$_mU\quad (h) = , X \frac{1}{(\frac{n}{m})} h(\sum_{1\ll\ll <\cdots<i} , \quad X \quad X_{i_m\cdot}),$$

The f o s n >0 r

$$\mathbb{E}\ (\exp \quad (h UX)\|_S) \leq \quad e\mathbb{E} \quad _S \|h\|\left(\frac{s}{k}\right)\Big)\Big)_{i_1}^{k}$$

$P\left(\quad n \right.$ m i $P\Big($ $\Big)$

Sk ch of th ce of p R re te U_n (h, X_i) as ave n dep d nt averages of

wh re $:=$ $\lfloor \cdot \rfloor$

$-$ 1 $km-m+1$

$$1 \qquad n \qquad i \qquad 1 \qquad i_1 \quad n \qquad n$$

$$1 \qquad 1$$

p o i s one □

By sy m $[U_m(x^m(h), X_i$ $=h\sum_{i_1,\ldots,i_m} W_m(\cdot)X_{i_1}\ldots,$ we $), X_p$ *d* *n* *h* *s* the sum] ma-
in the w ak p ological n l gs. If con M ditioni s re(m)ue if we replace R by

$+$

tion o v al ! l per n u s i $(i_n \ldots i_n)$ of $(B \quad \ldots$ ensem s i equality the d

$\exp$ n d $\int_{\mathcal{M})(1\ S} \nu(\ \nu) = \int_n$ ex V dp R $(d\nu)$

$$\int_R \qquad 1 \qquad n$$

R_n we get to fall $\mathcal{B}(S \quad \mathcal{B}(\ )$-M asurable f n ed n sc v c $S \to [0, \infty)$

$$\int \quad \left[\quad \right] \widetilde{R}_n \qquad n \widehat{\ } \int - \Big]_n \quad \sim \quad -$$

Th orem 3.h 9 et pr of o s e, n since y Co ol a L ^{2}) 3 ra S 3 ma e m 5 transform our
result to $\{R_n \quad n \in \mathbb{N}\}$.

m

where $V(x, :) = V(x, \cdot)$ l $(x, y) \in \mu S$ d and s en marginal distribution
of μ o n th res fi comm po te Thus the ex en po t ni s i of $\{R \quad , n \in \mathbb{N}\}$
i.id. rando m ariab e v is a $\leq p\, M$ $[\beta e \quad d\mu\, V]$

S 1 2

S

To verify n diti M f) R_i set
is pro en, e s ke ma k . a d, th refore $, n\{$ ob ys a LDP and via

$$- \qquad \int$$

Proof of Th o r M. l v check that $\{ m \quad n \in_n$ en s d is ies a full L P

$$-$$

$$F \nleq \nu \quad :) \quad V\,(x\,,\,y\,)\,d\nu\,(x\,,\in \qquad \text{fo}\,y) \quad (S\mathcal{M}$$

and

$$A(\ )F, \delta\{\ := :\ |F(V_n(\nu)) + \quad _n(\ \ \}.\geq \nu\delta)\ |$$

Th e we gen t tw u |V| a 2 M d, δ > 0: i

$$\int_{\mathcal{M}\ (S)} \text{p}\ \text{x}[F(\nu)]R\ \ (d\ \ ) \leq \int_{\mathcal{M}_1\ \ )} (\ \ \exp[\ ]\ \ (U_n\ R\ \ ]R(\nu))d(\nu_1, \ldots, \nu\ \ ) \qquad)$$

$$+ e(\ \ (\ R(\ )\ \ \ F)\ \delta A\ \ \ \ ) .$$

Now m sup $R(n)\ A(\ \ F\delta) \frac{n}{n} R|\ (\ (\ (F_n(\nu))\|\!>\ \ \ \ ) \quad R(n\ \ \ |h(\ )\ \nu\ \ )| > \qquad)$ eac r nd ha s condition (M) i

$$\frac{n\delta}{\ } \quad \frac{n\delta}{2}\left(\right) \quad -\frac{1}{n}\sum_{i=}^{n} \quad _j \quad \frac{n\delta}{2}\Big)$$

$$_j \quad _i \ \nu \quad \otimes\nu \ \frac{+\nu}{\ \ \ \ \ } \qquad \left(\ \ \ \ \ \right) \qquad \left(\ \ \right)$$

posi i ve θ we get

$$\leq n^2 \quad (n\|\ |\!> -n \ + nR^n\ \ ) \ |h(\ \ \ \frac{\delta}{\sim}| > \qquad h \qquad 2 \qquad 1$$

$$R(n) \quad |h \ \frac{n\delta}{>|} \leq \mathbb{E} -\theta\frac{n}{2}\text{p} \quad _{R(n \quad |h|)}(\theta\ \text{x}$$

$$\left(\ \ \ \right)\Big)\Big(\ \ \ \ \ \

where

$$K(\nu) := \iint \Big(\frac{V}{2}\Big)\, d(\mu_i \otimes \nu_i) = K(\nu_i), \quad \int_S \otimes d(\mu_i \nu_i) \Big).$$

Thu sing sa maa3.4 n (d ,) = K( ), w t e g

Since g is $\mathcal{M}(S^n)$ exp $\Big[\int \Big(\frac{1}{2}\Big)\, d(\mu_i \nu_i)\Big] \le 2$ $^2\big(\mathrm{e}^{i \cdot K_i(\mu)}\big)[\,/2]\cdot$

$$\mu_i Q(\Gamma) := \mathcal{M}\, {}^2 g(\nu_i;\nu_i()(\Gamma)) \quad \nu_i^S Q_\nu \in \mathcal{B}(S^n,\ ) \qquad)$$

$$\int_{n\in\mathbb{N}(S} \exp_{n} \int_{S^n} \int_{S^n} \Big[\int_V d\nu_i \Big] \bar{Q}\Big(\int_V d\nu_i\Big) \le \Big[\int_V \Big] \qquad {}^{[n/2]}$$

Hence w g e

Remarks 4.. 1

$$\Big(\int_{m}^{\mathcal{M}} (S \quad \Big[\int_{m}^{S^2} \Big)^{/} \int_{1\ n}^{2} \quad S \qquad {}^2$$

and th e ep Λ rs ea n pM ni (w Theorem (d ord u ii t e ou y. □

$$\text{p} \quad \text{u} \quad \text{e}_{R^n} \int_S \text{p} \quad W\, d\nu \Big] \bar{Q}^m \le \quad) \quad d \quad \exp\Big[\int_S V\Big] d\mu_Q^m,$$

of $V^m(S^n$ nd $\int_S$ is $\mathcal{M}(S^n$ by $\int_S$ a) m ble, rbounded, and)

$$Q \qquad (\mathcal{M}_S \quad) \qquad {}^1_m \qquad {}^m \qquad {}^m$$

and f efinedn s Remarkh3.7.dm 1 rti eion ofof $\{Q_n, n \in \mathbb{N}\}$

is wel ldefid e.Thus h Jy nse inequality

$$\text{esp i ve ly } \Big\{ \Big[Q \qquad \in \qquad \Big\} \quad \text{ur fec enote s the distribution}$$

$$X$$

(2)A rlc iteprcdof r G llacy 2.6hcannbd ont sn thi ease some argu-

esp dtely r

$$m \quad (\ ) \ \nu \ \text{ff} \quad \widetilde{H} \quad \nu \overline{+} \nu) : \tilde{\nu}, \mathcal{M} \quad , \tilde{\nu}\ HS) \}. \in \qquad (4.3)$$

(1)Rema k 3.7 lim yp t ur result ay e en bus fo m led to $\{Q \quad , n \in$

NW get

$$\Big[\qquad \Big)$$

$$n$$

sio . 2200 inm e a 2 0, . i ou v s

§5 APRLICA T O

Let X_n, $n \in \mathbb{N}$, be an i.i.d. sequence of S-valued random variables defined on a common probability space $(\Omega, \mathcal{A}, P)$, with common law μ and $h : S \times S \to \mathbb{R}$, $\mathbb{N} \in$, a measurable map symmetric in its arguments.

Denote by $\|\cdot\|_{L^p(\mu)}$ the L^p-norm on $\mathbb{R}$ and U_n as defined in Lemma 3.4, for $P \circ U_n^{-1}$ the n-th empirical measure. Under weak moment assumptions we derive a LDP for

Theorem 5.1 *Assume that*

$$\int_S e^{\theta \|h(x,\cdot)\|} \, d\mu(x) < \infty \quad \text{for all } \theta > 0.$$

Then $\{P \circ U_n^{-1}\}$ satisfies a LDP with rate function I given by

$$I(\varrho) := \inf_{\sigma \in \mathcal{M}_1(S)} \{ H(\varrho, \sigma) \mid F(\varrho) = F(\sigma) \}, \quad \varrho \in \mathcal{M}_1(S),$$

where $F(\varrho) := \int h \, d\varrho \otimes \varrho$, $\varrho \in \mathcal{M}_1(S)$, and H is as in (4.3).

$\mathcal{M}_1(S)$ is endowed with the topology of weak convergence. The proof is based on the following general setting. The proof is based on the following general setting. The proof is based on the following general setting

Th e fe oe ⊜ { ϱ G(r M)mSH (ϱ) ϱ∈ sL im } N∈ or a level set of m

H ·|μ(m). i cSen { ϱ(: Hμ)m|≤ N ϱ is ub rml f utesgl continuous with

respectto μf or ever> Ny , see Lemma 8 [11] , t e geo

unifor my o ev ϱ rwith(1) H≤ Iϱ∫S thus we h ave m

$$\int_S \left(\quad {}^{1}_{i} \quad {}^{r}_{n} \right) \quad {}^{r}_{1} \qquad {}^{m}_{n} \quad {}^{m}_{=} \quad {}^{1}_{,\ldots,i}$$

the asoscíatuēon Mlan - s stk. Assume t h(

Proof. Usn analogoc d mposition (3.8 V)mo inXtqrms of $U_n^m(h, X_i)$

and des K ∈ {e .e .}e yevnt Bcorrepma ∈ ℋ .Tm P∈ X ∈ Tm 2∈ ℕ Mrsatisfiee

a LD ith t eaU w

$$\frac{m}{\quad} \quad {}^{S} \qquad {}^{m}$$

$$Q \; {}^{\varrho}_{n} \; L \; [F(\quad)(\quad (\varrho) \| \geq \| \delta = P \quad (\; UX \;) \| \geq \delta \; .$$

Fina ll y Il s m{ up ∫‖h − hL‖ dϱ (ϱH = λ0} ≤

give . m n (□ L n m L)

Lemma 3.4 Ws t th a mponents code n in 5.3) and the proof is

Pr , ..., r ∈ ({1, . . .} m Set ℋ ds bda erations or h depending

exac l jon t mables . Ige ∈ ℋ iff

1 ≤ i≤ r . eWmera ef function in ℋ t a f s onoS c rather than a

Let ℙd e eothe s l e ddecomp ositions tse o. . . {ine} into r subsets

such that x =yi, if anl ny i∈ (P tio n Part a Pitiod 1 ≤ j ≤ m,

fu cmo S t .

*Remakr 5.*JU. $_n(x) = \frac{1}{m}$ HA $\varrho|\mu^m)$ wh $_m$ e e (

$A := A \cap A^2$
$:= \left\{ \varrho \in \mathcal{M}(\ ^m) : \int \ \varrho \ = x_d \right\} \cap \left\{ \varrho \in \mathcal{M}S\ S\ )_1\ ^m = \text{wi}\ \varrho\ \varrho\ \in \mathcal{M}\ (S) \right\}$.
$_1$ $_2$ $_1$ $_1$

closed). By ma 3.] 2 if $_1 \neq \emptyset$ isthere ts x A st hat $I_{\ }(x) =$ l
For to ad rmeft mr, A ds τ-clo $_x$ (A d kly ased and thus τ-
$_1$ $_x$ $_{1x}$

Theorem 3. f1 [7] $\in \mu_x$ A itw $_m$ $_n$ $_m$ U^m
$_m H\ \varrho\ |\mu)$ (. f thers ex tsi least one lw t $\in \nu A$ Hi h $|\mu\ ) < \infty$, by (
$d\ \mu^m$ $_x$ $_x$ d μ
$_m d\ \mu = c$ ex hp. $_1(\ , \frac{}{m}) >$ if ^{x} (0 othe wise),
$I(x) \geq |\mu\ H\ \mu)$.

minimizes $(H\ \mu\cdot)$ onT $_1$ hus for bounde i nde kons, we obtain

U_n x $_m$ $_m$

For $\nu = \nu\ (\ ,\ldots,\ e_n)$ ((S.M) l con sider

$(\nu\ _{1\cdot,}, \nu_m) : h(x\ \nu, x\ ) d\ \otimes (\ _1)(x_\nu; \cdot \otimes, x\ )$.
$_m$ $_m$ $_1$ $_n$ $_i$
If h is ot bo ded se o e eme 3.3 i [7]. $\frac{\sum}{h}$ $\left(\ \right)$

where $Q\ (\varrho^m : q\|F\ )(\ (\varrho)) \| \geq (\delta U = h Q - \ ^n \|, \nu\ ) \| \geq \delta$)
$\overline{h}$ Q S $\int$ $_1$ $_m$ $\left(\ ^n\right.$ ^{s} $_m$ $\left(\ \right.$ $_m$
$_1$

With $F(\varrho) = F: (\varrho, \overline{h})^1$ t wge e
$H\ \nu|\mu(\ ) \leq$ $(\beta\ (\tilde{\ })$ gs $M\ (o))$ 2 $\nu \in$ f $\mathcal{M}_1(S)$ ν som $\in$, [We ∞), choosen via
condition B.. 0 The rew re can imita te th oo e por Tfm o 5.1 and get the e
and

$\frac{}{n}$ χ_n ($\|$ pe(^m_n U s $(h, \nu_i Q)\|$)) $\leq$ EE $_m$ x $\|\Big)\frac{k}{k}\Big\|\frac{h}{L}$

$_1$ $_1$ Q Q

ϕ

for $k = \lfloor \frac{}{m} \rfloor$ (apt th fpe r of o Ioemme a 3d 4) $_Q$ ϱ tic N $\neq$ (∞) f for $\varrho \in \mathcal{M}\ (S^m)$
with $\varrho = \varrho^m_{\varrho}$ $\in (\mathcal{M}$ t), S h $\tilde{M}\ (\varrho)) = M$ e $(\varrho$ o see n Le (ov ma 3.2.19, [9])
$_1$ $_1$

$$F(\ \ U(\nu)) = \ \ ,\ \ \ _i(\ \ \ldots \nu\ \ : U) = (h, \nu\ ),$$
$_m$

$$\leq i_1 < \cdots < i\ \ \leq n$$

fo lowhig ge ma lized vn sie :o

Theem. 5 6 *Assum eEQ^m (exp(θ‖·‖ r ‖∞ l θ zal 0. Then {Q^n ∘ U^m(h , ν)^{−1} , ∈n ℕ} i sis sfta LDP with rat e*

$$\inf \quad I(\varrho), \qquad \in \mathbb{R} \qquad d$$

Remakr 5.7 . LA(= ϱ ; hd.ϱ ∫ = A}i = 0. s τ fored bounded kernel. The τsemi c inuityQa If dthe propertyp τ-cof levactnets { Q(ϱ) ≤ L} I

impliesinf{ I (ϱ), ϱϱ E =)(}x= I s(E) foE e E o1(SM) (see Lemma 3.2,

[11]. REFERENCE

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