Constructing large modules over artin algebras

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Let Λ be an artin algebra and denote by $\operatorname{mod}(\Lambda)$ the category of finitely generated Λ-modules. Apart from those we are also interested in Λ-modules which are not finitely generated. These are called large. Recall that the algebra Λ is of infinite representation type provided that $\operatorname{mod}(\Lambda)$ has infinitely many isomorphism classes of indecomposable objects. A theorem of Auslander asserts that there exists a large indecomposable Λ-module whenever Λ is of infinite representation type [2]. In this note we use his method to construct specific large indecomposable modules which arise naturally from certain infinite families of morphisms in $\operatorname{mod}(\Lambda)$ having isomorphic kernels. We apply this result as follows. Suppose there is given a chain

$$X_1 \xrightarrow{\psi_1} X_2 \xrightarrow{\psi_2} \ldots \xrightarrow{\psi_{s-1}} X_s \xrightarrow{\psi_s} \ldots$$

of monomorphisms between finitely generated indecomposable Λ-modules. Denote by $X_\infty = \varinjlim X_i$ the corresponding direct limit. It is natural to ask for conditions that ensure the indecomposability of X_∞. In fact, chains of monomorphisms of the above form occur for quasi-serial components of the Auslander-Reiten quiver of Λ. Ringel calls any module $X = X_s$ belonging to such a component quasi-serial of quasi-length s, since it determines, up to isomorphism, uniquely a chain of irreducible monomorphisms $\psi_i\colon X_i \to X_{i+1}$, $i \in \mathbb{N}$, such that there exists no irreducible monomorphism ending in X_1 [8]. We show that the corresponding direct limit $X_\infty = \varinjlim X_i$ is a large indecomposable module. Moreover, we obtain an exact sequence

$$0 \longrightarrow X \xrightarrow{\psi_X} X_\infty \longrightarrow (\operatorname{Tr} D)^s(X)_\infty \longrightarrow 0$$

where $(\operatorname{Tr} D)^s(X)$ denotes the s-th power of the transpose of the dual of X which is again a quasi-serial module. This exact sequence has the following property. Every morphism $\theta\colon X \to Y$ in $\operatorname{mod}(\Lambda)$ belongs to the infinite radical of $\operatorname{mod}(\Lambda)$ if and only if θ factors through ψ_X, and moreover, θ induces a commutative diagram of the following form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\psi_X} & X_\infty & \longrightarrow & (\operatorname{Tr} D)^s(X)_\infty & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\theta} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \xrightarrow{\psi_Y} & Y_\infty & \longrightarrow & (\operatorname{Tr} D)^t(Y)_\infty & \longrightarrow & 0
\end{array}$$

provided that Y is quasi-serial of quasi-length t.

We would like to point out as an application that modules of the form X_∞ can be used to construct generic modules in the sense of Crawley-Boevey [5]. This will be discussed in a forthcoming publication. Note that the existence of a generic module is crucial for the validity of the second Brauer-Thrall conjecture for artin algebras [5].

The construction

Let Λ be an artin algebra and denote by $\mathrm{Mod}(\Lambda)$ the category of right Λ-modules. Given two morphisms $\varphi_i: X \to Y_i$, $i = 1, 2$, in $\mathrm{Mod}(\Lambda)$ we define $\varphi_2 \preceq \varphi_1$ if there exists a morphism $\psi: Y_1 \to Y_2$ such that $\varphi_2 = \psi \circ \varphi_1$. We write $\varphi_2 \prec \varphi_1$ if $\varphi_2 \preceq \varphi_1$ and $\varphi_1 \not\preceq \varphi_2$. Recall that a morphism $\varphi: X \to Y$ in $\mathrm{mod}(\Lambda)$ is *irreducible* provided that φ is neither a split monomorphism nor a split epimorphism and if for any factorization $\varphi = \varphi_2 \circ \varphi_1$ in $\mathrm{mod}(\Lambda)$ either φ_1 is a split monomorphism or φ_2 is a split epimorphism.

Theorem 1 *Let* $0 \to X \xrightarrow{\varphi_i} Y_i \xrightarrow{\chi_i} Z_i \to 0$, $i \in \mathbb{N}$, *be a family of exact sequences in* $\mathrm{mod}(\Lambda)$. *Suppose that* χ_i *is irreducible and that* $\varphi_{i+1} \prec \varphi_i$ *for all* $i \in \mathbb{N}$. *Then there exists a family of morphisms* $\psi_i: Y_i \to Y_{i+1}$ *satisfying* $\varphi_{i+1} = \psi_i \circ \varphi_i$ *for all* $i \in \mathbb{N}$ *such that* $Y = \varinjlim Y_i$ *is a large denumerably generated indecomposable module.*

We postpone a proof and discuss some preliminaries. We shall use the category $(\mathrm{mod}(\Lambda), \mathrm{Ab})$ of additive functors from $\mathrm{mod}(\Lambda)$ to the category Ab of abelian groups. Note that for a pair of morphisms $\varphi_i: X \to Y_i$, $i = 1, 2$, in $\mathrm{mod}(\Lambda)$ we have $\varphi_2 \preceq \varphi_1$ iff $\mathrm{Im}(\mathrm{Hom}_\Lambda(\varphi_2, \ )) \subseteq \mathrm{Im}(\mathrm{Hom}_\Lambda(\varphi_1, \ ))$ as subfunctors of $\mathrm{Hom}_\Lambda(X, \ )$. We shall also use the fully faithful functor

$$\mathrm{Mod}(\Lambda) \to (\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab}), \quad X \mapsto X \otimes_\Lambda -$$

which identifies the pure-injective Λ-modules with the injective objects in $(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$. Note that $\mathrm{Ext}^1(\ , X \otimes_\Lambda -)$ vanishes on finitely presented functors for every Λ-module X. Moreover, every finitely generated Λ-module is automatically pure-injective. The following characterization of an irreducible morphism in $\mathrm{mod}(\Lambda)$ is needed.

Lemma 2 *For a non-split exact sequence* $0 \to X \xrightarrow{\varphi} Y \xrightarrow{\chi} Z \to 0$ *in* $\mathrm{mod}(\Lambda)$ *the following are equivalent.*

(1) χ *is irreducible.*

(2) $\varphi \preceq \theta$ *or* $\theta \preceq \varphi$ *for each morphism* $\theta: X \to Y'$ *in* $\mathrm{mod}(\Lambda)$.

(3) $\varphi \preceq \theta$ *or* $\theta \preceq \varphi$ *for each morphism* $\theta: X \to Y'$ *in* $\mathrm{Mod}(\Lambda)$.

Proof: (1) $\Leftrightarrow$ (2) See [3, Proposition 2.7].

(2) $\Leftrightarrow$ (3) One direction is trivial. Therefore suppose (2). We apply the well-known duality $\mathrm{fp}(\mathrm{mod}(\Lambda), \mathrm{Ab}) \to \mathrm{fp}(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ between the categories of finitely presented functors which sends $\mathrm{Hom}_\Lambda(X, \ )$ to $X \otimes_\Lambda -$ (e.g. see [6]). It follows from this duality that (2) is equivalent to the property that any subobject G of $X \otimes_\Lambda -$ in $(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ either contains or is contained in $F = \mathrm{Ker}(\varphi \otimes_\Lambda -)$. Taking $G = \mathrm{Ker}(\theta \otimes_\Lambda -)$ for a morphism $\theta: X \to Y'$ we obtain either a morphism $Y \otimes_\Lambda - \to Y' \otimes_\Lambda -$ if $F \subseteq G$ since $\mathrm{Ext}^1(Z \otimes_\Lambda -, Y' \otimes_\Lambda -) = 0$, or we obtain a morphism $Y' \otimes_\Lambda - \to Y \otimes_\Lambda -$ if $G \subseteq F$ since $\mathrm{Ext}^1(\ , Y \otimes_\Lambda -) = 0$. Thus we have $\varphi \preceq \theta$ or $\theta \preceq \varphi$.

Let $\mathcal{C}$ be an abelian category and $F: \mathcal{C} \to \mathrm{Ab}$ be an additive functor. A non-zero element $x \in F(X)$ is called *minimal* if for each $\varphi: X \to Y$ in $\mathcal{C}$ the property $F(\varphi)(x) \neq 0$ implies that φ is a monomorphism. The following facts are easily verified.

Lemma 3 (1) *If* $x \in F(X)$ *is minimal, then* X *is indecomposable.*

(2) *If* X *is a noetherian object in* $\mathcal{C}$ *and* $x \in F(X)$ *is a non-zero element, then there exists an epimorphism* $\varphi: X \to Y$ *such that* $F(\varphi)(x) \in F(Y)$ *is minimal.*

Proof: See [1, Lemma 3.2].

Proof of Theorem 1: Let $F = \mathrm{Hom}_\Lambda(X, \ )/\cap_{i \in \mathbb{N}} \mathrm{Im}(\mathrm{Hom}_\Lambda(\varphi_i, \ ))$ and denote by $\pi: \mathrm{Hom}_\Lambda(X, \ ) \to F$ the canonical morphism in $(\mathrm{mod}(\Lambda), \mathrm{Ab})$. Using the preceding lemmas one constructs easily a family of morphisms $\mu_i: X \to W_i$, $i \in \mathbb{N}$, together with a family of natural numbers $1 = n_0 < n_1 < \ldots$ such that

(i) $\pi(\mu_i) \in F(W_i)$ is minimal for all $i \in \mathbb{N}$;

(ii) $\varphi_{n_i} \prec \mu_i \preceq \varphi_{n_i - 1}$ for all $i \in \mathbb{N}$.

Applying (ii) we obtain the following chain of morphisms.

$$Y_1 \xrightarrow{\psi_1} \ldots \xrightarrow{\psi_{n_1-2}} Y_{n_1-1} \longrightarrow W_1 \longrightarrow Y_{n_1} \xrightarrow{\psi_{n_1}} \ldots \xrightarrow{\psi_{n_2-2}} Y_{n_2-1} \longrightarrow W_2 \longrightarrow Y_{n_2} \xrightarrow{\psi_{n_2}} \ldots$$

Taking the corresponding compositions in this chain we obtain morphisms $\psi_i\colon Y_i \to Y_{i+1}$ and $\nu_i\colon W_i \to W_{i+1}$, $i \in \mathbb{N}$, such that

(iii) $\varphi_{i+1} = \psi_i \circ \varphi_i$ and $\mu_{i+1} = \nu_i \circ \mu_i$ for all $i \in \mathbb{N}$;

(iv) $Y = \varinjlim Y_i$ and $W = \varinjlim W_i$ are isomorphic.

To show that Y is indecomposable we extend the functor $F\colon \mathrm{mod}(\Lambda) \to \mathrm{Ab}$ to a functor $F'\colon \mathrm{Mod}(\Lambda) \to \mathrm{Ab}$ which commutes with direct limits and denote by $\pi'\colon \mathrm{Hom}_\Lambda(X, \ ) \to F'$ the corresponding morphism between functors from $\mathrm{Mod}(\Lambda)$ to Ab. Using (i) it is not hard to check that the morphism $\mu = \varinjlim \mu_i$ has the property that $\pi'(\mu) \in F'(W)$ is minimal. We refer to the proof of [2, Theorem 1.5] for details and conclude that Y is indecomposable.

Our next aim is to discuss some properties of the morphism $\varphi = \varinjlim \varphi_i\colon X \to Y$ which arises in Theorem 1. We will use the following general fact.

Proposition 4 *Let $\varphi_i\colon X \to Y_i$, $i \in I$, be a directed family of morphisms in* $\mathrm{Mod}(\Lambda)$ *and denote by $\varphi = \varinjlim \varphi_i\colon X \to Y$ its direct limit. Suppose also that $Y' \in$* $\mathrm{Mod}(\Lambda)$ *is pure-injective. Then a morphism $\theta\colon X \to Y'$ factors through φ if and only if θ factors through φ_i for all $i \in I$.*

Proof: One direction is clear. Therefore suppose that θ factors through φ_i for all $i \in I$. We obtain for each i an exact commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & F_i & \longrightarrow & X \otimes_\Lambda - & \longrightarrow & Y_i \otimes_\Lambda - \\
 & & \big\downarrow & & \big\| & & \big\downarrow \\
0 & \longrightarrow & G & \longrightarrow & X \otimes_\Lambda - & \longrightarrow & Y' \otimes_\Lambda -
\end{array}$$

and therefore also an exact commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \varinjlim F_i & \longrightarrow & X \otimes_\Lambda - & \longrightarrow & Y \otimes_\Lambda - \\
 & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & G & \longrightarrow & X \otimes_\Lambda - & \longrightarrow & Y' \otimes_\Lambda -
\end{array}$$

since direct limits in $(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ are exact. Using the injectivity of $Y' \otimes_\Lambda -$ there exists a morphism $Y \otimes_\Lambda - \to Y' \otimes_\Lambda -$ making the above diagram commutative. Thus the assertion follows.

Corollary 5 *Let* $0 \to X \xrightarrow{\varphi_i} Y_i \xrightarrow{\chi_i} Z_i \to 0$, $i \in I$, *be a directed family of exact sequences in* $\mathrm{mod}(\Lambda)$ *and suppose that* χ_i *is irreducible for all* $i \in I$. *Suppose also that* $\theta: X \to Y'$ *is a morphism in* $\mathrm{Mod}(\Lambda)$ *where* Y' *is pure-injective. Then the direct limit* $\varphi = \varinjlim \varphi_i: X \to Y$ *has the following properties.*

(1) θ *factors through* φ *if and only if* θ *factors through* φ_i *for all* $i \in I$.

(2) *Either* θ *factors through* φ *or* φ *factors through* θ.

Proof: Combine Proposition 4 and Lemma 2.

Quasi-serial modules

Families of irreducible epimorphisms having isomorphic kernels arise quite often for artin algebras of infinite representation type. In fact one may ask whether any artin algebra of infinite representation type admits such a family. We devote the second half of this paper to a class of finitely generated modules where such families naturally occur. To this end recall the following concept which was introduced in [8]. A component $\mathcal{C}$ of the Auslander-Reiten quiver of Λ is called *quasi-serial* if

(i) $\mathcal{C}$ contains neither projective nor injective modules;

(ii) if $0 \to X \to \coprod_{i=1}^{k} Y_i \to Z \to 0$ is an almost split sequence of modules in $\mathcal{C}$, then $k \leq 2$ and $k = 2$ with $\ell(Y_1) \leq \ell(Y_2)$ implies that $\ell(Y_1) < \ell(X) < \ell(Y_2)$.

Examples of such components are the regular components over a hereditary artin algebra. An indecomposable module X belonging to a quasi-serial component is called *quasi-serial*, and X is *quasi-simple* if there exists no irreducible monomorphism ending in X. We shall use the following properties of a quasi-serial module.

Lemma 6 *To each quasi-serial module* X *corresponds a unique family* $(X_i)_{i \in \mathbb{N}}$ *of indecomposables such that*

(1) X_1 *is quasi-simple and* $X = X_s$ *for some* $s \in \mathbb{N}$;

(2) *there exists an irreducible monomorphism* $\psi_i: X_i \to X_{i+1}$ *for all* $i \in \mathbb{N}$.

Proof: See [8].

The number $s \in \mathbb{N}$ with $X = X_s$ is called the *quasi-length* of X and it is convenient to put $X_0 = 0$.

Lemma 7 *Let X be a quasi-serial module of quasi-length s and let $0 \leq i \leq s$. Then there is an exact sequence $0 \to X_i \stackrel{\psi_{s-1}\circ\ldots\circ\psi_i}{\to} X_s \to (\mathrm{Tr}\,D)^i(X_{s-i}) \to 0$. If $1 = i < s$, then the induced morphism $\chi_s \colon X_s \to \mathrm{Tr}\,D(X_{s-1})$ is irreducible.*

Proof: Straightforward.

Each quasi-simple module determines uniquely a large indecomposable module in the following sense.

Theorem 8 *Let X be a quasi-simple module. Then there exists a chain*

$$X = X_1 \xrightarrow{\psi_1} X_2 \xrightarrow{\psi_2} \ldots$$

of irreducible monomorphisms such that each X_i is indecomposable and $X_\infty = \varinjlim X_i$ is a large denumerably generated indecomposable module. The module X_∞ depends, up to isomorphism, only on X. If Y is a second quasi-simple module, then $Y_\infty \cong X_\infty$ if and only if $Y \cong X$.

Proof: The existence of a chain $X = X_1 \stackrel{\psi_1}{\to} X_2 \stackrel{\psi_2}{\to} \ldots$ of irreducible monomorphisms follows from Lemma 6. For each $i > 1$ define $\varphi_i = \psi_{i-1}\circ\ldots\circ\psi_1$. We obtain a family $0 \to X \stackrel{\varphi_i}{\to} X_i \stackrel{\chi_i}{\to} \mathrm{Tr}\,D(X_{i-1}) \to 0$ of exact sequences where χ_i is irreducible for all i by Lemma 7. Thus the existence of a large denumerably generated indecomposable module $X_\infty = \varinjlim X_i$ follows from Theorem 1 and it remains to show its uniqueness. First one observes that the X_i's are uniquely determined by X and that any morphism $\psi_i' \colon X_i \to X_{i+1}$ satisfying $\varphi_{i+1} = \psi_i'\circ\varphi_i$ is irreducible. The latter guarantees that the module $\varinjlim X_i$ constructed in Theorem 1 is actually a limit where each morphism $X_i \to X_{i+1}$ is irreducible. Now assume an arbitrary choice of irreducible morphisms $\psi_i' \colon X_i \to X_{i+1}$. It follows from [8, 4.1] that there exist isomorphisms $\mu_i \colon X_i \to X_i$ satisfying $\mu_{i+1}\circ\psi_i = \psi_i'\circ\mu_i$ for all i. Taking direct limits the morphism $\varinjlim \mu_i$ gives an isomorphism between $\varinjlim X_i$ taken over the ψ_i's and $\varinjlim X_i$ taken over the ψ_i''s. Finally suppose that Y is a second quasi-simple such that $Y_\infty \cong X_\infty$. The morphism $\varphi = \varinjlim \varphi_i \colon X \to X_\infty$ composed with the isomorphism $X_\infty \to Y_\infty$ factors through some Y_j since X is finitely generated. We denote the corresponding morphism by $\chi \colon X \to Y_j$. It is not hard to check that there exists an isomorphism $\nu \colon X_j \to Y_j$ such that $\chi = \nu\circ\varphi_j$ and therefore $X \cong Y$.

Let C be the center of the artin algebra Λ and denote by I an injective envelope of $C/\operatorname{rad}(C)$. Using the functor $D = \operatorname{Hom}_C(\ ,I)$ between $\operatorname{Mod}(\Lambda)$ and $\operatorname{Mod}(\Lambda^{\mathrm{op}})$ which induces a duality between $\operatorname{mod}(\Lambda)$ and $\operatorname{mod}(\Lambda^{\mathrm{op}})$ we obtain the following consequence of Theorem 8.

Corollary 9 *Let X be a quasi-simple module. Then there exists a chain*

$$\ldots \xrightarrow{\psi^2} X^2 \xrightarrow{\psi^1} X^1 = X$$

of irreducible epimorphisms such that each X^i is indecomposable and $X^\infty = \varprojlim X^i$ is a large module which depends, up to isomorphism, only on X.

Proof: The Λ^{op}-module $Y = D(X)$ is quasi-simple and admits a chain $Y = Y_1 \xrightarrow{\varphi_1} Y_2 \xrightarrow{\varphi_2} \ldots$ of irreducible monomorphisms. Applying D we obtain a chain $\ldots \xrightarrow{D(\varphi_2)} D(Y_2) \xrightarrow{D(\varphi_1)} D(Y_1) \cong X$ of irreducible epimorphisms with $X^\infty = \varprojlim D(Y_i) \cong D(\varinjlim Y_i) = D(Y_\infty)$.

Remark 10 We will show in a subsequent paper that the module X_∞ is Σ-pure-injective provided that $(\operatorname{Tr} D)^n(X) \cong X$ for some $n \in \mathbb{N}$. Of course, the module X^∞ is pure-injective since it is of the form $D(Y)$ for some Λ^{op}-module Y. We do not know of any example where the module X^∞ decomposes.

Let X be a quasi-serial module of quasi-length s and fix a chain $X = X_s \xrightarrow{\psi_s} X_{s+1} \xrightarrow{\psi_{s+1}} \ldots$ of irreducible monomorphisms such that X_i is indecomposable for all $i > s$. We denote by $\psi_X\colon X \to \varinjlim X_i = X_\infty$ the induced monomorphism. The module X_∞ depends only on X according to Lemma 6 and Theorem 8. Moreover, the morphism ψ_X is unique in the sense that for any different choice of irreducible monomorphisms $\psi_i'\colon X_i \to X_{i+1}$, $i \geq s$, there exists an isomorphism $\mu\colon X_\infty \to X_\infty$ with $\psi_X' = \mu \circ \psi_X$. Finally note that ψ_X induces an exact sequence

$$0 \longrightarrow X \xrightarrow{\psi_X} X_\infty \longrightarrow (\operatorname{Tr} D)^s(X)_\infty \longrightarrow 0.$$

This follows from Lemma 7. Analogously, there is an induced epimorphism $\psi^X\colon X^\infty \to X$ for $X^\infty = \varprojlim X^i$ which is part of an exact sequence

$$0 \longrightarrow (D \operatorname{Tr})^s(X)^\infty \longrightarrow X^\infty \xrightarrow{\psi^X} X \longrightarrow 0.$$

We proceed with a discussion of the morphisms ψ_X and ψ^X. For each $n \in \mathbb{N}$ and each pair of modules X, Y in $\operatorname{mod}(\Lambda)$ we denote by $\operatorname{rad}^n(X,Y)$ the morphisms in $\operatorname{Hom}_\Lambda(X,Y)$

which belong to the n-th power of the Jacobson radical of $\mathrm{mod}(\Lambda)$. As usual $\mathrm{rad}^\infty(X,Y) = \bigcap_{n\in\mathbb{N}} \mathrm{rad}^n(X,Y)$. Using the above notation we have the following lemma.

Lemma 11 *Let $X = X_s$ be a quasi-serial module and Y be indecomposable. If $\varphi \in \mathrm{rad}^n(X,Y)$ and $n \geq s$, then there exists a morphism $\chi \in \mathrm{rad}^{n-s}(X_{s+1}, Y)$ such that $\varphi = \chi \circ \psi_s$.*

Proof: First suppose that $n = s$. For each i, $0 \leq i \leq s$, there is an exact sequence $0 \to X_i \xrightarrow{\psi_{s-1}\circ\ldots\circ\psi_i} X \xrightarrow{\pi_i} Z_i \to 0$ with $Z_i = (\mathrm{Tr}\, D)^i(X_{s-i})$ according to Lemma 7. Let i be maximal such that there are morphisms $\alpha\colon X_{s+1} \to Y$ and $\beta\colon Z_i \to Y$ with $\varphi = \alpha\circ\psi_s + \beta\circ\pi_i$. The assertion follows if $i = s$ since $\pi_s = 0$. Therefore assume that $i < s$. Using the property of the almost split sequence starting in Z_i we deduce that β is an isomorphism. Thus $\varphi \notin \mathrm{rad}^{i+1}(X,Y)$ since $\pi_i \notin \mathrm{rad}^{i+1}(X, Z_i)$ by [7, Theorem 13.3] and $\mathrm{Hom}_\Lambda(X_{s+1}, Z_i) = \mathrm{rad}^{i+1}(X_{s+1}, Z_i)$. This contradiction finishes the first part of our proof. Now suppose that $n \geq s$. There are morphisms $\varphi_i\colon X \to Y_i$ and $\chi_i \in \mathrm{rad}(Y_i, Y_{i+1})$ for each $i \in \mathbb{N}$ such that $\mathrm{Im}(\mathrm{Hom}_\Lambda(\varphi_i, \ )) = \mathrm{rad}^i(X, \ )$ and $\varphi_{i+1} = \chi_i\circ\varphi_i$ for all $i \in \mathbb{N}$ [4, V, Lemma 7.10]. Using the first part we find a morphism $\chi_0\colon X_{s+1} \to Y_s$ such that $\varphi_s = \chi_0\circ\psi_s$. Thus $\varphi_n = \chi_{n-1}\circ\ldots\circ\chi_s\circ\chi_0\circ\psi_s$ with $\chi_{n-1}\circ\ldots\circ\chi_s\circ\chi_0 \in \mathrm{rad}^{n-s}(X_{s+1}, Y_n)$ and the proof is complete.

We are now in position to prove the following property of the morphism $\psi_X\colon X \to X_\infty$.

Theorem 12 *Let $\varphi\colon X \to Y$ be a morphism between finitely generated indecomposable modules and suppose that X is quasi-serial.*

(1) *$\varphi \in \mathrm{rad}^\infty(X,Y)$ if and only if there exists a morphism $\chi\colon X_\infty \to Y$ such that $\varphi = \chi\circ\psi_X$.*

(2) *If $\varphi \notin \mathrm{rad}^\infty(X,Y)$, then there exists a morphism $\chi\colon X_\infty \to Y_\infty$ such that $\psi_Y\circ\varphi = \chi\circ\psi_X$.*

Proof: (1) The inclusion $\mathrm{Im}(\mathrm{Hom}_\Lambda(\psi_X, Y)) \subseteq \mathrm{rad}^\infty(X,Y)$ is obvious from the definition of ψ_X. Therefore let $\varphi \in \mathrm{rad}^\infty(X,Y)$. We fix a chain $X_1 \xrightarrow{\psi_1} X_2 \xrightarrow{\psi_2} \ldots$ of irreducible monomorphisms with X_1 quasi-simple and $X = X_s$ for some $s \in \mathbb{N}$. It follows from the previous lemma that for all $i \in \mathbb{N}$ any morphism $\chi_i \in \mathrm{rad}^\infty(X_i, Y)$ can be extended to a morphism $\chi_{i+1} \in \mathrm{rad}^\infty(X_{i+1}, Y)$ such that $\chi_i = \chi_{i+1}\circ\psi_i$. Here one uses the fact that $\mathrm{rad}^\infty(X_{i+1}, Y) = \mathrm{rad}^n(X_{i+1}, Y)$ for some $n \in \mathbb{N}$. Taking $\chi_s = \varphi$ we obtain a morphism $\chi = \varinjlim \chi_i$ which has the desired property.

(2) Similar to (1).

Using again the functor $D = \mathrm{Hom}_C(\ ,I)$ we obtain the analogous property of the morphism $\psi^X\colon X^\infty \to X$.

Corollary 13 *Let $\varphi\colon Y \to X$ be a morphism between finitely generated indecomposable modules and suppose that X is quasi-serial.*

(1) *$\varphi \in \mathrm{rad}^\infty(Y,X)$ if and only if there exists a morphism $\chi\colon Y \to X_\infty$ such that $\varphi = \psi^X \circ \chi$.*

(2) *If $\varphi \notin \mathrm{rad}^\infty(Y,X)$, then there exists a morphism $\chi\colon Y^\infty \to X^\infty$ such that $\varphi \circ \psi^Y = \psi^X \circ \chi$.*

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